A CLASS OF COLLOCATION HYBRID BACKWARD DIFFERENTIATION METHOD FOR INITIAL VALUE PROBLEMS OF ORDINARY DIFFERENTIAL EQUATIONS

BY

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CERTIFICATION

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DEDICATION

This work is dedicated to my darling wife, Mrs. Nurat Ibidunni Ajileye and my beloved children – Semiu, Latifat, Rafiat, and Waheed.

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ABSTRACT

The development of linear multistep methods for initial value problems of ordinary differential equations (ODEs) has been the subject of investigation for some time. In this work, a class of hybrid backward differentiation methods with step numbers k = 1, 2, 3 and 4 for initial value problems of first order ordinary differential equations were presented.

The schemes have been tested and found to be consistent and zero stable. Numerical examples are given to demonstrate the efficiency of the new methods over the existing methods.

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CHAPTER ONE

INTRODUCTION

1.1 ORDINARY DIFFERENTIAL EQUATIONS

A large variety of scientific problems arise in which one tries to determine something from its rate of change. There are lot of scientific problems that result into determination of unknown function from the prescribed information expressed in the form of an equation involving at least one of the derivatives of the unknown functions. These equations are called *differential equations.* Differential equations originate whenever a universal law is expressed by means of variables and their derivatives.

Differential equations are classified into two main headings, Ordinary and Partial, depending on whether the unknown is a function of just one variable or two or more variables. Their study forms one of the most challenging branches of Mathematics.

Ordinary differential equation is an equation containing one independent and one dependent variable and at least one of its derivatives with respect to the independent variable.

The order of a differential equation is the order of the highest order derivative of the unknown function appearing in the equation.

1.2 LINEAR AND NON-LINEAR DIFFERENTIAL EQUATIONS

A differential equation is said to be linear if it can be expressed as a summation of unknown function and its derivative with only known functions of independent variable as coefficient. Otherwise, it is said to be non-linear if it is impossible.

If an equation is linear, then the independent variable $y(x)$ may appear to be a first order or higher order degree or may involve the product of the dependent variable and its derivatives.

1.3 INITIAL VALUE PROBLEMS OF ORDINARY DIFFERENTIAL EQUATIONS

A first order differential equation $y' = f(x,y)$ has in general a function of the form $f(x, y, c) = 0$ where c is an arbitrary constant in the solution. The integral curves form a one-parameter family, where a special curve corresponds to a special choice of the constant c. Usually, we pick out any particular solution by prescribing an initial condition $y(x_o) = y_o$. Then the integration can begin at this point. We say that the differential equation together with an initial condition constitutes an initial value problem.

$$y' = f(x,y), \quad y(x_o) = y_o \tag{1.1}$$

In this work, we shall be concerned with the initial value problems in ordinary differential equations (O.D.Es) of the form (1.1). We seek solution in the interval $a \le x \le b$, a and b are finite real numbers.

1.4 CONDITIONS FOR UNIQUE SOLUTION OF INITIAL VALUE PROBLEM

The conditions on $f(x,y)$ which guarantee the existence of a unique solution of initial value problem (1.1) were stated by the following theorem:

Theorem 1.1: Let $f(x,y)$ be defined and continuous for all points (x, y) in the region D defined by $a \le x \le b$, $-\infty < y < \infty$, a and b finite, and let there exist a constant L such that, for every x, y, y^* such that (x, y), and (x, y^*) are both in D

$$| f(x,y) - f(x,y) | \le L \left| y - y^* \right| \text{---} (1.2)$$

Then if y_o is any given number, there exists a unique solution y(x) of the initial value problem (1.1), where y(x) is continuous and differentiable for all (x, y) in D.

The requirement (1.2) is known as Lipschitz condition, and the constant L as a Lipschitz constant.

1.5 BASIC CONCEPTS AND PRINCIPLE

It is important at this point to explain the basic concepts and principle used in this work.

Steplength or Meshsize

Most Numerical methods for the solution of problem (1.1) are based on the principle of discretization in which an approximation to an unknown function y = φ(x) are sought on a certain discrete points x_i, i = 0, 1, . . . of set x. Consider the sequence of points $\{x_n\}$ in the interval I = $[a,b]$ defined by

$$a = x_o < x_1 < x_2 < \ldots < x_n = b$$

$$h_i = x_{i+1} - x_i \qquad i = 0(1)n-1$$

The parameter h_i is called the steplength or meshsize of the method. The points x_i, $i = 0(1)n-1$ are called the grid points: The points x_{i+u} where $i < u < i+1$ is called the off grid point. Each grid point is given in terms of the previous point by the relation x_{i+1} [illegible] $)n$, $x_0 = a$, $x_n = x_0 + nh$.

Linear Multistep Method

A linear multistep method (LMM) for problem (1.1) is a computational method for determining $\{y_n\}$ which takes the form of a linear relationship between y_{n+j} and f_{n+j}, $j = 0(1)k$ in the form

$$\sum_{j=0}^{k} \alpha_j y_{n+j} \quad = \quad h^n \sum_{j=0}^{k} \beta_j f_{n+j}, \quad j = 0, 1, 2, \ldots, k. \text{ ---------------------- } (1.3)$$

where α_j and β_j are constants, n represents the order of the ordinary differential equation. We assume $\alpha_n \neq 0$ and that not both α_o and β_o are zero. The parameter h is called the steplength of the method and k is the step number of the method.

Several researches had carried out a lot of research on these methods. These include Gear (1971), Lambert (1973), Oladele (1991), Awoyemi (1992), Butcher (1997), Onumanyi, et. al. (1994)., Kayode (2004).

1.6 EXISTING METHODS

Before the advent of computer science, ordinary differential equation of type (1.1) is studied by integration process pioneered by Pioncare (1881). However, because of the problems which include the analytical evaluation of indefinite integrals, this method was abandoned. With the arrival of computer, various scientists picked on the problem using numerical approaches which simplify the problem. A great number of scholars have placed attention on methods of solving (1.1). Among them were Liel and Norsette (1989), Ademiluyi (1987), Fatula (1988), Awoyemi (1992), etc. With the work of Adeniyi (1991), some methods have been found where continuous solution can be obtained through collocation of a perturbed form of (1.1) based on canonical polynomials.

Gear (1971) worked on backward differentiation method for solving (1.1). Lambert (1973) improved on the work of Gear (1971) using the last grid point as collocation point and proposed the schemes

$$\sum_{j=0}^{k}\alpha_j y_{n+j} \quad = \quad h\beta_k f_{n+k} \text{ ---(1.4)}$$

for kth order, where k is the step number k = 1, 2, . . ., 6.

The method produced mono-function scheme which is expected to save computer time and human efforts.

1.7 HYBRID COLLOCATION METHODS

The evaluation of function at off grid points as proposed by Gregg and Stetter (1964), Butcher (1965) and Gear (1965) to improve accuracy were classified as hybrid methods.

The hybrid methods share with Runge-Kutta methods the property of utilizing data at points other than the grid points $(x_{n+j}, \quad j = 0, 1, \ldots)$. A k-step hybrid differential method is of the form

$$\sum_{j=0}^{k}\alpha_j y_{n+j} = h\sum_{j=0}^{k}\beta_j f_{n+j} + h\beta_r f_{n+r} \text{ ----------------------------------- } (1.5)$$

where $\alpha_k = 1$ and α_o and β_o are not both zero, $v \notin \{0,\ 1,\ \ldots,\ k\}$.

1.8 CONVERGENCE OF LINEAR MULTISTEP METHOD

Convergence expresses the property that by using a sufficiently small step, an accurate computation of the numerical solution can be made arbitrarily close to the true solution y,

Definition (1.1)

A linear multistep method (1.3) or (1.5) is said to converge if for all initial value problems (1.1) subject to the condition of existence and uniqueness of solution, we have that:

$$\lim_{h \to 0} y_n = y(x_n)$$

$$nh = x - a$$

holds for all $x \in [a,\ b]$ and for all solution $\{y_n\}$ of the differential equation (1.3) or (1.5) satisfying the starting condition $y(x_o) = y_o$.

1.9 RESEARCH OBJECTIVES

The main purpose of this study is to develop class of collocation hybrid linear multistep methods with continuous coefficients for numerical solution of first order equations. Specifically, the objectives of the study are to:

(1) derive a class of order $p = i$, $i = 2,\ 3,\ 4,\ 5$ methods with step number k = 1, 2, 3, 4 by interpolating the differential system at

the selected grid points and the additional off grid point with collocation at the last point.

(2) determine the zero stability, order and error term of the methods.

(3) determine the interval of absolute stability of the method.

(4) develop computer programs for the method

(5) apply the methods to solve numerical examples.

1.10 RESEARCH METHODOLOGY

For this study, interpolation of approximate solutions and collocation of the differential system were adopted. The interpolations of the approximate solutions were taken at various grid and off grid points of the step numbers considered except the last grid point where collocation is taken.

To estimate the order of the schemes obtained, Taylor series expansion was adopted. Boundary Locus method was adopted to determine the region of absolute stability.

1.11 COLLOCATION METHODS

Collocation method is one of the procedures used for determining a numerical solution to a given problem. It involves the determination of an approximate solution in a suitable set of functions called the basis functions. In this work, the basis function is taken to be power series in x of degree n. The approximate solution is required to satisfy the differential equation (1.1) and its supplementary conditions at certain points often referred to as collocation points.

Various researchers had adopted collocation methods for the solution of differential equations. They include Frazer et. al. (1938), Collatz (1960), Oladele (1991), and Awoyemi (1992),

In all these collocation methods, polynomial approximations were adopted as the basis functions, except in Awoyemi (1992) where canonical polynomial was adopted.

1.12 EXPECTED CONTRIBUTION TO KNOWLEDGE

Backward differentiation method was proposed in literature by Gear (1971), Lambert (1973) using only the grid points. The introduction of off-grid points in this work has produced schemes of improved order and accuracy.

CHAPTER TWO

DERIVATION OF THE METHODS

2.1 INTRODUCTION

The backward differentiation method for solving the initial value problems makes use of linear multistep method with a step number $k \geq 1$ to obtain a polynomial $y(x)$ of the form:

$$y(x) = \sum_{j=0}^{k+1} a_j x^j \text{ -- } (2.1)$$

where the real parameters a_j are to be determined. The first derivative of (2.1) is given as:

$$y'(x) = \sum_{j=0}^{k+1} j a_j x^{j-1} \text{ -- } (2.2)$$

where k = 1, 2, 3, 4, . . . respectively. For this work, we consider four cares of interpolation and collocation. An off-grid point x_{n+u} where $k-1 < u < k$ is introduced.

Case 1: k = 1. Interpolating the approximate solution (2.1) at x_n and x_{n+u} and collocating the differential system (2.2) at $x = x_{n+1}$.

Case 2: k = 2. Interpolating the approximate solution (2.1) at $x = x_{n+j}$, j= 0, 1, u and collocating the differential system (2.2) at $x = x_{n+2}$.

Case 3: k = 3. Interpolating the approximate solution (2.1) at $x = x_{n+j}$, j = 0, 1, 2, u and collocating the differential system (2.2) at $x = x_{n+3}$.

Case 4: k = 4. Interpolating the approximate solution (2.1) at $x = x_{n+j}$, j = 0, 1, 2, 3, u and collocating the differential system (2.2) at $x = x_{n+4}$.

Interpolation of (2.1) at $x = x_{n+j}$, $j = 0(1)k$ and at $x = x_{n+u}$ and collocation of (2.2) at $x = x_{n+k}$ give the following system of equations:

$$
\begin{aligned}
&a_o + a_1 x_n + a_2 x_n^2 + \cdots + a_{k+1} x_n^{k+1} = y_n \\
&a_o + a_1 x_{n+1} + a_2 x_{n+1}^2 + \cdots + a_{k+1} x_{n+1}^{k+1} = y_{n+1} \\
&\vdots \\
&a_o + a_1 x_{n+u} + a_2 x_{n+u}^2 + \cdots + a_{k+1} x_{n+u}^{k+1} = y_{n+u} \\
&a_o + a_1 x_{n+k-1} + a_2 x_{n+k-1}^2 + \cdots + a_{k+1} x_{n+k-1}^{k+1} = y_{n+k-1} \\
&a_o + a_1 x_{n+k} + a_2 x_{n+k}^2 + \cdots + a_{k+1} x_{n+k}^{k+1} = f_{n+k}
\end{aligned}
\tag{2.3}
$$

Equation (2.3) are solved for $a_i s$ and after some manipulations, we have our method in the form:

$$\sum_{j=0}^{k} \alpha_j y_{n+j} + \alpha_u y_{n+u} = h\beta_k f_{n+k} \text{ -- } (2.4)$$

2.2 DERIVATION OF 1-STEP METHOD

Adopting (2.3), we interpolate at x_n, x_{n+u} and collocate at x_{n+1} to obtain.

$$\begin{aligned} &a_o + a_1 x_n + a_2 {x_n}^2 = y_n \\ &a_o + a_1 x_{n+u} + a_2 {x_{n+u}}^2 = y_{n+u} \\ &a_1 + 2a_2 x_{n+1} = f_{n+1} \end{aligned} \quad \text{--------------------} (2.5)$$

where $f_{n+1} = f(x_{n+1}, \ y_{n+1})$ and $x_{n+j} = x_n + jh$

Solving the system of equations in (2.5) to obtain the real parameters $a_i s$ as

$$\begin{aligned} a_2 &= \frac{1}{(2-u)h} f_{n+1} - \frac{1}{(2-u)uh^2}(y_{n+u} - y_n) \\ a_1 &= \frac{-(2x_n + uh)}{(2-u)h} f_{n+1} + \frac{2(x_n + h)}{(2-u)uh^2} y_{n+u} - \frac{2(x_n + h)}{(2-u)uh^2} y_n \\ a_0 &= \frac{{x_n}^2 + uhx_n}{(2-u)h} f_{n+1} - \frac{{x_n}^2 + 2hx_n}{(2-u)uh^2} y_{n+u} - \frac{{x_n}^2 - 2hx_n + (2-u)uh^2}{(2-u)uh^2} y_n \end{aligned} \qquad (2.6)$$

substituting the values of $a_i s$ in the approximate solution (2.1), we obtain

$$y(x) = \left[\frac{(x - x_n)^2 - uh(x - x_n)}{(2-u)h}\right] f_{n+1} - \left[\frac{(x - x_n)^2 - 2h(x - x_n)}{(2-u)uh^2}\right] y_{n+1} +$$

$$\left[\frac{(x - x_n)^2 - 2h(x - x_n) + (2-u)uh^2}{(2-u)uh^2}\right] y_n \quad \text{--------------------} (2.7)$$

where (2.7) is evaluated at $x = x_{n+1}$ we have the scheme

$$y_{n+1} - \frac{1}{u(2-u)} y_{n+u} + \frac{(u-1)^2}{u(2-u)} y_n = \frac{(1-u)h}{2-u} f_{n+1} \quad \text{--------------------} (2.8)$$

Since $0 < u < 1$, let $u = \frac{1}{2}$, we have a discrete hybrid method (2.4) obtained as

$$y_{n+1} - \tfrac{4}{3} y_{n+\frac{1}{2}} + \tfrac{1}{3} y_n = \tfrac{h}{3} f_{n+1} \quad \text{--------------------} (2.9)$$

$u = \frac{1}{4}$

$$y_{n+1} - \frac{16}{7} y_{n+\frac{1}{4}} + \frac{9}{7} y_n = \frac{3h}{7} f_{n+1}$$

2.3 DERIVATION OF 2-STEP METHOD

From equation (2.3) in section (2.1), we obtain the system of equation:

$$\begin{aligned} &a_0 + a_1 x_n + a_2 x_n^2 + a_3 x_n^3 = y_n \\ &a_0 + a_1 x_{n+1} + a_2 x_{n+1}^2 + a_3 x_{n+1}^3 = y_{n+1} \\ &a_0 + a_1 x_{n+u} + a_2 x_{n+u}^2 + a_3 x_{n+u}^3 = y_{n+u} \\ &a_1 + 2a_2 x_{n+2} + 3a_3 x_{n+2}^2 = f_{n+2} \end{aligned} \qquad \text{-------------------------------} \quad (2.10)$$

Solving the system of equation (2.10) to obtain the real parameters $a_i s$ as:

$$a_3 = \frac{3}{h(8-u)}\left[\frac{1}{3h} f_{n+2} - \frac{1}{uh^3(u+1)} y_{n+u} + \frac{4-u}{3h^2(u-1)} + \frac{u-3}{3uh^2} y_n\right]$$

$$\begin{aligned} a_2 = &-\left(\frac{3x_n + (u+1)}{h^2(8-3u)}\right) f_{n+2} + \left(\frac{9x_n + 11h}{uh^3(u-1)(8-3u)}\right) y_{n+u} \\ &-\left(\frac{3(4-u)x_n + h(12-u^2)}{h^2(u-1)(8-3u)}\right) y_{n+1} - \left(\frac{3(u-3)x_n + h(u^2+u-11)}{uh^3(8-3u)}\right) y_n \end{aligned}$$

$$\begin{aligned} a_1 = &\left[\frac{3{x_n}^2 + 2h(u+1)x_n + uh^2}{h^2(8-3u)}\right] f_{n+2} - \left[\frac{9{x_n}^2 + 22hx_n + 8h^2}{uh^3(u-1)(8-3u)}\right] y_{n+u} \\ &+\left[\frac{3(4-u){x_n}^2 + 2h(12-u^2)x_n + 4uh^2(3-u)}{h^2(u-1)(8-3u)}\right] y_{n+1} \\ &+\left[\frac{3(u-3){x_n}^2 + 2h(u^2+u-11) + 4h^2(u^2-2u-2)}{uh^3(8-3u)}\right] y_n \end{aligned}$$

$$a_o = -\left(\frac{x_n{}^3 + h(u+1)x_n{}^2 + uh^2x_n}{h^2(8-3u)}\right)f_{n+2} + \left(\frac{3x_n{}^3 + 11hx_n{}^2 + 8h^2x_n}{uh^3(u-1)(8-3u)}\right)y_{n+u}$$
$$-\left(\frac{(4-u)x_n{}^3 + h\left(12-u^2\right)x_n{}^2 + 4uh^2(3-u)x_n}{h^3(u-1)(8-3u)}\right)y_{n+1}$$
$$-\left(\frac{(u-3)x_n{}^3 + h\left(u^2+u-11\right)x_n{}^2 + 4h^2\left(u^2-2u-2\right)x_n - uh^3(8-3u)}{uh^3(8-3u)}\right)y_n$$

(2.11)

Substituting the values of $a_i s$ in the approximate solution (2.1), we obtain:

$$y(x) = \frac{1}{h^2(8-3u)}\left((x-x_n)^3 + h(u+1)(x-x_n)^2 + uh^2(x-x_n)\right)f_{n+2}$$
$$-\frac{1}{uh^3(u-1)(8-3u)}\begin{pmatrix}3(x-x_n)^3 - 11h(x-x_n)^2\\ +8h^2(x-x_n)\end{pmatrix}y_{n+u}$$
$$+\frac{1}{h^3(u-1)(8-3u)}\begin{pmatrix}(4-u)(x-x_n)^3 - h\left(12u-u^2\right)(x-x_n)^2\\ +4uh^2(3-u)(x-x_n)\end{pmatrix}y_{n+1}$$
$$+\frac{1}{uh^3(8-3u)}\begin{pmatrix}(u-3)(x-x_n)^3 - h\left(u^2+u-11\right)(x-x_n)^2\\ +4h^2(u-2u-2)(x-x_n) + uh^3(8-3u)\end{pmatrix}y_n$$

(2.12)

Evaluating (2.12) at $x = x_{n+2}$ we have the scheme

$$y_{n+2} - \frac{4}{u(u-1)(8-3u)}y_{n+u} + \frac{4(u-2)^2}{(u-1)(8-3u)}y_{n+1}$$
$$-\frac{(u-2)^2}{u(8-3u)}y_n = \frac{h}{8-3u}f_{n+2}$$

(2.13)

Put u = $\frac{3}{2}$ in (2.13), we obtain the method (2.4) as

$$y_{n+2}-\frac{32}{21}y_{n+\frac{3}{2}}+\frac{4}{7}y_{n+1}-\frac{1}{21}y_n=\frac{2h}{7}f_{n+2} \text{ ---------------------------------- } (2.14)$$

2.4 DERIVATION OF 3-STEP METHOD

When k = 3, the system of equation (2.3) becomes:

$$\begin{aligned}
&a_o+a_1x_n+a_2x_n^2+a_3x_n^3+a_4x_n^4=y_n\\
&a_o+a_1x_{n+1}+a_2x_{n+1}^2+a_3x_{n+1}^3+a_4x_{n+1}^4=y_{n+1}\\
&a_o+a_1x_{n+2}+a_2x_{n+2}^2+a_3x_{n+2}^3+a_4x_{n+2}^4=y_{n+2}\\
&a_o+a_1x_{n+u}+a_2x_{n+u}^2+a_3x_{n+u}^3+a_4x_{n+u}^4=y_{n+u}\\
&a_1+2a_2x_{n+3}+3a_3x_{n+3}^2+4a_4x_{n+3}^3=f_{n+3}
\end{aligned}\tag{2.15}$$

Solving the system of equation (2.15) to obtain the real parameter a_is as

$$\begin{aligned}
a_4 \quad = \quad & \frac{1}{(39-11u)h^3}f_{n+3}-\frac{11}{u(u-1)(u-2)(39-11u)h^4}y_{n+u}\\
+ \quad & \frac{21-5u}{2(u-2)(39-11u)h^4}y_{n+2}-\frac{15-4u}{(u-1)(39-11u)h^4}y_{n+1}\\
+ \quad & \frac{11-3u}{2u(39-11u)h^4}y_n
\end{aligned}$$

$$\begin{aligned}
a_3 \quad = \quad & -\left(\frac{4x_n+(u+3)h}{(39-11u)h^3}\right)f_{n+3}+\left(\frac{44x_n+72h}{u(u-1)(u-2)(39-11u)h^4}\right)y_{n+u}\\
- \quad & \left(\frac{(84-20u)x_n-(5u^2+5u-102)h}{2(u-2)(39-11u)h^4}\right)y_{n+2}\\
+ \quad & \left[\frac{(60-16u)x_n(4u^2+8u+84)h}{(u-1)(39-11u)h^4}\right]y_{n+1}\\
- \quad & \left(\frac{(44-12u)x_n-(3u^2+9u-72)h}{2u(39-11u)h^4}\right)y_n
\end{aligned}$$

$$a_2 = \left[\frac{6x_n2 + (3u+9)hx_n + (2+3u)h^2}{(39-11u)h^3}\right] f_{n+3}$$

$$- \left[\frac{66{x_n}^2 + 216hx_n + 139h^2}{u(u-1)(u-2)(39-11u)h^4}\right] y_{n+u}$$

$$+ \left[\frac{\begin{array}{c}(126-30u){x_n}^2 - \left(15u^2+15u-306\right)hx_n \\ -\left(26u^2-81u-81\right)h^2\end{array}}{u(u-1)(u-2)(39-11u)h^4}\right] y_{n+2}$$

$$- \left[\frac{\begin{array}{c}(90-24u){x_n}^2 - \left(12u^2+24u-252\right)hx_n \\ -\left(23u^2-54u-108\right)h^2\end{array}}{(u-1)(39-11u)h^4}\right] y_{n+1}$$

$$+ \left[\frac{\begin{array}{c}(66-18u){x_n}^2 - \left(9u^2+27u-216\right)hx_n \\ -\left(20u^2-33u-139\right)h^2\end{array}}{2u(39-11u)h^4}\right] y_n$$

$$a_1 = -\left[\frac{4{x_n}^3 + (9+3u)hxn^2 + (4+6u)h^2x_n + 2uh^3}{(39-11u)h^3}\right] f_{n+3}$$

$$+ \left[\frac{44{x_n}^3 + 216h{x_n}^2 + 278h^2x_n + 78h^3}{u(u-1)(u-2)(39-11u)h^4}\right] y_{n+u}$$

$$- \left[\frac{\begin{array}{c}(84-20u){x_n}^3 - \left(15u^2+15u-306\right)h{x_n}^2 \\ -\left(52u^2-162u-162\right)h^2x_n - \left(21u^2-81u\right)h^3\end{array}}{2(u-2)(39-11u)h^4}\right] y_{n+2}$$

$$+ \left[\frac{\begin{array}{c}(60-16u)x_n{}^3-\left(12u^2+24u-252\right)hx_n{}^2\\ -\left(46u^2-108u-216\right)h^2x_n-\left(30u^2-108u\right)h^3\end{array}}{(u-1)(39-11u)h^4}\right] y_{n+1}$$

$$- \left[\frac{\begin{array}{c}(44-12u)x_n{}^3-\left(9u^2+27u-216\right)hx_n{}^2\\ -\left(40u^2-66u-278\right)h^2x_n-\left(39u^2-117u-78\right)h^3\end{array}}{2u(39-11u)h^4}\right] y_n$$

$$a_o = \left[\frac{x_n{}^4-(u+3)hx_n{}^3+(2+3u)h^2x_n{}^2+2uh^3x_n}{(39-11u)h^3}\right] f_{n+3}$$

$$- \left[\frac{11x_n{}^4+72hx_n{}^3+139h^2x_n{}^2+78h^3x_n}{u(u-1)(u-2)(39-11u)h^4}\right] y_{n+u}$$

$$+ \left[\frac{\begin{array}{c}(21-5u)x_n{}^4+\left(5u^2+5u-102\right)hx_n{}^3\\ -\left(26u^2-81u-81\right)h^2x_n{}^2-\left(21u^2-81u\right)h^3x_n\end{array}}{2(u-2)(39-11u)h^4}\right] y_{n+2}$$

$$- \left[\frac{\begin{array}{c}(15-4u)x_n{}^4-\left(4_n{}^2+8u-84\right)hx_n{}^3\\ -\left(23u^2-54u-108\right)h^2x_n{}^2-\left(30u^2-108u\right)h^3x_n\end{array}}{(u-1)(39-11u)h^4}\right] y_{n+1}$$

$$+ \left[\begin{array}{c}(11-3u)\,x_n{}^4-\left(3u^2+9u-72\right)hx_n{}^3\\ -\left(20u^2-33u-139\right)h^2x_n{}^2\\ -\left(39u^2-117u-78\right)h^3x_n\left(78u-22u^2\right)h^4\end{array}\right] y_n$$

Substituting the values of a_is in the approximate solution (2.1), we have

$$
\begin{aligned}
y(x) = {} & \left[\frac{\begin{array}{c}(x-x_n)^4-(u+3)h(x-x_n)^3+(2+3u)h^2(x-x_n)^2\\ -2uh^3(x-x_n)\end{array}}{(39-11u)h^3}\right]f_{n+3}\\
& -\left[\frac{\begin{array}{c}11(x-x_n)^4-72h(x-x_n)^3+139h^2(x-x_n)^2\\ -78h^3(x-x_n)\end{array}}{u(u-1)(u-2)(39-11u)h^4}\right]y_{n+u}\\
& +\left[\frac{\begin{array}{c}(21-5u)(x-x_n)^4+\left(5u^2+5u-102\right)h(x-x_n)^3\\ -\left(26u^2-81u-81\right)h^2(x-x_n)^2\\ +\left(21u^2-81u\right)h^2(x-x_n)\end{array}}{2(u-2)(39-11u)h^4}\right]y_{n+2}\\
& -\left[\frac{\begin{array}{c}(15-4u)(x-x_n)^4+\left(4u^2+8u-84\right)h(x-x_n)^3\\ -\left(23u^2-54u-108\right)h^2(x-x_n)^2\\ +(30u2-108u)h^3(x-x_n)\end{array}}{(u-1)(39-11u)h^4}\right]y_{n+1}\\
& +\left[\frac{\begin{array}{c}(11-3u)(x-x_n)^4-\left(3u^2+9u-72\right)h(x-x_n)^3\\ -\left(20u^2-33u-139\right)h^2(x-x_n)^2\\ +\left(39u^2-111u-78\right)h^3(x-x_n)+\left(78u-22u^2\right)h^4\end{array}}{2u(39-11u)\,h^4}\right]y_n
\end{aligned}
\tag{2.17}
$$

Evaluating (2.17) at $x = x_{n+3}$, we have the scheme

$$
\begin{aligned}
y_{n+3} \quad = \quad & \left(\frac{(18-6u)}{39-11u}\right)f_{n+3} + \left(\frac{36}{u(u-1)(u-2)(39-11u)}\right)y_{n+u} \\
- \quad & \left(\frac{18(u-3)^2}{(u-2)(39-11u)}\right)y_{n+2} + \left(\frac{9(u-3)^2}{(u-1)(39-11u)}\right)y_{n+1} \\
- \quad & \left(\frac{2(u-3)^2}{u(39-11u)}\right)y_n
\end{aligned}
$$

(2.18)

Let $u = \frac{5}{2}$ to obtain the discrete scheme

$$
y_{n+3} - \left(\frac{192}{115}\right)y_{n+5/2} + \left(\frac{18}{23}\right)y_{n+2} - \left(\frac{3}{25}\right)y_{n+1} + \left(\frac{2}{115}\right)y_n = \left(\frac{6h}{23}\right)f_{n+3}
$$

(2.19)

2.5 DERIVATION OF 4-STEP METHOD

For k=4, the system of equation (2.3) becomes

$$
\begin{aligned}
& a_o + a_1x_n + a_2{x_n}^2 + a_3{x_n}^3 + a_4{x_n}^4 + a_5{x_n}^5 = y_n \\
& a_o + a_1x_{n+1} + a_2x_{n+1}^2 + a_3x_{n+1}^3 + a_4x_{n+1}^4 + a_5x_{n+1}^5 = y_{n+1} \\
& a_o + a_1x_{n+2} + a_2x_{n+2}^2 + a_3x_{n+2}^3 + a_4x_{n+2}^4 + a_5x_{n+2}^5 = y_{n+2} \\
& a_o + a_1x_{n+3} + a_2x_{n+3}^2 + a_3x_{n+3}^3 + a_4x_{n+3}^4 + a_5x_{n+3}^5 = y_{n+3} \\
& a_o + a_1x_{n+u} + a_2x_{n+u}^2 + a_3x_{n+u}^3 + a_4x_{n+u}^4 + a_5x_{n+u}^5 = y_{n+u} \\
& a_1 + 2a_2x_{n+4} + 3a_3x_{n+4}^2 + 4a_4x_{n+4}^3 + 5a_5x_{n+4}^4 = f_{n+4}
\end{aligned}
$$

(2.20)

Solving the system of equation (2.20) to obtain the real parameters a_is as

$$
\begin{aligned}
a_5 \quad = \quad & \frac{1}{2(112-25u)h^4}f_{n+4} - \frac{25}{u(u-1)(u-2)(u-3)(112-25u)h^5}y_{n+u} \\
+ \quad & \frac{(64-13u)}{6(u-3)(112-25u)}y_{n+3} - \frac{88-19u}{4(u-2)(112-25u)h^5}y_{n+2}
\end{aligned}
$$

$$+ \quad \frac{32-7u}{2(u-1)(112-25u)h^5}\,y_{n+1} - \frac{(50-11u)}{12u(112-25u)h^5}\,y_n$$

$$a_4 \quad = \quad -\left[\frac{5xn+(u+6)h}{2(112-25u)h^4}\right] f_{n+4}$$

$$+ \quad \left[\frac{125xn+262h}{u(u-1)(u-2)(u-3)(112-25u)h^5}\right] y_{n+u}$$

$$- \quad \left[\frac{5(64-13u)x_n+(496-39u-13u)h}{6(u-3)(112-25u)h^5}\right] y_{n+3}$$

$$+ \quad \left[\frac{5(88-19u)x_n+\left(752-76u-19u^2\right)h}{4(u-2)(112-25u)h^5}\right] y_{n+2}$$

$$- \quad \left[\frac{5(32-7u)x_n+\left(304-35u-7u^2\right)h}{2(u-1)(112-25u)h^5}\right] y_{n+1}$$

$$+ \quad \left[\frac{5(50-11u)x_n+\left(524-66u-11u^2\right)h}{12u(112-25u)h^5}\right] y_n$$

$$a_3 \quad = \quad \left[\frac{10{x_n}^2+4(6+u)hx_n+(11+6u)h^2}{2(112-25u)h^4}\right] f_{n+4}$$

$$- \quad \left[\frac{250{x_n}^2+108hx_n+947h^2}{u(u-1)(u-2)(u-3)(112-25u)h^3}\right] y_{n+u}$$

$$+ \quad \left[\frac{\begin{gathered}10(64-13u){x_n}^2+4\left(496-39u-13u^2\right)hx_n\\ +\left(1040+278u-103u^2\right)h^2\end{gathered}}{6(u-3)(112-25u)h^5}\right] y_{n+2}$$

$$+\left[\frac{\begin{array}{c}10(32-7u)x_n{}^2+4\left(304-35u-7u^2\right)hx_n\\ +\left(912+102u+67u^2\right)\end{array}}{2(u-1)(112-25u)h^5}\right]y_{n+1}$$

$$-\left[\frac{\begin{array}{c}10(50-11u)x_n{}^2+4\left(524-66u-11u^2\right)hx_n\\ -\left(1894+103u-116u^2\right)h^2\end{array}}{12u(112-25u)h^5}\right]y_n$$

$$\alpha_2 = -\left[\frac{10x_n{}^3+6(6+u)hx_n{}^2+3(11+6u)h^2x_n+(6+11u)h^3}{2(112-25u)h^4}\right]f_{n+4}$$

$$+\left[\frac{250\,x_n{}^3+1572\,hx_n{}^2+2841\,h^2x_n+1382\,h^3}{u(u-1)(u-2)(u-3)(112-25u)h^5}\right]y_{n+u}$$

$$-\left[\frac{\begin{array}{c}10(64-13u)x_n{}^3+6\left(496-39u-13u^2\right)hx_n{}^2\\ +3\left(1040+278u-103u^2\right)h^2x_n\\ +\left(608+912u-218u^2\right)h^3\end{array}}{6(u-3)(112-25u)h^5}\right]y_{n+3}$$

$$+\left[\frac{\begin{array}{c}10(88-19u)x_n{}^3+6\left(752-76u-19u^2\right)hx_n{}^2\\ +3\left(1864+343u-164u^2\right)h^2x_n\\ +\left(1200+1600u-409u^2\right)h^3\end{array}}{4(u-2)(112-25u)h^5}\right]y_{n+2}$$

$$-\left[\frac{\begin{array}{c}10(32-7u){x_n}^3+6\left(304-35u-7u^2\right)h{x_n}^2\\+3\left(912+102u-67u^2\right)h^2x_n\\+\left(864+720u-202u^2\right)h^3\end{array}}{2(u-1)(112-25u)h^5}\right]y_{n+1}$$

$$+\left[\frac{\begin{array}{c}10(50-11u)x_n3+6\left(524-66u-11u^2\right)h{x_n}^2\\+3\left(1894+103u-116u^2\right)h^2x_n\\+\left(276+1278u-421u^2\right)h^3\end{array}}{12u(112-25u)h^5}\right]y_n$$

$$a_1=\left[\frac{\begin{array}{c}5{x_n}^4+4(u+6)h{x_n}^3+3(11+6u)h^2{x_n}^2\\+2(6+11u)h^3x_n+6uh^4\end{array}}{2(112-25u)h^4}\right]f_{n+4}$$

$$-\left[\frac{\begin{array}{c}125{x_n}^4+1048h{x_n}^3+2841h^2{x_n}^2\\+2764h^3x_n+672h^4\end{array}}{u(u-1)(u-2)(u-3)(112-25u)h^5}\right]y_{n+u}$$

$$+\left[\frac{\begin{array}{c}5(64-13u){x_n}^4+4\left(496-39u-13u^2\right)h{x_n}^3\\+3\left(1040+278u-103u^2\right)h^2{x_n}^2\\+2\left(608+912u-218u^2\right)h^3x_n+\left(608u-128u^2\right)h^4\end{array}}{6(u-3)(112-25u)h^5}\right]y_{n+3}$$

$$-\left[\frac{\begin{gathered}5(88-19u)+4\left(752-76u-19u^2\right)hx_n{}^3\\ +3\left(1864+343u-164u^2\right)h^2x_n{}^2\\ +2\left(1200+1600u-409u^2\right)h^3x_n\\ +(1200u+264u2)h^4\end{gathered}}{4(u-2)(112-25u)h^5}\right]y_{n+2}$$

$$+\left[\frac{\begin{gathered}5(32-7u)x_n{}^4+4\left(304-35u-7u^2\right)hx_n{}^3\\ +3\left(912+102u-67u^2\right)h^2x_n\\ +2\left(864+720u-202u^2\right)h^3x_n+\left(864u-192u^2\right)h^4\end{gathered}}{2(u-1)(112-25u)h^5}\right]y_{n+1}$$

$$-\left[\frac{\begin{gathered}5(50-11u)x_n{}^4+4\left(524-66u-11u^2\right)hx_n{}^3\\ +3\left(1894+103u-116u^2\right)h^2x_n\\ +2\left(2764+1278u-421u^2\right)h^3x_n\\ +\left(1344+2464u-616u^2\right)h^4\end{gathered}}{12u(112-25u)h^5}\right]y_n$$

$$a_o = -\left[\frac{\begin{gathered}x_n{}^5+(u+6)hx_n{}^4+(11+6u)h^2x_n{}^3\\ +(6+11u)h^3x_n{}^2+6uh^4x_n\end{gathered}}{2(112-25u)h^4}\right]f_{n+4}$$

$$+\left[\frac{\begin{array}{c}25{x_n}^5+262\,h{x_n}^4+947\,h^2{x_n}^3\\+1382\,h^3x_n2+672\,h^4x_n\end{array}}{2(112-25u)h^4}\right]y_{n+u}$$

$$-\left[\frac{\begin{array}{c}(64-13u)x_n^5+\left(496-39u-13u^2\right)hx_n^4\\+\left(912+102u-67u^2\right)h^2x_n^3\\+\left(864+720u-202u^2\right)h^3x_n^2\\+\left(864u-192u^2\right)h^4x_n\end{array}}{2(u-1)(112-25u)h^5}\right]y_{n+1}$$

(2.21)

Substituting (2.21) in the approximate solution (2.1), we have

$$y(x)=\left[\frac{\begin{array}{c}(x-x_n)^5-(u+6)h(x-x_n)^4+(11+6u)h^2(x-x_n)^3\\-(6+11u)h^3(x-x_n)^2+64h^4(x-x_n)\end{array}}{2(112-25u)h^4}\right]f_{n+4}$$

$$-\left[\frac{\begin{array}{c}25(x-x_n)^5-262h(x-x_n)^4+947h^2(x-x_n)^3\\-1382h^3(x-x_n)^2+672(x-x_n)\end{array}}{u(u-1)(u-2)(u-3)(112-25u)h^5}\right]y_{n+u}$$

$$+\left[\frac{\begin{array}{c}(64-3u)(x-x_n)^5-\left(496-39u-13u^2\right)h(x-x_n)^4\\ +\left(1040+278u-103u^2\right)h^2(x-x_n)^3\\ -\left(608+912u-218u^2\right)h^3(x-x_n)^2\\ +\left(790u-128u^2\right)h^4(x-x_n)\end{array}}{6(u-3)(112-25u)h^5}\right]y_{n+3}$$

$$-\left[\frac{\begin{array}{c}(88-19u)(x-x_n)^5-\left(752-76u-19u^2\right)h(x-x_n)^4\\ +\left(1864+343u-164u^2\right)h^2(x-x_n)^3\\ -\left(1200+1600u-409u^2\right)h^3(x-x_n)^2\\ +\left(1200u-264u^2\right)h^4(x-x_n)\end{array}}{4(u-2)(112-25u)h^5}\right]y_{n+2}$$

$$+\left[\frac{\begin{array}{c}(32-7u)(x-x_n)^5-\left(304-35u-7u^2\right)h(x-x_n)^4\\ +\left(912+44u-67u^2\right)h^2(x-x_n)^3\\ +\left(864+282u-202u^2\right)h^3(x-x_n)^2\\ +\left(484u-192u^2\right)h^4(x-x_n)\end{array}}{2(u-1)(112-25u)h^5}\right]y_{n+1}$$

$$-\left[\frac{\begin{array}{c}(50-11u)(x-x_n)^5-\left(254-66u-11u^2\right)h(x-x_n)^4\\ +\left(1894+103u-116u^2\right)h^2(x-x_n)^3\\ -\left(2764+1278u-421u^2\right)h^3(x-x_n)^2\\ +\left(1344+2464u-616u^2\right)h^4(x-x_n)\\ -\left(1344u+300u^2\right)h^5\end{array}}{12u\left(112-25u\right)}\right]y_n \tag{2.22}$$

Evaluating (2.22) at $x = x_{n+4}$, we obtain the scheme

$$\begin{aligned} y_{n+4} &= \frac{288}{u(u-1)(u-2)(u-3)(112-25u)}y_{n+u} \\ &+\frac{48(u-4)^2}{(u-3)(112-25u)}y_{n+3} \\ &-\frac{36(u-4)^2}{(u-2)(112-25u)}y_{n+2}+\frac{16(u-4)^2}{(u-1)(112-25u)}y_{n+1} \\ &-\frac{3(u-4)^2}{u(112-25u)}y_n=\frac{(48-12u)h}{112-25u}f_{n+4} \end{aligned} \tag{2.23}$$

Put $u=\frac{7}{2}$ to obtain a discrete scheme

$$y_{n+4}-\frac{9216}{5145}y_{n+\frac{7}{2}}+\frac{48}{49}y_{n+3}-\frac{12}{49}y_{n+2}+\frac{16}{245}y_{n+1}-\frac{3}{343}y_n=\frac{12h}{49}f_{n+4} \tag{2.24}$$

It should be noted that $u \in (k-1,k)$ is arbitrary. Thus, any value of u chosen in this interval will produce an independent scheme. In other words, an infinite number of schemes can be obtained for each k.

CHAPTER THREE

ANALYSIS OF BASIC PROPERTIES OF THE METHODS

3.1 ANALYSIS OF BASIC PROPERTIES OF THE METHODS (2.9) : K=1

(1) ORDER AND ERROR TERM

Here we shall estimate the order and the error term of our method k=1. Let us define a linear operator L associated with (2.4) as

$$L[y(x),\ h] = \sum_{j=0}^{k} \alpha_j y_{n+j} + \alpha_u y_{n+u} - h\beta_k f_{n+k} \text{ ----------------- } (3.1.1)$$

Adopting the Taylor series expansion of $y(x_{n+j}),\ j = 0(i)k$ about $x = x_n$. Using the result of Ademiluyi and Kayode (2001), reproduced as:

$$y(x_{n+k}) = \sum_{r=0}^{q} \frac{(kh)^r y^{(r)}(x_n)}{r!} \text{ -------------------------------------- } (3.1.2)$$

$$y'(x_{n+k}) = \sum_{r=1}^{q} \frac{(kh)^{r-1} y^r(x_n)}{(r-1)!} \text{ ------------------------------------ } (3.1.3)$$

In (2.4), for k = 1 and combining terms of equal powers of h, we obtain

$$L[y(x),\ h] = C_o y(x_n) + C_1 h y^1(x_n) + C_2 h^2 y^{(2)}(x_n) + C_3 h^3 y^{(3)}(x_n) + \dots \qquad (3.1.4)$$

where

$$C_o = 1 - \tfrac{4}{3} + \tfrac{1}{3} = 0$$
$$C_1 = 1 - \tfrac{4}{5}\left(\tfrac{1}{2}\right) - \tfrac{1}{3} = 0$$
$$C_2 = \tfrac{1}{2} - \tfrac{4}{3}\left(\tfrac{1}{8}\right) - \tfrac{1}{3} = 0$$

$$C_3 = \frac{1}{6} - \frac{4}{3}\left(\frac{1}{8 \cdot 3!}\right) - \frac{1}{3}\left(\frac{1}{2}\right) = \frac{-1}{36}$$

Thus,

$$C_o = C_1 = C_2 = 0, \qquad C_3 = -\frac{1}{36} \text{ ------------------------------ } (3.1.5)$$

Definition (3.1): A linear multistep method (2.4) for first order equation is of order p and the principal error constant is C_{p+1} if $C_o = C_1 = C_2 = \cdots = C_p = 0$ and $C_{p+1} \neq 0$.

[See Lambert (1973, 1991), Fatunla (1988)]

Hence, the scheme (2.9) is of order p = 2 and the principal error constant $C_{p+1} = -\frac{1}{36}$.

(2) CONSISTENCY AND ZERO STABILITY

Definition (3.2): A linear multistep method is consistent if

(i) the order $p \geq 1$

[illegible] $\rho(1) = 0$ [illegible]

(ii) $\sum_{j=0}^{k} \alpha_j = 0$

(iii) $\sum_{j=0}^{k} j\alpha_j = \sum_{j=0}^{k} \beta_j$

[See Lambert (1973), Awoyemi (1992]

Now for the scheme (2.9), we have

(i) p = 2 > 1

(ii) $\sum_{j=0}^{1} j\alpha_j = 1 - 4 + 3 = 0$

(iii) $\sum_{j=0}^{1} j\alpha_j = 0(1) + \frac{1}{2}(-4) + 1(3) = 1$

(iv) $\sum_{j=0}^{1} j\beta_j = \beta_1 = 1$

Hence, $\sum_{j=0}^{1} j\alpha_j = \sum_{j=0}^{1} j\beta_j = 1$

Thus, the conditions for consistency are satisfied. Hence, the method (2.9) is consistent.

Definition (3.3): A linear multistep method is zero stable if no root of the first characteristics polynomial p has modulus greater than one.

For the method (2.9), the first characteristics equation is

$3r - 4r^{\frac{1}{2}} + 1 = 0$ -- (3.1.6)

or $9r - 10r + 1 = 0$

$r = 1 \; or \;\; 0.11$

Since $r \le 1$, the method is zero stable.

Theorem (Dahlquist (1963))

The necessary and sufficient conditions for a linear multistep method to be convergent are for it to be consistent and zero stable.

Since the method (2.9) is consistent and zero stable, the method converges.

(3) REGION OF ABSOLUTE STABILITY

To determine the region of absolute stability of the method (2.9), we adopt the boundary locus method (Lambert (1973), Fatula (1988))

$$\bar{h}(r) = \frac{\rho(r)}{\sigma(r)}$$

where ρ is the first characteristic polynomial and σ is the second characteristics polynomial. For the method (2.9), we have

$$h(r)=\frac{3r-4r^{\frac{1}{2}}+1}{r} \tag{3.1.7}$$

Let $r=e^{i\theta}$, $0\le\theta\le\pi$ where $e^{i\theta}=\cos\theta+i\sin\theta$. Hence,

$$h(\theta)=\frac{3e^{i\theta}-4e^{i\frac{\theta}{2}}+1}{e^{i\theta}}$$

$$\text{or } h(\theta)=\frac{3(\cos\theta+i\sin\theta)-4\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)+1}{\cos\theta+i\sin\theta} \tag{3.1.8}$$

Rationalizing and simplifying (3.1.8) by multiplying the numerator and denominator by the conjugate of $\cos\theta+i\sin\theta$ yields:

$$h(\theta)=x(\theta)+iy(\theta) \tag{3.1.9}$$

where

$$x(\theta)=3-4\cos\frac{\theta}{2}+\cos\theta$$

$$y(\theta)=4\sin\frac{\theta}{2}-\sin\theta$$

Evaluating $x(\theta)$, $y(\theta)$, $0\le\theta\le180$ at intervals of 30^0 gives the following result as shown in the table below:

Table 1:

θ	0	30	60	90	120	150	180
x(θ)	0	0.002322	0.0359	0.1716	0.5	1.0987	2.000
y(θ)	0	-0.4302	1.1340	1.8284	2.598	3.3637	4.000

The results shown in the above table can be used to plot the curve of the region of absolute stability of (2.9). The interval of absolute stability along the real axis is (0, 2) which is contained in the positive x-axis of the complex plane.

3.2 ANALYSIS OF BASIC PROPERTIES OF THE METHODS (2.14) : k = 2

In this section, the basic properties of method (2.14) were examined by adopting similar approach as in section (3.1).

(1) ORDER AND ERROR TERM

The linear operator L associated with (2.14) as defined by equation (3.1.4) for k = 2. Adopting Taylor series expansion

$$L[y(x), h] = C_o y(x_n) + C_1 h y^1(x_n) + C_2 h^2 y^{(2)}(x_n) + C_3 h^3 y^{(3)}(x_n) + \ldots \tag{3.1.4}$$

where

$$C_o = 1 - \frac{32}{21} + \frac{4}{7} - \frac{1}{21} = 0$$

$$C_1 = 2 = \frac{32}{21}\left(\frac{3}{2}\right) + \frac{4}{7}(1) - \frac{2}{7}(1) = 0$$

$$C_2 = 2 - \frac{32}{21}\left(\frac{9}{8}\right) + \frac{4}{7}\left(\frac{1}{2}\right) - \frac{2}{7}(2) = 0$$

$$C_3 = \frac{8}{6} - \frac{32}{21}\left(\frac{27}{48}\right) + \frac{4}{7}\left(\frac{1}{6}\right) - \frac{2}{7}(2) = 0$$

$$C_4 = \frac{16}{4!} - \frac{32}{21}\left(\frac{81}{16(4!)}\right) + \frac{4}{7}\left(\frac{1}{4!}\right) - \frac{2}{7}\left(\frac{8}{3!}\right) = 0.25$$

Thus,

$$C_o = C_1 = C_2 = C_3 = 0, \ C_4 = 0.25$$

Hence, the scheme (2.14) is of order p = 3, and the principal error constant $C_{p+1} = 0.25$.

(2) CONSISTENCY AND ZERO STABILITY

For the method (2.14)

(i) p > 1

(ii) $$\sum_{j=0}^{1} \alpha_j = 1 - \frac{32}{21} + \frac{4}{7} - \frac{1}{21} = 0$$

(iii) $$\sum_{j=0}^{1} j\alpha_j = 0\left(-\frac{1}{21}\right) + 1\left(\frac{4}{7}\right) - \frac{3}{2}\left(\frac{32}{21}\right) + 2(1) = \frac{2}{7}$$

(iv) $$\sum_{j=0}^{1} \beta_j = \beta_2 = \frac{2}{7}$$

Hence, $\sum_{j=0}^{2} j\alpha_j = \sum_{j=0}^{1} \beta_j = \frac{2}{7}$

Thus, the method (2.14) is also consistent as stated in definition (3.1).

For the method (2.14), the first characteristic equation is:

$$r^2 - \frac{32}{21} r^{3/2} + \frac{4}{7} r - \frac{1}{21} = 0$$

or $$21r^2 - 32r^{3/2} + 12r - 1 = 0$$

or $$441r^4 - 520r^3 + 102r^2 - 24r + 1 = 0$$

$r = 1$ or $r < 1$

Hence, the method (2.14) is also zero stable.

Like the method (2.9), method (2.14) is consistent and zero stable, hence it also converges.

(3) REGION OF ABSOLUTE STABILITY

Here, the same line of argument used for method (2.9) in section (3.1) above will be used to discuss the region of absolute stability for method (2.14). The stability polynomial for the method (2.14) is:

$$h(r) = \frac{r^2 - \frac{32}{21} r^{3/2} + \frac{4}{7} r - \frac{1}{21}}{\frac{2}{7} r} \text{ ------------------------------------ } (3.1.6)$$

We put $r=e^{i\theta}$, $0\le\theta\le\pi$, where $e^{i\theta}=\cos\theta+i\sin\theta$. Hence we have

$$h(\theta)=\frac{(\cos 2\theta+i\sin 2\theta)-\frac{32}{21}\left(\cos\frac{3}{2}\theta+i\sin\frac{3}{2}\theta\right)+\frac{4}{7}(\cos\theta+i\sin\theta)-\frac{1}{21}}{\frac{2}{7}(\cos 2\theta+i\sin 2\theta)}$$

(3.1.7)

Rationalizing the denominator by multiplying both the numerator and denominator by the conjugate of the denominator, we have

$h(\theta)=x(\theta)+iy(\theta)$ -- (3.1.8)

where

$$x(\theta)=\frac{1-\frac{32}{21}\cos\frac{\theta}{2}+\frac{4}{7}\cos\theta-\frac{1}{21}\cos 2\theta}{\frac{2}{7}}$$

$$y(\theta)=\frac{\frac{32}{21}\sin\frac{\theta}{2}+\frac{4}{7}\sin\theta+\frac{1}{21}\sin 2\theta}{\frac{2}{7}}$$

Evaluating $x(\theta)$ and $y(\theta)$, $0\le\theta\le 180$ we have

Table 2

θ	0	30	60	90	120	150	180
x(θ)	0	0.00289	0.03547	-0.10457	-0.0833	0.30425	1.333
y(θ)	0	2.5347	4.543	5.7712	6.2965	6.0073	5.333

The above table could be used for plotting the graph of region of stability of (x, y). Also, the interval of absolute stability is (0, 1.333) which is also contained in the positive x-axis of the complex plane.

3.3 ANALYSIS OF BASIC PROPERTIES OF THE METHOD (2.19) : k = 3

Here, we shall examine the basic properties of the method (2.19) following similar procedure as in section (3.1).

(1) ORDER AND ERROR TERM

Here, we look into the estimation of the order and error term of the method k = 3. We define a linear operator L associated with (2.19) as defined by the equation (3.1.4) for k = 3. Adopting Taylor series expansion $y(x_{n+j})$ and $y'(x_{n+j}), j=1,2,3$ about $x = x_n$ and combining the terms of equal powers of h, we obtain:

$$L[y(x), h] = C_o y(x_n) + C_1 h y^1(x_n) + C_2 h^2 y^2(x_n) + \ldots \text{------} \quad (3.1.4)$$

where

$$C_o = 1 - \frac{192}{115} + \frac{18}{23} - \frac{3}{23} + \frac{2}{115} = 0$$

$$C_1 = 3 - \frac{192}{115}\left(\frac{5}{2}\right) + \frac{18}{23}(2) - \frac{3}{23} - \frac{6}{23}(1) = 0$$

$$C_2 = \frac{9}{2} - \frac{192}{115}\left(\frac{25}{8}\right) + \frac{18}{23}(2) - \frac{3}{23} - \frac{6}{23}(3) = 0$$

$$C_3 = \frac{9}{2} - \frac{192}{115}\left(\frac{125}{48}\right) + \frac{18}{23}\left(\frac{8}{6}\right) - \frac{3}{23}\left(\frac{1}{2}\right) - \frac{6}{23}\left(\frac{9}{2}\right) = 0$$

$$C_4 = \frac{27}{8} - \frac{192}{115}\left(\frac{625}{384}\right) + \frac{18}{23}\left(\frac{16}{24}\right) - \frac{3}{23}\left(\frac{1}{24}\right) - \frac{6}{23}\left(\frac{27}{6}\right) = 0$$

$$C_5 = \frac{81}{40} - \frac{192}{115}\left(\frac{3125}{3840}\right) + \frac{18}{23}\left(\frac{32}{120}\right) - \frac{3}{23}\left(\frac{1}{120}\right) - \frac{6}{23}\left(\frac{81}{24}\right) = -0.75$$

Thus,

$$C_o = C_1 = C_2 = C_3 = C_4 = 0, \quad C_5 \neq 0$$

Hence, the scheme (2.19) is of order p = 4 and the principal error constant is $C_{p+1} = -0.75$.

(2) CONSISTENCY AND ZERO STABILITY

For the method (2.19)

(i) P > 1

(ii) $\sum_{j=0}^{3} \alpha_j = 1 - \frac{192}{115} + \frac{18}{23} - \frac{3}{23} + \frac{2}{115} = 0$

(iii) $\sum_{j=0}^{3} j\alpha_j = 0\left(\frac{2}{115}\right) + 1\left(\frac{-3}{23}\right) + 2\left(\frac{18}{23}\right) + \frac{5}{2}\left(\frac{-192}{115}\right) + 3(1) = \frac{6}{23}$

(iv) $\sum_{j=0}^{3}$ [illegible] $\beta_j = \beta_3 = \frac{6}{23}$

Since $\sum_{j=0}^{3} j\alpha_j = \beta_j = \frac{6}{23}$, then method (2.19) is also consistent.

The characteristic equation for method (2.19) is

$$r^3 - \frac{192}{115} r^{5/2} + \frac{18}{23} r^2 - \frac{3}{23} r + \frac{2}{115} = 0$$

$\rho(r)$

or $\quad 115r^3 - 192r^{5/2} + 90r^2 - 15r + 2 = 0$

or $\quad 13225r^6 - 16164r^3 + 4650r^4 + 2240r^3 + 585r^2 - 60r + 4 = 0$

$r = 1$ or $r < 1$

Hence, the method (2.19) is also zero stable. Since the method is consistent and zero stable, it also converges.

(3) REGION OF ABSOLUTE STABILITY

As for the methods (2.9) and (2.14), the same procedure is used to discuss the region of stability for method (2.19). The stability polynomial for the method (2.19) is:

$$h(r) = \frac{r^3 - \frac{192}{115}r^{5/2} + \frac{18}{23}r^2 - \frac{3}{23}r + \frac{2}{115}}{\frac{6}{23}r^3}$$

Let $r = e^{i\theta}$, $0 \le \theta \le \pi$, where $e^{i\theta} = \cos\theta + i\sin\theta$, we have

$$h(\theta) = \frac{(\cos 3\theta + i\sin 3\theta) - \frac{192}{115}\left(\cos\frac{5\theta}{2} i\sin\frac{5\theta}{2}\right) + \frac{18}{23}(\cos 2\theta + i\sin 2\theta) - \frac{3}{23}(\cos\theta + i\sin\theta) + \frac{2}{115}}{\frac{6}{23}(\cos 3\theta + i\sin 3\theta)}$$

Rationalizing the denominator by multiplying both numerator and denominator by the conjugate of the denominator, we have

$$h(\theta) = x(\theta) + iy(\theta)$$

where

$$x(\theta) = \frac{1 - \frac{192}{115}\cos\frac{\theta}{2} + \frac{18}{23}\cos\theta - \frac{3}{23}\cos 2\theta + \frac{2}{115}\cos 3\theta}{\frac{6}{23}}$$

$$y(\theta) = \frac{\frac{192}{115}\sin\frac{\theta}{2} + \frac{18}{23}\sin\theta + \frac{3}{23}\sin 2\theta \frac{2}{115}\sin 3\theta}{\frac{6}{23}}$$

Evaluating $x(\theta)$ and $y(\theta)$, $0 \le \theta \le 180$, we have

Table 3:

θ	0	30	60	90	120	150	180
x(θ)	0	-0.000276	-0.0259	-0.145	-0.55	-0.6712	1.2667
y(θ)	0	3.5228	6.2311	7.5922	7.7076	7.1872	6.400

From the table above, the interval of absolute stability is (0, 1.2667) which is also contained in the positive axis of the complex plane.

3.4 ANALYSIS OF BASIC PROPERTIES OF THE METHOD (2.24) : k = 4

In this section, we shall examine the basic properties of the method (2.23) following the same procedure as in section (3.1).

(1) ORDER AND ERROR TERM

We are looking at the estimation of the order and error term of the method k = 4. We define a linear operator L associated with (2.24) as defined by the equation (3.1.4) for k = 4. Adopting Taylor series expansion $y(x_{n+j})$ and $y'(x_{n+j})$, $j = 1,\ 2,\ 3,\ 4$ about $x = x_n$, and combining the terms of equal powers of h, we obtain

$$L[y(x),\ h] = C_o y(x_n) + C_1 h y'(x_n) + C_2 h^2 y''(x_n) + C_3 h^3 y'''(x_n) + \ldots \tag{3.1.4}$$

where

$$C_o = 1 - \frac{9216}{5145} + \frac{48}{49} - \frac{12}{49} + \frac{16}{245} - \frac{3}{343} = 0$$

$$C_1 = 4 - \frac{9216}{5145}\left(\frac{7}{2}\right) + \frac{48}{49}(3) - \frac{12}{49}(2) + \frac{16}{245}(1) + \frac{12}{49}(1) = 0$$

$$C_2 = 8 - \frac{9216}{5145}\left(\frac{49}{8}\right) + \frac{48}{49}\left(\frac{9}{2}\right) - \frac{12}{49}\left(\frac{4}{2}\right) + \frac{16}{245}\left(\frac{1}{6}\right) + \frac{12}{49}(4) = 0$$

$$C_3 = \frac{32}{3} - \frac{9216}{5145}\left(\frac{343}{48}\right) + \frac{48}{49}\left(\frac{27}{6}\right) - \frac{12}{49}\left(\frac{8}{6}\right) + \frac{16}{245}\left(\frac{1}{6}\right) + \frac{12}{49}(8) = 0$$

$$C_4 = \frac{32}{3} - \frac{9216}{5145}\left(\frac{2401}{384}\right) + \frac{48}{49}\left(\frac{81}{24}\right) - \frac{12}{49}\left(\frac{16}{24}\right) + \frac{16}{245}\left(\frac{1}{24}\right) + \frac{12}{49}\left(\frac{32}{3}\right) = 0$$

$$C_5 = \frac{128}{15} - \frac{9216}{5145}\left(\frac{16807}{3840}\right) + \frac{48}{49}\left(\frac{81}{40}\right) - \frac{12}{49}\left(\frac{4}{15}\right) + \frac{16}{245}\left(\frac{1}{120}\right) + \frac{12}{49}\left(\frac{128}{15}\right) = 0$$

$$C_6 = \frac{256}{45} - \frac{11649}{25725} + \frac{729}{735} - \frac{16}{735} + \frac{1}{11025} - \frac{625}{98} = -0.17$$

Thus,

$$C_o = C_1 = C_2 = C_3 = C_4 = C_5 = 0 \qquad C_6 \neq 0$$

Hence, the scheme (2.24) is of order p = 5 and the principal error constant is $C_{p+1} = -0.17$.

(2) CONSISTENCY AND ZERO STABILITY

For the method (2.24),

(i) $P > 1$

(ii) $$\sum_{j=0}^{1} \alpha_j = 1 - \frac{9216}{5145} + \frac{48}{49} - \frac{12}{49} + \frac{16}{245} - \frac{3}{343} = 0$$

(iii) $$\sum_{j=0}^{4} j\alpha_j = 0\left(-\frac{33}{343}\right) + 1\left(\frac{16}{245}\right) + 2\left(-\frac{12}{49}\right) + 3\left(\frac{48}{49}\right) + \frac{7}{2}\left(\frac{-9216}{5145}\right) + 4(1) = \frac{12}{49}$$

(iv) $$\beta_4 = \frac{12}{49}$$

Since $\sum_{j=0}^{4} j\alpha_j = \beta_j = \frac{12}{49}$, then the method (2.24) is also constant.

The characteristic equation is

$$r^4 - \frac{9216}{5145} r^{7/2} + \frac{48}{49} r^3 - \frac{12}{49} r^2 + \frac{16}{245} r - \frac{3}{343} = 0$$

or $$5145r^4 - 9216r^{7/2} + 5040r^3 + 1260r^2 + 336r - 45 = 0$$

or $$26471025r^8 - 33073056r^7 + 12436200r^6 - 9243360r^5 + 4511430r^5 - 1300320r^3 + 226296r^2 - 30240r + 2025 = 0$$

Solving the characteristic equation, we, obtain $r \leq 1$. Hence, method (2.24) is also zero stable. Since it is consistent and zero stable, it also converges like other methods (2.9), (2.14), and (2.19).

(3) REGION OF ABSOLUTE STABILITY

As for the other methods (2.9), (2.14), and (2.19), the same line of argument will be used to discuss the region of absolute stability for method (2.24). The stability polynomial is:

$$h(r) = \frac{r^4 - \frac{9216}{5145} r^{7/2} + \frac{48}{49} r^3 - \frac{12}{49} r^2 + \frac{16}{245} r - \frac{3}{343}}{\frac{12}{49} r^4}$$

Expressing $r = e^{i\theta}$, $0 \le \theta \le \pi$, where $e^{i\theta} = \cos\theta + i\sin\theta$, we have

$$h(\theta) = \frac{\begin{array}{c}(\cos 4\theta + i\sin 4\theta) - \frac{9216}{5145}\left(\cos\frac{7}{2}\theta + i\sin\frac{7}{2}\theta\right) \\ + \frac{48}{49}(\cos 3\theta + i\sin 3\theta) - \frac{12}{49}(\cos 2\theta + i\sin 2\theta) \\ + \frac{16}{245}(\cos\theta + i\sin\theta) - \frac{3}{343}\end{array}}{\frac{12}{49}(\cos 4\theta + i\sin 4\theta)}$$

Rationalizing the denominator by multiplying both numerator and denominator by the conjugate of the denominator, we have

$$h(\theta) = x(\theta) + iy(\theta)$$

where

$$x(\theta) = \frac{5145 - 9216\cos\frac{\theta}{2} + 5040\cos\theta - 1260\cos 2\theta + 336\cos 3\theta - 45\cos 4\theta}{1260}$$

$$y(\theta) = \frac{9216\sin\frac{\theta}{2} + 5040\sin\theta + 1260\sin 2\theta + 336\sin 3\theta + 45\sin 4\theta}{1260}$$

Evaluating $x(\theta)$ and $y(\theta)$, $0 < \theta < 180^o$, we have

Table 4:

θ	0	30	60	90	120	150	180
x(θ)	0	0.00023	-0.00167	-0.1244	-0.7893	-1.7556	-1.2190
y(θ)	0	4.9946	8.0182	8.9053	8.9015	8.4966	7.3143

Here, the interval of absolute stability is (-1.219, 0) which is contained in the negative axis of the complex plane.

CHAPTER FOUR

NUMERIC EXPERIMENT

Problem 1:

$$y' = 5y, \quad y(0) = 1, \quad 0 \le x \le 0.5$$

The theoretical solution is $y(x) = e^{5x}$, using the method (2.9)

Table 5: Results for Problem 1, h = 0.1

X	EXACT SOLUTION	NUMERICAL SOLUTION	ERROR
0.0	1.000000000	1.000000	0.0000000
0.1	1.648721271	1.652777778	4.0565071×10^{-3}
0.2	2.2718281828	2.2720196759	$1.19149308 \times 10^{-3}$
0.3	4.481689070	4.476990500	4.6985707×10^{-3}
0.4	7.389056099	7.368380197	2.0675042×10^{-2}
0.5	12.18249396	12.12712574	5.3368219×10^{-2}

Table 6: Results for Problem 1, h = 0.05

X	EXACT SOLUTION	NUMERICAL SOLUTION	ERROR
0.0	1.000000000	1.0000000	0.00000000
0.1	1.648721271	1.649117364	$3.960931237 \times 10^{-4}$
0.2	2.2718281828	2.718209677	$7.215126716 \times 10^{-5}$
0.3	4.481689070	4.480374782	$1.314288493 \times 10^{-3}$
0.4	7.389056099	7.384918961	$4.137137662 \times 10^{-3}$
0.5	12.18249396	12.17242546	$1.006850079 \times 10^{-2}$

Table 7: Results for Problem 1, h = 0.025

X	EXACT SOLUTION	NUMERICAL SOLUTION	ERROR
0.0	1.000000000	1.00000000	0.00000000
0.1	1.648721271	1.648759226	$3.79557929 \times 10^{-5}$
0.2	2.2718281828	2.718244304	$3.752490774 \times 10^{-5}$
0.3	4.481689070	4.481462166	$2.2269046915 \times 10^{-5}$
0.4	7.389056099	7.388409907	$6.461918878 \times 10^{-4}$
0.5	12.18249396	12.180979999	$1.513973201 \times 10^{-3}$

Problem 2:

$$y' = x + y, \quad y(0) = 1, \quad 0 \le x \le 1$$

The theoretical solution is $y(x) = 2e^{x} - x - 1$. With our method (2.9), we have

Table 8: Results for Problem 2, h = 0.1

X	EXACT SOLUTION	NUMERICAL SOLUTION	ERROR
0.0	1.0000000	1.000000000	0.000000
0.1	1.110341836	1.110400000	5.8638487×10^{-5}
0.2	1.242805516	1.242860400	$5.488367966 \times 10^{-5}$
0.3	1.399717615	1.399767885	$5.027024799 \times 10^{-5}$
0.4	1.583649395	1.583693475	$4.407939869 \times 10^{-5}$
0.5	1.797442541	1.797478572	$3.603036829 \times 10^{-5}$
0.6	2.044237601	2.044263402	$2.580078518 \times 10^{-5}$
0.7	2.327505415	2.327518436	$1.302102329 \times 10^{-5}$
0.8	2.65108157	2.651079125	$2.732171782 \times 10^{-5}$
0.9	3.019206222	3.019184279	$2.194287456 \times 10^{-5}$
1.0	3.436563657	0.3436518493	$4.516409105 \times 10^{-5}$

Table 9: Results for Problem 2, h = 0.05

X	EXACT SOLUTION	NUMERICAL SOLUTION	ERROR
0.0	1.0000000	1.00000000	0.00000000
0.1	1.110341836	1.110348761	6.924424515 x 10^{-6}
0.2	1.242805516	1.242811947	6.430220900 x 10^{-6}
0.3	1.399717615	1.399723371	5.755475747 x 10^{-6}
0.4	1.583649395	1.583654263	4.867680034 x 10^{-6}
0.5	1.797442541	1.797446271	3.72948369 x 10^{-6}
0.6	2.044237601	2.044239899	2.298036938 x 10^{-6}
0.7	2.327505415	2.2327505939	5.242465182 x 10^{-7}
0.8	2.65108157	2.651080209	1.648063529 x 10^{-6}
0.9	3.019206222	3.019201939	4.283098675 x 10^{-6}
1.0	3.436563657	3.436556203	7.45416149 x 10^{-6}

Table 10: Results for Problem 2, h = 0.025

X	EXACT SOLUTION	NUMERICAL SOLUTION	ERROR
0.0	1.000000	1.0000000	0.0000000
0.1	1.110341836	1.110342678	8.413922674 x 10^{-7}
0.2	1.242805516	1.242806290	7.7399323 x 10^{-7}
0.3	1.399717615	1.399718298	6.831107939 x 10^{-7}
0.4	1.583649395	1.583649960	5.645509114 x 10^{-7}
0.5	1.797442541	1.797442955	4.134971023 x 10^{-7}
0.6	2.044237601	2.044237825	2.244259218 x 10^{-7}
0.7	2.327505415	2.327505406	8.988481515 x 10^{-9}
0.8	2.65108157	2.651081563	2.939820409 x 10^{-7}
0.9	3.019206222	3.019205583	6.388222253 x 10^{-7}
1.0	3.436563657	3.436562604	1.052944995 x 10^{-6}

Table 11: Results for Problem 2, h = 0.0125

X	EXACT SOLUTION	NUMERICAL SOLUTION	ERROR
0.0	1.000000	1.0000000	0.000000
0.1	1.110341836	1.110341940	$1.035809269 \times 10^{-7}$
0.2	1.242805516	1.242805611	$9.479278962 \times 10^{-8}$
0.3	1.399717615	1.39917698	$8.301043941 \times 10^{-8}$
0.4	1.583649395	1.583649463	$6.770127193 \times 10^{-8}$
0.5	1.797442541	1.797442590	$4.825377253 \times 10^{-8}$
0.6	2.044237601	2.044237625	$2.396681564 \times 10^{-8}$
0.7	2.327505415	2.327505409	$5.962435878 \times 10^{-9}$
0.8	2.65108157	2.651081815	$4.245215601 \times 10^{-8}$
0.9	3.019206222	3.019206136	$8.655123995 \times 10^{-8}$
1.0	3.436563657	3.436563517	$1.394566476 \times 10^{-7}$

Problem 3:

$$y' = 2y^2 + 4x^2, \quad y(1) = -2$$

The theoretical solution is $-2x\sqrt{2x^2 - 1}$

Table 12: Results for Problem 3, h = 0.1

X	EXACT SOLUTION	NUMERICAL SOLUTION	ERROR
1.0	-2.0000000	-2.00000000	0.0000000
1.1	-2.621602563	-2.621846035	$2.434713990 \times 10^{-4}$
1.2	-3.290714208	-3.291269324	$5.5551161287 \times 10^{-4}$
1.3	-4.011084641	-4.011798112	$7.134706411 \times 10^{-4}$
1.4	-4.784642097	-4.785479878	$8.377807207 \times 10^{-4}$
1.5	-5.612486080	-5.613442175	$9.56094811 \times 10^{-4}$
1.6	-6.495290602	-6.496367498	$1.076895984 \times 10^{-4}$
1.7	-7.43391777	-7.434694981	$1.203203701 \times 10^{-3}$
1.8	-8.427383936	-8.428720155	$1.336219444 \times 10^{-2}$
1.9	-9.477172574	-9.478649034	$1.476459548 \times 10^{-2}$
2.0	-10.58300524	-10.58462940	$1.624157342 \times 10^{-3}$

Table 13: Results for Problem 3, h = 0.05

X	EXACT SOLUTION	NUMERICAL SOLUTION	ERROR
1.0	-2.0000000	-2.00000000	0.0000000
1.1	-2.621602563	-2.621653704	$5.114046236 \times 10^{-5}$
1.2	-3.290714208	-3.290784699	$7.04908947 \times 10^{-5}$
1.3	-4.011084641	-4.011169750	$8.410903360 \times 10^{-5}$
1.4	-4.784642097	-4.784738797	$9.669963625 \times 10^{-5}$
1.5	-5.612486080	-5.612595594	$1.095138547 \times 10^{-4}$
1.6	-6.495290602	-6.495413557	$1.229557603 \times 10^{-4}$
1.7	-7.43391777	-7.433628950	$1.371733066 \times 10^{-4}$
1.8	-8.427383936	-8.427536161	$1.522251645 \times 10^{-4}$
1.9	-9.477172574	-9.477340710	$1.681355895 \times 10^{-4}$
2.0	-10.58300524	-10.58319016	$1.849146002 \times 10^{-4}$

Table 14: Results for Problem 3, h = 0.025

X	EXACT SOLUTION	NUMERICAL SOLUTION	ERROR
1.0	-2.0000000	-2.00000000	0.0000000
1.1	-2.621602563	-2.621609436	$6.872551613 \times 10^{-6}$
1.2	-3.290714208	-3.260722836	$8.627333923 \times 10^{-6}$
1.3	-4.011084641	-4.011094712	$1.149393995 \times 10^{-5}$
1.4	-4.784642097	-4.784853591	$1.149393995 \times 10^{-5}$
1.5	-5.612486080	-5.612499060	$1.297998086 \times 10^{-5}$
1.6	-6.495290602	-6.495305156	$1.455480311 \times 10^{-5}$
1.7	-7.43391777	-7.433508005	$1.622.794646 \times 10^{-5}$
1.8	-8.427383936	-8.427401939	$1.800300010 \times 10^{-6}$
1.9	-9.477172574	-9.477192455	$1.988128808 \times 10^{-5}$
2.0	-10.58300524	-10.58302711	$2.186322921 \times 10^{-5}$

Table 15: Results for Problem 3, h = 0.0125

X	EXACT SOLUTION	NUMERICAL SOLUTION	ERROR
1.0	-2.0000000	-2.00000000	0.0000000
1.1	-2.621602563	-2.621603437	8.738211275 x 10^{-7}
1.2	-3.290714208	-3.290715270	1.061735584 x 10^{-6}
1.3	-4.011084641	-4.01185870	1.228418948 x 10^{-6}
1.4	-4.784642097	-4.784643495	1.39777763 x 10^{-6}
1.5	-5.612486080	-5.612487657	1.576508597 x 10^{-6}
1.6	-6.495290602	-6.495292368	1.766773173 x 10^{-6}
1.7	-7.43391777	-7.433493746	1.969320978 x 10^{-6}
1.8	-8.427383936	-8.427388120	2.184411242 x 10^{-6}
1.9	-9.477172574	-9.477174986	2.412121642 x 10^{-6}
2.0	-10.58300524	-10.58300790	2.652461140 x 10^{-6}

The first two examples used for this work are linear problems while the last example is non-linear. It will be observed that the method is suitable for both linear and non-linear problems.

From the tables, the numerical solutions have negligible errors. It is observed from the tables that as the step length is reducing, the solutions continue to improve, although the intervals of absolute stability are also decreasing with increasing k.

CHAPTER FIVE

CONCLUSION AND RECOMMENDATION

In this work, a new collocation hybrid backward differentiation method for first order ordinary differential equation was proposed. Four cases of k = 1, 2, 3, 4 of our method which give four different independent schemes were considered. Each scheme obtained is one-function evaluation per iteration. Hence, the computer time as well as human efforts will not be wasted. Also the orders of the methods are greater than the orders of the methods obtained by Lambert (1973) for the same k. The methods were shown to be consistent, zero stable, and hence convergent. It was observed that as the step number k increases, the order of accuracy of the methods also increases with small error constant. Numerical results of the implementation of the schemes are very close to the exact solutions. This confirmed the suitability of the methods for general purpose use.

Finally, it is recommended that further work should be done to produce better methods with the introduction of more than one off-grid points since the introduction of an off-grid point in this work produced more accurate, robust, and reliable methods than the existing methods produced when only the grid points are used for interpolation and collocation respectively.

REFERENCES

Awoyemi, D. O. (1992): "Some continuous Linear Multistep Methods for Initial Value Problems". Ph.D. Thesis (Unpublished), University of Ilorin, Ilorin.

Ademiluyi, R. A. (1987): "New Hybrid Methods for System of Stiff Ordinary Differential Equations". Ph.D. dissertation, University of Benin, Benin City.

Butcher, J. C. (1965): "A Modified Method for Numerical Integration of Ordinary Differential Equations", Journal of Association of Computer, 124-135.

Collatz, L. (1960): The Numerical Treatment of Ordinary Differential Equations, Springer Verlag. Berlin.

Dahlquist G. (1963): "A Special Stability Problem for Linear Multistep Methods, B17, 3, 27-43.

Fatula, S. O. (1988): Numerical Methods for Initial Value Problems in Ordinary Differential Equations. Academic Press Inc. New York.

Frazer, R. A., Duncan W. J., and Collar A. R. (1938): Elementary Matrices and Some Applications to Dynamics and Differential Equations. Cambridge University Press, England.

Gear, C. W. (1971): Algorithm 407 DIFUB for Solution of Ordinary Differential Equations Communication of ACM 14, 185-190.

Gragg, W. B. and Stetter H. J. (1964): Generalised Multistep Predictor–Corrector Methods. Journal of Association of Computer. March 11, 108-209.

Kayode, S.J. (2004): "A class of Multi derivative method for solving second order initial value problem of ordinary differential equation" Ph.D Thesis (unpublished), Federal University of Technology, Akure.

Lambert, J. D. (1973): "Computational Methods in Ordinary Differential Equations". New York: John Wiley & Sons.

Lie, I., and Norsette S. P. (1989): Super Convergence for Multistep Collocation. Maths. Comp 52:69-79.

Oladele, J. O. (1991): "Some New Collocation Formulae for the Continuous Numerical Solutions of Initial Value Problems, M.Sc. Degree Dissertation (Unpublished), University of Ilorin, Ilorin.

Onumanyi P., Awoyemi D. O., Jator S. N., and Sirisena U. W. (1994): New Linear Multistep Methods with Continuous Coefficients for First Order Difference Approximations for Solving Differential Equations, Comp, Maths. 72, 15-27.

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    NAME OF FILE: AJILEYE1
    SOLUTION OF FIRST ORDER INITIAL VALUE PROBLEMS Y'=F(X,Y)
    BY A FAMILY OF ONE-STEP SYMMETRIC HYBRID METHODS
    IMPLICIT DOUBLE PRECISION(A-H,O-Z)
    DIMENSION YN1C(80,80),YEX(80,80),ERC(80,80),TT(80,80)
    F(X,Y)=(2.D0*Y/X)+(4.D0*X/Y)
    Y(X)=-2.D0*X*DSQRT(2.D0*X*X-1.D0)
    OPEN(6,FILE='AJI1.OUT')
    N=80
    NSTEP=80
    A=1.D0
    H=0.0125D0
    B=A+H
    DX=H/FLOAT(N)
    D=2.D0
    XN=A
    YN=-2.D0
    U=0.5D0
    XN1=XN+H
    XNU=XN+0.5D0*U
    WRITE(6,5)
    FORMAT(8X,'X',12X,'EXACT',20X,'YC',20X,'EC',22X,'YD',22X,'ED'/)
    CALCULATE PREDICTOR
    DO 1 I=1,N
    CALCULATE FP
    FF=F(XN,YN)
    DFX=-2.D0*YN/(XN*XN)+4.D0/YN
    DFY=2.D0/XN-4.D0*XN/(YN*YN)
    FP=DFX+FF*DFY
    DFXX=4.D0*YN/(XN**3)
    DFYY=8.D0*XN/(YN**3)
    DFXY=-2.D0/(XN*XN)-4.D0/(YN*YN)
    FPP=DFXX+2.D0*FF*DFXY+FF*FF*DFYY+DFX*DFY+FF*DFY*DFY
    YN1=YN+H*FF+((H*H)/2.D0)*FP+((H**3)/6.D0)*FPP
    F1=F(XN1,YN1)
    YNU=YN+U*H*FF+U*U*H*H*FP/2.D0+((U*H)**3)*FPP/6.D0
    CALCULATE COEFFICIENTS OF CONTINUOUS METHOD
    DO 2 J=1,NSTEP
    TT(I,J)=XN+DX*FLOAT(J)
    X=TT(I,J)
    P=X-XN
    A1=(P*P-U*H*P)/((2.D0-U)*H)
    A2=(P*P-2.D0*H*P)/((2.D0-U)*U*H*H)
    A3=(P*P-2.D0*H*P+(2.D0-U)*U*H*H)/((2.D0-U)*U*H*H)
    YN1C(I,J)=A1*F1-A2*YNU+A3*YN
    YC=YN1C(I,J)
    CALCULATE EXACT SOLUTION AND ERROR OF THE METHOD
    IF(X.GE.B) THEN
    YEX(I,J)=Y(X)
    YE=YEX(I,J)
    ERC(I,J)=DABS(YN1C(I,J)-YEX(I,J))
    ER=ERC(I,J)
    WRITE(6,10)X,YE,YC,ER
0   FORMAT(1X,F8.2,3X,3D20.10)
    CHANGE VARIABLES
    XN=XN1
    YN=YN1
    XN1=XN1+H
    ELSE
    GO TO 2
```

```
      ENDIF
      IF(B.GE.D) GO TO 1
      B=B+H
      GO TO 1
  2   CONTINUE
  1   CONTINUE
      STOP
      END
```