

**A CLASS OF EXPLICIT LINEAR MULTISTEP METHOD FOR DIRECT  
SOLUTION OF THIRD ORDER  
INITIAL VALUE PROBLEMS OF ORDINARY DIFFERENTIAL  
EQUATIONS**

**BY**



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## CERTIFICATION

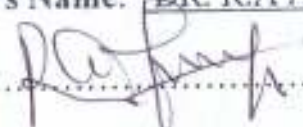
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## DEDICATION

This project is dedicated to God (The father of our Lord Jesus Christ); who by His Mercy and Grace supply the strength for the completion of this work.



## ACKNOWLEDGEMENT

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## ABSTRACT

In this thesis, a class of discrete Linear Multistep Methods for Direct Solution of third order initial value problems of Ordinary differential Equations is developed using Taylor series expansion technique.

The method is motivated by the variety of application areas of third order Ordinary Differential equations, which include Engineering, Science, Management and Technology. The analysis of the basic properties of the methods were carried out using Boundary Locus method. The result shows that the methods are Consistent, Zero-Stable, and Convergent.

The methods are programmed in Fortran Programming Language and used to solve some sample third order initial value problems in Ordinary differential equations to demonstrate the practical feasibility and accuracy of the methods.

## CHAPTER ONE



### INTRODUCTION

#### 1.1 PREAMBLE

Differential Equations play a major role in the analysis of problems arising from sciences, Engineering, Medicine and Management. Many of the basic laws governing these problems are sometime defined in mathematical equations that contain unknown functions and their derivatives of 3<sup>rd</sup> orders. Any equation involving an unknown function, its derivatives of 1<sup>st</sup>, 2<sup>nd</sup> and 3<sup>rd</sup> order is called a 3rd order differential equation. Third order equations are often expressed in form

$$y''' = F(x, y, y', y'') \quad y(x_0) = y_0, y'(x_0) = y'_0, y''(x_0) = y''_0 \dots\dots\dots(1.1)$$

Differential equations are of two types namely, Ordinary Differential Equations (ODES) and Partial Differential Equations ( PDES) respectively. It is ordinary if the function depends on one independent variable  $x$  and partial if it depends on two or more variables.

Third order differential equation can be classified into Linear and Non-linear equations. A third order differential equation is said to be Linear if no product of the dependent variable  $y(x)$  with itself or any of its derivatives occurs. While it is non-Linear otherwise.

The subject of third order differential equations constitutes a large and very important branch of Mathematics. Its study is a wide field in both Pure and applied Mathematics. Pure Mathematicians study the types and properties of third order differential equations to know whether or not the solutions exist, and should they exist, whether they are unique. Applied Mathematicians, Physicists and Engineers are usually

more interested in how to compute solutions to them. These solutions are then used to simulate real life problems in Engineering, Sciences and Management. Very often, these equations do not have closed form solutions and they are solved using Numerical methods.

One of the conditions that such equations must satisfy in order to ensure the existence and uniqueness of solution is contained in the theorem postulated by Coddington and Levinson (1965)

**Definition 1:** A solution of an ordinary differential equation of order three is a function that possesses at least three derivatives in some interval  $I$  and satisfies the equation and associated initial conditions.

Basically, there are two main methods of solution of 3<sup>rd</sup> order ordinary differential equation. They are Analytical and numerical methods respectively.

**Analytical method** consists of expressing the unknown variable  $y$  in terms of the independent variable  $x$  or as a linear combination of elementary functions such as Bessel functions, exponential functions and trigonometric functions respectively.

However, there are many problems of this kind, which do not possess a closed form solution. In these situations, an approximate solution using numerical methods becomes an important approach:

**Numerical Method** is an approximate method to get a solution to a particular problem at some discrete points in the interval of definition of the problem. The advent of digital computer has influenced the use and the development of Numerical methods. This helps to find solution to differential equations at the desired points. It also helps in

the creation of new methods, modification of existing methods to make them more effective and give room for theoretical / practical analysis of algorithms for the standard computational process and pointing out those algorithms which are satisfactory in various situations. There are several algorithms, amongst which are single-step algorithm and linear multistep algorithms (method); which form the object of this work.

## 1.2 THE EXISTING NUMERICAL METHODS

It has been observed in literature that the commonest method for solving a 3<sup>rd</sup> order ordinary differential equations of the form

$$y''' = F\{x, y, y', y''\} \dots\dots\dots(1.2)$$

is by reduction of the problem into a system of first order ordinary differential equations of the form  $y' = f(x, y), y(x_0) = y_0 \dots\dots\dots(1.3)$

And then adopts any appropriate numerical method for solving the first order system of ordinary differential equations. More information on this can be obtained in Fatunla(1988), Lambert(1973), Jain (1984), Jain et al (1984), Enright (1974), Jeltsch (1976), Jacques and Judd (1987).

The methods had been classified into:

(i) The expansion methods, where by, the theoretical solution  $y(x)$  is approximated by the first few terms of an expansion involving orthogonal functions  $H_j(x)$  in the form

$$y_k(x) = \sum_{j=1}^k \alpha_j H_j(x) \dots\dots\dots(1.4)$$

where  $\alpha_j$  can depend on  $k$ .

(ii) The discrete variable methods in which numerical approximations are obtained at some specified points in the integration interval .It seems to be more amenable to computer implementation; hence it is more universally applicable.

The conventional method which involves the reduction of (1.3) to a system of first order ordinary differential equation suffers some major setbacks. The setbacks include computational burden and cost implication.

Furthermore, a more serious drawback to such technique or approach arises when the given system of equations to be solved cannot be solved explicitly for the derivatives of the highest order [Bun and Vsil'yer, (1992)]. The method then becomes inefficient and uneconomical for a general – purpose application.

The existing methods can also be classified into Continuous and discrete methods. Continuous method makes use of collocation and interpolation approach in the derivation of the methods. Though, this method is acceptable and unique in terms of accuracy but their method of derivation is complex and the implementation is difficult. Discrete methods based on Taylor series expansion method are less difficult to derive and apply.

In this work, a class of discrete linear multistep method of the form

$$y_{n+k} = \sum_{j=0}^{k-1} \alpha_j y_{n+j} + h^3 \sum_{j=0}^{k-1} \beta_j y'''_{n+j} \dots\dots\dots(1.5)$$

is to be developed for direct solution of third order initial value problems of ordinary differential equations of the form

$$y''' = f(x,y,y',y''), y(x_0) = y_0, y'(x_0) = y'_0, y''(x_0) = y''_0 \dots\dots\dots(1.6)$$

### 1.3

### MOTIVATION

Third order ordinary differential equations have large areas of application such as Engineering, Management and Sciences where they serve as model equations for analysis of real life problems like electrical network. The adoption of these Numerical methods in solving the real life problems which includes Controls and Population Dynamics in Biology and Blasious equation in fluid dynamics), directly without reducing the modeled equation to a system of first order equation motivated the research work.

The Existing Methods of reduction of higher order equations to lower order are inadequate due to their inefficiency and heavy computations. The proposed method will therefore reduce these inadequacies by reducing computational efforts and inefficiency associated with the existing method.

### 1.4

### RESEARCH OBJECTIVES

The objectives of this work are to: -

- (i) develop a Discrete Computational formular of the form (1.5) for direct solution of 3<sup>rd</sup> order Ordinary differential equations.
- (ii) develop necessary predictors for determination of starting values.
- (iii) analyze the basic properties of the methods which includes the order of accuracy, error constants, consistency, zero-stability convergence and check for A- stability
- (iv) apply the method to solve some sample problems on computer with a view of accessing its applicability and accuracy.

To achieve the above objectives we adopt:

- (i) Taylor series expansion of the variables  $y_{n+k}$ ,  $y_{n+j}$  and  $f_{n+j}$  about point  $(x_n, y_n)$  are adopted and substituted into equation (1.5) and combine terms in equal power of  $h$  with a view to determine the unknown coefficients of the methods.
- (ii) the system of linear equations resulting from the above procedure were solved by Gaussian elimination technique.
- (iii) analysis of the basic properties of the methods were also carried out using Boundary Locus method.
- (iv) the formulas (or methods) were coded in Fortran programming language.
- (v) the programme was implemented on a digital computer with sample 3<sup>rd</sup> order initial value problems.



**PRELIMINARY CONCEPTS AND PRINCIPLES**

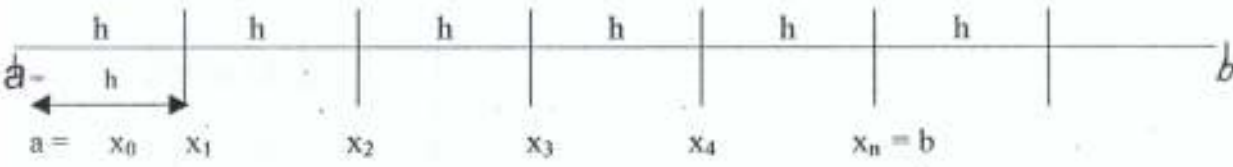
In this chapter, the basic concepts and principles that are relevant to the development, analysis and implementation of the scheme are discussed.

**2.1 PRELIMINARY CONCEPTS**

The basic concepts of interest include step length or meshsize, explicit and implicit method, characteristic polynomial. These are discussed in turns below.

**A. STEP LENGTH OR MESH SIZE.**

Most numerical methods for the solution of (1.1) are based on the principle of discretization. Discretization is a process of dividing an interval into some small subintervals; in which an approximation  $y_n$  to the theoretical solution  $y(x)$  are sought on a discrete points set  $x_i, i = 0(1)n$  defined by  $a = x_0 < x_1 < x_2 \dots \dots \dots < x_n = b$  in the interval  $I = [a,b]$  with  $h = x_i - x_{i-1}, i = [i(1)n-1]$  as shown in the figure below.



The parameter  $h$  is called the step length or mesh size for the method while the  $x_i$ 's are called mesh-sizes.

**B. Implicit and Explicit method: A Linear multistep method**

$$\sum_{j=0}^k \alpha_j y_{n+j} = h \sum_{j=0}^k \beta_j f_{n+j} \dots \dots \dots (2.1)$$

for initial value problem (1.1) is said to be implicit if the value  $\beta_k \neq 0$  and explicit otherwise.

**C. Characteristic polynomial:** A characteristic polynomial of a linear multistep method of type (2.1) is defined by  $\Pi(r, h) = \rho(r) - h\sigma(r)$  .....(2.2)

where

$$\rho(\tau) = \sum_{j=0}^k \alpha_j \tau^j$$

and

$$\sigma(\tau) = \sum_{j=0}^k \beta_j \tau^j$$

are called first and second characteristic polynomials respectively.

## 2.2 PRINCIPLES

**2.2.1 Power series:** In the derivation of numerical analysis, power series technique such as Maclaurin's series and Taylor series expansion are used in the derivation of the method. Taylor series is a one-step method, which can be used to obtain an approximation to the value of the solution of an initial value problem at specific values of  $x$ . Assuming that the function  $y(x_{n+h})$  is continuously differentiable in a certain interval  $[a,b]$  then its Taylor expansion

$$y(x_{n+h}) = \sum_{r=0}^{\infty} \frac{(h)^r y^{(r)}}{r!} \dots\dots\dots(2.3)$$

For further information, see (John L. Van Iwaarden, 1985)

**A Interpolation:** This is the process of finding a solution at a particular or some specified points inside a given interval  $[a,b]$

**B Predictor –Corrector procedure:** This is a process in which an explicit method is used in estimating the unknown function in the given problem and this predicted value is used as starting value for the corrector method adopted. Suppose an implicit linear k-step method of the form

$$y_{n+k} + \sum_{j=0}^{k-1} \alpha_j y_{n+j} = h^3 \beta_k f(x_{n+k}, y_{n+k}) + h \sum_{j=0}^{k-1} \beta_j y''_{n+j} \dots\dots\dots(2.4)$$

is to be used to solve initial value problem, At each step  $y_{n+k}$  must be calculated. The initial guess or value  $y_{n+k}^{(0)}$  will be required. This is done by using a separate explicit method to estimate  $y_{n+k}$ , and taking this predicted value to be initial guess  $y_{n+k}^{(0)}$ . The explicit method is called the predictor, and the implicit method the corrector.

**C Linear multistep method (LMM):** A linear multistep method is a method that tends to approximate  $y_{n+k}$  from a linear combination of the previous estimates  $y_{n+j}$ ,  $j = 0(1)k-1$ .

It is of the form

$$y_{n+k} = \sum_{j=0}^{k-1} \alpha_j y_{n+j} - h^m \sum_{j=0}^{k-1} \beta_j f_{n+j}, m = 1,2 \dots\dots\dots(2.5)$$

Where  $\alpha_j$  and  $\beta_j$  are constants and  $\alpha_k \neq 0$ , and that both  $\alpha_0$  and  $\beta_0$  are not both zero.(For details see Lambert,1973)

## CHAPTER THREE

### DERIVATION OF THE PROPOSED METHODS



The method discussed here is motivated by the linear multistep method

$$\sum_{j=0}^k \alpha_j y_{n+j} = h \sum_{j=0}^k \beta_j y'_{n+j} \dots\dots\dots(3.1)$$

Proposed by Adams (1883) for numerical solution of 1<sup>st</sup> order ordinary differential equation (1.3)

In this work, formula (3.1) is redefined for the 3rd order ordinary differential equations.

$y''' = f(x, y, y', y'')$  as

$$y_{n+k} = \sum_{j=0}^{k-1} \alpha_j y_{n+j} + h^3 \sum_{j=0}^{k-1} \beta_j y'''_{n+j} \dots\dots\dots(3.2)$$

Where the parameters  $\alpha_j$  and  $\beta_j$  are to be determined as to ensure that the resultant methods are consistent, zero-stable and convergent. Like it is done for the Linear multistep method (Lambert, 1973)  $\alpha_k = 1$  and  $\alpha_0$  and  $\beta_0$  are not both zero.

Assuming that a truncation error  $T_{n+k}$  is defined as

$$T_{n+k} = \sum_{j=0}^{k-1} \alpha_j y_{n+j} - h^3 \sum_{j=0}^{k-1} \beta_j y'''_{n+j} \dots\dots\dots(3.3)$$

is made at each step of application of the method for solution of (1.1) Then the parameter  $\alpha_j$ 's and  $\beta_j$ 's,  $j=0(1)k$  can be determined as to ensure that  $T_{n+k}$  is minimum.

To achieve this, we adopt the following steps.

### 3.1 PROCEDURES

(i) Adopting Taylor series expansion of  $y_{n+k}$ ,  $y_{n+j}$ , and  $y_{n+j}^{(j)}$ ,  $j = 0(1)k-1$  about point  $(x_n, y_n)$  to have;

$$y_{n+k} = y_n + (kh)y_n^{(1)} + \frac{(kh)^2}{2!}y_n^{(2)} + \frac{(kh)^3}{3!}y_n^{(3)} + \frac{(kh)^4}{4!}y_n^{(4)} + \dots + \frac{(kh)^p}{p!}y_n^{(p)} + \frac{(kh)^{p+1}}{(p+1)!}y_n^{(p+1)} + \frac{(kh)^{p+2}}{(p+2)!}y_n^{(p+2)} + O(h^{p+3}) \dots \dots \dots (3.4)$$

$$y_{n+j} = y_n + (jh)y_n^{(1)} + \frac{(jh)^2}{2!}y_n^{(2)} + \frac{(jh)^3}{3!}y_n^{(3)} + \frac{(jh)^4}{4!}y_n^{(4)} + \dots + \frac{(jh)^p}{p!}y_n^{(p)} + \frac{(jh)^{p+1}}{(p+1)!}y_n^{(p+1)} + \frac{(jh)^{p+2}}{(p+2)!}y_n^{(p+2)} + O(h^{p+3})$$

$$y_{n+j}^{(j)} = y_n^{(j)} + (jh)y_n^{(j+1)} + \frac{(jh)^2}{2!}y_n^{(j+2)} + \frac{(jh)^3}{3!}y_n^{(j+3)} + \dots + \frac{(jh)^p}{p!}y_n^{(p+j)} + \frac{(jh)^{p+1}}{(p+1)!}y_n^{(p+j+1)} + \frac{(jh)^{p+2}}{(p+2)!}y_n^{(p+j+2)} + O(h^{p+3})$$

(ii) Inserting the above expansions into the error equation (3.3) and combine terms in equal power of h to have,

$$T_{n+k} = \left[ 1 - \sum_{j=0}^{k-1} \alpha_j \right] y_n + \left[ k - \sum_{j=0}^{k-1} j\alpha_j \right] hy_n^{(1)} + \left[ \frac{k^2}{2!} - \sum_{j=0}^{k-1} \frac{(j)^2 \alpha_j}{2!} \right] h^2 y_n^{(2)} + \left[ \frac{k^3}{3!} - \sum_{j=0}^{k-1} \frac{(j)^3 \alpha_j}{3!} - \sum_{j=0}^{k-1} \beta_j \right] h^3 y_n^{(3)} + \left[ \frac{k^4}{4!} - \sum_{j=0}^{k-1} \frac{(j)^4 \alpha_j}{4!} - \sum_{j=0}^{k-1} j\beta_j \right] h^4 y_n^{(4)} + \dots + \left[ \frac{k^p}{p!} - \sum_{j=0}^{k-1} \frac{j^p \alpha_j}{p!} - \sum_{j=0}^{k-1} \frac{j^{(p-2)} \beta_j}{(p-2)!} \right] h^p y_n^{(p)} + \dots + O(h^{p+1}) \dots \dots \dots (3.5)$$

Which is compactly written as

$$T_{n+k} = C_0 y_n + C_1 h y_n' + C_2 h^2 y_n'' + \dots + C_p h^p + C_{p+1} h^{p+1} + C_{p+2} h^{p+2} + O(h^{p+3})$$

Where

$$C_0 = 1 - \sum_{j=0}^{k-1} \alpha_j y_n$$

$$C_1 = k - \sum_{j=0}^{k-1} j \alpha_j h y_n'$$

$$C_2 = \frac{k^2}{2!} - \sum_{j=0}^{k-1} (j)^2 \alpha_j h^2 y_n''$$

$$C_p = \frac{k^p}{p!} - \sum_{j=0}^{k-1} \frac{j^p \alpha_j}{p!} - \sum_{j=0}^{k-1} \frac{j^{(p-2)} \beta_j}{(p-2)!} h^p y_n^{(p)} + \dots + O(h^{p+1}) \dots \dots \dots (3.6)$$

**Definition 1:** The numerical method (3.1) and the local truncation error  $T_{n+k}$  are said to

be of order  $p$  if  $C_0 = C_1 = C_2 = \dots = C_{p+1} = C_{p+2} = 0$  but  $C_{p+1} \neq 0$ .

(iii) Imposing accuracy of order  $p$  on  $T_{n+k}$  we have:

$$C_i = 0, i=0(1)p+2 \dots \dots \dots (3.7)$$

$$C_{p+3} = \frac{k^{p+3}}{(p+3)!} - \sum_{j=0}^{k-1} \frac{j^{p+3} \alpha_j}{(p+3)!} - \sum_{j=0}^{k-1} \frac{j^{(p+1)} \beta_j}{(p+1)!}$$

(iv) Solving equation (3.7) for the unknown parameters  $\alpha_j$ 's and  $\beta_j$ 's by using Gaussian elimination process.

The parameters  $\alpha_j$ 's and  $\beta_j$ 's are then chosen to ensure that one or more of the following conditions are satisfied.

(a) Adequate high order of accuracy of the scheme

- (b) Minimum bound of local truncation error
- (c) Minimum storage requirement
- (d) Maximum interval of absolute stability.

These will be done for the cases  $k=3, 4, 5$  and  $k=6$ .

### THREE STEP SCHEME

Setting  $k = 3$  in equation (3.2) we have a 3-step scheme of the form

$$y_{n+3} = \alpha_0 y_n + \alpha_1 y_{n+1} + \alpha_2 y_{n+2} + h^3 (\beta_0 y_n''' + \beta_1 y_{n+1}''' + \beta_2 y_{n+2}''') \dots\dots\dots(3.8)$$

with local truncation error

$$\begin{aligned} T_{n+3} = & (1 - \alpha_0 - \alpha_1 - \alpha_2) y_n + (3 - \alpha_1 - 2\alpha_2) h y_n' + (9/2 - \alpha_1/2 - 2\alpha_2) h^2 y_n'' + (27/6 - \alpha_1/6 \\ & - 8\alpha_2/6 - \beta_0 - \beta_1 - \beta_2) h^3 y_n''' + (81/24 - \alpha_1/24 - 16\alpha_2/24 - \beta_1 - 2\beta_2) h^4 y_n'''' + (243/120 - \alpha_1 \\ & /120 - 32\alpha_2/120 - \beta_1/2 - 2\beta_2) h^5 y_n'''' + (729/720 - \alpha_1/720 - 64\alpha_2/720 - \beta_1/6 - 8\beta_2/6) \\ & h^6 y_n'''' + (2187/5040 - \alpha_1/5040 - 128\alpha_2/5040 - \beta_1/24 - 16\beta_2/24) h^7 y_n'''' + O(h^8) \end{aligned} \dots\dots\dots(3.9)$$

Imposing accuracy of order 4 on  $(T_{n+3})$ , we have the following system of Linear equation involving the coefficients of the method

$$\begin{aligned} 1 - \alpha_0 - \alpha_1 - \alpha_2 &= 0 \\ 3 - \alpha_1 - 2\alpha_2 &= 0 \\ 9/2 - \alpha_1/2 - 2\alpha_2 &= 0 \\ 27/6 - \alpha_1/6 - 8\alpha_2/6 - \beta_0 - \beta_1 - \beta_2 &= 0 \\ 81/24 - \alpha_1/24 - 16\alpha_2/24 - \beta_1 - 2\beta_2 &= 0 \dots\dots\dots(3.10) \\ 243/120 - \alpha_1/120 - 32\alpha_2/120 - \beta_1/2 - 2\beta_2 &= 0 \\ 729/720 - \alpha_1/720 - 64\alpha_2/720 - \beta_1/6 - 8\beta_2/6 &= 0 \\ 2187/5040 - \alpha_1/5040 - 128\alpha_2/5040 - \beta_1/24 - 16\beta_2/24 &= 0 \end{aligned}$$

In matrix notations (3.9) modifies into

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 1/2 & 4/2 & 0 & 0 & 0 \\ 0 & 1/6 & 8/6 & 1 & 1 & 1 \\ 0 & 1/24 & 16/24 & 0 & 1 & 2 \\ 0 & 1/120 & 32/120 & 0 & 1/2 & 4/2 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 9/2 \\ 27/6 \\ 81/24 \\ 243/120 \end{pmatrix} \dots\dots\dots(3.11)$$

Solving (3.11) by Gaussian elimination technique, the value of the parameters are obtained as

$$\alpha_0 = 1, \alpha_1 = -3, \alpha_2 = 3, \beta_0 = 0, \beta_1 = 1/2, \beta_2 = 1/2, \beta_3 = 0 \dots\dots\dots(3.12)$$

Substituting the values of these parameters into the expansion in (3.8) above, the general 3-step linear multistep-method is

$$y_{n+3} = 3y_{n+2} - 3y_{n+1} + y_n + \frac{h^3}{2} [f_{n+2} + f_{n+1}] \dots\dots\dots(3.13)$$

**Definition 2:-** A method is said to be of order p, if p is the largest positive integer for which

$$C_0 = C_1 = C_2 = C_3 = \dots\dots\dots C_{p+2} = 0 \text{ but } C_{p+3} \neq 0$$

According to definition 2, method (3.12) is said to be of order  $P+3 = 7$  where  $P = 4$  and principal truncation error  $C_7 = 1/240$

## FOUR – STEP SCHEME

Setting  $k = 4$  in (3.2) we have a 4-step scheme of the form

$$y_{n+4} = \alpha_0 y_n + \alpha_1 y_{n+1} + \alpha_2 y_{n+2} + \alpha_3 y_{n+3} + h^3 (\beta_0 y''_n + \beta_1 y''_{n+1} + \beta_2 y''_{n+2} + \beta_3 y''_{n+3}) \quad (3.14)$$

with local truncation error

$$\begin{aligned} T_{n+4} = & (1-\alpha_0 - \alpha_1 - \alpha_2 - \alpha_3) y_n + (4 - \alpha_1 - 2\alpha_2 - 3\alpha_3) h y'_n \\ & + (16/2 - \alpha_1/2 - 4/2\alpha_2 - 9/2\alpha_3) h^2 y''_n + (64/6 - \alpha_1/6 - 8\alpha_2/6 - 27\alpha_3/6 \cdot \beta_0 - \beta_1 - \beta_2 - \beta_3) h^3 y'''_n + \\ & \{256/24 - 1/24[\alpha_1 + 16\alpha_2 + 81\alpha_3] + (\beta_1 + 2\beta_2 + 3\beta_3)\} h^4 y''''_n + \\ & \{1024/120 - \{1/120[\alpha_1 + 32\alpha_2 + 243\alpha_3] - 1/2[\beta_1 + 4\beta_2 + 9\beta_3]\}\} h^5 y''''''_n + \\ & \{4096/720 - 1/720[\alpha_1 + 64\alpha_2 + 729\alpha_3] - 1/6[\beta_1 + 8\beta_2 + 27\beta_3]\} h^6 y''''''''_n + \\ & \{16384/5040 - 1/5040[\alpha_1 + 128\alpha_2 + 2187\alpha_3] - 1/24[\beta_1 + 16\beta_2 + 81\beta_3]\} h^7 y''''''''''_n + \\ & \{65536/40320 - 1/40320[\alpha_1 + 256\alpha_2 + 6561\alpha_3] - 1/120[\beta_1 + 32\beta_2 + 243\beta_3]\} h^8 y''''''''''''_n + \dots + O(h^9) \dots \quad (3.15) \end{aligned}$$

Imposing accuracy of order 5 on  $T_{n+4}$  we have

$$\begin{aligned} 1 - \alpha_0 - \alpha_1 - \alpha_2 - \alpha_3 &= 0 \\ 4 - \alpha_1 - 2\alpha_2 - 3\alpha_3 &= 0 \\ 16/2 - \alpha_1/2 - 4/2\alpha_2 - 9/2\alpha_3 &= 0 \\ 64/6 - \alpha_1/6 - 8\alpha_2/6 - 27\alpha_3/6 \cdot \beta_0 - \beta_1 - \beta_2 - \beta_3 &= 0 \\ \{256/24 - 1/24[\alpha_1 + 16\alpha_2 + 81\alpha_3] + (\beta_1 + 2\beta_2 + 3\beta_3)\} &= 0 \quad \dots \quad (3.16) \\ \{1024/120 - \{1/120[\alpha_1 + 32\alpha_2 + 243\alpha_3] - 1/2[\beta_1 + 4\beta_2 + 9\beta_3]\}\} &= 0 \\ \{4096/720 - 1/720[\alpha_1 + 64\alpha_2 + 729\alpha_3] - 1/6[\beta_1 + 8\beta_2 + 27\beta_3]\} &= 0 \\ \{16384/5040 - 1/5040[\alpha_1 + 128\alpha_2 + 2187\alpha_3] - 1/24[\beta_1 + 16\beta_2 + 81\beta_3]\} &= 0 \\ \{65536/40320 - 1/40320[\alpha_1 + 256\alpha_2 + 6561\alpha_3] - 1/120[\beta_1 + 32\beta_2 + 243\beta_3]\} &\neq 0 \end{aligned}$$

which in matrix notation, becomes,

$$\begin{pmatrix}
 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
 0 & 1 & 2 & 3 & 0 & 0 & 0 & 0 \\
 0 & 1/2 & 4/2 & 9/2 & 0 & 0 & 0 & 0 \\
 0 & 1/6 & 8/6 & 27/6 & 1 & 1 & 1 & 1 \\
 0 & 1/24 & 16/24 & 81/24 & 0 & 1 & 2 & 3 \\
 0 & 1/120 & 32/120 & 243/120 & 0 & 1/2 & 4/2 & 9/2 \\
 0 & 1/720 & 64/720 & 729/720 & 0 & 1/6 & 8/6 & 27/6 \\
 0 & 1/5040 & 128/5040 & 2187/5040 & 0 & 1/24 & 16/24 & 81/24
 \end{pmatrix}
 \begin{pmatrix}
 \alpha_0 \\
 \alpha_1 \\
 \alpha_2 \\
 \alpha_3 \\
 \beta_0 \\
 \beta_1 \\
 \beta_2 \\
 \beta_3
 \end{pmatrix}
 \begin{pmatrix}
 1 \\
 4 \\
 16/2 \\
 64/6 \\
 256/24 \\
 1024/120 \\
 4096/720 \\
 1681/5040
 \end{pmatrix}
 \tag{3.17}$$

Solving (3.17) with Gaussian elimination technique, we have

$$\alpha_0 = -1, \alpha_1 = 4, \alpha_2 = -6, \alpha_3 = 4, \beta_0 = 0, \beta_1 = -1/2, \beta_2 = 0, \beta_3 = 1/2, \beta_4 = 0 \dots \dots \tag{3.18}$$

Substituting the values of these parameters into the expansion in (3.14) above, we have the general Explicit 4-step linear multistep method of the form

$$y_{n+4} = 4y_{n+3} - 6y_{n+2} + 4y_{n+1} - y_n + \frac{h^3}{2} [f_{n+3} - f_{n+1}] \tag{3.19}$$

## FIVE -STEP SCHEME

Setting k =5 in (3.5) we have a 5-step scheme of the form

$$y_{n+5} = \alpha_0 y_n + \alpha_1 y_{n+1} + \alpha_2 y_{n+2} + \alpha_3 y_{n+3} + \alpha_4 y_{n+4} + h^3 (\beta_0 y''_n + \beta_1 y''_{n+1} + \beta_2 y''_{n+2} + \beta_3 y''_{n+3} + \beta_4 y''_{n+4}) \tag{3.20}$$

with local truncation error

$$\begin{aligned}
 T_{n+5} = & (1 - \alpha_0 - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4) y_n + (5 - \alpha_1 - 2\alpha_2 - 3\alpha_3 - 4\alpha_4) h y'_n \\
 & + (25/2 - \alpha_1/2 - 4/2\alpha_2 - 9/2\alpha_3 - 16/2\alpha_4) h^2 y''_n + \\
 & \{125/6 - 1/6(\alpha_1 - 8\alpha_2 - 27\alpha_3 - 64\alpha_4 - \beta_0 - \beta_1 - \beta_2 - \beta_3 - \beta_4)\} h^3 y'''_n + \\
 & \{625/24 - 1/24[\alpha_1 + 16\alpha_2 + 81\alpha_3 + 256\alpha_4] - (\beta_1 + 2\beta_2 + 3\beta_3 + 4\beta_4)\} h^4 y''''_n + \\
 & \{3125/120 - 1/120[\alpha_1 + 32\alpha_2 + 243\alpha_3 + 1024\alpha_4] - 1/2[\beta_1 + 4\beta_2 + 9\beta_3 + 16\beta_4]\} h^5 y''''''_n + \\
 & \{15625/720 - 1/720[\alpha_1 + 64\alpha_2 + 729\alpha_3 + 4096\alpha_4] - 1/6[\beta_1 + 8\beta_2 + 27\beta_3 + 64\beta_4]\} h^6 y''''''''_n + \dots
 \end{aligned}$$

$$\begin{aligned}
& \{78125/5040 - 1/5040 [\alpha_1 + 128\alpha_2 + 2187\alpha_3 + 16384\alpha_4] - 1/24 [\beta_1 + 16\beta_2 + 81\beta_3 + 243\beta_4]\} h^7 y_n^{(7)} + \\
& \{390625/40320 - 1/40320 [\alpha_1 + 256\alpha_2 + 6561\alpha_3 + 65536\alpha_4] - 1/120 [\beta_1 + 32\beta_2 + 243\beta_3 + 1024\beta_4]\} h^8 y_n^{(8)} + \\
& \{1953125/362880 - 1/362880 [\alpha_1 + 512\alpha_2 + 19683\alpha_3 + 262144\alpha_4] - 1/720 [\beta_1 + 64\beta_2 + 729\beta_3 \\
& + 4096\beta_4]\} h^9 y_n^{(9)} + \{9756525/3628800 - 1/3628800 [\alpha_1 + 1024\alpha_2 + 5904\alpha_3 + 1048576\alpha_4] - 1/5040 [\beta_1 + \\
& 128\beta_2 + 2187\beta_3 + 16384\beta_4]\} h^{10} y_n^{(10)} + \dots \dots \dots O(h^{11}) \dots \dots \dots (3.21)
\end{aligned}$$

Imposing accuracy of order 6 on  $T_{n+5}$ , we have the following system of linear equations involving the unknown coefficients of the method

$$\begin{aligned}
1 - \alpha_0 - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 &= 0 \\
5 - \alpha_1 - 2\alpha_2 - 3\alpha_3 - 4\alpha_4 &= 0 \\
25/2 - \alpha_1/2 - 4/2\alpha_2 - 9/2\alpha_3 - 16/2\alpha_4 &= 0 \\
\{125/6 - 1/6 [\alpha_1 + 8\alpha_2 + 27\alpha_3 + 64\alpha_4] - \beta_0 - \beta_1 - \beta_2 - \beta_3 - \beta_4\} &= 0 \\
\{625/24 - 1/24 [\alpha_1 + 16\alpha_2 + 81\alpha_3 + 256\alpha_4] - (\beta_1 + 2\beta_2 + 3\beta_3 + 4\beta_4)\} &= 0 \\
\{3125/120 - \{1/120 [\alpha_1 + 32\alpha_2 + 243\alpha_3 + 1024\alpha_4] - 1/2 [\beta_1 + 4\beta_2 + 9\beta_3 + 16\beta_4]\}\} &= 0 \\
\{15625/720 - 1/720 [\alpha_1 + 64\alpha_2 + 729\alpha_3 + 4096\alpha_4] - 1/6 [\beta_1 + 8\beta_2 + 27\beta_3 + 64\beta_4]\} &= 0 \\
\{78125/5040 - 1/5040 [\alpha_1 + 128\alpha_2 + 2187\alpha_3 + 16384\alpha_4] \\
- 1/24 [\beta_1 + 16\beta_2 + 81\beta_3 + 256\beta_4]\} &= 0 \\
\{390625/40320 - 1/40320 [\alpha_1 + 256\alpha_2 + 6561\alpha_3 + 65536\alpha_4] - 1/120 [\beta_1 + 32\beta_2 + 243\beta_3 \\
+ 1024\beta_4]\} &= 0 \\
\{1953125/362880 - 1/362880 [\alpha_1 + 512\alpha_2 + 19683\alpha_3 + 262144\alpha_4] \\
- 1/720 [\beta_1 + 64\beta_2 + 729\beta_3 + 4096\beta_4]\} &\neq 0 \dots \dots \dots (3.22)
\end{aligned}$$

In matrix notation, equation, (3.22) modifies into the matrix equation



Solving (3.23) with Gaussian elimination technique, we have

$$\alpha_0 = -1, \alpha_1 = -5, \alpha_2 = 10, \alpha_3 = -10, \alpha_4 = -5, \alpha_5 = -1, \beta_0 = 0, \beta_1 = 1/2, \beta_2 = -1/2, \beta_3 = -1/2, \beta_4 = 1/2, \beta_5 = 0 \dots\dots\dots(3.24)$$

Using these values in equation (3.20) we have the general fifth step method of the form

$$y_{n+5} = 5y_{n+4} - 10y_{n+3} + 10y_{n+2} - 5y_{n+1} + y_n + \frac{h^5}{2} [f_{n+4} - f_{n+3} - f_{n+2} + f_{n+1}] \dots\dots\dots(3.25)$$

### SIX – STEP SCHEME

Setting k = 6 in (3.5) we have a 6-step scheme of the form

$$y_{n+6} = \alpha_0 y_n + \alpha_1 y_{n+1} + \alpha_2 y_{n+2} + \alpha_3 y_{n+3} + \alpha_4 y_{n+4} + \alpha_5 y_{n+5} + h^3 (\beta_0 y''_n + \beta_1 y''_{n+1} + \beta_2 y''_{n+2} + \beta_3 y''_{n+3} + \beta_4 y''_{n+4} + \beta_5 y''_{n+5}) \dots\dots\dots(3.26)$$

with local truncation error

$$\begin{aligned} T_{n+6} = & (1 - \alpha_0 - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5) y_n + (6 - \alpha_1 - 2\alpha_2 - 3\alpha_3 - 4\alpha_4 - 5\alpha_5) h y'_n \\ & + \{36/2 - 1/2 [\alpha_1 + 4\alpha_2 + 9\alpha_3 + 16\alpha_4 + 25\alpha_5]\} h^2 y''_n + \\ & \{216/6 - 1/6(\alpha_1 + 8\alpha_2 + 27\alpha_3 + 64\alpha_4 + 125\alpha_5 - \beta_0 - \beta_1 - \beta_2 - \beta_3 - \beta_4 - \beta_5)\} h^3 y'''_n + \\ & \{1296/24 - 1/24[\alpha_1 + 16\alpha_2 + 81\alpha_3 + 256\alpha_4 + 625\alpha_5] - (\beta_1 + 2\beta_2 + 3\beta_3 + 4\beta_4 + 5\beta_5)\} h^4 y''''_n + \\ & \{7776/120 - 1/120[\alpha_1 + 32\alpha_2 + 243\alpha_3 + 1024\alpha_4 + 3125\alpha_5] - 1/2[\beta_1 + 4\beta_2 + 9\beta_3 + 16\beta_4 + 16\beta_5]\} h^5 y''''''_n + \\ & \{46656/720 - 1/720[\alpha_1 + 64\alpha_2 + 729\alpha_3 + 4096\alpha_4 + 15625\alpha_5] - 1/6[\beta_1 + 8\beta_2 + 27\beta_3 + 64\beta_4 + 125\beta_5]\} h^6 y''''''''_n \\ & \{279936/5040 - 1/5040[\alpha_1 + 128\alpha_2 + 2187\alpha_3 + 16384\alpha_4 + 78125\alpha_5] \\ & - 1/24[\beta_1 + 16\beta_2 + 81\beta_3 + 256\beta_4 + 625\beta_5]\} h^7 y''''''''''_n + \\ & \{1679616/40320 - 1/40320[\alpha_1 + 256\alpha_2 + 6561\alpha_3 + 65536\alpha_4 + 390625\alpha_5] \\ & - 1/120[\beta_1 + 32\beta_2 + 243\beta_3 + 1024\beta_4 + 3125\beta_5]\} h^8 y''''''''''''_n + \dots\dots\dots + \\ & \{10077696/362880 - 1/362880[\alpha_1 + 512\alpha_2 + 19683\alpha_3 + 262144\alpha_4 + 1953125\alpha_5] \\ & - 1/720[\beta_1 + 64\beta_2 + 729\beta_3 + 4096\beta_4 + 15625\beta_5]\} h^9 y''''''''''''''_n + \\ & \{60466176/3628800 - 1/3628800[\alpha_1 + 1024\alpha_2 + 59049\alpha_3 + 1048576\alpha_4 + 9765625\alpha_5] \\ & - 1/5040[\beta_1 + 128\beta_2 + 2187\beta_3 + 16384\beta_4 + 78125\beta_5]\} + \\ & h^{10} y''''''''''''''''_n + \dots\dots\dots O(h^{11}) \dots\dots\dots (3.27) \end{aligned}$$

Imposing accuracy of order 7 on  $T_{n+6}$ , we have the following system of linear equation involving the unknown coefficients of the method as:

$$\begin{aligned}
 1 - \alpha_0 - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5 &= 0 \\
 6 - \alpha_1 - 2\alpha_2 - 3\alpha_3 - 4\alpha_4 - 5\alpha_5 &= 0 \\
 36/2 - 1/2 [\alpha_1 + 4\alpha_2 + 9\alpha_3 + 16\alpha_4 + 25\alpha_5] &= 0 \\
 \{216/6 - 1/6 [\alpha_1 + 8\alpha_2 + 27\alpha_3 + 64\alpha_4 + 125\alpha_5 - \beta_0 - \beta_1 - \beta_2 - \beta_3 - \beta_4 - \beta_5]\} &= 0 \\
 \{1296/24 - 1/24 [\alpha_1 + 16\alpha_2 + 81\alpha_3 + 256\alpha_4 + 625\alpha_5] - (\beta_1 + 2\beta_2 + 3\beta_3 + 4\beta_4 + 5\beta_5)\} &= 0 \\
 \{7776/120 - \{1/120 [\alpha_1 + 32\alpha_2 + 243\alpha_3 + 1024\alpha_4 + 3125\alpha_5] - 1/2[\beta_1 + 4\beta_2 + 9\beta_3 + 16\beta_4 + 25\beta_5]\}\} &= 0 \\
 \{46656/720 - 1/720 [\alpha_1 + 64\alpha_2 + 729\alpha_3 + 4096\alpha_4 + 15625\alpha_5] \\
 - 1/6[\beta_1 + 8\beta_2 + 27\beta_3 + 64\beta_4 + 125\beta_5]\} &= 0 \\
 \{279936/5040 - 1/5040 [\alpha_1 + 128\alpha_2 + 2187\alpha_3 + 16384\alpha_4 + 78125\alpha_5] \\
 - 1/24 [\beta_1 + 16\beta_2 + 81\beta_3 + 256\beta_4 + 625\beta_5]\} &= 0 \\
 \{1679616/40320 - 1/40320 [\alpha_1 + 256\alpha_2 + 6561\alpha_3 + 65536\alpha_4 + 390625\alpha_5] \\
 - 1/120 [\beta_1 + 32\beta_2 + 243\beta_3 + 1024\beta_4 + 3125\beta_5]\} &= 0 \\
 \{10077696/362880 - 1/362880 [\alpha_1 + 512\alpha_2 + 19683\alpha_3 + 262144\alpha_4 + 1953125\alpha_5] \\
 - 1/720 [\beta_1 + 64\beta_2 + 729\beta_3 + 4096\beta_4 + 15625\beta_5]\} &\neq 0 \dots\dots\dots(3.28)
 \end{aligned}$$

In matrix notation, Equations (3.28) modifies into equation



Solving (3.29) with Gaussian elimination technique, we have

$$\alpha_0 = -1, \alpha_1 = 6, \alpha_2 = -15, \alpha_3 = 20, \alpha_4 = -15, \alpha_5 = 6, \alpha_6 = 1, \beta_0 = 0, \beta_1 = -1/2, \beta_2 = 1, \beta_3 = 0, \beta_4 = -1, \beta_5 = 1/2, \beta_6 = 0 \quad \dots\dots\dots (3.30)$$

Substituting the values of these parameters into the expansion (3.26) we have the general sixth step method of the form

$$y_{n+6} = 6y_{n+5} - 15y_{n+4} + 20y_{n+3} - 15y_{n+2} + 6y_{n+1} - y_n + \frac{h^3}{2} [f_{n+5} - 2f_{n+4} + 2f_{n+2} - f_{n+1}] \quad \dots\dots\dots (3.31)$$

The results of all these calculations are displayed in table 1. below

**TABLE 1** :- Coefficients of the proposed k – Step Linear Multistep Methods ( $3 \leq k \leq 6$ )

Step Nos (k)	Coefficients of the Formula $\alpha_j$ 's and $\beta_j$ 's														Order of the method	Error Constant
	$\alpha_0$	$\alpha_1$	$\alpha_2$	$\alpha_3$	$\alpha_4$	$\alpha_5$	$\alpha_6$	$\beta_0$	$\beta_1$	$\beta_2$	$\beta_3$	$\beta_4$	$\beta_5$	$\beta_6$		
3	1	-3	3	1				0	1/2	1/2	0				4	1/240
4	-1	4	-6	4	1			0	-1/2	0	1/2	0			5	1/240
5	-1	-5	10	-10	5	1		0	1/2	-1/2	-1/2	1/2	0		6	1/240
6	-1	6	-15	20	-15	6	1	0	-1/2	1	0	-1	1/2	0	7	1/240

In general, the order of the methods is  $p = k+1$

### DEVELOPMENT OF PREDICTORS

The characteristic of Linear Multistep Methods (3.2) shows that, they require two or more starting values before they can function. Thus, to implement these methods, there is need to have a helping method to compute these starting values and these helping methods are called the predictors

### PREDICTORS FOR THE 3-STEP METHOD

To implement the scheme (3.12) for the solution of third order initial value problem of type (1.1) there is need to develop the predictors for the evaluation of  $y_{n+2}$ ,  $y_{n+1}$  and its derivatives  $y'_{n+2}$ ,  $y'_{n+1}$  in order to evaluate  $f_{n+j} = f(x_{n+j}, y_{n+j}, y'_{n+j})$ ,  $j = 1, 2$ . In order not to have initial data error in our computation, there is need for these predictors to have the same order of accuracy as the main method. Consequently, Taylor series expansion method is as the predictor that has the same order of accuracy as the main methods. This yields

$$y_{n+1} = y_n + hy_n^{(1)} + \frac{(h)^2}{2!} y_n^{(2)} + \frac{(h)^3}{3!} y_n^{(3)} + \frac{(h)^4}{4!} y_n^{(4)} + \frac{(h)^5}{5!} y_n^{(5)} + \dots + O(h^6)$$

.....(3.32)

$$y_{n+2} = y_n + 2hy_n^{(1)} + \frac{(2h)^2}{2!} y_n^{(2)} + \frac{(2h)^3}{3!} y_n^{(3)} + \frac{(2h)^4}{4!} y_n^{(4)} + \frac{(2h)^5}{5!} y_n^{(5)} + \dots + O(h^6)$$

.....(3.33)

Differentiating (3.31) and (3.32) once; yields

$$y'_{n+1} = y_n^{(1)} + hy_n^{(2)} + \frac{(h)^2}{2!} y_n^{(3)} + \frac{(h)^3}{3!} y_n^{(4)} + \frac{(h)^4}{4!} y_n^{(5)} + \dots + O(h^6)$$

$$y'_{n+2} = y_n^{(1)} + 2hy_n^{(2)} + \frac{(2h)^2}{2!} y_n^{(3)} + \frac{(2h)^3}{3!} y_n^{(4)} + \frac{(2h)^4}{4!} y_n^{(5)} + \dots + O(h^6)$$

.....(3.34)

Equation (3.32, 3.33, 3.34) are used for the evaluation of  $y_{n+2}$  and  $y'_{n+2}$  respectively in the function  $f_{n+1} = f(x_{n+1}, y_{n+1}, y'_{n+1})$  and  $f_{n+2} = f(x_{n+2}, y_{n+2}, y'_{n+2})$ . Predictors for higher step numbers will be adopting the lower step schemes. For example, the 4-step method will adopt the 3-step scheme as predictor for  $y_{n+3}$  appearing in its equation.

## CHAPTER FOUR



### ANALYSIS OF THE BASIC PROPERTIES OF THE METHODS

A characteristic of Numerical Schemes for solution of ordinary differential equations is that errors are generated as a result of the process of derivation and implementation. The magnitude of these errors determines the degree of accuracy, consistency, zero-stability convergence and stability of the Schemes.

This is the reason why the analysis of these basic properties of the Method is highly essential. The analysis forms the major objective of this chapter.

#### 4.1 THE BASIC PROPERTIES OF 3-STEP METHOD

##### 4.1.1 ACCURACY OF THE METHOD

Accuracy is determined by the magnitude of the error arising from the process of derivation and implementation. If the magnitude is small, then the method is said to be accurate and inaccurate if otherwise. Some of the major sources of these errors are:-

- (i) Truncation error
- (ii) Discretization error
- (iii) Round off error
- (iv) Error propagation

Truncation error can be defined as the error that occurs because of the replacement of infinite process by a finite one. While Discretization error can be defined as the error introduced by the replacement of continuous equation by its finite difference equivalences.

Round off error can be defined as error that occurs as a result of the adoption of number with infinite number of digits by a finite number of digits.

Error propagation can be defined as the error that occurs as a result of errors inheritance from step to step.

The degree of accuracy of a method is measured from the order of accuracy and error constant of the method. These concepts are examined below.

#### 4.1.1 ORDER OF ACCURACY AND ERROR CONSTANT

As mentioned earlier, the local truncation error of the 3-step method is

$$T_{n+k} = \sum_{j=0}^3 \{ \alpha_j y(x_{n+j}) - h^3 \beta_j y'''(x_{n+j}) \} \dots\dots\dots(4.1)$$

where  $\alpha_3 = 1$ ,  $\alpha_0$  and  $\beta_0$  are not both zero.

Adopting Taylor series expansion of  $y(x_{n+j})$ ,  $j = 0(1)3$  about  $x = x_n$  as in (3.5) and

combining terms in equal powers of  $h$ , we obtain

$$T_{n+3} = c_0 y(x_n) + c_1 h y'(x_n) + c_2 h^2 y''(x_n) + c_3 h^3 y'''(x_n) + \dots\dots\dots + c_p h^p y^{(p)}(x_n) + c_{p+1} h^{p+1} y^{(p+1)}(x_n) + c_{p+2} h^{p+2} y^{(p+2)}(x_n) + O(h^{p+3})$$

Where

$$\begin{aligned} C_0 &= 1 - \alpha_0 - \alpha_1 - \alpha_2 \\ C_1 &= 3 - \alpha_1 - 2\alpha_2 \\ C_2 &= 9/2 - \alpha_1/2 - 2\alpha_2 \\ C_3 &= 27/6 - \alpha_1/6 - 8\alpha_2/6 - \beta_0 - \beta_1 - \beta_2 \\ C_4 &= 81/24 - \alpha_1/24 - 16\alpha_2/24 - \beta_1 - 2\beta_2 \dots\dots\dots(4.2) \\ C_5 &= 243/120 - \alpha_1/120 - 32\alpha_2/120 - \beta_1/2 - 2\beta_2 \\ C_6 &= 729/720 - \alpha_1/720 - 64\alpha_2/720 - \beta_1/6 - 8\beta_2/6 \\ C_7 &= 2187/5040 - \alpha_1/5040 - 128\alpha_2/5040 - \beta_1/24 - 16\beta_2/24 \end{aligned}$$

Using the results obtained in (3.11) and substitutes into (4.2) to have

$$\begin{aligned}
 C_0 &= 1 - 3 + 3 - 1 = 0 \\
 C_1 &= 3 - 6 + 3 = 0 \\
 C_2 &= \frac{9}{2} - 6 + \frac{3}{2} = 0 \\
 C_3 &= \frac{27}{6} - \frac{24}{6} + \frac{3}{6} - \frac{1}{2} - \frac{1}{2} = 0 \\
 C_4 &= \frac{81}{24} + \frac{3}{24} - \frac{48}{24} - \frac{1}{2} - 1 = 0 \dots\dots\dots(4.3) \\
 C_5 &= \frac{243}{120} + \frac{3}{120} - \frac{96}{120} - \frac{1}{4} - 1 = 0 \\
 C_6 &= \frac{729}{720} + \frac{3}{720} - \frac{192}{720} - \frac{8}{12} - \frac{1}{12} = 0 \\
 C_7 &= \frac{2187}{5040} + \frac{3}{5040} - \frac{384}{5040} - \frac{1}{48}(-16 - 1) = \frac{1}{240}
 \end{aligned}$$

Since  $C_0=C_1=C_2=C_3=C_4=C_5=C_6=0$  but  $C_7 \neq 0$  then, by Lambert (1973, 1991), Jain (1979), Fatunla (1988), Awoyemi(2002), and Kayode(2004) it implies that the order  $p$  of accuracy of the 3-step method is  $p+3 = 7$ . That is  $p = 4$ . Hence the method is of order 4 and the principal error constant  $T_{n+3} = 1/240$

**4.1.2. CONSISTENCY**

According to Lambert (1973), a linear multistep method is said to be consistent if and only if its order of accuracy  $P \geq 1$  and its characteristic polynomial functions  $\rho(r)$  and

$\sigma(r)$  given by  $\rho(r) = r^3 - 3r^2 + 3r - 1$  and  $\sigma(r) = r^2 + r$  satisfy the conditions:

- (i)  $\sum_{j=0}^k \alpha_j r^j = 0$
- (ii).  $\rho'(r) = \rho''(r) = 0$ , for  $r = 1$
- (iii).  $\rho'''(r) = 3! \sigma(r)$ , for  $r = 1$  i.e.  $\rho'''(1) = 3! \sigma(1)$

Where  $\rho$  and  $\sigma$  are called first and second characteristic polynomials of the method respectively. The three-step method is of order 4, and therefore satisfies the order condition.

Consider the stability equation:

$\Pi(r, h) = \rho(r) - h\sigma(r)$  where  $\rho(r)$  and  $\sigma(r)$  are first and second characteristic polynomials of the method respectively. They are given as

$$\rho(r) = r^3 - 3r^2 + 3r - 1$$

$$\sigma(r) = r^2 + r$$

respectively

Now,

$$(i) \quad \sum_{j=0}^3 \alpha_j r^j = (\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3) = -1 + 3 - 3 + 1 = 0$$

and

$$(ii) \quad \rho(r) = r^3 - 3r^2 + 3r - 1 \\ = (r-1)^3 = 0$$

That is, the roots of the first characteristic polynomial are  $r = 1, 1, 1$

Also,

$$\rho'(r) = 3r^2 - 6r + 3$$

$$\rho''(r) = 6r - 6$$

$$= 6(r-1)$$

$$\rho(1) = 6(1) - 6$$

$$= 0$$

$$(iii) \quad \rho(r) = r^3 - 3r^2 + 3r - 1$$

$$\rho'(r) = 3r^2 - 6r + 3$$

$$\rho''(r) = 6r - 6$$

$$= 6(r-1)$$

$$= 3!(r-1)$$

$$\rho'''(r) = 6$$

$$= 3!$$

$$\sigma(r) = \frac{1}{2} [r^2 + r]$$

$$r(1) = \frac{1}{2} (1 + 1)$$

$$= \frac{1}{2} (2)$$

$$= 1$$

$$\sigma(1) = 1$$

$$\text{Hence } \rho'''(1) = 3! \sigma(1)$$

Therefore, since the conditions for Consistency are met, then the method is consistent.

#### 4.1.3 STABILITY ANALYSIS

There are three types of stability; namely zero stability, absolute stability and p-stability. They are discussed below.

##### 4.1.3.1 ZERO STABILITY

**Definition 3:-** A Linear Multistep method for a given initial Value problem is said to be zero stable if no root of the First Characteristic Polynomial  $\rho(r)$  has modulus greater than one and if every root of modulus one is simple. The roots with modulus one are called the principal roots and the other roots are called the spurious roots.

The first characteristic polynomial equation of method (3.12) is

$$\rho(r) = r^3 - 3r^2 + 3r - 1$$

Which simplifies into  $(r-1)^3 = 0$

and its roots are  $r = (1, 1, 1)$

Since the roots condition is satisfied, hence the 3-step method is zero stable.

##### 4.1.3.2 INTERVAL OF ABSOLUTE STABILITY.

To determine the interval or region of absolute stability of the methods, we adopt the Boundary Locus Method as reported in [Lambert, (1973, 1991) and Fatunla, (1988)].

A Linear Multistep method is said to be absolutely stable in a region  $C$  of the complex plane  $C = \{h \mid \forall h \in C, \pi(r, h) = 0\}$ , if for all  $h \in C$ , all roots ( $r_s$ ) of the stability Polynomial  $\pi(r, h)$  associated with the method satisfy  $|r_s| < 1, s = 1, 2, \dots, k$ . That is, all the roots  $r_s$  of the stability polynomials lie within a unit circle.

**Definition 4:-** A numerical method is said to be A-stable if its region of absolute stability contains the whole of the left-hand half-plane  $\text{Re}h < 0$ . An interval  $(\alpha, \beta)$  of the real line is said to be an interval of absolute stability if the method is absolutely stable for all  $h \in (\alpha, \beta)$ .

**Definition 5:- P - STABILITY**

A Numerical method is said to be P-stable if its region of absolute stability contains the whole of the right – hand side of the complex half plane. That is the entire positive half part of the complex plane.  $(0, \infty)$

**Definition 6:-** A linear multistep method is said to be stable if it is consistent, zero stable and convergent.

**Definition 7:-** A linear multistep method is said to be convergent if it is consistent and zero stable. [Lambert, 1973].

Since the three-step method is consistent and zero stable; then the method is convergent.

Hence  $\lim_{n \rightarrow \infty} y_n = y(x_n)$

From the stability equation

$\Pi(r, h) = \rho(r) - h\sigma(r) = 0$

we have

$$\frac{h(r) = \rho(r)}{\sigma(r)} \dots \dots \dots (4.4)$$

Substituting the values of  $\rho(r)$  and  $\sigma(r)$  as in (4.4) we have

$$\bar{h}(r) = \frac{2[r^3 - 3r^2 + 3r - 1]}{r^2 + r} \dots\dots\dots(4.5)$$

Using the transformation  $r = e^{i\theta}$ ,  $0 \leq \theta \leq \pi$ , which by Euler is given as

$$e^{i\theta} = \cos \theta + i \sin \theta$$

Therefore,

$$r = \cos \theta + i \sin \theta. \dots\dots\dots(4.6)$$

Substituting this into (4.6), we get

$$\bar{h}(\theta) = \frac{2 \{(\cos 3\theta - 3\cos 2\theta + 3\cos \theta - 1) + i(\sin 3\theta - 3\sin 2\theta + 3\sin \theta)\}}{\{( \cos 2\theta + \cos \theta) + i(\sin 2\theta + \sin \theta)\}} \dots\dots\dots(4.7)$$

Rationalizing and simplifying (4.7) to get

$$\bar{h}(\theta) = x(\theta) + iy(\theta)$$

Where

$$x(\theta) = 0$$

And

$$y(\theta) = \frac{2 [6\sin 2\theta - 3\sin 3\theta]}{2 + 2\cos \theta} \dots\dots\dots(4.8)$$

Evaluation of  $x(\theta)$  and  $y(\theta)$ ,  $0 \leq \theta \leq 180^\circ$  at intervals of  $30^\circ$  gives the following results as shown in the table 2 below:

**TABLE 2: REGION OF ABSOLUTE STABILITY OF 3-STEP METHOD**

$\theta$	0	30	60	90	120	150	180
$X(\theta)$	0	0	0	0	0	0	0
$y(\theta)$	0	2.1339	2.5980	-3.0000	-15.5880	-49.9700	$\infty$

The result from the table above shows that the three-step method has no interval of absolute stability that implies that it is neither A-stable nor P-stable.

#### 4.2.5 THE BASIC PROPERTIES OF THE OF FOUR -STEP METHOD

##### 4.2.5.1 ORDER OF ACCURACY AND ERROR CONSTANT

Recalling equation (3.17) we have

$$y_{n+4} = 4y_{n+3} - 6y_{n+2} + 4y_{n+1} - y_n + \frac{h^3}{2} [f_{n+3} - f_{n+1}]$$

with the local truncation error

$$T_{n+4} = c_0 y(x_n) + c_1 h y'(x_n) + c_2 h^2 y''(x_n) + c_3 h^3 y'''(x_n) + \dots + c_p h^p y^{(p)}(x_n) + c_{p+1} h^{p+1} y^{(p+1)}(x_n) + c_{p+2} h^{p+2} y^{(p+2)}(x_n) + O(h^{p+3})$$

Where

$$C_0 = 1 - \alpha_0 - \alpha_1 - \alpha_2 - \alpha_3$$

$$C_1 = 4 - \alpha_1 - 2\alpha_2 - 3\alpha_3$$

$$C_2 = 16/2 - \alpha_1/2 - 4/2\alpha_2 - 9/2\alpha_3$$

$$C_3 = 64/6 - \alpha_1/6 - 8\alpha_2/6 - 27\alpha_3/6, \beta_0 - \beta_1 - \beta_2 - \beta_3$$

$$C_4 = \{256/24 - 1/24[\alpha_1 + 16\alpha_2 + 81\alpha_3] + (\beta_1 + 2\beta_2 + 3\beta_3)\} \dots \dots \dots (4.9)$$

$$C_5 = \{1024/120 - \{1/120[\alpha_1 + 32\alpha_2 + 243\alpha_3] - 1/2[\beta_1 + 4\beta_2 + 9\beta_3]\}\}$$

$$C_6 = \{4096/720 - 1/720[\alpha_1 + 64\alpha_2 + 729\alpha_3] - 1/6[\beta_1 + 8\beta_2 + 27\beta_3]\}$$

$$C_7 = \{16384/5040 - 1/5040[\alpha_1 + 128\alpha_2 + 2187\alpha_3] - 1/24[\beta_1 + 16\beta_2 + 81\beta_3]\}$$

$$C_8 = \{65536/40320 - 1/40320[\alpha_1 + 256\alpha_2 + 6561\alpha_3] - 1/120[\beta_1 + 32\beta_2 + 243\beta_3]\}$$

The results obtained in (3.16) is substituted back into (4.9) to have

$$C_0 = 1 - 4 + 6 - 4 + 1 = 0$$

$$C_1 = 4 - 12 + 12 - 4 = 0$$

$$C_2 = 8 - 18 + 12 - 2 = 0$$

$$C_3 = \frac{64}{6} - \frac{108}{6} + \frac{48}{6} - \frac{4}{6} - \frac{1}{2} + \frac{1}{2} = 0 \dots\dots\dots(4.10)$$

$$C_4 = \frac{256}{24} - \frac{324}{24} + \frac{96}{24} - \frac{4}{24} - \frac{3}{2} + \frac{1}{2} = 0$$

$$C_5 = \frac{1024}{120} - \frac{972}{120} - \frac{4}{120} + \frac{192}{120} - \frac{9}{4} + \frac{1}{4} = 0$$

$$C_6 = \frac{4096}{720} - \frac{2916}{720} + \frac{384}{720} - \frac{4}{720} - \frac{27}{12} + \frac{1}{12} = 0$$

$$C_7 = \frac{16384}{5040} - \frac{8748}{5040} + \frac{768}{5040} - \frac{4}{5040} - \frac{81}{48} + \frac{1}{48} = 0$$

$$C_8 = \frac{65536}{40320} - \frac{26244}{40320} + \frac{1536}{40320} - \frac{4}{40320} - \frac{243}{240} + \frac{1}{240} = \frac{1}{240}$$

Thus  $C_0=C_1=C_2=C_3=C_4=C_5=C_6=C_7=0$  but  $C_8 = 1/240 \neq 0$ . Thus the principal local truncation error constant of the method is  $c_{p+3} = 1/240$  and

$$p+3 = 8$$

That is, the order  $p$  of accuracy of the method is 5

#### 4.2.5.2 CONSISTENCY

Adopting steps (i) to (iii) as in 3-step scheme the order of the method is 5 which is greater than 1.

$$(i) \quad \sum_{j=0}^4 \alpha_j r^j = (\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)$$

Substituting the values of  $\alpha_j$ 's as in table 1 we have

$$\sum_{j=0}^4 \alpha_j r^j$$

$$= -1 + 4 - 6 + 4 - 1$$

$$= 0$$

(ii)  $\rho(r) = r^4 - 4r^3 + 6r^2 - 4r + 1$

which simplifies into  $\rho(r) = (r-1)^4$

and its roots are  $r = (1, 1, 1, 1)$

$$\rho'(r) = 4r^3 - 12r^2 + 12r - 4$$

$$\rho''(r) = 12r^2 - 24r + 12$$

$$\rho''(1) = 12 - 24 + 12$$

$$= 0$$

as expected

(iii)  $\rho(r) = r^4 - 4r^3 + 6r^2 - 4r + 1$

$$\rho'(r) = 4r^3 - 12r^2 + 12r - 4$$

$$\rho''(r) = 12r^2 - 24r + 12$$

$$\rho'''(r) = 24r - 24$$

$$\rho'''(1) = 24(r-1)$$

$$= 4!(r-1)$$

$$= 24(1-1)$$

$$= 0$$

$$\sigma(r) = \frac{1}{2}(r^3 - r)$$

$$= \frac{1}{2}r(r^2 - 1)$$

$$\sigma(1) = \frac{1}{2}(1^3 - 1)$$

$$= 0$$

Which implies that  $\rho'''(1) = 4! \sigma(1)$

### 4.2.5.3

### ZERO STABILITY

The first characteristic polynomial equation of the Four -step scheme is

$$\rho(r) = r^4 - 4r^3 + 6r^2 - 4r + 1 = 0$$

$$(r-1)^4 = 0$$

Solving, we have that  $r = (1,1,1,1)$ .

Since the roots of the 1<sup>st</sup> characteristic polynomial is simple and lie within a unit circle, then the method is zero stable. Since the method is consistent and zero -stable, then it is Convergent.

### 4.2.5.4

### REGION OF ABSOLUTE STABILITY

$$\bar{h}(r) = \frac{\rho(r)}{\sigma(r)} \dots\dots\dots (4.11)$$

$$\rho(r) = r^4 - 4r^3 + 6r^2 - 4r + 1$$

$$\sigma(r) = \frac{1}{2}(r^3 - r)$$

Substituting these into (4.11) we have

$$\bar{h}(r) = \frac{2[r^4 - 4r^3 + 6r^2 - 4r + 1]}{r^3 - r} \dots\dots\dots (4.12)$$

Using the transformation  $r = e^{i\theta}$ ,  $0 \leq \theta \leq \pi$  which by Euler is given as

$$e^{i\theta} = \cos \theta + i \sin \theta$$

That is,

$$r = \cos \theta + i \sin \theta.$$

On substitution (4.12) becomes

$$\bar{h}(\theta) = \frac{[(\cos 4\theta - 4\cos 3\theta + 6\cos 2\theta - 4\cos \theta + 1) + i(\sin 4\theta - 4\sin 3\theta + 6\sin 2\theta - 4\sin \theta)]}{\{(\cos 3\theta - \cos \theta) + i(\sin 3\theta + \sin \theta)\}} \dots\dots\dots (4.13)$$

## 4.2.5.3

ZERO STABILITY

The first characteristic polynomial equation of the Four -step scheme is

$$\rho(r) = r^4 - 4r^3 + 6r^2 - 4r + 1 = 0$$

$$(r-1)^4 = 0$$

Solving, we have that  $r = (1,1,1,1)$ .

Since the roots of the 1<sup>st</sup> characteristic polynomial is simple and lie within a unit circle, then the method is zero stable. Since the method is consistent and zero -stable, then it is Convergent.

## 4.2.5.4

REGION OF ABSOLUTE STABILITY

$$\bar{h}(r) = \frac{\rho(r)}{\sigma(r)} \dots\dots\dots (4.11)$$

$$\rho(r) = r^4 - 4r^3 + 6r^2 - 4r + 1$$

$$\sigma(r) = \frac{1}{2}(r^3 - r)$$

Substituting these into (4.11) we have

$$\bar{h}(r) = 2 \frac{[r^4 - 4r^3 + 6r^2 - 4r + 1]}{r^3 - r} \dots\dots\dots (4.12)$$

Using the transformation  $r = e^{i\theta}$ ,  $0 \leq \theta \leq \pi$  which by Euler is given as

$$e^{i\theta} = \cos \theta + i \sin \theta$$

That is,

$$r = \cos \theta + i \sin \theta.$$

On substitution (4.12) becomes

$$\bar{h}(\theta) = \frac{[(\cos 4\theta - 4\cos 3\theta + 6\cos 2\theta - 4\cos \theta + 1) + i(\sin 4\theta - 4\sin 3\theta + 6\sin 2\theta - 4\sin \theta)]}{\{(\cos 3\theta - \cos \theta) + i(\sin 3\theta + \sin \theta)\}} \dots\dots\dots (4.13)$$

Rationalizing and simplifying we have

$$\bar{h}(\theta) = x(\theta) + iy(\theta)$$

Where

$$x(\theta) = \frac{2 [\cos \theta - \cos 2\theta]}{2+2\cos 2\theta} \dots\dots\dots(4.14)$$

and

$$y(\theta) = \frac{2 [10\sin\theta + 8\sin 2\theta]}{2+2\cos 2\theta} \dots\dots\dots(4.15)$$

Evaluating  $x(\theta)$  and  $y(\theta)$ ,  $0 \leq \theta \leq 180^\circ$

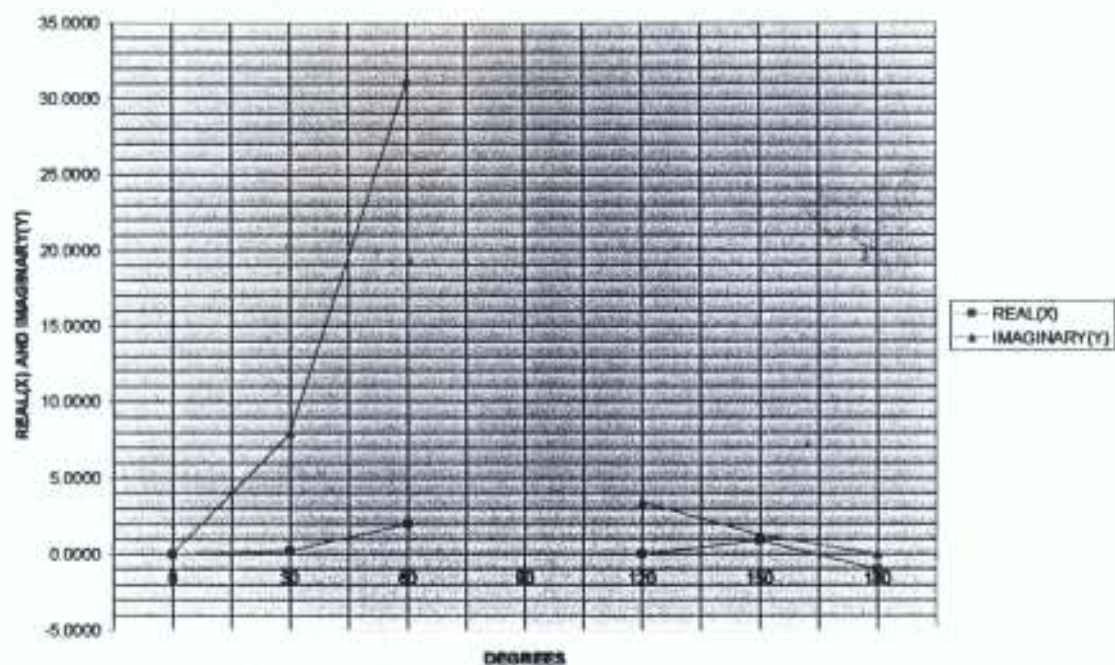
The result is given in the table below

**TABLE 3 : REGION OF ABSOLUTE STABILITY OF 4-STEP METHOD**

$\theta$	0	30	60	90	120	150	180
X ( $\theta$ )	0	0.2440	2.000	$\infty$	0	0.91067	-1
Y ( $\theta$ )	0	7.9520	31.1769	$\infty$	3.4642	1.2855	0

The result from the table above shows that the 4<sup>th</sup> step method have interval of absolute Stability of  $(-1, \infty)$ . Hence the method is weakly p-stable. The graph is shown below

FIG 4.2 REGION OF ABSOLUTE STABILITY OF 4-STEP METHOD



## 4.2.6. THE BASIC PROPERTIES OF FIVE -STEP METHOD

### 4.2.6.1 ORDER OF ACCURACY AND ERROR CONSTANT

Recall that the fifth step scheme is

$$y_{n+5} = 5y_{n+4} - 10y_{n+3} + 10y_{n+2} - 5y_{n+1} + y_n + \frac{h^5}{2} [f_{n+4} - f_{n+3} - f_{n+2} + f_{n+1}] \dots\dots\dots(4.16)$$

with local truncation error

$$T_{n+5} = c_0 y(x_n) + c_1 h y^1(x_n) + c_2 h^2 y^2(x_n) + c_3 h^3 y^3(x_n) + \dots\dots\dots + c_p h^p y^p(x_n) + c_{p+1} h^{p+1} y^{p+1}(x_n) + c_{p+2} h^{p+2} y^{p+2}(x_n) + O(h^{p+3})$$

where

$$C_0 = 1 - \alpha_0 - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4$$

$$C_1 = 5 - \alpha_1 - 2\alpha_2 - 3\alpha_3 - 4\alpha_4$$

$$C_2 = 25/2 - \alpha_1/2 - 4/2\alpha_2 - 9/2\alpha_3 - 16/2\alpha_4$$

$$C_3 = \{125/6 - 1/6 [\alpha_1 + 8\alpha_2 + 27\alpha_3 + 64\alpha_4 - \beta_0 - \beta_1 - \beta_2 - \beta_3 - \beta_4]$$

$$C_4 = \{625/24 - 1/24 [\alpha_1 + 16\alpha_2 + 81\alpha_3 + 256\alpha_4] - (\beta_1 + 2\beta_2 + 3\beta_3 + 4\beta_4)\}$$

$$C_5 = \{3125/120 - \{1/120 [\alpha_1 + 32\alpha_2 + 243\alpha_3 + 1024\alpha_4] - 1/2[\beta_1 + 4\beta_2 + 9\beta_3 + 16\beta_4]\}\}$$

$$C_6 = \{15625/720 - 1/720[\alpha_1 + 64\alpha_2 + 729\alpha_3 + 4096\alpha_4] - 1/6[\beta_1 + 8\beta_2 + 27\beta_3 + 64\beta_4]\}$$

$$C_7 = \{78125/5040 - 1/5040 [\alpha_1 + 128\alpha_2 + 2187\alpha_3 + 16384\alpha_4] - 1/24 [\beta_1 + 16\beta_2 + 81\beta_3 + 256\beta_4]\}$$

$$C_8 = \{390625/40320 - 1/40320 [\alpha_1 + 256\alpha_2 + 6561\alpha_3 + 65536\alpha_4] - 1/120[\beta_1 + 32\beta_2 + 243\beta_3 + 1024\beta_4]\}$$

$$C_9 = \{1953125/362880 - 1/362880 [\alpha_1 + 512\alpha_2 + 19683\alpha_3 + 262144\alpha_4] - 1/720 [\beta_1 + 64\beta_2 + 729\beta_3 + 4096\beta_4]\} \dots\dots\dots(4.17)$$

The results obtained in (3.21) is substituted back into (4.17) to have

$$C_0 = 1 - 1 - 5 - 10 + 10 - 5 = 0$$

$$C_1 = 5 + 5 - 20 + 30 - 20 = 0$$

$$C_2 = 25/2 - [5 + 20 - 90 + 40] = 0$$

$$C_3 = \{^{125}/_6 - 1/6[-5 + 80 - 270 + 320]$$

$$- [1/2 - 1/2 - 1/2 + 1/2] = 0$$

$$C_4 = \{^{625}/_{24} - 1/24[-5 + 160 - 810 + 1280] - [1/2 - 1 - 3/2 + 2]\} = 0$$

$$C_5 = \{^{3125}/_{120} - 1/120[-5 + 320 - 2430 + 5120] - [^{1/4} - 1 - 9/4 + 4]\} = 0$$

$$C_6 = \{^{15625}/_{720} - 1/720[-5 + 640 - 7290 + 16384] - 1/6[^{1/2} - 4 - 27/2 + 32]\} = 0$$

$$C_7 = \{^{78125}/_{5040} - 1/5040[-5 + 1280 - 21870 + 81920] - 1/24[^{1/2} - 8 - 81/2 + 128]\} = 0$$

$$C_8 = \{^{390625}/_{40320} - 1/40320[-5 + 2560 - 65610 + 327680]$$

$$- 1/120[1/2 - 16 - 243/2 + 512]\} = 0$$

$$C_9 = \{^{1953125}/_{362880} - 1/362880[-5 + 5120 - 196830 + 1310720] \dots\dots\dots(4.18)$$

$$- 1/720[1/2 - 32 - 729/2 + 2048]\} = 1/240$$

Thus  $C_0=C_1=C_2=C_3=C_4=C_5=C_6=C_7=0$   $C_8=0$  but  $C_9 \neq 0$  that is

$$C_{p+3} = 9$$

$$p + 3 = 9$$

$$p = 9 - 3$$

$$p = 6$$

Thus the fifth stage scheme is of order 6, and the principal error constant is 1/240

#### 4.2.6.2 CONSISTENCY

Applying the same procedure as in 3-step method above, the order of the method is 6, which is greater than 1; and the polynomial equations satisfy the following;

$$(i) \quad \sum_{j=0}^5 \alpha_j = (\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5)$$

Substituting the values of  $\alpha_j$ 's as in table 1, we have

$$\begin{aligned} \sum_{j=0}^5 \alpha_j &= -1 - 5 + 10 - 10 + 5 + 1 \\ &= 0 \end{aligned}$$

Also, the first characteristic polynomial  $\rho(r)$  of the scheme is

$$\rho(r) = r^5 - 5r^4 + 10r^3 - 10r^2 + 5r - 1$$

Which simplifies into  $(r-1)^5 = 0$

with roots  $r = (1, 1, 1, 1, 1)$

$$(ii) \quad \rho'(r) = 5r^4 - 20r^3 + 20r^2 - 20r + 5$$

$$\rho''(r) = 20r^3 - 60r^2 + 60r - 20$$

$$\rho''(1) = 20 - 60 + 60 - 20$$

$$= 0$$

$$(iii) \quad \rho(r) = r^5 - 5r^4 + 10r^3 - 10r^2 + 5r - 1$$

$$\rho'(r) = 5r^4 - 20r^3 + 20r^2 - 20r + 5$$

$$\rho''(r) = 20r^3 - 60r^2 + 60r - 20$$

$$\rho'''(r) = 60r^2 - 120r + 60$$

$$= 60(r^2 - 2r + 1)$$

$$\rho'''(1) = 60 - 120 + 60$$

$$= 0$$

$$(r) = \frac{1}{2}(r^4 - r^3 - r^2 + r)$$

$$\sigma(1) = \frac{1}{2}(1 - 1 - 1 + 1)$$

$$= 0$$

### 4.2.6.3

### ZERO STABILITY

The first characteristic polynomial equation of the Fifth step scheme is

$$\rho(r) = r^5 - 5r^4 + 10r^3 - 10r^2 + 5r - 1$$

$$(r-1)^5 = 0$$

Solving we have that  $r = (1, 1, 1, 1, 1)$ .

Hence the method is zero stable. Since all the conditions are satisfied.

### 4.2.6.4

### REGION OF ABSOLUTE STABILITY

Using boundary locus method as considered in other cases, the stability equation is given by

$$\bar{h}(r) = \frac{\rho(r)}{\sigma(r)} \dots\dots\dots(4.19)$$

where  $\rho(r) = r^5 - 5r^4 + 10r^3 - 10r^2 + 5r - 1$

and  $\sigma(r) = \frac{1}{2}(r^4 - r^3 - r^2 + r)$

substituting these into (4.19) we have

$$\bar{h}(r) = 2 \frac{[r^5 - 5r^4 + 10r^3 - 10r^2 + 5r - 1]}{r^4 - r^3 - r^2 + r} \dots\dots\dots(4.20)$$

Using the transformation  $r = e^{i\theta}$ ,  $0 \leq \theta \leq \pi$  which by Euler is given as

$$e^{i\theta} = \cos \theta + i \sin \theta$$

That is,

$$r = \cos \theta + i \sin \theta.$$

On substitution into (4.20) becomes

$$\bar{h}(\theta) = \frac{2}{r^4 - r^3 - r^2 + r} [( \cos 5\theta - 5 \cos 4\theta + 10 \cos 3\theta - 10 \cos 2\theta - 5 \cos \theta - 1 ) + i ( \sin 5\theta - 5 \sin 4\theta + 10 \sin 3\theta - 10 \sin 2\theta + 5 \sin \theta )]$$

$$\{ (\cos 4\theta - \cos 3\theta - \cos 2\theta + \cos \theta) + i (\sin 4\theta - \sin 3\theta - \sin 2\theta + \sin \theta) \} \dots\dots\dots(4.21)$$

where

$$x(\theta) = \frac{2[(2\cos\theta - 2\cos 2\theta - 2\cos 3\theta + 2\cos 4\theta)]}{[(4 - 3\cos\theta - 3\cos 2\theta + 2\cos 3\theta)]} \dots\dots\dots (4.22)$$

$$y(\theta) = \frac{2[(12\sin\theta - 2\sin 2\theta - 12\sin 3\theta + \sin 4\theta)]}{[(4 - 3\cos\theta - 3\cos 2\theta + 2\cos 3\theta)]} \dots\dots\dots (4.23)$$

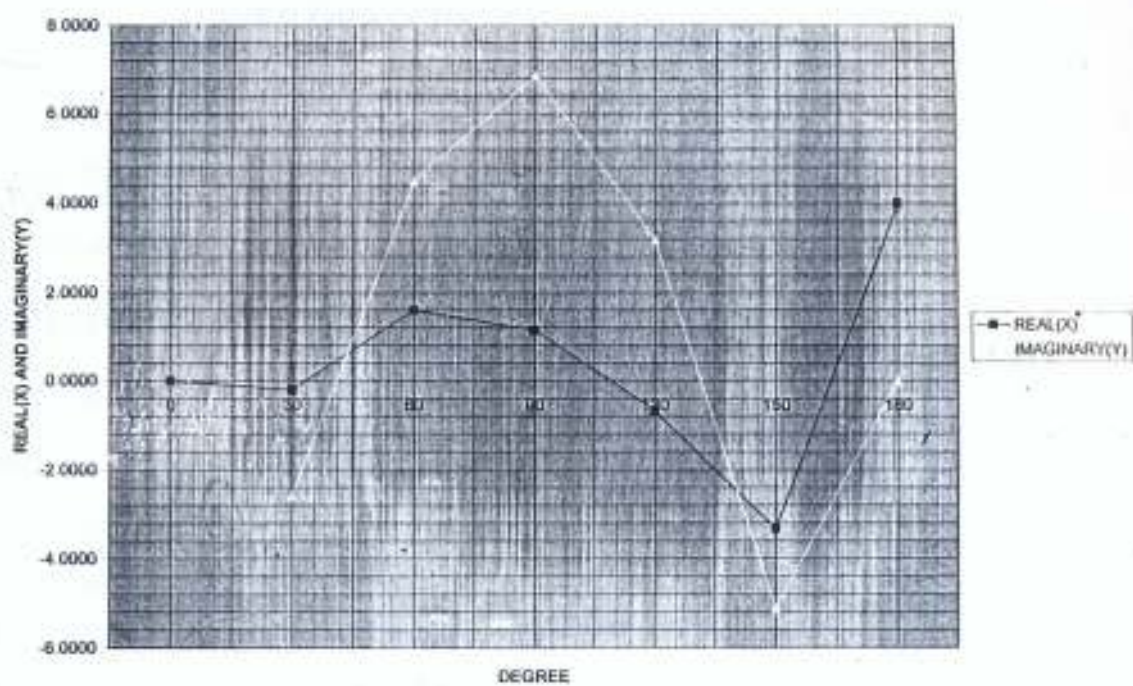
Evaluating  $x(\theta)$  and  $y(\theta)$ ,  $0 \leq \theta \leq 180^\circ$  as for other cases we have

**TABLE 4: REGION OF ABSOLUTE STABILITY OF 5-STEP METHOD**

$\theta$	0	30	60	90	120	150	180
$X(\theta)$	0	-0.2035	1.6000	1.1429	-0.6667	-3.3262	4
$y(\theta)$	$\infty$	-2.5831	4.5034	6.8571	3.1754	-5.1088	0

The result in the table above shows that the fifth step method has an interval of absolute stability of  $(-3.3262, 4)$ . The graph is shown below.

FIG. 4.3 REGION OF ABSOLUTE STABILITY OF 5-STEP METHOD



## 4.2.7 THE BASIC PROPERTIES OF SIX-STEP METHOD

### 4.2.7.1 ORDER OF ACCURACY AND ERROR CONSTANT

Recall that the sixth-step method is of the form

$$y_{n+6} = 6y_{n+5} - 15y_{n+4} + 20y_{n+3} - 15y_{n+2} + 6y_{n+1} - y_n + \frac{h^3}{2} [f_{n+5} - 2f_{n+4} + 2f_{n+2} - f_{n+1}] \quad (4.24)$$

with local truncation error  $T_{n+6}$  defined as

$$T_{n+6} = c_0 y(x_n) + c_1 h y'(x_n) + c_2 h^2 y''(x_n) + c_3 h^3 y'''(x_n) + \dots + c_p h^p y^{(p)}(x_n) + c_{p+1} h^{p+1} y^{(p+1)}(x_n) + c_{p+2} h^{p+2} y^{(p+2)}(x_n) + O(h^{p+3}),$$

where

$$C_0 = 1 - \alpha_0 - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5$$

$$C_1 = 6 - \alpha_1 - 2\alpha_2 - 3\alpha_3 - 4\alpha_4 - 5\alpha_5$$

$$C_2 = 36/2 - 1/2 [\alpha_1 + 4\alpha_2 + 9\alpha_3 + 16\alpha_4 + 25\alpha_5] \quad \dots \dots \dots (4.25)$$

$$C_3 = \{216/6 - 1/6 [\alpha_1 + 8\alpha_2 + 27\alpha_3 + 64\alpha_4 + 125\alpha_5 - \beta_0 - \beta_1 - \beta_2 - \beta_3 - \beta_4 - \beta_5]$$

$$C_4 = \{1296/24 - 1/24 [\alpha_1 + 16\alpha_2 + 81\alpha_3 + 256\alpha_4 + 625\alpha_5] - (\beta_1 + 2\beta_2 + 3\beta_3 + 4\beta_4 + 5\beta_5)\}$$

$$C_5 = \{7776/120 - \{1/120 [\alpha_1 + 32\alpha_2 + 243\alpha_3 + 1024\alpha_4 + 3125\alpha_5] - 1/2[\beta_1 + 4\beta_2 + 9\beta_3 + 16\beta_4 + 25\beta_5]\}$$

$$C_6 = \{46656/720 - 1/720 [\alpha_1 + 64\alpha_2 + 729\alpha_3 + 4096\alpha_4 + 15625\alpha_5]$$

$$- 1/6[\beta_1 + 8\beta_2 + 27\beta_3 + 64\beta_4 + 125\beta_5]\}$$

$$C_7 = \{279936/5040 - 1/5040 [\alpha_1 + 128\alpha_2 + 2187\alpha_3 + 16384\alpha_4 + 78125\alpha_5]$$

$$- 1/24 [\beta_1 + 16\beta_2 + 81\beta_3 + 256\beta_4 + 625\beta_5]\}$$

$$C_8 = \{1679616/40320 - 1/40320 [\alpha_1 + 256\alpha_2 + 6561\alpha_3 + 65536\alpha_4 + 390625\alpha_5]$$

$$- 1/120[\beta_1 + 32\beta_2 + 243\beta_3 + 1024\beta_4 + 3125\beta_5]\}$$

$$C_9 = \{10077696/362880 - 1/362880 [\alpha_1 + 512\alpha_2 + 19683\alpha_3 + 262144\alpha_4 + 1953125\alpha_5]$$

$$- 1/720 [\beta_1 + 64\beta_2 + 729\beta_3 + 4096\beta_4 + 15625\beta_5]\}$$

$$C_{10} = \{60466176/3628800 - 1/3628800 [\alpha_1 + 1024\alpha_2 + 59049\alpha_3 + 1048576\alpha_4 + 9765625\alpha_5]$$

$$- 1/5040[\beta_1 + 128\beta_2 + 2187\beta_3 + 16384\beta_4 + 78125\beta_5]\} \quad \dots \dots \dots (4.26)$$

The values of the parameters  $\alpha_j$ 's and  $\beta_j$ 's as in table 1 are substituted back into  $c_i$ 's. We have

$$C_0 = 1 - \{-1 + 6 - 15 + 20 - 15 + 6\} = 0$$

$$C_1 = 6 - \{6 - 30 + 60 - 60 + 30\} = 0$$

$$C_2 = 18 - \{6 - 60 + 270 - 240 + 150\} = 0$$

$$C_3 = \{2^{16}/6 - 1/6[6 - 120 + 540 - 960 + 750] - [1/2 + 1 - 1 + 1/2]\} = 0$$

$$C_4 = \{1^{296}/24 - 1/24[6 - 240 + 1620 - 3840 + 3750] - [1/2 + 2 - 4 - 5/2]\} = 0$$

$$C_5 = 7776/120 - 1/120[-6 - 480 + 4860 - 15360 + 18750] - [-30 + 240 - 960 + 750] = 0$$

$$C_6 = 46656/720 - 1/720[6 - 960 + 14580 - 61440 + 93750] - [-60 + 960 - 7680 + 7500] = 0$$

$$C_7 = 279936/5040 - 1/5040[6 - 1920 + 43740 - 245760 + 468750]$$

$$-1/24[-1/2 + 16 - 256 + 625/2] = 0$$

$$C_8 = 16779616/40320 - 1/40320[6 - 3840 + 131220 - 983040 + 2343750]$$

$$-1/120[-1/2 + 32 - 1024 + 3125/2] = 0 \quad \dots\dots\dots(4.27)$$

$$C_9 = 10077696/362880 - 1/362880[6 - 7680 + 393660 - 1171872]$$

$$-1/720[-1/2 + 64 - 4096 + 15625/2] = 0$$

$$C_{10} = 60466176/3628800 - 1/3628800[6 - 15360 + 1180980 - 15728640 + 58593750]$$

$$-1/5040[1/2 + 128 - 16384 + 78125/2] = 1/240$$

Thus  $C_0 = C_1 = C_2 = C_3 = C_4 = C_5 = C_6 = C_7 = 0$ ,  $C_8 = C_9 = 0$  but  $C_{10} \neq 0$  that is

$$C_{p+3} = 10$$

$$p + 3 = 10$$

$$p = 10 - 3$$

$$p = 7$$

Thus the sixth -step scheme is of order 7, and the principal error constant is 1/240

Applying the same procedure as used in other schemes, the order of the method is 7 which is greater than 1; and the characteristic polynomial equations satisfy the following

$$(i) \quad \sum_{j=0}^6 \alpha_j = (\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)$$

$$\begin{aligned} \sum_{j=0}^6 \alpha_j &= 1 - 6 + 15 - 20 + 15 - 6 + 1 \\ &= 32 - 32 \\ &= 0 \end{aligned}$$

While the first characteristic polynomial equation

$$(ii) \quad \rho(r) = r^6 - 6r^5 + 15r^4 - 20r^3 + 15r^2 - 6r + 1$$

$$\text{Simplifies into } (r-1)^6 = 0$$

With roots  $r = (1, 1, 1, 1, 1, 1)$

$$\rho(r) = 6r^5 - 30r^4 + 60r^3 - 60r^2 + 30r - 6$$

$$\rho'(r) = 30r^4 - 120r^3 + 180r^2 + 120r + 30$$

$$\rho''(1) = 30 - 120 + 80 - 120 + 30$$

$$= 240 - 240$$

$$= 0$$

Also,

$$(iii) \quad \rho(r) = 6r^5 - 30r^4 + 60r^3 - 60r^2 + 30r - 6$$

$$\rho'(r) = 30r^4 - 120r^3 + 180r^2 + 120r + 30$$

$$\rho''(1) = 30 - 120 + 80 - 120 + 30$$

$$\rho'''(r) = 120r^3 - 360r^2 + 360r - 120 \text{ for } r = (1, 1, 1, 1, 1)$$

$$= 120(r^3 - 30r^2 + 30r - 1)$$

$$= 5!(r^3 - 30r^2 + 30r - 1)$$

$$\rho(1) = 120 - 360 + 360 - 120$$

$$= 0$$

$$\sigma(r) = \frac{1}{2} [r^5 - 2r^4 + 2r^2 - r]$$

$$\sigma(1) = \frac{1}{2} (1 - 2 + 2 - 1)$$

$$= 0$$

Which implies that  $\rho'''(1) = 5! \sigma(1)$

#### 4.2.7.3 ZERO STABILITY

From the stability equation above, the root of the first characteristic polynomial equation,  $\rho(r) = r^6 - 6r^5 + 15r^4 - 20r^3 + 15r^2 - 6r + 1$ ,  $(r-1)^6 = 0$  of the sixth-step scheme are  $r = (1, 1, 1, 1, 1, 1)$  i.e.  $r \leq 1$ . Since the method is consistent and zero stable, then it is convergent.

#### 4.2.7.4 REGION OF ABSOLUTE STABILITY

Using boundary locus method as considered in other cases, the stability equation is given by

$$\bar{h}(r) = \frac{\rho(r)}{\sigma(r)} \dots \dots \dots (4.28)$$

Where  $\rho(r) = r^6 - 6r^5 + 15r^4 - 20r^3 + 15r^2 - 6r + 1$

And  $\sigma(r) = \frac{1}{2} [r^5 - 2r^4 + 2r^2 - r]$

Substituting these into (4.28) we have

$$\bar{h}(r) = \frac{2[r^6 - 6r^5 + 15r^4 - 20r^3 + 15r^2 - 6r + 1]}{[r^5 - 2r^4 + 2r^2 - r]} \dots \dots \dots (4.29)$$

Using the transformation  $r = e^{i\theta}$ ,  $0 \leq \theta \leq \pi$  which by Euler gives

$$e^{i\theta} = \cos \theta + i \sin \theta$$

Therefore,

$$r = \cos \theta + i \sin \theta.$$

On substitution, (4.29) becomes

$$h(\theta) = 2 \frac{[\cos 6\theta - 6\cos 5\theta + 15\cos 4\theta - 20\cos 3\theta + 15\cos 2\theta + 6\cos \theta - 1] + i [\sin 6\theta - 6\sin 5\theta + 15\sin 4\theta - 20\sin 3\theta + 15\sin 2\theta + 6\sin \theta]}{(\cos 5\theta - 2\cos 4\theta + 2\cos 2\theta - \cos \theta) + i (\sin 5\theta - 2\sin 4\theta + 2\sin \theta - \sin \theta)} \dots \dots \dots (4.30)$$

where

$$x(\theta) = 2 \frac{[(2\cos \theta - 4\cos 2\theta - 24\cos 3\theta + 16\cos 4\theta - 2\cos 5\theta - 12)]}{[(2 - 8\cos \theta - 8\cos 2\theta + 8\cos 3\theta - 2\cos 4\theta)]} \dots \dots \dots (4.31)$$

$$y(\theta) = 6 \frac{[(-30\sin \theta + 7\sin 2\theta - 13\sin 3\theta)]}{[(2 - 8\cos \theta - 8\cos 2\theta + 8\cos 3\theta - 2\cos 4\theta)]} \dots \dots \dots (4.32)$$

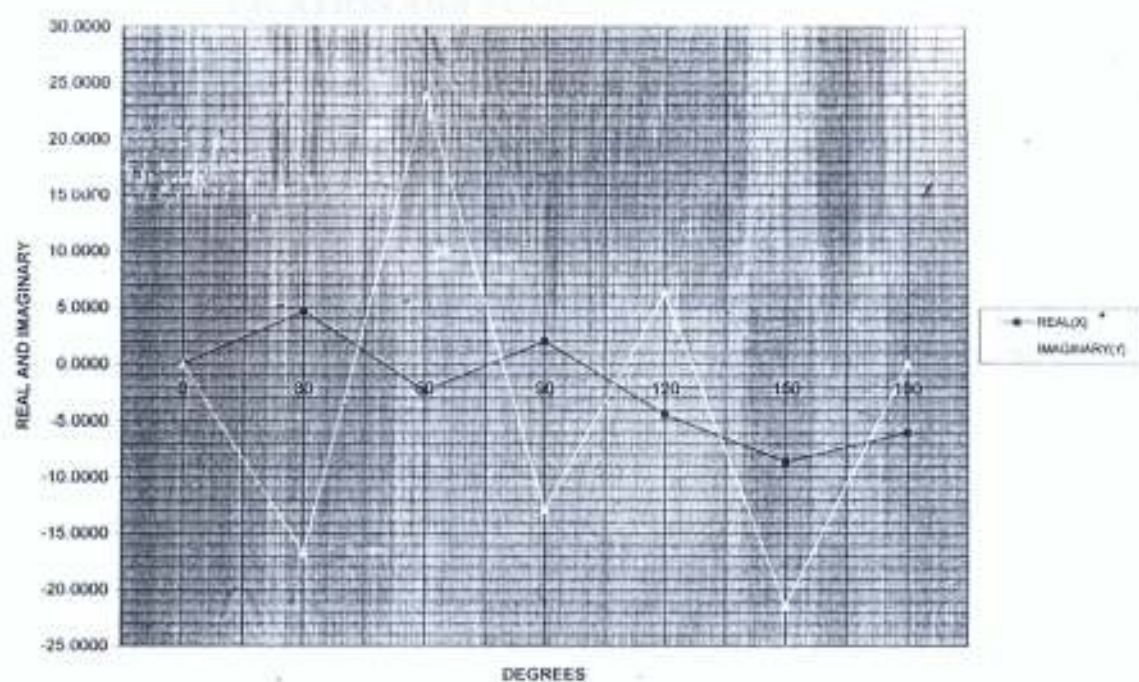
Evaluating  $x(\theta)$  and  $y(\theta)$ ,  $0 \leq \theta \leq 180^\circ$  as for other cases we have

**TABLE 5: REGION OF ABSOLUTE STABILITY OF 6-STEP METHOD**

$\theta$	0	30	60	90	120	150	180
$X(\theta)$	0	4.6759	-2.4000	2.0000	-4.4200	-8.5908	-6
$y(\theta)$	0	-16.6024	23.9022	-12.7500	6.2900	-21.3127	0

From the table above, the sixth-step scheme has an interval of absolute stability of  $(-8.5908, 4.6759)$  which shows that the method is weakly stable. The graph is shown below

FIG. 4.3 REGION OF ABSOLUTE STABILITY FOR SIX STEP METHOD



## CHAPTER FIVE

### APPLICATION AND NUMERICAL RESULTS



In order to demonstrate the applicability and assess the performance of the new schemes, there is need to apply the method for solution of some sample initial value problems in ordinary differential equations. To achieve this, there is need to implement the method on a digital computer. This will require adopting the following steps; namely:

- (i) Re-write the formula in an algorithmic form.
- (ii) Translate the algorithm into a computer flowchart.
- (iii) Translate the flowchart into computer code.
- (iv) Implement the code with sample problems on a digital computer.

#### 5.1 COMPUTATIONAL ALGORITHM

A set of steps taken to obtain the solution of a given problem is defined as the algorithm of that problem. The numerical algorithm for implementing the methods derived is given below

In this section, we develop the numerical algorithm for implementing the methods derived in Chapter three. The algorithm is given below:-

- STEP 1:- Declaration of variable
- STEP 2:- Define functions  
DIM Y, DIM X, DIM Yp, Ypp, ERROR
- STEP 3:- Selection of input values  
 $Y_0, V_0, V_1, V_2, h$
- STEP 4:- Initialise variables by setting  
 $f_0 = 0$

$$x = X_0$$

$$y = Y_0$$

$$x_{(i+1)} = x_{(i)} + h$$

STEP 5:- Compute the predictors

$$Y_p^{(1)} = Y_p^{(0+1)} = Y_p^{(0)} + (i+1) * h * Y_{pp}^{(0)} \\ + (((i+1) * h) ** 2) * f(0)/2$$

$$Y_p^{(2)} = Y_p^{(i+1)} = (((i+1) * h) ** 3) * f_p(0)/6 + (((i+1) * h) ** 4) * f_{pp}(0)/24$$

STEP 6 Compute the approximate values of

$$Y_1^{(i+1)} \text{ and } Y_2^{(i+2)}$$

STEP 7 Print  $Y(i+1)$ ,  $Y(i+2)$

STEP 8 Calculate  $y_n$  using

$$\text{Do } 20 \text{ } j = 0(1) 300$$

$$Y_k(j+3) = 3 * y(j+2) - 3 * y(j+1) + y(j)$$

$$V(j+3) = y(j+3) \dots$$

STEP 9 Compute YE, YC, ERR

$$ER(k) = V_1(C) - V(k)$$

IF  $ER(k) < 0$ , THEN

$$ER(k) = -ER(k)$$

CONTINUE

END

## 5.2 PROGRAM FLOWCHART

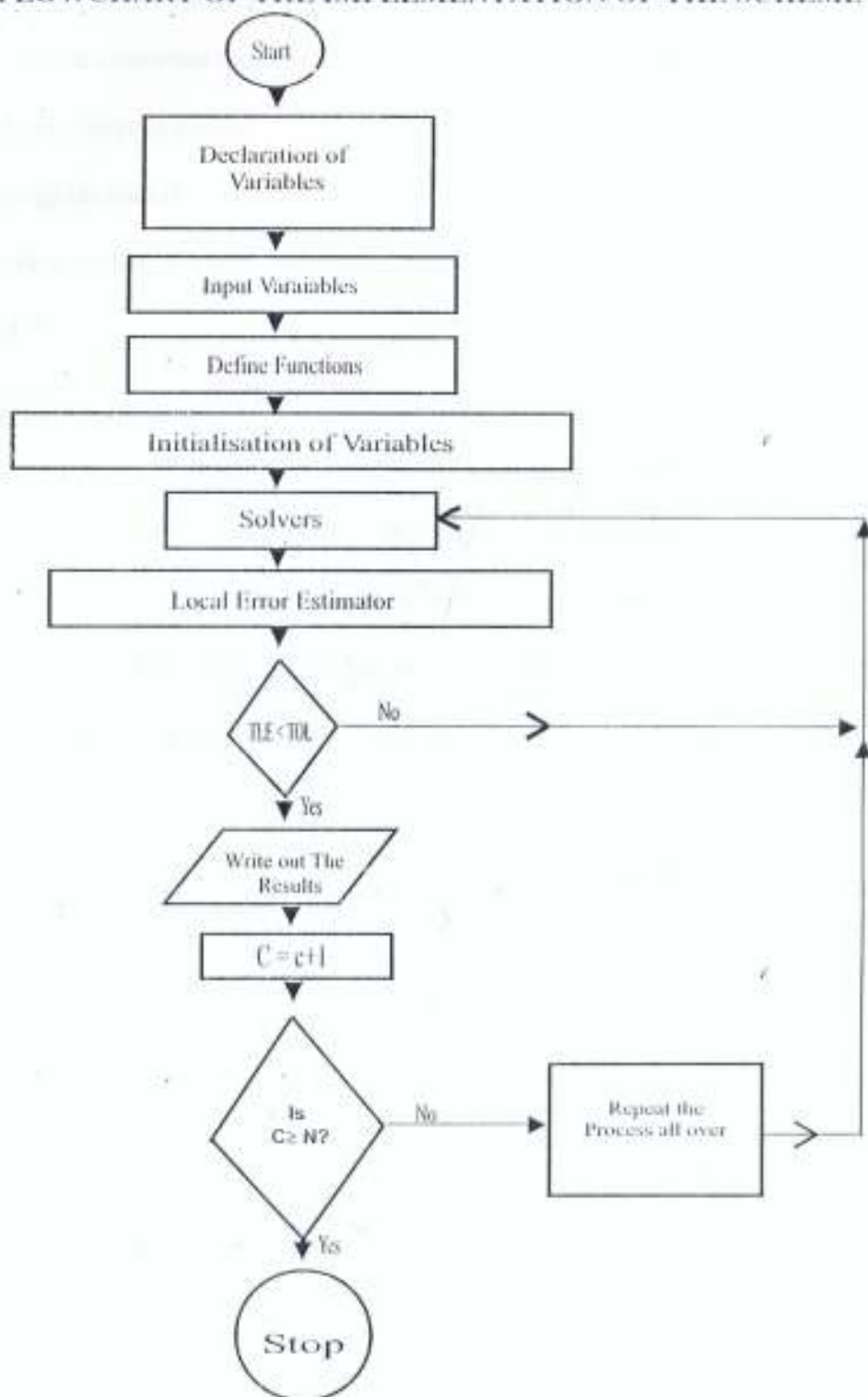
A computer flowchart is a diagrammatic representation of the algorithm or the plan of solution of a problem. It indicates the process of solution, the relevant operation and computations, the point of decision and other information at a point of solution.

Flowcharts are of particular interest because of its documenting feature. They are constructed by using special geometrical symbols, such as squares, rectangles, diamonds shapes or circles.

Each symbol represent some activities which could be input/output of data, taking a decision, terminating the solution process and soon.

The symbols are joined by directed lines segments to indicate direction of flow. The flowchart of the above algorithms is given in the fig (5.1) below

Fig 5.1 FLOWCHART OF THE IMPLEMENTATION OF THE SCHEME



### 5.3 PROGRAMMING IMPLEMENTATION

This section considers the computer implementation of the above algorithm. The implementation is done in a fixed step size order method.

The program starts by declaring the value of variables in double precision mode in order to reduce round off error. After this, the program chooses initial estimate for the variables; and the function subprogram to evaluate  $f(x, y)$ .

After this, the approximate solution  $y_{n+1}$  to  $y(x_n)$  at  $X_{n+1}$  was generated and used to estimate the values of the error (LTE) associated with the computation.

On the receipt of the error estimate from the subroutines, the solution was now tested for convergence by comparing the magnitude of the error with allowable error tolerance. As soon as the condition  $LTE < TOL$  is met the program will ask whether the upper point is being reached, if yes, it will output result. Otherwise, it will go to the next step to generate the next round of approximation to the solution. The process will be repeated and continued until the upper end point is reached. When the program reached the end point ( $X_{\text{last}}$ ) it will stop the process.

## 5.4 NUMERICAL COMPUTATIONS AND RESULTS

To demonstrate the applicability of the schemes developed, the following sample problems were solved using the developed methods.

### Problem 1:

$$y''' = x - 4y' \quad y(0) = y'(0) = 0, \quad y''(0) = 1$$

whose exact solution is  $y(x) = (3/16)(1 - \cos 2x) + (1/8)x^2$

The step size used is 0.0025

### Problem 2

$$y''' = -y' \quad y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 2$$

whose exact solution is  $y(x) = 2(1 - \cos x) + \sin x$

The step size used is 0.0025

### Problem 3

$$y''' = -y y'' \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 1$$

This is a Blasius equation in fluid dynamics.

The computer results and the graphs of problems 1,2 and 3 adopting the 3-step and 4-step methods with a fixed step size ( $h=0.0025$ ) is shown in Table 6,7,8,9 and 10 below.

## 5.5 COMPARATIVE ANALYSIS

The methods were used to solve some sample problems on third order initial value problems of ordinary differential equation and the comparative analysis of the proposed 3-step method and that of continuous 3-step method of Awoyemi was

undertaking by using them to solve the same problem. The comparison shows that the new method is competitive. The result and the graph are shown respectively in table 11 and figure below.

## 5.6 DISCUSSION OF RESULTS

From the results in tables 6, 7,8,9,10,11, it can be seen that the numerical solution  $y_n$  and the Exact solution  $y(x_n)$  are relatively close to each other. This shows that the schemes are accurate and convergent. The results in table 11 show that the 3-step scheme compared favorably with the Continuous method of Awoyemi, (2002).

TABLE 6: NUMERICAL SOLUTION OF PROBLEM 1

$$y''' = x - 4y' \quad y(0) = y'(0) = 0, \quad y''(0) = 1$$

Using the 3-step method with mesh size  $h = 2.500000E-003$ 

S/N	Mesh pt $x_n$	Exact solu. $y(x)$	Numerical solu. $y_{(n)}$ COMPUTED	ERROR ( $e_n$ )
0	.00000000D+00	.00000000D+00	.00000000D+00	.00000000D+00
1	.25000000D-02	.31281830D-05	.31249950D-05	.31880060D-08
2	.50000000D-02	.12501560D-04	.12499920D-04	.16361810D-08
3	.75000000D-02	.28120120D-04	.28124600D-04	.44801710D-08
4	.10000000D-01	.49995050D-04	.49998730D-04	.36852730D-08
5	.12500000D-01	.78126340D-04	.78121920D-04	.44165060D-08
6	.15000000D-01	.11249160D-03	.11249360D-03	.19936120D-08
7	.17500000D-01	.15311330D-03	.15311320D-03	.72759580D-10
8	.20000000D-01	.19998020D-03	.19998000D-03	.18917490D-09
9	.22500000D-01	.25309220D-03	.25309300D-03	.75669960D-07
10	.25000000D-01	.31244950D-03	.31245130D-03	.18044370D-08
11	.27500000D-01	.37805190D-03	.37805370D-03	.18335410D-08
12	.30000000D-01	.44989950D-03	.44989910D-03	.40745360D-09
13	.32500000D-01	.52798110D-03	.52798620D-03	.50058590D-08
14	.35000000D-01	.61230800D-03	.61231330D-03	.53551050D-08
15	.37500000D-01	.70288000D-03	.70287910D-03	.87311490D-09
16	.40000000D-01	.79967490D-03	.79968180D-03	.68685040D-08
17	.42500000D-01	.90271490D-03	.90271950D-03	.45984050D-08
18	.45000000D-01	.10119890D-02	.10119900D-02	.13969840D-08
19	.47500000D-01	.11274860D-02	.11274920D-02	.62864270D-08
20	.50000000D-01	.12492170D-02	.12492230D-02	.58207660D-08
21	.52500010D-01	.13771710D-02	.13771800D-02	.88475640D-08
22	.55000010D-01	.15113580D-02	.15113600D-02	.18626450D-08
23	.57500010D-01	.16517580D-02	.16517620D-02	.45401980D-08
24	.60000010D-01	.17983800D-02	.17983830D-02	.30267980D-08
25	.62500010D-01	.19512140D-02	.19512200D-02	.58207660D-08
26	.65000010D-01	.21102710D-02	.21102700D-02	.11641530D-08
27	.67500000D-01	.22755280D-02	.22755300D-02	.16298150D-08
28	.70000000D-01	.24469970D-02	.24469970D-02	.23283060D-09
29	.72500000D-01	.26246770D-02	.26246680D-02	.95460560D-08
30	.75000000D-01	.28085470D-02	.28085400D-02	.72177500D-08
31	.77499990D-01	.29986170D-02	.29986090D-02	.83819030D-08
32	.79999990D-01	.31948880D-02	.31948720D-02	.16298150D-07
33	.82499990D-01	.33973370D-02	.33973250D-02	.12107190D-07
34	.84999990D-01	.36059760D-02	.36059640D-02	.11175870D-07

35	.87499980D-01	.38208036D-02	.38207860D-02	.16996640D-07
36	.89999980D-01	.40418090D-02	.40417860D-02	.22817400D-07
37	.92499980D-01	.42689820D-02	.42689600D-02	.21886080D-07
38	.94999980D-01	.45023320D-02	.45023030D-02	.28871000D-07
39	.97499970D-01	.47418390D-02	.47418130D-02	.25611370D-07
40	.99999976D-01	.49875120D-02	.49874840D-02	.28405340D-07
41	.10250000D+00	.52393410D-02	.52393110D-02	.30267980D-07
42	.10500000D+00	.54973270D-02	.54972900D-02	.36321580D-07
43	.10750000D+00	.57614560D-02	.57614170D-02	.39581210D-07
44	.11000000D+00	.60317200D-02	.60316850D-02	.34458940D-07
45	.11250000D+00	.63081270D-02	.63080900D-02	.36787240D-07
46	.11500000D+00	.65906680D-02	.65906280D-02	.40512530D-07
47	.11750000D+00	.68793320D-02	.68792910D-02	.40046870D-07
48	.12000000D+00	.71741290D-02	.71740770D-02	.52154060D-07
49	.12250000D+00	.74750250D-02	.74749770D-02	.47963110D-07
50	.12499990D+00	.77820440D-02	.77819880D-02	.56345020D-07
51	.12750000D+00	.80951520D-02	.80951020D-02	.49360100D-C7
52	.13000000D+00	.84143710D-02	.84143160D-02	.54948030D-07
53	.13249990D+00	.87396780D-02	.87396220D-02	.55879350D-07
54	.13499990D+00	.90710740D-02	.90710150D-02	.59604640D-07
55	.13749990D+00	.94085480D-02	.94084870D-02	.61467290D-07
56	.13999990D+00	.97520990D-02	.97520320D-02	.67055230D-07
57	.14249990D+00	.10101720D-01	.10101640D-01	.72643160D-07
58	.14499990D+00	.10457390D-01	.10457320D-01	.73574480D-07
59	.14749990D+00	.10819120D-01	.10819040D-01	.76368450D-07
60	.14999990D+00	.11186900D-01	.11186810D-01	.86613000D-07
61	.15249990D+00	.11560720D-01	.11560630D-01	.89406970D-07
62	.15499990D+00	.11940570D-01	.11940480D-01	.91269610D-07
63	.15749990D+00	.12326450D-01	.12326350D-01	.98720190D-07
64	.15999990D+00	.12718350D-01	.12718240D-01	.10896470D-06
65	.16249990D+00	.13116250D-01	.13116150D-01	.10710210D-06
66	.16499990D+00	.13520170D-01	.13520060D-01	.11175870D-06
67	.16749990D+00	.13930080D-01	.13929960D-01	.11827800D-06
68	.16999990D+00	.14345980D-01	.14345860D-01	.12386590D-06
69	.17249990D+00	.14767870D-01	.14767730D-01	.13597310D-06
70	.17499990D+00	.15195730D-01	.15195590D-01	.13969840D-06
71	.17749990D+00	.15629540D-01	.15629400D-01	.14156100D-06
72	.17999990D+00	.16069330D-01	.16069180D-01	.15273690D-06
73	.18249990D+00	.16515060D-01	.16514910D-01	.15459950D-06
74	.18499990D+00	.16966730D-01	.16966580D-01	.15646220D-06
75	.18749990D+00	.17424330D-01	.17424180D-01	.14714900D-06
76	.18999990D+00	.17887850D-01	.17887710D-01	.14156100D-06
77	.19249990D+00	.18357300D-01	.18357160D-01	.14342370D-06
78	.19499990D+00	.18832660D-01	.18832510D-01	.14901160D-06
79	.19749990D+00	.19313910D-01	.19313760D-01	.14342370D-06
80	.19999990D+00	.19801040D-01	.19800900D-01	.13969840D-06

Fig. 5.2: Graphs of Numerical and exact Solutions of Problem 1 using formular  $y_{n+3} = 3y_{n+2} - 3y_{n+1} + y_n + \frac{h^3}{2} [f_{n+2} + f_{n+1}]$

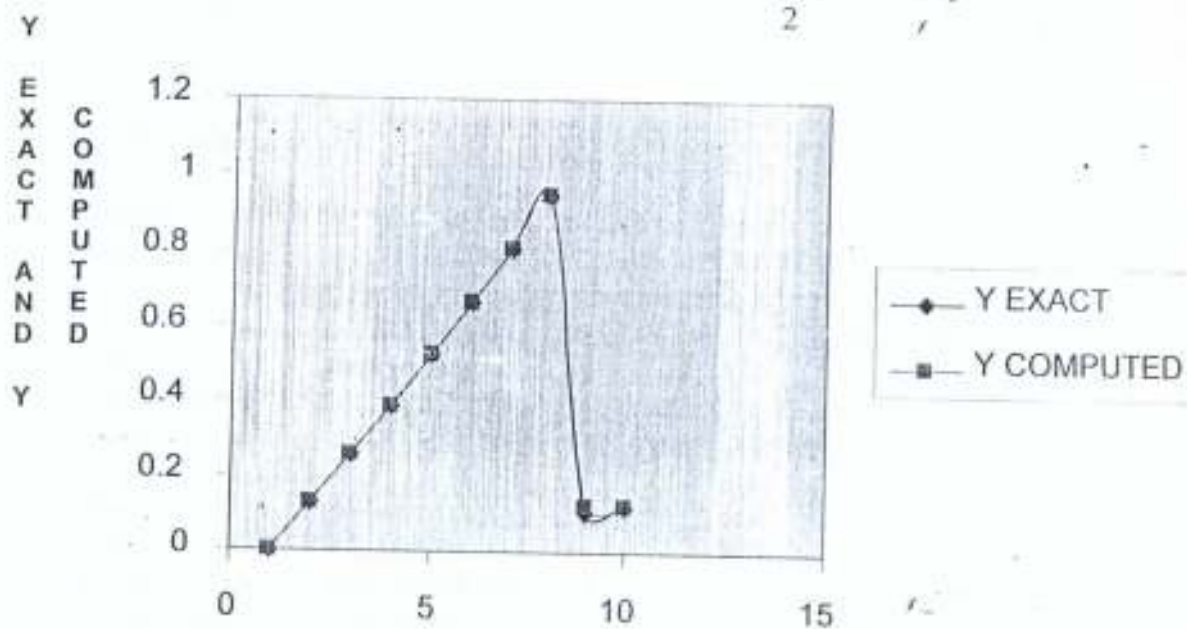


TABLE 7: NUMERICAL SOLUTION OF PROBLEM 1

$$y''' = x - 4y' \quad y(0) = y'(0) = 0, \quad y''(0) = 1$$

Using the 4 -step method with Mesh size  $h = 2.500000E-003$

S/N	$X_n$	EXACT $Y(x_n)$	COMPUTED $(Y_n)$	ERROR $(e_n)$
*****				
**				
0	.00000000D+00	.00000000D+00	.00000000D+00	.00000000D+00
1	.25000000D-02	.31281830D-05	.31249950D-05	.31880060D-08
2	.50000000D-02	.12501560D-04	.12499920D-04	.16361810D-08
3	.75000000D-02	.28120120D-04	.28124600D-04	.44819900D-08
4	.10000000D-01	.49995050D-04	.49998740D-04	.36961860D-08
5	.12500000D-01	.78126340D-04	.78121940D-04	.43946780D-08
6	.15000000D-01	.11249160D-03	.11249370D-03	.20227160D-08
7	.17500000D-01	.15311330D-03	.15311330D-03	.43655750D-10
8	.20000000D-01	.19998020D-03	.19998000D-03	.18917490D-09
9	.22500000D-01	.25309220D-03	.25309290D-03	.66938810D-09
10	.25000000D-01	.31244950D-03	.31245100D-03	.15425030D-08
11	.27500000D-01	.37805190D-03	.37805320D-03	.12805690D-08
12	.30000000D-01	.44989950D-03	.44989810D-03	.13969840D-08
13	.32500000D-01	.52798110D-03	.52798450D-03	.33760440D-08
14	.35000000D-01	.61230800D-03	.61231080D-03	.28521750D-08
15	.37500000D-01	.70288000D-03	.70287540D-03	.45401980D-08
16	.40000000D-01	.79967490D-03	.79967660D-03	.16880220D-08
17	.42500000D-01	.90271490D-03	.90271240D-03	.25029290D-08
18	.45000000D-01	.10119890D-02	.10119810D-02	.80326570D-08
19	.47500000D-01	.11274860D-02	.11274800D-02	.58207660D-08
20	.50000000D-01	.12492170D-02	.12492080D-02	.93132260D-08
21	.52500010D-01	.13771710D-02	.13771610D-02	.96624720D-08
22	.55000010D-01	.15113580D-02	.15113380D-02	.20372680D-07
23	.57500010D-01	.16517580D-02	.16517360D-02	.21769670D-07
24	.60000010D-01	.17983800D-02	.17983520D-02	.27706850D-07
25	.62500010D-01	.19512140D-02	.19511840D-02	.29685910D-07
26	.65000010D-01	.21102710D-02	.21102290D-02	.41909520D-07
27	.67500000D-01	.22755280D-02	.22754830D-02	.44703480D-07
28	.70000000D-01	.24469970D-02	.24469450D-02	.52386890D-07
29	.72500000D-01	.26246770D-02	.26246100D-02	.67288060D-07
30	.75000000D-01	.28085470D-02	.28084770D-02	.69849190D-07
31	.77499990D-01	.29986170D-02	.29985430D-02	.74505810D-07
32	.79999990D-01	.31948880D-02	.31948050D-02	.83819030D-07
33	.82499990D-01	.33973370D-02	.33972590D-02	.77998270D-07
34	.84999990D-01	.36059760D-02	.36059040D-02	.71479010D-07
35	.87499980D-01	.38208030D-02	.38207360D-02	.66589560D-07

35	.87499980D-01	.38208030D-02	.38207360D-02	.66589560D-07
36	.89999980D-01	.40418090D-02	.40417540D-02	.54948030D-07
37	.92499980D-01	.42689820D-02	.42689540D-02	.27939680D-07
38	.94999980D-01	.45023320D-02	.45023340D-02	.13969840D-08
39	.97499970D-01	.47418390D-02	.47418920D-02	.53085390D-07
40	.99999970D-01	.49875120D-02	.49876250D-02	.11315570D-06
41	.10250000D+00	.52393410D-02	.52395320D-02	.19092110D-06
42	.10500000D+00	.54973270D-02	.54976100D-02	.28358770D-06
43	.10750000D+00	.57614560D-02	.57618570D-02	.40046870D-06
44	.11000000D+00	.60317200D-02	.60322690D-02	.54994600D-06
45	.11250000D+00	.63081270D-02	.63088460D-02	.71898100D-06
46	.11500000D+00	.65906680D-02	.65915850D-02	.91642140D-06
47	.11750000D+00	.68793320D-02	.68804820D-02	.11506490D-05
48	.12000000D+00	.71741290D-02	.71755370D-02	.14081600D-05
49	.12250000D+00	.74750250D-02	.74767460D-02	.17210840D-05
50	.12499990D+00	.77820440D-02	.77841080D-02	.20638110D-05
51	.12750000D+00	.80951520D-02	.80976190D-02	.24670740D-05
52	.13000000D+00	.84143710D-02	.84172770D-02	.29057260D-05
53	.13249990D+00	.87396780D-02	.87430780D-02	.34002590D-05
54	.13499990D+00	.90710740D-02	.90750210D-02	.39469450D-05
55	.13749990D+00	.94085480D-02	.94131030D-02	.45541670D-05
56	.13999990D+00	.97520990D-02	.97573190D-02	.52200630D-05
57	.14249990D+00	.10101720D-01	.10107670D-01	.59520830D-05
58	.14499990D+00	.10457390D-01	.10464150D-01	.67586080D-05
59	.14749990D+00	.10819120D-01	.10826760D-01	.76368450D-05
60	.14999990D+00	.11186900D-01	.11195490D-01	.85849310D-05
61	.15249990D+00	.11560720D-01	.11570340D-01	.96214940D-05
62	.15499990D+00	.11940570D-01	.11951310D-01	.10743740D-04
63	.15749990D+00	.12326450D-01	.12338400D-01	.11949800D-04
64	.15999990D+00	.12718350D-01	.12731600D-01	.13247130D-04
65	.16249990D+00	.13116250D-01	.13130910D-01	.14655290D-04
66	.16499990D+00	.13520170D-01	.13536330D-01	.16160310D-04
67	.16749990D+00	.13930080D-01	.13947850D-01	.17771500D-04
68	.16999990D+00	.14345980D-01	.14365480D-01	.19496310D-04
69	.17249990D+00	.14767870D-01	.14789200D-01	.21331940D-04
70	.17499990D+00	.15195730D-01	.15219020D-01	.23297970D-04
71	.17749990D+00	.15629540D-01	.15654940D-01	.25393440D-04
72	.17999990D+00	.16069330D-01	.16096940D-01	.27611850D-04
73	.18249990D+00	.16515060D-01	.16545040D-01	.29977410D-04
74	.18499990D+00	.16966730D-01	.16999220D-01	.32486390D-04
75	.18749990D+00	.17424330D-01	.17459490D-01	.35157430D-04
76	.18999990D+00	.17887850D-01	.17925840D-01	.37983060D-04
77	.19249990D+00	.18357300D-01	.18398270D-01	.40967020D-04
78	.19499990D+00	.18832660D-01	.18876780D-01	.44120480D-04
79	.19749990D+00	.19313910D-01	.19361370D-01	.47465790D-04
80	.19999990D+00	.19801040D-01	.19852040D-01	.50997360D-04

Fig. 5.3: Graphs of Numerical and Exact Solutions of Problem 1  
 Using Formula  $y_{n+4} = 4y_{n+3} - 6y_{n+2} + 4y_{n+1} - y_n + \frac{h^3}{2} [f_{n+3} - f_{n+1}]$

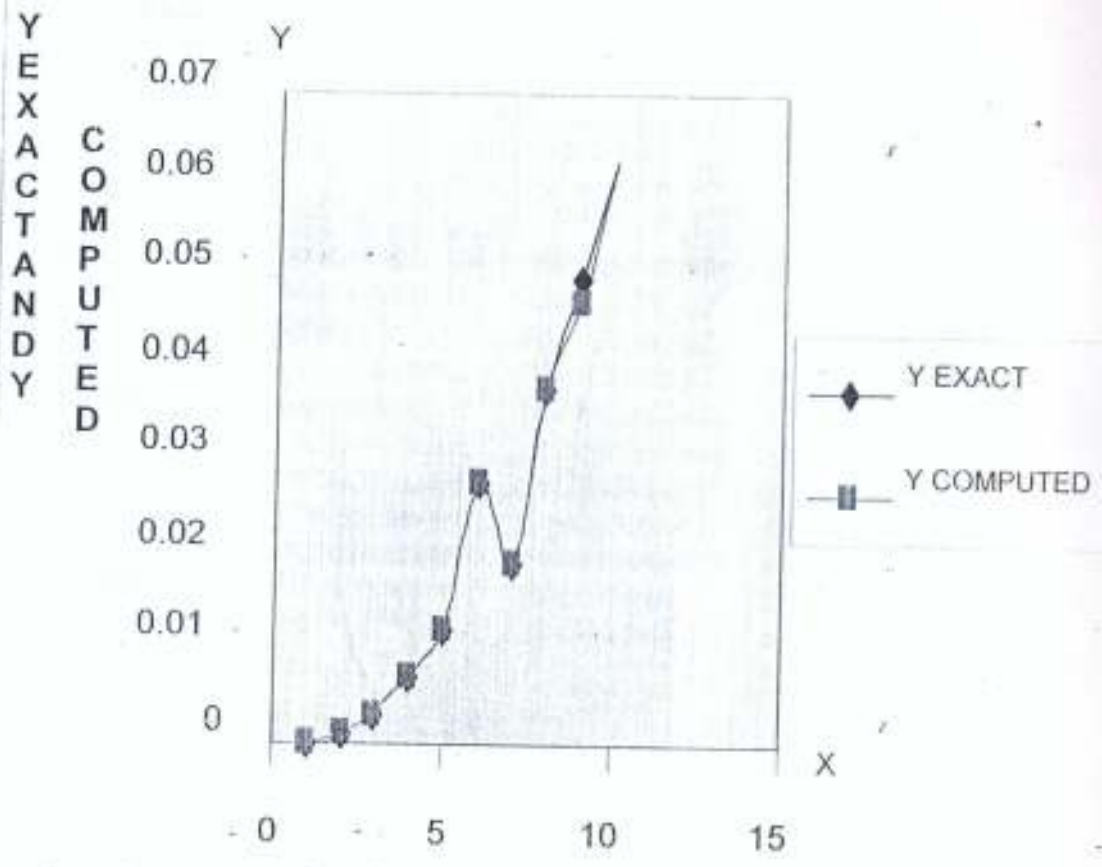


TABLE 8: NUMERICAL SOLUTION OF PROBLEM 2

$$y''' = -y' \quad y(0) = 0, y'(0) = 1, y''(0) = 2$$

Using the 3- step method with mesh size  $h = 2.500000E-003$

S/N	Mesh pt. $x_n$	Exact soln $Y(x_n)$	Numerical soln. $y_n$	ERROR ( $e_n$ )
*****				
0	.00000000D+00	.00000000D+00	.00000000D+00	.00000000D+00
1	.25000000D-02	.25061960D-02	.25062500D-02	.53551050D-07
2	.50000000D-02	.50250130D-02	.50249790D-02	.33993270D-07
3	.75000000D-02	.75561960D-02	.75561240D-02	.72177500D-07
4	.10000000D-01	.10099850D-01	.10099620D-01	.22724270D-06
5	.12500000D-01	.12655960D-01	.12655410D-01	.54575500D-06
6	.15000000D-01	.15224380D-01	.15223430D-01	.95739960D-06
7	.17500000D-01	.17805350D-01	.17803610D-01	.17471610D-05
8	.20000000D-01	.20398610D-01	.23000190D-01	.27287750D-05
9	.22500000D-01	.23004380D-01	.23000190D-01	.27287750D-05
10	.25000000D-01	.25622410D-01	.25616470D-01	.59381130D-05

TABLE 9 : NUMERICAL SOLUTION OF PROBLEM 3

$$y''' = -y y'' \quad y(0) = 0, y'(0) = 0, y''(0) = 1$$

Using the 3-step method with Mesh Size  $h = 2.500000E-003$

S/N	$X_n$	$(Y_n)$ COMPUTED
0	.00000000D+00	.00000000D+00
1	.25000000D-01	.31249950D-05
2	.50000000D-01	.12499920D-04
3	.75000000D-01	.28125260D-04
4	.10000000D+00	.50002240D-04
5	.12500000D+00	.78132570D-04
6	.15000000D+00	.11251840D-03
7	.17500000D+00	.15316260D-03
8	.20000000D+00	.20006810D-03
9	.22500000D+00	.25323870D-03
10	.25000000D+00	.31267850D-03
11	.27500000D+00	.37839230D-03
12	.30000000D+00	.45038500D-03
13	.32500000D+00	.52866230D-03
14	.35000010D+00	.61323030D-03
15	.37500010D+00	.70409560D-03
16	.40000010D+00	.80126520D-03
17	.42500010D+00	.90474670D-03
18	.45000010D+00	.10145480D-02
19	.47500010D+00	.11306780D-02
20	.50000010D+00	.12531460D-02
21	.52500000D+00	.13819610D-02
22	.55000000D+00	.15171330D-02
23	.57500000D+00	.16586730D-02
24	.60000000D+00	.18065920D-02
25	.62499990D+00	.19609000D-02
26	.64999990D+00	.21216100D-02
27	.67499990D+00	.22887340D-02
28	.69999990D+00	.24622860D-02
29	.72499980D+00	.26422770D-02
30	.74999980D+00	.28287240D-02
31	.77499980D+00	.30216390D-02
32	.79999980D+00	.32210380D-02
33	.82499970D+00	.34269370D-02

34	.84999970D+00	.36393500D-02
35	.87499970D+00	.38582950D-02
36	.89999970D+00	.40837880D-02
37	.92499970D+00	.43158460D-02
38	.94999960D+00	.45544870D-02
39	.97499960D+00	.47997280D-02
40	.99999960D+00	.50515890D-02
41	.10250000D+01	.53100880D-02
42	.10500000D+01	.55752460D-02
43	.10750000D+01	.58470820D-02
44	.11000000D+01	.61256170D-02
45	.11250000D+01	.64108730D-02
46	.11499990D+01	.67028720D-02
47	.11749990D+01	.70016360D-02
48	.11999990D+01	.73071880D-02
49	.12249990D+01	.76195510D-02
50	.12499990D+01	.79387490D-02
51	.12749990D+01	.82648060D-02
52	.12999990D+01	.85977460D-02
53	.13249990D+01	.89375950D-02
54	.13499990D+01	.92843790D-02
55	.13749990D+01	.96381230D-02
56	.13999990D+01	.99988550D-02
57	.14249990D+01	.10366600D-01
58	.14499990D+01	.10741390D-01
59	.14749990D+01	.11123250D-01
60	.14999990D+01	.11512210D-01
61	.15249990D+01	.11908300D-01
62	.15499990D+01	.12311540D-01
63	.15749990D+01	.12721980D-01
64	.15999990D+01	.13139630D-01
65	.16249990D+01	.13564530D-01
66	.16499990D+01	.13996720D-01
67	.16749990D+01	.14436210D-01
68	.16999990D+01	.14883050D-01
69	.17249990D+01	.15337270D-01
70	.17499990D+01	.15798900D-01
71	.17749990D+01	.16267980D-01
72	.17999990D+01	.16744530D-01
73	.18249990D+01	.17228600D-01

- 74 .18499990D+01 .17720220D-01
- 75 .18749990D+01 .18219420D-01
- 76 .18999990D+01 .18726240D-01
- 77 .19249990D+01 .19240730D-01
- 78 .19499990D+01 .19762910D-01
- 79 .19749990D+01 .20292820D-01
- 80 .19999990D+01 .20830500D-01

TABLE 10: NUMERICAL SOLUTION OF PROBLEM 3

$$y''' = -y y'' \quad y(0) = 0, y'(0) = 0, y''(0) = 1$$

Using 4-step method with Mesh Size  $h = 2.500000E-003$ 

S/N	$X_n$	$(Y_n)$ COMPUTED
0	.00000000D+00	.00000000D+00
1	.25000000D-01	.31249950D-05
2	.50000000D-01	.12499920D-04
3	.75000000D-01	.28124600D-04
4	.10000000D+00	.49999590D-04
5	.12500000D+00	.78125940D-04
6	.15000000D+00	.11250520D-03
7	.17500000D+00	.15313930D-03
8	.20000000D+00	.20003080D-03
9	.22500000D+00	.25318280D-03
10	.25000000D+00	.31259860D-03
11	.27500000D+00	.37828240D-03
12	.30000000D+00	.45023850D-03
13	.32500000D+00	.52847190D-03
14	.35000010D+00	.61298800D-03
15	.37500010D+00	.70379270D-03
16	.40000010D+00	.80089210D-03
17	.42500010D+00	.90429310D-03
18	.45000010D+00	.10140030D-02
19	.47500010D+00	.11300290D-02
20	.50000010D+00	.12523800D-02
21	.52500000D+00	.13810640D-02
22	.55000000D+00	.15160910D-02
23	.57500000D+00	.16574690D-02
24	.60000000D+00	.18052090D-02
25	.62499990D+00	.19593220D-02
26	.64999990D+00	.21198180D-02
27	.67499990D+00	.22867090D-02
28	.69999990D+00	.24600080D-02
29	.72499980D+00	.26397280D-02
30	.74999980D+00	.28258800D-02
31	.77499980D+00	.30184790D-02
32	.79999980D+00	.32175380D-02
33	.82499970D+00	.34230730D-02

34	.84999970D+00	.36350970D-02
35	.87499970D+00	.38536270D-02
36	.89999970D+00	.40786780D-02
37	.92499970D+00	.43102660D-02
38	.94999960D+00	.45484090D-02
39	.97499960D+00	.47931220D-02
40	.99999960D+00	.50444250D-02
41	.10250000D+01	.53023360D-02
42	.10500000D+01	.55668720D-02
43	.10750000D+01	.58380530D-02
44	.11000000D+01	.61158980D-02
45	.11250000D+01	.64004290D-02
46	.11499990D+01	.66916640D-02
47	.11749990D+01	.69896250D-02
48	.11999990D+01	.72943340D-02
49	.12249990D+01	.76058120D-02
50	.12499990D+01	.79240830D-02
51	.12749990D+01	.82491700D-02
52	.12999990D+01	.85810990D-02
53	.13249990D+01	.89198960D-02
54	.13499990D+01	.92655870D-02
55	.13749990D+01	.96182000D-02
56	.13999990D+01	.99777650D-02
57	.14249990D+01	.10344310D-01
58	.14499990D+01	.10717870D-01
59	.14749990D+01	.11098470D-01
60	.14999990D+01	.11486150D-01
61	.15249990D+01	.11880940D-01
62	.15499990D+01	.12282870D-01
63	.15749990D+01	.12691980D-01
64	.15999990D+01	.13108310D-01
65	.16249990D+01	.13531890D-01
66	.16499990D+01	.13962760D-01
67	.16749990D+01	.14400960D-01
68	.16999990D+01	.14846530D-01
69	.17249990D+01	.15299520D-01
70	.17499990D+01	.15759960D-01
71	.17749990D+01	.16227890D-01
72	.17999990D+01	.16703360D-01
73	.18249990D+01	.17186420D-01

74	.18499990D+01	.17677110D-01
75	.18749990D+01	.18175470D-01
76	.18999990D+01	.18681550D-01
77	.19249990D+01	.19195400D-01
78	.19499990D+01	.19717050D-01
79	.19749990D+01	.20246550D-01
80	.19999990D+01	.20783960D-01

TABLE 11: COMPARATIVE ANALYSIS OF THE 3-STEP

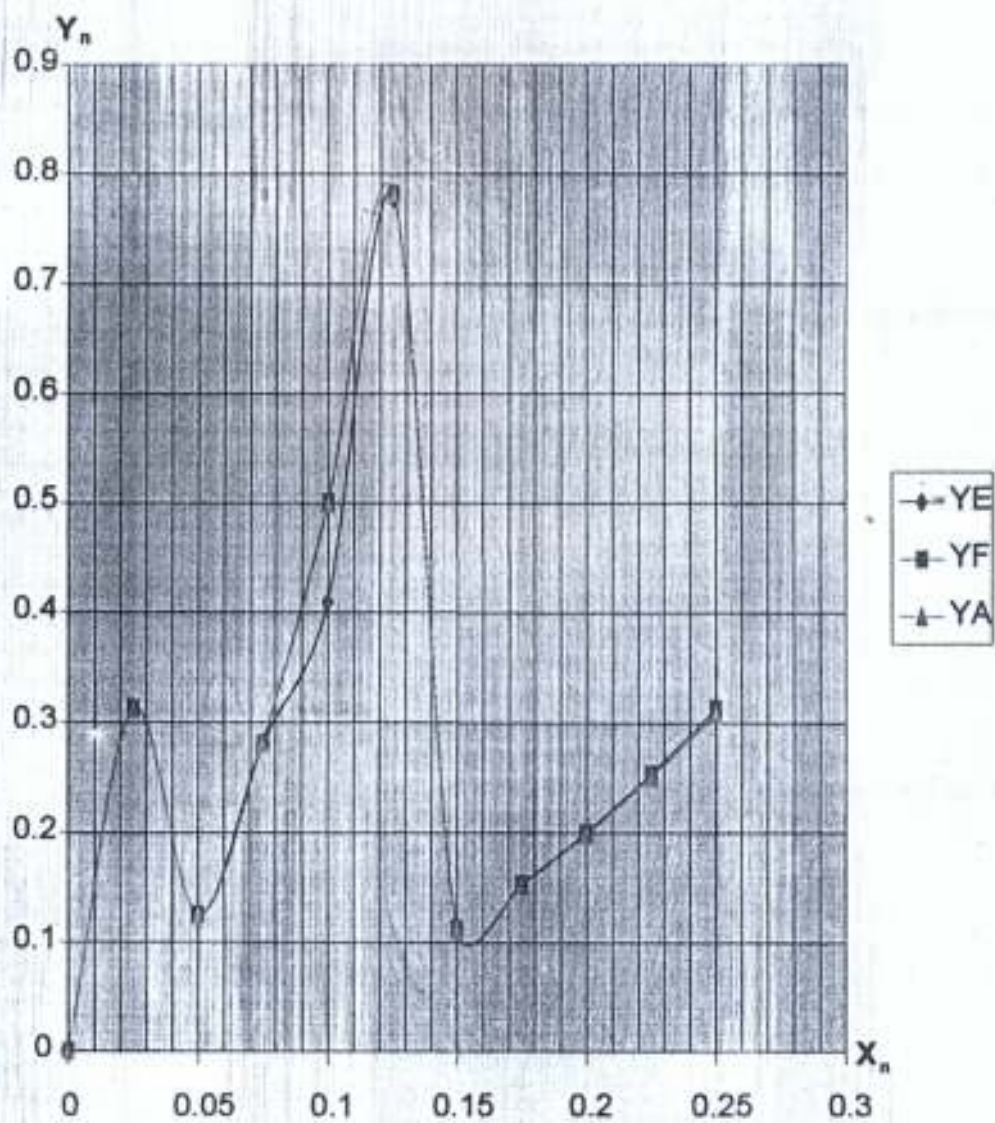
METHODS ON  $y''' - x - 4y' y(0) = y'(0) = 0, y''(0) = 1$

When  $h = 2.500000E-002$

S/N	$x_n$	EXACT SOLUTION ( $y_{xn}$ )	PROPOSED METHOD ( $y_n$ )	AWOYEMI ( $y_n$ )
*****				
0	.00000000D+00	.00000000D+00	.00000000D+00	0.0000000000D+00
1	.25000000D-01	.31244950D-03	.31249950D-05	0.3124511719D-03
2	.50000000D-01	.12492170D-02	.12499920D-04	0.1249218917D-02
3	.75000000D-01	.28085470D-02	.28124770D-04	0.2808547497D-02
4	.90464420D-02	.40918710D-04	.49999560D-04	0.4987515640D-02
5	.12500000D+00	.77820460D-02	.78124270D-04	0.7782043843D-02
6	.15000000D+00	.11186900D-01	.11249890D-03	0.1118690437D-01
7	.17500000D+00	.15195740D-01	.15312350D-03	0.1519573531D-01
8	.20000000D+00	.19801070D-01	.19999790D-03	0.1980105449D-01
9	.22500000D+00	.24994300D-01	.25312230D-03	0.2499428017D-01
10	.25000000D+00	.30765770D-01	.31249660D-03	0.3076575151D-01

## COMPARISON OF EXACT AND NUMERICAL SOLUTIONS

**Fig. 5. 4: Graphs of Numerical and Exact Solutions of Problem 1 using the Continuous three-step Method and the proposed three step method**





## CHAPTER SIX CONCLUSION

### 6.1 SUMMARY

In this work, the design, analysis and implementation of a class of Explicit Linear Multistep Methods for direct solution of 3rd order initial value problem of ordinary Differential equations are discussed. The derivation, analysis and implementation adopt Taylor series techniques, Boundary Locus method, The results show that the methods are Consistent, zero- stable and Convergence. The formulas is rewritten in computer algorithm and implemented using Fortran programming language to solve some sample problems of third order differential equation. The result is accurate and compare favourably with existing methods.

### 6.2 SUGGESTIONS FOR FURTHER STUDIES

Since the method is linear Multistep, there is need for a predictor to determine the starting values which are not at that instant available. The predictor should be of the same order of accuracy as the main method.

### 6.3 THE ACHIEVEMENTS

The proposed method can be used to analyze problems on Control and Population dynamics in Biology.



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## APPENDIX ONE

```

DIMENSION Y(300),yp(300),ypp(300),ypp1(300),ypp2(300)
DIMENSION x(300),f(300),fp(300),yp1(300),yp2(300),v1(300)
DIMENSION v2(300),v3(300),ER(300),Y1(300),Y2(300),Yk(300)
DIMENSION Yn(300), v(300), EE(300),fpp(300)
OPEN(2,FILE='adebi.OUT',STATUS='NEW')

```

```
DO 10 i = 0,80
```

```
Y(0)=0
```

```
v(0)=Y(0)
```

```
v(1)=.000003124995
```

```
v(2)=.00001249992
```

```
x(0)=0
```

```
yp(0)=0
```

```
h = 1.0/40.0
```

```
x(i+1)=x(i)+h
```

```
  x(i+2)=x(i+1)+h
```

```
  WRITE(*,*)X(i+1),x(i+2)
```

```
f(0)=0
```

```
ypp(0)=1.0
```

```
fp(0)=-yp(0)*ypp(0)
```

```
fpp(0)=-ypp(0)*f(0)
```

```
ypp1(i+1)=ypp(0)+i*h*f(0)+(i**2)*(h**2)*fp(0)/2
```

```
ypp2(i+1)=(i**3)*(h**3)*fpp(0)/6
```

```
ypp(i+1)=ypp1(i+1)+ypp2(i+1)
```

```
yp1(i+1)=yp(0)+(i+1)*h*ypp(0)+(((i+1)*h)**2)*f(0)/2
```

```
yp2(i+1)=(((i+1)*h)**3)*fp(0)/6+(((i+1)*h)**4)*fpp(0)/24
```

```
yp(i+1)=yp1(i+1)+yp2(i+1)
```

```
f(i+1)=y(i+1)*yp(i+1)
```

```
  PRINT f(i)
```

```
  SLEEP
```

```
fp(i+1)=-y(i+1)*yp(i+1)
```

```
ypp1(i+2)=ypp(0)+(i+1)*h*f(0)+((i+1)**2)*(h**2)*fp(0)/2
```

```
ypp2(i+2)=((i+1)**3)*(h**3)*fpp(0)/6
```

```

ypp(i+2)=ypp1(i+2)+ypp2(i+2)
f(i+2)=-y(i+2)*yp(i+2)
Y1(i+1)=Y(0)+h*yp(0)+((h**2)*ypp(0)/2)
Y2(i+1)=((h**3)*f(0)/6)+(h**4)*fp(0)/24
Y(i+1)=Y1(i+1)+Y2(i+1)

```

```

Y1(i+2)=Y(0)+2*h*yp(0)+((4*(h**2))*ypp(0)/2)
Y2(i+2)=((8*(h**3))*f(0)/6)+(16*(h**4))*fp(0)/24
Y(i+2)=Y1(i+2)+Y2(i+2)

```

```

C PRINT Y(i + 1), Y(i + 2)

```

```

10 CONTINUE

```

```

v(1)=.000003124995
v(2)=.00001249992
v(3)=.000028124600
Y(1)=v(1)
Y(2)=v(2)
Y(3)=v(3)

```

```

C SLEEP

```

```

DO 20 j = 0,80
Yk(j+4)=4*Y(j+3)-6*Y(j+2)+4*Y(j+1)-Y(j)
Yn(j+4)=(h**3)*(f(j+3)-f(j+1))/2
Y(j+4)=Yk(j+4)+Yn(j+4)
v(j+4)=Y(j+4)

```

```

20 CONTINUE

```

```

C REM

```

```

C REM

```

```

WRITE(2,*)'THIS IS THE NUMERICAL RESULT FOR MESH SIZE

```

```

H =',h

```

```

WRITE(2,*)

```

```

WRITE(2,*)

```

```

WRITE(2,30)

```

```

C H1XN=(-(H(0)/Y(0)**2)*(-X(0)/Y(0))

```

```

C DO 15 K=0,80

```

```

C WRITE(*,*)X(K)

```

```

C15 CONTINUE

```

```

30  FORMAT(2X,'NT',10X,'X',8X,'Y COMPUTED')
    WRITE(2,40)
40
FORMAT(4X,'*****')
*****
      I*****')

      WRITE(2,*)
C   PRINT #21, " X "; "      Y EXACT"; "      Y COMPUTED"; "
ERROR"
C   REM
C   REM

C   REM

      DO 50 K =0,80
      v1(K)=(3*(1-COS(2*x(K)))/16)+(x(K)**2)/8
      WRITE(*,*)x(K),v1(K)
C   REM
C   REM
C   WRITE(*,*)v1(K)
      ER(K)= v1(K)-v(K)
C   'IF ER(k) < 0 THEN
C   'ER(k) = -ER(k)
C   'ELSE
C   'ER(k) = ER(k)
C   'END IF
      EE(K)=ABS(ER(K))
      WRITE(2,60)K,x(K),v(K)
60  FORMAT(1X,I3,4(D16.8,2X))

C   PRINT #21, K,x(K), v(K)
C   REM
C   REM
50  CONTINUE
C   PRINT #21, "At h = "; h
      END

```

## APPENDIX TWO

```

DIMENSION Y(300),yp(300),ypp(300),ypp1(300),ypp2(300)
DIMENSION x(300),f(300),fp(300),yp1(300),yp2(300),v1(300)
DIMENSION v2(300),v3(300),ER(300),Y1(300),Y2(300),Yk(300)
DIMENSION Yn(300), v(300), EE(300),fpp(300)
OPEN(2,FILE='PRO1K4.OUT',STATUS='NEW')

DO 10 i = 0,80
Y(0)=0
v(0)=Y(0)
v(1)=.000003124995
v(2)=.00001249992

x(0)=0
yp(0)=0

h = 1.0/40.0
x(i+1)=x(i)+h
C x(i+2)=x(i+1)+h
C WRITE(*,*)X(i+1),x(i+2)

f(0)=0
ypp(0)=1.0
fp(0)=1-4*ypp(0)
fpp(0)=-4*f(0)

ypp1(i+1)=ypp(0)+i*h*f(0)+(i**2)*(h**2)*fp(0)/2
ypp2(i+1)=(i**3)*(h**3)*fpp(0)/6
ypp(i+1)=ypp1(i+1)+ypp2(i+1)
yp1(i+1)=yp(0)+(i+1)*h*ypp(0)+(((i+1)*h)**2)*f(0)/2
yp2(i+1)=(((i+1)*h)**3)*fp(0)/6+(((i+1)*h)**4)*fpp(0)/24
yp(i+1)=yp1(i+1)+yp2(i+1)
f(i+1)=x(i+1)-4*yp(i+1)
C PRINT f(i)
C SLEEP

fp(i+1)=1-4*ypp(i+1)
ypp1(i+2)=ypp(0)+(i+1)*h*f(0)+((i+1)**2)*(h**2)*fp(0)/2
ypp2(i+2)=((i+1)**3)*(h**3)*fpp(0)/6

```

```

ypp(i+2)=ypp1(i+2)+ypp2(i+2)
f(i+2)=x(i+2)-4*ypp(i+2)
Y1(i+1)=Y(0)+h*yp(0)+((h**2)*ypp(0)/2)
Y2(i+1)=((h**3)*f(0)/6)+(h**4)*fp(0)/24
Y(i+1)=Y1(i+1)+Y2(i+1)

Y1(i+2)=Y(0)+2*h*yp(0)+((4*(h**2))*ypp(0)/2)
Y2(i+2)=((8*(h**3))*f(0)/6)+(16*(h**4))*fp(0)/24
Y(i+2)=Y1(i+2)+Y2(i+2)
PRINT Y(i + 1), Y(i + 2)

```

0) CONTINUE

```

v(1)=.000003124995
v(2)=.00001249992
v(3)=.000028124600
Y(1)=v(1)
Y(2)=v(2)
Y(3)=v(3)
SLEEP

```

```

DO 20 j = 0,80
Yk(j+4)=4*Y(j+3)-6*Y(j+2)+4*Y(j+1)-Y(j)
Yn(j+4)=(h**3)*(f(j+3)-f(j+1))/2
Y(j+4)=Yk(j+4)+Yn(j+4)
v(j+4)=Y(j+4)
CONTINUE
REM
REM

```

WRITE(2,\*)THIS IS THE NUMERICAL RESULT FOR MESH SIZE  
=,h

```

WRITE(2,*)
WRITE(2,*)
WRITE(2,30)
H1XN=-(H(0)/Y(0)**2)*(-X(0)/Y(0))
DC 15 K=0,80
WRITE(*,*)X(K)
5 CONTINUE

```

```
30 FORMAT(2X,'NT',10X,'X',15X,'Y EXACT',8X,'Y  
COMPUTED',10X,'ERROR')
```

```
WRITE(2,40)
```

```
40
```

```
FORMAT(4X,'*****')
```

```
*****
```

```
[*****')
```

```
WRITE(2,*)
```

```
C PRINT #21, "X "; " Y EXACT"; " Y COMPUTED"; "  
ERROR"
```

```
C REM
```

```
C REM
```

```
C REM
```

```
DO 50 K =0,80
```

```
v1(K)=(3*(1-COS(2*x(K)))/16)+(x(K)**2)/8
```

```
WRITE(*,*)x(K),v1(K)
```

```
C REM
```

```
C REM
```

```
C WRITE(*,*)v1(K)
```

```
ER(K)=v1(K)-v(K)
```

```
C 'IF ER(k) < 0 THEN
```

```
C 'ER(k) = -ER(k)
```

```
C 'ELSE
```

```
C 'ER(k) = ER(k)
```

```
C 'END IF
```

```
EE(K)=ABS(ER(K))
```

```
WRITE(2,60)K,x(K),v1(K),v(K),EE(K)
```

```
60 FORMAT(1X,I3,4(D16.8,2X))
```

```
C PRINT #21, K,x(K), v1(K), v(K), EE(K)
```

```
C REM
```

```
C REM
```

```
50 CONTINUE
```

```
C PRINT #21, "At h = "; h
```

```
END
```