A CLASS OF SEMI-IMPLICIT RATIONAL RUNGE-KUTTA SCHEME FOR SOLVING ORDINARY DIFFERENTIAL EQUATIONS WITH DERIVATIVE DISCONTINUITIES.

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IMC/00/8279

A THESIS IN THE DEPARTMENT OF MATHEMATICAL SCIENCES SUBMITTED TO THE SCHOOL OF POSTGRADUATE STUDIES TOWARDS A PARTIAL FULFILMENT OF THE REQUIREMENT FOR THE AWARD OF THE DEGREE OF MASTERS OF TECHNOLOGY (M.TECH) IN INDUSTRIAL MATHEMATICS, OF FEDERAL UNIVERSITY OF TECHNOLOGY, AKURE, NIGERIA.

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This work is dedicated to the Almighty God, and my caring mother,

Chief (Mrs.) **ESTHER ODUNAYO BOLAJI,**

a mother in a million .

ACKNOWLEDGEMENT

Foremost, thanks and praises to God almighty for sparing my *life and* making me worthy of completing this work. His protection, guidance and blessing over me is enormous.

My gratitude to my supervisor, Dr. R. A. Ademiluyi is without bound. He readily and freely gave me his books and ideas. His enormous contributions through guidance, valuable corrections and suggestions towards making the work a good piece of work is highly appreciated; indeed, I thank him for his patience and tolerance throughout the course of this study. I appreciate the efforts and contributions of my co supervisor Dr. Awoyemi ,and that of my Head of Department ,Professor S. T. Oni as appreciated.

My special thanks goes to other lecturers in the department for their contributions and support during the course of the Programme. Dr. Alao's contribution is highly appreciated.

The support of my course mate Mr. Babatope Omolofe is highly appreciate. To Ife Ajiwoye, I say a big thank you for his support. The effort of my colleague; Mr. Ogundele, Ayo Arobieke.his appreciated. I also appreciate the effort of my younger brother, Emmanuel Abimbola for the typewritten of the manuscript.

The support and understanding of my wife, Temitayo and my Children, Bolade and Obasanmi during the course of the Programme is appreciated. To others who contributed in one way or the other, I say thank you.

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ABSTRACT

In this thesis, a class of semi-implicit Rational runge-Kutta schemes were developed, analyzed and computerized for solving deferential equations with derivative discontinuities. The method is motivated by variety of application areas of this class of ordinary differential equations, which include electrical transmission networks, nuclear reactions, delay problems and computer aided designs, economy affected by inflation as well as perturbation problems or dynamic processes in industries and technological fields and the need to cater for the deficiencies identified in the adoption of the existing methods of solving this class of differential equations.

The development of the schemes, its analysis and implementation on a microcomputer adopt the power series (Taylor and Binomial) expansion and Pade's approximation techniques and Fortran programming respectively. The consistency, convergence and stability properties were investigated, it was discovered that these schemes converge and were stable. Numerical results of the adoption of the schemes on some sample problems shows that it is effective and efficient.

INTRODUCTION

An equation connecting the ***derivatives*** of a differentiable function of one independent variable with respect to itself is known as ***Ordinary Differential Equation (ODE).***

The general form of an n^{th} ***order*** ordinary differential equation is

$$F\{x, y, y^{I}, y^{II} \text{ ———————— } y^{(n)}\} = 0 \qquad \text{.................................} \quad (1.1)$$

Where x is the independent variable and, y is the dependent variable. with y^{I}, y^{II}, y^{III} ———————— $y^{(n)}$ as its differential coefficients.

Often, equation (1.1) is expressed in the canonical form

$$y^{(n)} = f\{x, y, y^{I}, y_{1}^{II} \text{ -------- } y^{(n-1)}\} \qquad \text{.................................} \quad (1.2)$$

$$\text{were } y^{(i)}(xo) = \eta_{i} \qquad i = 0(1)\ n-1 \qquad \text{.................................} \quad (1.3)$$

This class of equation form a major model of real life problems arising from the field of Sciences, Management and Technology. It can be classified according to its structure or behaviour.

A *Differential Equation* is *linear* when it does not contain a product of the idependent variable or its derivative, and, it is of highest order 1; otherwise it is *non-linear.*

The ***Order*** of a Differential Equation is the order of the highest derivative contained in the equation, while its ***degree*** is the exponent or power to which the highest order derivatives in the equation is raised after rationalization.

The Differential equation (1.1) with the initial condition (1.3) as stated above is called ***Initial Value Problem (IVP)***.

Any nth order differential equation of the form (1.2) and (1.3) can be reduced to a first order vector equation of;

$y' = f(x,y), \quad y(xo) = \eta$ .... (1.4)

where

$\eta_i = (\eta_0, \eta_1, \eta_2 \text{———} \eta_{n-1})^T$

$f = (f_1, f_2, f_3 \text{———} f_n)^T$

$y = (y_1, y_2, y_3 \text{———} y_{n-1})^T$

by setting $y_1 = y$, $y_2 = y^1$, $y_3 = y^{(11)}$ ------- $y_n = y^{(n-1)}$

To solve the given initial value problem (1.4) its structure plays an important role, because its solution must be seen to satisfy the associated initial conditions.

However, the situations that can arise in connection with the given initial value problem (1.4) are that:

(i) f(x,y) is continuous and real valued.

(ii) f(x,y) contains Discontinuities in the form of finite jumps in the components of f itself in the region D of (x,y) – plane defined by:

$D = \{(x,y) \text{ such that } a \leq x \leq b, -\infty < y < \infty\}$

Or discontinuities of low, order in some derivatives of the function f.

(iii) f(x,y) is ***non linear***

(iv) Eigen value λ of the Jacobean J =df/dy of f is large, that is, stiff problem.

(v) There exist low, order discontinuities in the solution.

Differential equations constitute a large and important branch of mathematics with various applications in nearly all fields of human endeavours. Several researchers had contributed to its study and solution both ***analytically*** and ***numerically.***

Apostate (1965) worked on analytical solution of differential equation whose f(x,y) is continuous and satisfy Lipschitz condition. Lambert (1965,1973), Fatunla (1982), Ademiluyi (1987), Okunbor (1985) and many others worked on the numerical solution of this kind of differential equations.

However, when f(x,y) contains discontinuities in the form of finite jumps in the component of f or the solution; few researchers are known to have worked in this area, they include Feldstein and Neves (1984) Okosun (2003). Okosun proposed a k^{th} – order inverse polynomial methods for its numerical integration. The limitation to his work is the complexity of the determination of the k parameters of the methods, which increases with the stage K of the scheme.

We say that the differential equation has a low order Discontinuous derivatives when f(x,y) is undefined or unbounded and the partial derivatives f_x, f_y are large and unbounded at the point(x,y). Few literature exist in this area, however the concern of the thesis is to describe the development, analysis and implementation of some semi implicit Rational Runge-Kutta schemes for the integration of Ordinary Differential Equations whose derivatives are discontinuous.

The feature of this differential equation (1.4) that require consideration in this work is the phenomenon of ***Derivative Discontinuities.***

This class of Ordinary Differential Equations arises in variety of application areas such as:

(i) Electrical transmission network,

(ii) Nuclear reactions,

(iii) Delay problems and computer aided designs

(iv) Economy affected by inflation, as well as perturbation problems or dynamical processes in industries and technological fields.

Next, we consider the nature of differential equation with derivative discontinuities.

1.1.1 NATURE OF DERIVATIVE DISCONTINUOUS ORDINARY DIFFERENTIAL EQUATIONS

Derivative Discontinuities as it is used in the content of Ordinary Differential Equation is the concept describing the nature of some subset of ordinary differential equation whose derivatives contains discontinuities in the form of finite jumps or and whose partial derivatives is unbounded. At these points where there is low order derivative discontinuities, a numerical method adopted for its solution may become either in accurate or inefficient or both in this region of discontinuities.

For ***example;*** the first order differential equations.

$$y^{1}(x) = \begin{cases} 3-y(x) & \text{if } y(x) \leq 2 \\ 5-y(x) & \text{if } y(x) > 2 \end{cases} \quad \text{with } y(0) = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1.5)$$

whose theoretical solution

$$y(x) = \begin{cases} 3-2e^{-x} & \text{for } 0 \leq x < \ln(2) \\ 5-6e^{-x} & \text{for } \ln(2) \leq x < \infty \end{cases}$$

Clearly, $y^{1}(x)$ jumps when the solution $y(x)$ passes through the value $x = 2$.

Another example is first order initial value problem.

$$y^{1} = -x/y \qquad y(x_0) = y_0 \qquad \text{----------------- (1.6)}$$

whose theoretical solution is

$$y(x) = \sqrt{c - x^2}$$

represents a family of circles with centre (0,0). Clearly, the derivative y^{1} is discontinuous at $(\pm\sqrt{c},0)$, showing that the differential equation posses derivative discontinuity at point (0.0). The discontinuity is ***State-dependent,*** that is, it depends on the solution y (x).

1.1.2PROBLEM ASSOCIATED WITH SOLVING ORDINARY DIFFERENTIAL EQUATIONS WITH DERIVATIVE DISCONTINUITIES

Since the differential equations we are considering does not satisfy uniqueness and existence theorem, that is, f(x,y) and its partial derivatives f_x, f_y are non continuous and unbounded in the region of integration; conventional algorithm (that are based on polynomial representation which preassumes that the solution and its derivatives are sufficiently continuous throughout the region of integration,) will be deficient in solving initial value problem that violates this uniqueness theorem, because they give rise to a solution whose solution "explodes" in the neighbourhood of this discontinuity, Carver (1978).

1.1.3 EXISTING METHODS FOR SOLVING ORDINARY DIFFERENTIAL EQUATION WITH DERIVATIVE DISCONTINUITIES.

The existing methods designed for solving ordinary differential equations with low order discontinuities evolved from researchers who were motivated to work in this area after having discovered that the accuracy and efficiency of the existing methods vary with the location of points of discontinuities that is, ***State dependent*** (Paul, (1999).

Such existing methods includes:

(i) Fraction step method by Fatunla & Evans (1975)

(ii) The switching function techniques by Fatunla and Evans (1975)

(iii) The inverse interpolation method by Hay et al (1974)

(iv) Use of discontinuity tracking equation proposed by Paul (1999)

(v) Use of Defect error control method proposed by Paul (1999)

(vi) The local error estimator techniques by Gear and Qsterly (1984)

The local error estimator technique entails locating and detecting of discontinuities of the derivatives in the **Ordinary Differential Equations**.

Hind-Marsh (1974) incorporated this into an existing automatic code called GEAR. GEAR provided an efficient vehicle to restart the integration beyond the point of discontinuity and the algorithm also provides estimates of the magnitude and order of the discontinuity

The Detector

$$d = \left|\frac{Tol}{tn}\right|^{\frac{1}{(p+1)}} \geq 1 \qquad \text{.......................... (1.7)}$$

where "tol" is the allowable error tolerance, "p" is the order of the method and t_η is the estimate of the local truncation error; this equation (1.7) is the basis for identifying the presence of discontinuity in the derivative of the ODE. The procedure for doing this, is by repeatedly halving the step size h.

Lambert and Shaw (1966) proposed an algorithm in which the theoretical solution to (1.4) is represented by the perturbed polynomial interpolant

$$F(x) = P_m(x) + \begin{cases} b/A+x/^N; \; N \notin \{0, 1 \text{-----------} L\} \\ b/A+x/^N \log /A+x/; \; \notin \{0, 1 \text{------} L\} \end{cases} \text{................. (1.8)}$$

with $P_m(x)$ being a polynomial of degree m defined as

$$P_m(x) = \sum_{r=0}^{m} a_r x^r \qquad \text{..................(1.9)}$$

and the second term on the right hand side is the perturbed term. A and N are the discontinuity parameters. "A" controls the location of the discontinuities while "N" determines its nature. The shortcoming of this scheme is that, it is efficient only for initial value problems whose discontinuities are restricted to those of (1.8),[B0, *H-L (1983)*]

An alternative procedure include the Rational functions approximation technique of the form.

$$f(x) = \frac{P_m(x)}{b+x} \qquad (P_m \text{ being a polynomial of degree m.}) \qquad \text{........................(1.10)}$$

Suggested by Lambert and Shaw (1965). However, its limitation includes the need to classify the nature of the singularity involved in the ordinary differential equations.

However, Luke et al (1975) extended (1.10) to a more general Rational polynomial functions of the form.

$$R(x) = \frac{P_u(x)}{Q_v(x)} \qquad \text{......................(1.11)}$$

where P_u and Q_v are polynomilas of degree u and v respectively

His suggestion, perhaps must have been motivated by satisfactory performance of the formula (1.10) despite its limitations. Formula (1.11) eliminates the need to characterize the nature of singularities; the singularities are specified by the zeros of the $Q_v(x)$. Also, because of the complexity in the derivation of the resultant numerical method, Fatunla (1982) suggested the adoption of a special variant of (1.11) in the form.

$$y(x) = \frac{A}{1 + \sum_{j=1}^{k} b_j x^j} \qquad k \geq 1 \qquad \text{......................(1.12)}$$

where the parameters A and b_j are real coefficients as the form of the solution to the differential equation. He considered the case k = 1 and came up with first order computational method:

$$y_{n+1} = \frac{y_n^2}{y_n - hy_n^1} \qquad \text{......................(1.13)}$$

Called ***inverse Euler formula***. However, the difficulty that may be associated with this formula (1.13) is that, if

$$y_n - hy^1_n = 0$$

$$\text{Or } h = \frac{y_n}{y_n^1}$$

The method will break down.

Another computational method is that of ***Hong Yuan fu (1982)*** who proposed a rationalised Runge - Kutta Schemes of the form.

$$y_{n+1} = \frac{y_n + \sum_{i=1} W_i k_i}{1 + y_n \sum_{i=1}^{s} V_i K_i} \qquad \text{.................... (1.14)}$$

Where

$k_1 = hf(x_n, y_n)$

$k_i = hf(x_n + c_i h, y_n + \sum_{j=1}^{i} a_{ij} k_j)$

$H_1 = hg(x_n, z_n)$

$H_i = hg(x_n + d_i h, z_n + \sum_{j=1}^{i} b_{ij} H_j)$

$g(x_n, z_n) = -z_n^2 f(x_n, y_n)$

and $z_n = 1/y_n$

a_{ij}, b_{ij} are real valued constants.

He developed families of explicit methods of orders two and three. He found out that the schemes were 'A - stable.

Perhaps, the A - stability property of these explicit schemes stimulated Okunbor (1985) in extending the schemes to the family of order four. It was observed that the higher the stage of the methods the poorer the stability of the method. Four years later, Babatola (1999) considered a performance evaluation of this methods and found that it is accurate but inefficient for stiff-oscillatory problems.

This deficiency of explicit Rational Runge Kutta schemes motivated Babatola (1999) to consider the implicit Rational Runge-Kutta schemes and generated a family of one stage scheme of order two, two stage scheme of order three and four for numerical solution of stiff and stiffly oscillatory ordinary differential equations. The schemes were found to be A stable.

The A - stability and accuracy properties of this formula, stimulate the extension of the method to ***semi implicit*** method which will be less cumbersome. We shall exploit the A-stability, accuracy properties, and its structure to develop a computational method.

We shall at the neighbourhood of singularities select step size that will step over it. Consequently, we are proposing in chapter three of this thesis, a numerical methods that will be suitable for the solution of ordinary differential equation with low order derivative discontinuities.

1.2 AIMS AND OBJECTIVES

In this thesis, we consider the development, analysis and implementation of a family of semi implicit Rational Runge-Kutta schemes for numerical integration of ordinary differential equations with derivative discontinuities.

Of particular consideration are one stage schemes of order two, two stage schemes of order three and four . The method will be computerized in ***Fortran programming*** language and implemented on a digital computer adopting double precision arithemetic using sample problems arising from real life situations.

1.3 MOTIVATION

The motivation for this work are:

(a) The large areas of applications of this class of ordinary differential equations, which includes: electrical transmission network, nuclear reactions, delay problems and computer aided designs, economy affected by inflation as well as perturbation problems or dynamic processes in industries and technological fields.

(b) The need to cater for the deficiencies (associated with) the existing methods in solving differential equation of this class.

(c) The structure of the new method will enable us to select appropriate step sizes that will over step the points of discontinuities in the ordinary differential equations.

1.4 RESEARCH METHODOLOGY

Power series expansion techniques (Taylor and Binomial expansions) were adopted to generate the parameters of the schemes as stated in equation (1.14).

Pade's approximation and Richardson extrapolation techniques plus Felhberg error control method wereadopted for the control and analysis of the convergence and stability properties of the proposed schemes.

The applicability, suitability and accuracy of the schemes were established by translating the formula into a computer code using **fortran programming language** and Implemented on a digital computer using sample problems arising from real life situations.

1.5 STRUCTURE OF THE THESIS

The remaining chapters of the thesis were organized as described below:

In *Chapter Two,* the general principles of one step schemes and other concepts used in the development of the proposed schemes were discussed.

In *Chapter Three,* we consider the derivation of the proposed schemes viz: one stage scheme of order two, two stage scheme of order three and four methods respectively.

Chapter Four contains consistency, convergence and stability properties analysis of the schemes.

Chapter Five discusses the computer implementation of the methods and its application for solution of some sample problems with the result recorded in tables.

The *last Chapter* contains, the summary of the work, its limitation, contribution to knowledge and recommendations.

CHAPTER TWO

PRELIMINARY CONCEPTS AND PRINCIPLES

The proposed schemes are based on one step approach, therefore, it is important to discuss some of the techniques involved, particularly, the conventional Runge-Kutta Schemes that form the basis of the proposed schemes.

2.1 CONVENTIONAL RUNGE-KUTTA SCHEME

This classical schemes was proposed by Runge (1901) and later improved by Kutta (1915) and it is known as Runge-Kutta schemes. It is one of the oldest generation of numerical methods for solution of Ordinary Differential Equations (ODEs).

It is an example of one – step methods for solving differential equation of type (1.4), because, the approximation y_{n+1} to the solution at point x_{n+1} can be obtained from the knowledge of y_n at point x_n. Thus,
an R- stage Runge-Kutta schemes is of the form.

$$y_{n+1} = y_n + \sum_{j=1}^{R} W_i K_i \qquad \text{.........................} (2.1)$$

where

$$K_i = hf\left(x_n + a_i h, y_n + \sum_{j=1}^{i} b_{ij} K_j\right) \qquad \text{................} (2.2)$$

With the constraints

$$C_i = \sum_{j=1}^{i} b_{ij}$$

$$\text{and } \sum_{j=1}^{i} W_i = 1 \qquad \text{..................} (2.3)$$

It is often divided into three classes namely;

(i) Explicit: If $B = (b_{ij}) = 0$ for $j \geq i$

(ii) Semi implicit if $B = (b_{ij}) = 0$ for $j > i$

(iii) Implicit: If $B = (b_{ij}) \neq 0$ for at least one $j > i$

a set of non linear equations generated from adoption of the following steps:

(i) Taylor series expansion of k_i about point (x_n, y_n) for $i = 1(1)s$

(ii) Insertion of the results of the expansion in (1) into equation (2.1)

(iii) Comparison of the final expansion with the Taylor series expansion of y_{n+1} about x_n in powers of h_i

The number of these parameters normally exceed the numbers of equations, hence, some important parameters were chosen as to ensure that the resultant methods have the following properties.

(i) minimum local truncation error bound (Ralston, 1962)

(ii) maximum attainable order of accuracy (King 1966)

(iii) Optimal interval of absolute stability (Lawson 1966, 1976)

(iv) optimal storage space requirement (Gill, 1951)

It was shown that R – stage implicit Runge-Kutta Scheme of orders 2 to 5 are all A- stable (Butcher, 1954). These properties shall be discussed later in this thesis.

Some examples of popular Runge Kutta schemes are as given below:

(i) The implicit Euler Scheme

$y_{n+1} = y_n + hk_1$ .... (2.4)

Where $k_1 = f(x_n + h, y_n + k_1)$

(ii) Two stage trapezoidal scheme of order two

$y_{n+1} = y_n + h/2\,(k_1 + k_2)$ .... (2.5)

Where

$k_1 = f(x_n + h, y_n + k_1)$

$k_2 = f(x_n + h, y_n + k_2)$

and Hollingsworth (1955)

$$y_{n+1} = y_n + h/2\,(k_1 + k_2) \quad \text{.................. (2.6)}$$

Where

$$k_1 = f(x_n + (1/2 - \sqrt{3}/6)h,\ y_n + \tfrac{1}{4}k_1 + (\tfrac{1}{4} - \sqrt{3}/6)k_1$$

$$k_2 = f(x_n + (1/2 + \sqrt{3}/6)h,\ y_n + (\tfrac{1}{4} + \sqrt{3}/6)hk + \tfrac{1}{4}hk_2)$$

(iv) Three stage implicit Runge Kutta Scheme of order 3 defined by Butcher (1964)

$$y_{n+1} = y_n + h/18(5k_1 + 8k_2 + 5k_3) \quad \text{.................. (2.7)}$$

Where

$$k_1 = f\left[x_n + \{1/2 - \sqrt{15}/10\}h,\ y_n + 5/36hk_1 + (2/9 - \sqrt{15}/15)k_2 + (5/36 - \sqrt{15}/30)hk_3\right]$$

$$k_2 = f\left[x_n + 1/2h, y_n + (5/3 - \sqrt{15}/34)hk_1, + 5/3hk_2 + (2/9 + \sqrt{15}/34)k_3\right]$$

$$k_3 = f\left[x_n + (1/2 + \sqrt{15}/10)h,\ y_n + (5/36 + \sqrt{15}/30)hk_1 + (2/9 + \sqrt{15}/18)hk_2 + 5/36hk_3\right]$$

The basis for measuring the reliability of numerical one-step scheme includes its ability to control the error it generates per step.Hence we consider the error, order of convergence and stability properties of our scheme in the next section.

2.2 ITS ERROR, ORDER OF ERROR, CONVERGENCE AND STABILITY PROPERTIES

For any numerical scheme, it is important that the scheme has ability to reliably control the global error given by

$$e_{n+1} = y_{n+1} - y(x_{n+1}) \quad \text{................ (2.8)}$$

where $y(x_{n+1})$ is the theoretical solution and

y_{n+1} is the numerical solution at step x_{n+1}.

One of the concepts of convergence of any numerical scheme is that, it is required that the global error (2.8) is made as small as possible by making h sufficiently close to zero.

Definition 1

The ***local truncation*** error T_{n+1} associated with one step schemes (2.1) is defined as

$$T_{n+1} = y(x_{n+1}) - y(x_n) - h\phi(x_n, y(x_n); h) \qquad \text{.................. (2.9)}$$

Which is the amount by which the theoretical solution $y(X_{n+1})$ of initial value problem (1.4) fails to satisfy difference equation (2.1)

Considering (2.8) and (2.9), it is obvious that there is a relationship between the global error defined in (2.8) and local truncation error defined in (2.9). The two are connected by this inequality.

$$\left|e_{n+1}\right| \leq k\left|T_{n+1}\right| \text{ (Lambert, 1973)} \qquad \text{.............. (2.19)}$$

Where k is a constant. Hence, the **local truncation error is directly** proportional to the global error introduced at each step, **particularly, when** the derivation and computation of the local truncation error is rigorous and all previous solutions are exact. The above ideas lead to establishment of convergence of one step schemes, which necessitate the following definitions.

Definition 2

One step scheme of type (2.1) is said to be ***convergent,*** if when applied to initial value problem of type (1.4) and the correspondent approximation y_n to the solution satisfies:

$y_n \rightarrow y(x_n)$ as $n \rightarrow \infty$

$$\text{Or } \lim_{n \rightarrow \infty} y_n = y(x_n) \qquad \text{.......................(2.11)}$$

Definition 3

The one step scheme (2.1) is said to be of order P, if P is the largest positive integer such that the local truncation error T_{n+1} satisfies

$$T_{n+1} = 0(h^{p+1}) \qquad \text{..................(2.12)}$$

where $0(h^{p+1})$ implies the existence of finite constant C and $h > 0$, such that

$$T_{n+1} = C(h^{p+1}) \qquad \text{..................(2.13)}$$

as $h \to 0$

Definition 4

The one step schemes (2.1) is said to be ***consistent*** if

$$Q(x,y;0) = f(x,y) \qquad \text{..................(2.14)}$$

As h tends to zero.

The consistency of one-step numerical scheme ensures that the scheme is at least of order one. That is, $P \geq 1$.

Definition 5

The one-step scheme (2.1) is said to be regular if function $\phi(x,y;h)$ is defined in the domain $\Delta = \{(x,y)/a \leq x \leq b, -\infty - /y/ < \infty \text{ and } 0 \leq h \leq h_0\}$ (h_0 being a positive constant) if there exist a constant L such that

$$|\phi(x,y,h) - \phi(x,z,h)| \leq L\,|y - z| \qquad \text{..................(2.15)}$$

for every $x \in [a,b]$ and $/z/ < \infty$, $h \in [0,h_0)$

We then state without proof a theorem which guarantees the convergence of one-step scheme, as follows :

THEOREM 1

A necessary and sufficient condition for *convergence* of a one-step scheme is that the scheme must be *consistent*.

2.2.1 ACCURACY OF NUMERICAL SCHEME

A numerical solution y_n to an initial value problem in ordinary differential equation is said to be accurate if it does not deviate significantly from the corresponding exact solution $y(x_n)$ otherwise it is inaccurate.It is

Important to mention and discuss in brief two major forms of instability viz:

(i) ***inherent instability***
(ii) ***induced instability***

Inherent instability is a property of the differential equation itself. While the induced instability is the characteristics of the numerical schemes.

For better explanation of these concepts, let us consider the Scalar initial value problem.

$$y^1 = \lambda y,\ y(x_0) = y_0 \qquad \text{.........................(2.16)}$$

With re $(\lambda) < 0$ over the interval $a \leq x \leq b$. its theoretical solution is $y(x) = y_0 e^{\lambda x}$ ....(2.17)

Suppose the initial value condition in (2.16) is changed to

$$y(x_0) = y_0 + \beta \qquad \text{.........................(2.18)}$$

Where $\beta > 0$,

Then, the solution of (2.17) having the new initial condition (2.18) yields.

$$Y(x) = y_0 e^{\lambda x} + \beta e^{\lambda x} \qquad \text{.........................(2.19)}$$

For B>0, no matter how small, the second term $\beta e^{\lambda x}$ in (2.19) can be seen to grow exponentially if $Re(\lambda) > 0$ as the computation proceeds, regardless of any integration scheme used. Therefore, the solution becomes ***unstable*** for slight change caused by a small perturbation of initial value called initial condition, even when the given differential equation is stable. This kind of instability is called inherent instability. On the other hand, finite iteration process is adopted as against infinite iteration process in the computer implementation of our scheme, which introduce build up truncation error that eventually causes an ***induced instability.*** To detect this, the integration scheme is applied to solve the Scalar test equation (2.16).

To minimize the instability which shows up in form of a spurious exponential as occured in (2.19), the step size h is reduced.

We state the following definitions on the stability of one-step scheme with respect to, the global error, for better understanding of the concept.

Definition 6

One step scheme is said to be stable, if for any initial error e_0, there exist a constant k and $h_o>o$, such that when (2.1) is applied to initial value problem (1.4) with step size $h \in (0,h_0)$ the ultimate error e_n satisfies the following inequality.

$$e_n \leq ke_{o}, \qquad k < \infty \qquad \text{.................... (2.20)}$$

One step schemes is said to be absolutely stable for a given step size and for initial value problem (1.4) if the errors tends to zero as the step size approaches zero.

To investigate the absolute stability property of the one-step scheme (2.1), we apply it to the Scalar test problem (2.16). This yields a first order difference equation.

$$y_{n+1} = \mu(\alpha)y_n \quad \alpha = \lambda h \qquad \text{............ (2.21)}$$

where $\mu(\lambda)$ is the so-called stability function of the one-step schemes ($\mu(\alpha)$ can be a polynomial or a rational function).

The stability function of an S-stage implicit Runge-Kutta Scheme of order 2 is given as:

$$\mu(\alpha) = 1 + \alpha W^T(I - \alpha A)^{-1} e \qquad \text{...............(2.22)}$$

Where

$A = \{a_{ij}\} \quad i, j = 1(1)m$

$e = [1,1,1,\text{----------} 1]^T$

$W = [W_1, W_2, W_3, \ldots\ldots\ldots W_m]^T$

I is a unit $m \times m$ matrix.

The *absolute stability* of implicit Runge-Kutta Schemes is relatively

Large compared with explicit counter part which is a quality that makes them suitable to solving differential equations with singularities and stiff oscillatory problems in ODEs.

We shall discuss in broad sense the Stability properties of the proposed Scheme in Chapter four of this thesis.

CHAPTER THREE

THE PROPOSED SCHEMES

3.1 RATIONAL RUNGE-KUTTA SCHEMES

Rational functions are quotients of polynomials, that is, functions of the form.

$$R(X) = \frac{S(X)}{T(X)} \quad \text{.......................................(3.1)}$$

Where S(x) and T(x) are polynomials of degrees ***m*** and ***n*** respectively (the coefficient of the highest power of x in T(x) is taken to be unity). Rational functions constitute a much better representation of a function in the region of singularity. This quality make them principal targets for function approximation.

Perhaps, this property of rational functions motivated Hong Yuan-Fu (1982) to propose the rational scheme.

$$y_{n+1} = \frac{y_n + \sum_{j=1}^{r} W_i K_i}{1 + y_n \sum_{j=1}^{s} V_i H_i} \quad \text{..........................(3.2)}$$

Where

$$K_i = hf(x_n + c_i h,\ y_n + \sum_{j=1}^{r} a_{ij} k_j)\, i = 1\ (1)\ r$$

$$H_i = hg\ (x_n + d_i h,\ z_n + \sum_{j=1}^{r} b_{ij} H_j) \qquad i = 1\ (1)\ s$$

$g(x_n, z_n) = - z_n 2\ f(x_n, y_n)$ and $z_n = 1/y_n$ and r is the stage of the method ....(3.3)

This formula forms the basis of the proposed scheme. Babatola and Ademiluyi (2000) classified the method into Explicit, semi-implicit and implicit family of methods. For details consult Ademiluyi (2000). The explicit family and the explicit families of orders one, two and three respectively were discussed in Hong Yuan Fu (1982), Okunbor (1985) while implicit class of the method were considered in Babatola (1999), and

A two stage explicit formula of order two which was obtained by setting r = s = 2 in (3.1) is given by

$$y_{n+1} = \frac{y_n + \frac{1}{2}(k_1 + k_2)}{1 + y_n/4\,(H_1 + H_2)} \qquad \text{.....................(3.4)}$$

Where

$$k_1 = hf(x_n, y_n)$$

$$k_2 = hf(x_n + h, y_n + hf_n)$$

$$H_1 = hg(x_n, z_n)$$

$$H_2 = hg(x_n + h, y_n + H_1)$$

He analyzed this schemes and discovered that the schemes were A-stable. The stability properties and its relative ease of programming probably stimulated Okunbor in extending the scheme's to family of order four in 1985.

These family of explicit rational Runge-Kutta schemes are known and confirmed to perform well on certain class of ordinary differential equations, especially stiff and stiff oscillatory equations but more satisfactorily if they are not sophisticated. The success history of the implicit, Runge-Kutta schemes over its explicit type, becomes the motivator for the extension of the schemes to semi-implicit formula which hopefully will be more efficient than the implicit method.

Of particular interest is the development of some semi implicit one-stage and two stage schemes of order two and four respectively for numerical approximation of differential equations with Derivative discontinuities.

3.2 THE DEVELOPMENT OF THE PROPOSED SCHEME

An r-stage rational Runge-Kutta Scheme is a numerical formula of the form

$$y_{n+1} = \frac{y_n + \sum_{i=1}^{r} W_i k_i}{1 + y_n \sum_{i=1}^{r} V_i H_i} \qquad \text{.....................(3.5)}$$

where

$$k_i = hf(x_n + C_i h,\ y_n + \sum_{j=1}^{i} a_{ij} k_j)\quad i = 1(1)r \qquad \text{..................(3.6)}$$

$$H_i = hg(x_n + d_i h,\ z_n + \sum_{j=1}^{i} b_{ij} H_j)\quad i = 1(1)r \qquad \text{..........(3.7)}$$

$$g(x_n, z_n) = -z_n^2 f(x_n, y_n) \qquad \text{..............(3.8)}$$

$$\text{and } z_n = 1/y_n \qquad \text{..............(3.9)}$$

With the constraints

$$C_i = \sum_{j=1}^{i} a_{ij} \quad i = 1(1)r \qquad \text{...............(3.10)}$$

$$d_i = \sum_{j=1}^{i} b_{ij} \quad i = 1(1)r \qquad \text{................(3.11)}$$

The formula is semi-implicit if $a_{ij} = 0$ and $b_{ij} = 0$ for $j > i$. The consistency of the schemes is ensured by the constraints(3.10) and (3.11).

3.2.1 PROCEDURE FOR THE DEVELOPMENT OF THE SCHEME

The value of parameters a_{ij}, b_{ij}, c_i, d_i, V_i and W_i of the scheme are determined from the systems of non-linear equations generated from the adoption of the following steps:

Step 1: (i) Expansion of k_i and H_i about point (x_n, y_n) for $i = 1(1)r$, using Taylor series approach.

(ii) Insertion of the result of the expansion in (1) into equation (3.5) and re-... in powers of h

(iii) Comparison of the final expansion with the Taylor series expansion of y_{n+1} about (x_n, y_n) in the powers of h.

Step 2: The resulting equations were solved bearing in mind that the number of parameters (are found to) exceed the numbers of equations, hence, some parameters were choosen arbitrarily as free parameters so that the resultant formulae have the following properties:

(a) Adequate order of accuracy.

(b) Minimum bound of local truncation error

(c) Relatively large interval of absolute stability.

(d) Minimum computer storage facilities requirement.

In the next unit, the parameters a_{ij}, b_{ij}, c_i, d_i, v_i and W_i are determined (for some typical family of semi-implicit rational Runge Kutta Scheme).

3.2.2 ONE STAGE SCHEMES

By Setting r = 1 in equation (3.7), the general one stage semi-implict rational Runge-Kutta scheme is of the form:

$$y_{n+1} = \frac{y_n + w_1 k_1}{1 + y_n V_1 H_1} \qquad \text{.................(3.12)}$$

Where

$$K_1 = hf(x_n + c_1 h, y_n + a_{11} k_1 \qquad \text{.................(3.13)}$$

$$H_1 = hg(x_n + d_1 h, z_n + b_{11} H_1) \qquad \text{.................(3.14)}$$

$$g(x_n, z_n) = -z_n^2 f(x_n, y_n) \qquad \text{.................(3.15)}$$

$$\text{and } z_n = 1/y_n \qquad \text{.................(3.16)}$$

With the constraints

$$C_1 = a_{11} \qquad \text{.................(3.17)}$$

$$d_1 = b_{11} \qquad \text{.................(3.18)}$$

Using binomial expansion on the right hand side of (3.12), and ignore terms of order higher than one, we have

$$y_{n+1} = y_n + W_1 k_1 - y_n^2 V_1 H_1 + \text{(higher order terms)} \qquad \text{..........(3.19)}$$

The Taylor series expansion of y_{n+1} about (x_n, y_n) gives:

$$y_{n+1} = y_n + hy'_n + h^2/2!\, y'' + h^3/3!\, y^{(3)} + h^4/4!\, y^{(4)} + 0(h^5) \qquad \text{.............(3.20)}$$

But, $y'_n = f(x_n, y_n) = f_n$

$$y'' = f_n + f_n f_y = Df_n$$

$$y^{(3)} = f_{xx} + 2f_n f_{xy} + f_n^2 f_{yy} + f_y(f_x + f_n f_y) = D^2 f_n + f_y^2 f_n$$

$$y^{(4)} = f_{xxx} + 3f_n f_{xxy} + 3f_n^2 f_{xyy} + f_n^3 f_{yyy} + f_y(f_{xx} + 2f_n f_{xy} + f_n^2 f) + (f_x + f_n f_y$$

$$(3f_{xy} + 3f_n f_y + f_y^2) = D^3 f_n + f_y D^2 f_n + 3Df_n Df_y + f_y^2 Df_n \qquad \text{.................(3.21)}$$

Where

$$D^2 f_n = f_{xx} + 2f_n f_{xy} + f_n^2 f_{yy}$$

$$Df_y = f_{xy} + f_n f_{yy} + f_y^2$$

$$D^3 f_n = f_{xxx} + 3f_n f_{xxy} + 3f_n^2 2f_{xyy} + f_n^3 f_{yy} \qquad \text{.................(3.22)}$$

Putting (3.21) into (3.20), we have

$$y_{n+1} = y_n + hf_n + h^2/2!\, Df_n + h^3/3!\,(D^2f_n + f_y Df_n) + h^4/4!$$
$$(D^3f_n + f_y D^2 f_n + 3Df_n\, Df_y) + f_y^2\, Df_n + 0(h^5) \qquad \text{.................(3.23)}$$

The Tylor Expansion of k_1 about (x_n, y_n) is given by

$$k_1 = h(f_n + C_1 f_x + a_{11} k_1 + fy) + 1/2(C_1^2 h^2 f_{xx} + 2C_1 h a_{11} k_1 f_{xy} + a_{11} k_1^2 f_{yy})$$
$$+ 0(h^4) \qquad \text{.................(3.24)}$$

Collecting coefficients of equal powers of h, equation (3.24) is expressed in the form

$$k_1 = hA_1 + h^2 B_1 + h^3 D_1 + 0(h^4)\ . \qquad \text{.................(3.25)}$$

Where

$$A_1 = f_n$$
$$B_1 = c_1(f_x + f_n f_y) = c_1 Df_n$$
$$D_1 = C_1 B_1 f_y + \tfrac{1}{2} C_1(f_{xx} + 2f_n f_{xy} + f_n^2 f_{yy}) = C_1^2(Df_n fy + 1/2 D^2 fn) \ \text{......(3.26)}$$

In a similar manner, expansion of H_1 about (x_n, z_n) yields

$$H_1 = hN_1 + h^2 M_1 + h^3 R_1 + 0(h^4) \qquad \text{.................(3.27)}$$

Where

$$N_1 = g(x_n, z_n) = g_n$$
$$M_1 = d_1(g_x + g_n g_z) = d_1 Dg_n$$
$$R_1 = d_1 m_1 g_z + 1/2\, d_1^2 (g_{xx} + 2g_n g_{xz} + g_n^2 g_{zz})$$
$$= d_1^2 (g_z Dg_n + \tfrac{1}{2} D^2 g_n) \qquad \text{.................(3.28)}$$

with $Dg_n = g_x + g_n g_z$

and $D^2 g_n = g_{xx} + 2g_n g_{xz} + g_n^2 g_{zz}$ ....(3.29)

From definition:

$A = (a_{ij}) = 0$ for $i,j = 1(1)s$, for $j>i$

and $B = b_{ij} = 0$ for $i,j = 1(1)r$, for $j>i$

This implies that when $b_{ij} = 0$ for $j > i$, the element of the matrix B is as given (represented in the matrix) below:

$$B = \begin{pmatrix} b_{11} & 0 & 0 & 0 \\ b_{21} & b_{22} & 0 & 0 \\ b_{31} & b_{32} & b_{33} & 0 \\ b_{41} & b_{42} & b_{43} & b_{44} \end{pmatrix}$$

Which is a lower triangular matrix.

Similarly matrix A is given by

$$A = \begin{pmatrix} a_{11} & 0 & 0 & 0, \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & 0 & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

Hence, from (3.19) and (3.20) and Matrix A

$C_1 = a_{11} \neq 0$

$D_1 = b_{11} \neq 0$

$$\qquad\qquad\qquad\qquad\qquad\qquad (3.30)$$

Assuming that f and g in (3.29) are sufficiently differentiable functions, then, we can express g and its partial derivatives in terms of f and its partial derivatives to facilitates the comparison of coefficients.

That is

$g_n = -f_n/y^2_n$, $\qquad g_x = -f_x/y^2_n$, $\qquad g_{xx} = f_{xx}/y_n^2$

$g_y = -2f_n/y_n + f_x$, $\qquad g_{xy} = -2f_x/y_n + f_{xx}$

$g_{nxy} = -2f_{xx}/y_n + f_{xxx}$, $g_{yy} = -2f_n - y^2_n f_{xx}$

$g_{xyy} = -2f_x - y_n^2 f_{xxx}$

$g_{yyy} = 4y^2_n fy + 6y^2_n f_{xx} + y_n^4 f_{xxx}$ $\qquad\qquad (3.31)$

Putting (3.31) into (3.28), we have

$N_1 = -f_x/y^2_n$

$M_1 = -d_1/y_n^2 (Df_n + 2f_n^2/y_n)$

$R_1 = -d_1^2/y_n^2 [-2f_n/y_n + f_x) + (Df_n + f_n^2/y_n) + 1/2D^2 f_n$

$\qquad -2f_n/y_n (f_n^2/y_n + f_x)]$ $\qquad\qquad (3.32)$

Adopting (3.25) and (3.27) in (3.19), we have

$$y_{n+1} = y_n + W_1(hA_1 + h_2B_1 + h^3R_1 + 0(h^3) - y_n^2[V_1(hN_1 + h^2M_1 + h^3R_1 + 0(h^4)]$$
$$+ (W_1R_1 - y^2_nV_1R_1)h^3 + 0(h^4)$$

$$y_{n+1} = y_n + (W_1A_1 - y_n^2V_1w_1)h + (W_1B_1 - y_n^2v_1m_1)h^2 \quad \text{.................(3.33)}$$

Comparing the coefficients of the powers of h in equation (3.23) and (3.33) we obtained

$$W_1A_1 - y_n^2V_1N_1 = f_n \quad \text{.................(3.34)}$$

from (3.26) and (3.32), we have

$$A_1 = f_n$$
$$N_1 = -f_n/y_n^2 \quad \text{.................(3.35)}$$

Putting these values into (3.36), we have

$$W_1 + V_1 = 1 \quad \text{.................(3.36)}$$

Comparing coefficients of h^2 in equations (3.23) and (3.33), we obtained

$$W_1B_1 = y_n^2V_1M_1 = Df_n/2 \quad \text{.................(3.37)}$$

and from (3.26) and (3.32) we have

$$B_1 = C_1Df_n$$
$$M_1 = -d_1/y_n^2(Df_n + 2f_n^2/y_n) \quad \text{.................(3.38)}$$

Putting (3.38) into (3.37) we have

$$(W_1C_1 + V_1d_1)Df_n + \frac{2f_n^2V_1d_1}{y_n} = \frac{Df_n}{2} \quad \text{.................(3.39)}$$

From (3.39) we obtain

$$W_1C_1 + V_1d_1 = ½ \quad \text{.................(3.40)}$$

Combining (3.36) and (3.40) we obtain the following systems of equations for family of one stage Runge-Kutta Scheme of order two.

$$\left.\begin{array}{l} W_1 + V_1 = 1 \\ W_1C_1 + V_1d_1 = ½ \end{array}\right\} \quad \text{.................(3.41)}$$

with

$a_{11} = C_1$

$b_{11} = d_1$ ....(3.42)

These can be solved to provide numerical values for W_1, V_1, C_1 and d_1: Imposing the conditions

(i) $W_1 = 0$, $V_1 = 1$ in equations (3.41)

we obtain, $C_1 = d_1 = ½$

$\Rightarrow a_{11} = b_{11} = ½$

Putting this values into (3.12), we have

$$y_{n+1} = \frac{y_n}{1 + y_n H_1} \qquad \text{..(3.43)}$$

Where

$H_1 = hg(x_n + ½h, z_n + ½ H_1)$

(ii) By setting $V_1 = W_1 = ½$ and $C_1 = 3/4$ in equation (3.42)

Then, $C_1 = a_{11} = ¾$

$\Rightarrow d_1 = b_{11} = ¼$

Using this in equation (3.12), we have

$$y_{n+1} = \frac{y_n + ½ k_1}{1 + y_{n_2} H_1} \qquad \text{..(3.44)}$$

Where $k_1 = hf(x_n + ¾h, y_n + ¾k_1)$

$H_1 = hg(x_n + ¼h, z_n + ¼H_1)$

$z_n = 1/y_n$

(iii) similarly by setting $W_1 = 1/5$, $V = 4/5$, $C_1 = d_1 = ½$

$\Rightarrow a_{11} = b_{11} = ½$

Equation (3.12) yields

$$y_{n+1} = \frac{y_n + 1/5k_1}{1+4/5y_n H_1} \quad \text{.......................................(3.45)}$$

Where

$K_1 = hf(x_n + ½h, y_n + ½k_1)$

$H_1 = hg(x_n + ½h, z_n + ½H_1)$

(iv) When $W = 1/3$, $V_1 = 2/3$, $a_{11} = C_1 = 1/3$

$B_{11} = d_1 = 7/12$

Equation (3.14) yields

$$y_{n+1} = \frac{y_n + 1/3\, k_1}{1 + 2/3\, y_n H_1} \quad \text{........................(3.46)}$$

Where

$K_1 = hf(x_n + 1/3h, y_n + 1/3k_1)$

$H_1 = hg(x_n + 7/12h, z_n + 7/12H_1)$

From here, it can be seen that many formulae can be derived from a one stage method. And this is the reason why we call it family of one stage method.

3.2.3 TWO STAGE SCHEME

A general two-stage semi-implicit rational Runge Kutta Scheme is of the form

$$y_{n+1} = \frac{y_n + \sum_{i=1}^{2} W_i k_i}{1 + y_n \sum_{i=1}^{2} V_i H_i} \quad \text{........................(3.47)}$$

Where

$$k_i = hf(x_n + c_i h, y_n + \sum_{j=1}^{2} a_{ij} k_j) \qquad i = 1(1)2 \quad \text{................(3.48)}$$

$$Hi = hg(x_n + d_i h, z_n + \sum_{j=1}^{2} b_{ij} H_j), \qquad i = 1(1)2 \quad \text{................(3.49)}$$

$g(x_n, z_n) = -z_n^2 f(x_n, y_n)$

and $z_n = 1/y_n$

With the constraints

$a_{11} + a_{12} = C_1$

$a_{21} + a_{22} = C_2$

$b_{11} + b_{22} = d_1$

$b_{21} + b_{22} = d_2$ ....(3.50)

Adopting binomial expansion theorem on the right hand side of equation(3.47), we have

$y_{n+1} = y_n + \sum_{i=1}^{2} W_i k_i \quad y_n^2 \sum_{i=1}^{2} V_i k_i$ + (higher other terms)..............(3.51)

Expanding k_i in equation (3.48) about (x_n, y_n)

We obtain

$K_i = h[f_n + (C_i h f_x + (a_{i1}k_1 + a_{i2}k_2)f_y) + ½ (c_i^2 h^2 f_{xx} + 2c_i h(a_{11}k_1 + a_{12}k^2)f_{xy} + (a_{11}k_1 + a_{12}k_2)^2 f_{yy} + 1/6 \quad (c_i^3 h^3 f_{xxx} + 3c_i^2 h^2(a_{11}k_1 + a_{i2}k_2)f_{xxy} + 3c_i h (a_{i1}k_1 + a_{i2}k_2)^2 f_{xxy} + (a_{i1}k_1 + a_{12}k_2)^3 f_{yy}]$ ....(3.52)

By rearranging and collecting coefficients of terms of equal powers of h, we have

$k_i = hA_i + h^2 B_i + h^3 E_i + h^4 D_i + 0(h^5)$ $\quad i=1(1)2$ ....(3.53)

Where

$A_i = f_n$

$B_i = c_i f_x + (a_{i1}A_1 + a_{i2}A_2)f_y$

$E_i = (a_{i1}B_1 + a_{12}B_2)f_y + ½ C_i^2 f_{xx} + (a_{i1}A_1 + a_{12}A_2)^2 f_{xy} + C_i (a_{i1}A_1 + a_{12}A_2)^2 f_{yy}$

$D_i = [(a_{i1}E_1 + a_{12}E_2)f_y + c_i (a_{i1}B_1 + a_{12}B_2)f_{xy} + (a_{ii}A_1 + a_{i2}A_1)(a_{ii}B_1 + a_{12}B_2)f_{yy} + 1/6 C_i^3 f_{xxx} + ½ C_i (a_{ii}A_1 + a_{12}A_2)f_{xxy} + ½ C_i (a_{ii}A_1 + a_{11}A_2)f_{xyy} + 1/6 (a_{i1}A_1 + a_{i2}A_2)^3 f_{yyy}$ ....(3.54)

Putting (3.21) into (3.54), we have

$A_i = f_n$

$B_i = C_i Df_n$

$E_i = (a_{i1}C_1 + a_{12}C_2) f_y Df_n + ½ C_i^2 D^2 f_n$.

$D_i = \{a_{i1}(a_{11}C_1+a_{12}C_2)+a_{12}(a_{21}C_1+a_{22}C_2)\} fy^2 Df_n + ½ C_1(a_{11}C_1+a_{12}C_2)Df_n$
$Df_y + ½ (a_{i1}C_1^2+a_{12}C_2^2) f_y D^2 f_n + 1/6 C_1^3 D^3 f_n$ ....(3.55)

In a similar manner, the expansion of H_i, i =1 (1) 2 about (x_n, z_n) is

$H_i = hN_i + h^2 m_i + h^3 R_i + h^4 h_i + 0(h^5)$. i= 1(1)2 .... (3.56)

Where

$N_i = g(x_n, z_n) = g_n$

$M_i = d_i g_x + (b_{i1}N_1 + b_{i2}N_2) = d_i^2 g_n$

$R_i = (b_{i1}m_1 + b_{12}m_2) g_z + 1/2\ d_i^2 g_{xx} + d_i\ (b_{i1}N_1 + b_{12}N_2) g_{xz} + 1/2$
$(b_{i1}N_1 + b_{i2}N_2)^2 g_{zz} = (b_{i1}d_1 + b_{i2}n_2) g_z Dg_n + 1/2\ diD^2 g_n$

$Li = (b_{i1}R_1 + b_{i2}R^2) g_z + (b_{i1}M_1 + b_{12}M_2) g_{xz} + \{(b_{i1}N_1 + b_{i2}N_2)(b_{i1}M_1 + h_{i2}M_2)\} g_{zz}$
$+ 1/6 d_i^3 g_{xxx} + 1/2\ d_i^2 (b_{i1}N_1 + b_{i2}N_2) g_{xxz} + 1/2\ d_i\ (b_{i1}N_1 + b_{i2}N_2) g_{xzz}$
$= \{b_{i1}(b_{i1}\ d_1 + b_{i2}d^2) + b_{i2}(b_{21}d_1 + b_{22}d_2) g^2_z 0g_n + 1/6\ (b_{i1}N_1 + b_{i2}N_2) g_{zzz}$
$+ \{d_i\ (b_{i1}d_1 + b_{i2}d_2\} Dg_n\ Dg_z + 1/2\ (b_{i1}\ d_i^2 + b_{12}d_2^2) g_z D^2 g_n$
$+ 1/6 d_1^3 D^2 g_n$ i = 1 (1) 2 ....(3.57)

Where

$Dg_n = g_x + g_n g_x$

$D^2 g_n = g_{xx} + 2g_n g_{xz} + g_n^2 g_{zz}$

$D^3 g_n = g_{xxx} + 3g_x g_{xxz} + 3g_n^2 g_{zzz} + g_n^3 g_{zzz}$

$Dg_z = g_{xz} + g_z^2 + g_n g_{zz}$ ....(3.58)

Putting equation (3.31) into (3.57), we have

$N_i = f_n / y_n^2$

$M_i = -d_i / y_n^2 (Df_n + 2f_n^2 / y_n)$

$R_i = 1/y_n^2 [(b_{i1}d_1 + b_{12}d_2)(-2f_n/y_n + f_y)](Df_n + 2f_n^2/y_n) + ½ (d_1^2 (D^2 f_n$
$+ 2f_n/y_n (f_x + 2f_n/y_n)]$

$L_i = 1/y^2_n \{b_{11}(b_{11}d_1 + b_{12}d_2) + b_{12}(b_{21}d_1 + b_{22}d_1^2)(-2f_n/y_n + f_y)(Df_n + 2f^2_n/yn$

$$+(d_1(b_{11}d_1+b_{12}d_2)Df_n + 2f^2_n/y_n(-2f_n/y_n+f_y)] + \tfrac{1}{2}[(b_{11}d^2+b_{12}d_2^2)$$
$$D^2f_n - 2f_n/y_n(f_y - f_n^2/y_n)] + 1/6\,(d_1^3 + f_{n}f_y) + 2f_n^3(2f_y+3f_{yy} \ldots\ldots (3.59)$$

Recalling that

$$y_{n+1} = y_n + (W_1K_1 + W_2K_2) - y_n^2(V_1H_1 + V_2H_2 + \text{(higher other terms)} \ldots (3.60)$$

Putting (3.53) and (3.56) into (3.60) we obtain

$$y_{n+1} = y_n + [W_1A_1 + W_2A_2) - y_n^2(V_1N_1 + V^2N_2)]h^2 + [(W_1B_1 + W_2B_2)$$
$$-y_n^2(V_1M_1 + V_2M_2)]h^2 + [(W_1E_1 + W_2E_2) - y_n^2(V_1R_1 + V_2R_2)]h^3$$
$$+[(W_1D_1 + W^2D^2) - y_n^2(V_1L_1 + V_2L_2)]h^4 + 0(h^5) \ldots\ldots\ldots\ldots (3.61)$$

Comparing the coefficient of h, h^2 and h^3 in equation (3.23) and (3.61).
We obtained the following systems of equations for family of two stage scheme of order three.

$$W_1 + W_2 + V_1 + V_2 = 1 \ldots\ldots\ldots\ldots (3.62)$$
$$W_1C_1 + W_2C_2 + V_1d_1 + V_2d_2 = \tfrac{1}{2} \ldots\ldots\ldots\ldots (3.63)$$
$$W_1(a_{11}C_1 + a_{12}C_2) + W_2(a_{21}C_1 + a_{22}C_2) + V_1(b_{11}d_1 + b_{12}d_2)$$
$$+ V_2(b_{21}d_1 + b_{22}d_2) = 1/6 \ldots\ldots\ldots\ldots (3.64)$$
$$W_1C_1^2 + W_2C_2^2 + V_1d_1^2 + V_2d_2^2 = 1/3 \ldots\ldots\ldots\ldots (3.65)$$

With constraints

$a_{11} + a_{12} = C_1 \Rightarrow a_{11} = C_1$ (Since $a_{12} = 0$ for semi-implicit method)
$a_{21} + a_{22} = C_2$
$b_{11} + b_{12} = d_1 \Rightarrow b_{11} = d_1$ (Since $b_{12} = 0$ for semi-implicit method)
$b_{21} + b_{22} = d_1 \ldots\ldots\ldots\ldots (3.66)$

Solving the above system of equations (3.62)- (3.65),
a family of two stage schemes of order three are obtained by Imposing the conditions

(i) $W_1 = W_2 = 0$, then $V_1 = ¼$, $V_2 = ¾$

$d_1 = b_{11} = b_{21} = 1$

$b_{12} = a_{12} = 0$, $d_2 = 1/3$

$B_{22} = -5/3$

Putting these values into (3.47), we have

$$y_{n+1} = \frac{y_n}{1 + y_n/4(H_1 + 3H_2)} \quad ....(3.68)$$

Where

$H_1 = hg(x_n + h, z_n + H_1)$

$H_2 = hg(x_n + 1/3h, z_n + (H_1 + 5/3\, H_2)$

(ii) $W_1 = W_2 = 0$, then $V_1 = 3/4$, $V_2 = 1/4$

$d_1 = b_{11} = 10/3$

$d_2 = -8$, $b_{21} = 1$

$a_{12} = b_{12} = 0$

$b_{22} = 9/2$

Putting these values into (3.47), we have

$$y_{n+1} = \frac{y_n}{1 + y_n/4\,(3H_1 + H_2)} \quad ....(3.69)$$

Where

$H_1 = hg(x_n + 10/3h, z_n + 10/3H_1)$

$H_2 = hg(x_n + 8h, z_n + (H_1 + 9/2\, H_2)$

(iii) $V_1 = W_1 = 0$, $V_2 = W_2 = ½$

$C_2 = ½ + 3\sqrt{6}$, $d_2 = ½ - 3\sqrt{6}$

$a_{12} = b_{12} = 0$ $\quad a_{21} = b_{21} = -½$

$C_1 = d_1 = 0$ $\quad a_{22} = b_{22} = 1/3$

$b_{11} = a_{11} = 1/3$, $\quad b_{22} = 2/3 - \sqrt{[illegible]}$

Putting these values into equation (3.47), we have

$$y_{n+1} = \frac{y_n + \frac{1}{2} k_2}{1 + y_n/2(H_2)} \quad \text{.......(3.70)}$$

Where

$k_1 = hf(x_n, y_n + 1/3k_1)$

$k_2 = hf(x_n + (\frac{1}{2} \sqrt{3/6})h, y_n - \frac{1}{2} k_1 + \frac{1}{3} k_2)$

$H_1 = hg(x_n, z_n + \frac{1}{2} H_1)$

$H_2 = hg(x_n + (\frac{1}{2} - 3/6)h, z_n \quad \frac{1}{2} H_1 + 1/3\, H_2)$

Also imposing accuracy of order four on(3.6▮), that is, $T_{n+1} = 0(h^5)$ ....(3.71)

we obtain the following set of equations for two stage method of order four:

$$V_1 + V_2 + W_1 + W_2 = 1 \quad \text{.......(3.72)}$$

$$W_1C_1 + W_2C_2 + V_1d_1 + V_2d_2 = \frac{1}{2} \quad \text{.......(3.73)}$$

$$W_1C_1^2 + W_2C_2^2 + V_1d_1^2 + V_2d_2^2 = 1/3 \quad \text{.......(3.74)}$$

$$W_1C_1^3 + W_2C_2^3 + V_1d_1^3 = V_2d_2^3 = \frac{1}{4}$$

$$W_1(a_{11}C_1 + a_{12}C_2) + W_2(a_{21}C_1 + a_{22}C_2) + V_1(b_{11}d_1 + b_{12}d_2) + V_2(b_{21}d_1 + b_{22}d_2) = 1/6 \quad \text{.......(3.75)}$$

$$W_1C_1(a_{11}C_1 + a_{12}C_2) + W_2C_2(a_{21}C_1 + a_{22}C_2) + V_1d_1(b_{11}d_1 + b_{12}d_2 + V_2d_2(b_{21}d_1 + b_{22}d_2) = \frac{1}{8} \quad \text{.......(3.76)}$$

$$W_1(a_{11}C_1^2 + a_{12}C_2^2) + W_2(a_{21}C_1^2 + a_{22}C_2^2) + V_1(b_{11}d_1^2 + b_{12}d_2) + V_2(b_{21}d_1^2 + b_{22}d_2^2) = \frac{1}{2} \quad \text{.......(3.77)}$$

$$W_1[a_{11}(a_{11}C_1 + a_{12}C_2) + a_{11}(a_{21}C_1 + a_{22}C_2)] + W_2[a_{21}(a_{11}C_1 + a_{12}C_2) + a_{22}(a_{22}c_1 + a_{22}c_1 + a_{22}c_2)] + V_1(b_{11}(b_{11}d_1 + b_{12}d_2) + a_1(b_{21}d_1 + b_{22}d_2)] + V_2(b_{21}(b_{11}d_1 + b_{12}d_1) + b_{22}(b_{21}d_1 + b_{22}d_2)] = 1/24 \quad \text{.......(3.78)}$$

with the following constraints

$a_{11} + a_{12} = C_1 \quad \Rightarrow \quad a_{11} = C_1$ (Since $a_{12} = 0$)

$a_{21} + a_{22} = C_2$

$b_{11} + b_{12} = d_1 \quad \Rightarrow \quad b_{11} = d_1$ (since $a_{12} = 0$)

$b_{21} + b_{22} = d_2$ ....(3.79)

Solving systems of equations (3.71) - (3.78) Simultaneously, and simplifying those with the constraints, the values of the parameters are gotten; for family of two stage schemes of order four as follows:

(i) $W_1 = W_2 = 0$, then $V_1 = V_2 = \frac{1}{2}$

$b_{11} = d_1 = \frac{1}{2} - \sqrt{3/6}$ $\quad d_2 = 1/2 + \sqrt{3/6}$

$b_{21} = \frac{1}{4} + \sqrt{3/6}$ $\quad b_{22} = 1/4$

Putting these values into (3.49), we have

$$y_{n+1} = \frac{y_n}{1 + y_n/2(H_1 + H_2)} \quad \text{.........................(3.80)}$$

Where

$H_1 = hg(x_n + (\frac{1}{2} - \sqrt{3/6})h, \; z_n + (\frac{1}{2} - \sqrt{3/6})H_1$

$H_2 = hg[x_n + (\frac{1}{2} + \sqrt{3/6})h, \; z_n + (\frac{1}{4} + \sqrt{3/6})H_1 + \frac{1}{4} H_2]$

similarly setting

(ii) $V_1 = V_2 = 0$, then $W_1 = W_2 = \frac{1}{2}$

$C_2 = \frac{1}{2} + \sqrt{3/6}$ $\quad C_1 = a_{11} = \frac{1}{2} - \sqrt{3/6}$

$a_{21} = \frac{1}{4} + \sqrt{3/6}$ $\quad a_{22} = \frac{1}{4}$

Putting these values into (3.44), we have

$$y_{n+1} = \frac{y_n + \frac{1}{2} k_1 + \frac{1}{2} k_2}{1 + y_n(0 + 0)}$$

$$\Rightarrow y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2) \quad \text{.................................(3.82)}$$

Where

$k_1 = hf[(x_n + (\frac{1}{2} - \sqrt{3/6})h, y_n + (\frac{1}{2} - \sqrt{3/6})k_1]$

$k_2 = hf(x_n + (\frac{1}{2} + \sqrt{3/6})h, y_n + (\frac{1}{4} - \sqrt{3/6})k_1 + \frac{1}{4} k_2)$

(iii) $V_1 = V_2 = \frac{1}{3}$ $\quad W_1 = W_2 = \frac{1}{6}$

$b_{11} = a_{11} = c_1 = d_1 = \frac{1}{2} - \sqrt{3/6}$ $\quad C_2 = d_2 = \frac{1}{2} - \sqrt{3/6}$

$a_{12} = b_{12} = 0$, $b_{21} = a_{21} = \frac{1}{4} - 3/6$

$b_{22} = a_{22} = \frac{1}{4}$

Putting these values into (3.47), we have

$$y_{n+1} = Y_n + \frac{1}{6}(K_1 + K_2) \Big/ \left(1 + z_n 3[H_1 + H_2]\right) \quad \text{.................................(3.81)}$$

where $K_1 = hf[x_n + (1/2 + \sqrt{3/6})h,\ y_n + (1/2 + \sqrt{3/6})k_1]$

$$k_2 = hf[x_n + (1/2 - \sqrt{3/6})h,\ y_n + (1/4 - \sqrt{3/6})k_1 + 1/4\,k_2]$$

$$H_1 = hg[x_n + (1/2 + \sqrt{3/6})h,\ z_n + (1/2 + \sqrt{3/6})H_1]$$

$$H_2 = hg[x_n + (1/2 - \sqrt{3/6})h,\ z_n + (1/4 - \sqrt{3/6})H_1 + 1/4\,H_2]$$

In the next chapter, we consider the analysis of the error, consistency and stability properties of the schemes.

CHAPTER FOUR

PROPERTIES OF THE NEW SCHEMES

From the development of the scheme it is natural to expect that errors will occur. It is therefore very important to analyze these errors and possibly the consistency, convergence and stability properties of the new schemes, so that we can know if the method will be capable, adequate and efficient in solving the differential equation of our interest, that is, Ordinary differential equations, with derivative discontinuities. In this chapter, we shall carefully examine these aforementioned properties of the schemes.

4.1 ERROR ANALYSIS

A major feature of numerical schemes is that errors (no matter how small) are generated when they are adopted for the approximation of solutions of Ordinary differential equations. The effect of these errors is that it can make the solution unstable because the magnitude of this error determines the degree of accuracy of the schemes.

Errors associated with numerical approximation techniques arise from different sources viz: discretization, truncation and round off errors respectively. Others include inherent errors which arise from modelling process and copying.

Discretization error is the error associated with the replacement of the differential equation (1.4) by its difference equivalence (3.1). Mathematically it can be expressed as:

$$e_{n+1} = y(x_{n+1}) - y_{n+1} \qquad \text{.......................} \ (4.1)$$

That is, the difference between the exact solution $y(x_{n+1})$ and the numerical solution y_{n+1} at x_{n+1}.

Numerical approximation involves iteration process, due to this, there will be propagation of error from step to step when interating with a numerical scheme. These propagated errors can subsequently grow to the extent of distorting the accuracy of the numerical result. The main feature of adequate numerical scheme is its ability to control the growth of such errors.

Of importance is the need to guarantee the quality of the integration scheme, hence, it is important to have estimate of these errors.

There are two major error estimation techniques that are relevant to these schemes. Felberg (1964) postulated a technique of error estimate that entails computation of two approximations to y using methods of order P and P+1 and then find the difference between the two to obtain the local error associated with the proposed schemes. On the other hand, Richardson extrapolation method which entails estimation of local truncation error as the difference between two predictions to y using differents step sizes. Richardson extrapolation method is adopted in this work because it involves just making use of two different step sizes in the same scheme which is less cumbersome as against Felberg's method that involves computation of two approximation to y using two different orders. Felberg technique is found to be more ***Time Consuming, Energy consuming*** and the process too ***cumbersome***. Hence, in this work, Richardson's extrapolation technique is made use of.

By Richardson's extrapolation technique we mean that if y_{n+1} designate the solution by our method using single step size h, the local error can be estimated from

$$e_{n+1} = y(x_{n+1}) - y_{n+1} = \Psi(x_n, y(x_n), h)h^{(P+1)} + O(h^{(P+2)}) \quad \ldots\ldots\ldots\ldots\ldots (4.2)$$

Similarly, by adopting step size h/2, the local error of the method is given by

$$y(x_{n+1}) - L_{n+1} = \Psi(x_n, y(x_n) h/2)(h/2)^{P+1} + O(h^{P+2}) \quad \ldots\ldots\ldots\ldots\ldots (4.3)$$

Where L_{n+1} is the computed solution by the method;

Subtracting (4.2) from (4.3) and simplifying, we have

$$\Psi(x_n, y(x_n))h^{p+1} = [y_{n+1} - L_{n+1}][1 - \tfrac{1}{2}^{p+1}]^{-1} \qquad \text{.............(4.4)}$$

The accuracy of the scheme is estimated form

$$A = \left| [y_{n+1} - L_{n+1}][1 - 1/2^{p+1}]^{-1} \right| \qquad \text{............................(4.5)}$$

Thus Setting

$$D = \left| \Psi(x_n, y(x_n))h \right| h^{p+1} \qquad \text{............................(4.6)}$$

Therefore, the local discretization error of the scheme can be estimated from

$$e_{n+1} = [y_{n+1} - L_{n+1}]\left[\frac{2^p}{1-2^p}\right] \qquad \text{.........................(4.7)}$$

Thus, the approximation y_{n+1} from step x_n to x_{n+1} is accepted as a good approximation to the exact solution if

$$\left| e_{n+1} \right| < \text{tolerance}$$

that is, if the global error is less than error tolerance.

This form of error estimate was found to be adequate [Lambert (1963), Gear(1971)and Hindmarch (1983)] for stiff and non-stiff initial value problem in ordinary differential equations. However, the use of this approach entails considerable amount of computating efforts, but it is necessary to choose a reasonable step size that can accelerate convergence.

Truncation Error on the other hand is the error introduced into the scheme as a result of ignoring of higher terms of the power series expansion by either Taylor or Binomial algorithm. Mathematically, it is defined as the amount by which the theoretical solution $y(x_{n+1})$ fails to satisfy the numerical formula(3.1), that is

$$T_{n+1} = y(x_{n+1}) - \frac{y(x_n) + \sum W_i k_i}{1 + y(x_n)\sum V_i K_i} \qquad \text{............................(4.8)}$$

Where

$$k_i = hf(x_n + C_i h, y_n + a_{ij} k_j)$$

$$H_i = hg(x_n + d_i h, z_n + b_{ij} H_j)$$

$$G(x_n + z_n) = -z_n^2 + f(x_n, y_n)$$

The local truncation error associated with our one stage method of order 2 is found by adopting Taylor and Binomial series expansion of $y(x_{n+1})$, H_1 and K_1 about $(x_n, y(x_n))$ in equation (4.8). Thus:

$$y(x_{n+1}) = y(x_n) + hf_n + h^2/2!\,Df_n + h^3/3!\,(D^2f_n + f_yDf_n) + 0(h^4) \quad .....(4.9)$$

$$\frac{y(x_n) + W_1K_1}{1 + y(x_n)V_1H_1} = y(xn) + W_1K_1 - y(x_n)^2V_1H_1 + \text{higher term} .... \quad (4.10)$$

putting (4.10) into (4.8), we have

$$T_{n+1} = y(x_{n+1}) - y(x_n) - W_1k_1 + y(x_n)^2V_1H_1 + \quad ....(4.11)$$

$$k_1 = hf(x_n + C_1h, y_n + a_{11}k_1) \quad .$$

When expanded yields

$$k_1 = hf_n + h^2C_1Df_n + h^3C_1^{\,3}Df_nf_y + \tfrac{1}{2}h^3C_1^{\,3}D^2f_n + 0(h^4) \quad .... \quad (4.12)$$

Similarly, when H_1 is expanded, we have

$$H_1 = hg_n + h^2d_1Dg_n + h^3d_1^{\,2}Dg_ng_y + \tfrac{1}{2}h^3d_1^{\,2}D^2g_n + 0(h^4) \quad .... \quad (4.13)$$

Simplifying (4.11) by putting (4.9), (4.12) and (4.13) into it, we have

$$T_{n+1} = Y(x_n) + hf_n + h^2/2!\,Df_n + h^3/3!\,(D^2f_n + f_yDf_n) + 0(h^4) - y(x_n) - W_1(hf_n + h^2C_1Df_n + h^3C_1^{\,3}Df_nf_y + \tfrac{1}{2}h^3C_1^{\,3}D^2f_n +) - y(x_n)^2\,V_1\,(hg_n + h^2d_1Dg_n + h^3d_1^{\,2}Dg_ng_y + \tfrac{1}{2}h^3d_1^{\,2}D^2g_n + --)$$

$$T_{n+1} = hf_n - W_1hf_n + h^2/2Df_n - h^2W_1C_1Df_n + h^3/6D^2f_n - h^3W_1C_1^{\,3}Df_nf_y + h^3/6Df_nf_y - \tfrac{1}{2}h^3W_1C_1^{\,3}D^2f_n + y(x_n)^2V_1hg_n + y(x_n)^2V_1h^2d_1Dg_n + y(x_n)^2V_1h^3d_1^{\,2}Dg_ng_y + y(x_n)^2V_1h^3/2d_1^{\,2}D^2g_n + 0(h^4)$$

$$\therefore T_{n+1} = (f_n - W_1f_n - y(x_n)^2V_1g_n)h + (Df_n/2 - W_1C_1Df_n + y(x_n)V_1d_1Dg_n)h^2 + (1/6\,D^2f_n - W_1C_1^{\,3}Df_nf_y + 1/6Df_nf_y - \tfrac{1}{2}W_1C_1^{\,3}D^2f_n + y(x_n)^2V_1d_1^{\,2}Dg_ng_y + \tfrac{1}{2}y(x_n)^2V_1d_1^{\,2}D^2g_n)h^3 + 0(h^4) \quad ....(4.14)$$

Simplifying further, we have

$$T_{n+1} = C_0y_n + C_1h + C_2h^2 + C_3h^3 + 0(h^4) \quad ....(4.15)$$

Where $C_0 = 0$

$C_1 = f_n - W_1f_n - y(x_n)^2V_1g_n$

$C_2 \; \frac{Df_n}{2} - W_1C_1Df_n + y(x_n)^2V_1d_1Dg_n$

$C_3 = 1/6\,D^2f_n - W_1C_1{}^3Df_nf_y - \tfrac{1}{2}W_1C_1{}^3D^2f_n$
$\qquad + y(x_n)^2V_1d_1{}^2Dg_ng_y + \tfrac{1}{2}y(x_n)^2d_1{}^2D^2g_n$

imposing accuracy of order 2 on T_{n+1}, we have

$C_1 = 0$

$C_2 = 0$

$C_3 \neq 0$

Using these in (4.15), it can be seen that the principal truncation error of the method is

$$T_{n+1} = (1/6D^2f_n - W_1C_1{}^3Df_nf_y + 1/6\,Df_nf_y - \tfrac{1}{2}W_1C_1{}^3D^2f_n + y(x_n)^2V_1d_1{}^2Dg_ng_y + \tfrac{1}{2}y(x_n)^2V_1d_1{}^2D^2g_n)h^3 \quad\text{.................}(4.16)$$

The bound of the principal local truncation error T_{n+1} can be found by adopting Lotkin's (1951) error bound

$$\left|\frac{d^{i+j} f(x,y)}{dx^i\,dy^j}\right| < \frac{N^{i+j}}{M^{j-i}} \quad \text{— — — —} \quad (4.16b)$$

$x \in (a,b),\; y \in [-\infty, \infty]$

Where M^{j-i}, N^{i+j}; $i, j = 0, 1, 2, \ldots$
are bounds of f and its partial derivatives. Thus, by adopting (4-16b) the Bound of T_{n+1} is given by

$/T_{n+1}/ = h^3D^2f_n(1/6 - \tfrac{1}{2}W_1C_1{}^3 - y_n{}^2/2V_1d_1{}^2) + Df_nf_y(1/6 - W_1C_1{}^3 - y_n{}^2V_1d_1{}^2)/$
$\qquad \leq h^3/D^2f_n/\,1/6 - \tfrac{1}{2}W_1C_1{}^3 - y_n{}^2/2V_1d_1{}^2/ + Df_nf_y/\,1/6 - W_1C_1{}^3 - y_n{}^2V_1d_1/$

$/T_{n+1}/ \leq [N^2/P_1/ + m/P_2/]h^3 \quad \text{......................} (4.17)$

Where

$P_1 = 1/6 - \tfrac{1}{2}W_1C_1{}^3 - y_n{}^2/2V_1d_1{}^2$

$P_2 = 1/6 - W_1C_1{}^3 - y_n{}^2V_1d_1{}^2$

Showing that the bound of local truncation error of one stage method exists.

Round off error is the error introduced as a result of computing

Mathematically, round off error can be estimated from $R_{n+1} = y_{n+1} - P_{n+1}$

Where y_{n+1} is the expected solution of the difference equation (3.1) while P_{n+1} is the computed output at the (n+1)th iteration. In order words, round off error is the amount by which the computed approximation P_{n+1} differs from the expected approximation y_{n+1} by the scheme at point x_{n+1}.

The magnitude of the round off errors are determined by the way and manners machine operations are performed in terms of storage and manipulation of numbers.

According to Fatunla (1987) and Lambert (1963) the effects of round off error can be disastrous because there may be inevitable loss of accuracy. In accordance with their postulate, double precision arithmetic can be employed to control its magnitude. Therefore, double precision arithmetic is adopted in our subsequent computations.

4.2 CONSISTENCY PROPERTY OF THE METHOD

Recall that the conventional one step scheme is said to be consistent for solving differential equation (1.4) if

$$\lim_{h \to 0} \left(\frac{y_{n+1} - y_n}{h} \right) = f(x_n, y_n)$$

We shall demonstrate the consistency of our proposed scheme in this section.

4.2.1 ONE STAGE SCHEME

The general one stage scheme is

$$y_{n+1} = \frac{y_n + w_1 k_1}{1 + y_n V_1 H_1} \qquad \text{.............................(4.18)}$$

Where $\quad k_1 = hf(x_n + c_1 h, y_n + a_{11} k_1)$

$H_1 = hg(x_n + d_1 h, z_n + b_{11} H_1)$

$g(x_n, z_n) = -z_n^2 f(x_n, y_n)$

and $z_n = 1/y_n$

using binomial theorem to expand the right hand side of (4.18) and ignoring terms of order higher than one, we have

$$y_{n+1} = y_n + W_1k_1 - y_n^2V_1H_1 + \text{(higher order terms)}$$

subtracting y_n from both sides we have,

$$y_{n+1} - y_n = y_n + W_1k_1 y_n^2 V_1H_1 + 0(h) - y_n$$

$$= W_1k_1 - y_n^2 V_1H_1$$

by putting the value of k_1 and H_1 into this, we have

$$y_{n+1} - y_n = W_1 hf(x_n + c_1h, y_n + a_{11}k_1) - y_n^2 V_1 hg(x_n + d_1h, z_n + b_{11}H_1)$$

$$\left(\frac{y_{n+1} - y_n}{h}\right) = (W_1 hf(x_n + c_1h, y_n + a_{11}k_1) - y_n^2 V_1 hg(x_n + d_1h, z_n + b_{11}H_1))/h$$

$$=(W_1 hf(x_n + c_1h, y_n + a_{11}\dot{k}_1) - y_n^2 V_1 (1/y_n^2) f(x_n + d_1h, y_n + b_{11}H_1))/h$$

$$(W_1 hf(x_n + c_1h, y_n + a_{11}k_1) + V_1 f(x_n + d_1h, Y_n + b_{11}H_1))/h$$

taking the limit as $h \rightarrow 0$, we have:

$$\lim_{h\rightarrow 0} \left(\frac{y_{n+1} - y_n}{h}\right) = W_1 f(x_n; y_n) + V_1 f(x_n, y_n)$$

$$= f(x_n, y_n) [W_1 + V_1]$$

but $W_1 + V_1 = 1$ (from equation 3.38)

$$\lim_{h\rightarrow 0} \left(\frac{y_{n+1} - y_n}{h}\right) = f(x_n, y_n)$$

Showing that the one stage method is consistent.

4.2.2 TWO STAGE SCHEME

The general two stage scheme is:

$$y_{n+1} = \frac{y_n + \sum_{i=1}^{2} W_i k_i}{1 + y(x_n)\sum_{i=1}^{2} V_i K_i} \quad \text{.............................(4.19)}$$

where

$$K_i = hf(x_n + C_ih, y_n + \sum_{j=1}^{2} a_{ij} K_j) \quad i = 1(1)2$$

$$H_i = hg(x_n + d_ih, z_n + \sum_{j=1}^{2} b_{ij} H_j) \quad i = 1(1)2$$

$g(x_n, z_n) = -z_n^2 f(x_n, y_n)$

and $z_n = 1/y_n$

Using binomial theorem to expand the right hand side of (4.19) and ignoring the terms of order higher than one, we have:

$$y_{n+1} = y_n + \sum_{i=1}^{2} W_i k_i - \quad y_n^2 \sum_{i=1}^{2} V_i H_i + \text{(higher order terms)}$$

Subtracting y_n from both sides of this equation, we have:

$$y_{n+1} - y_n = y_n + \sum_{i=1}^{2} W_i k_i - y_n^2 \sum_{i=1}^{2} V_i H_i + 0(h) - \qquad y_n$$

$$= \sum_{i=1}^{2} W_i k_i - y_n^2 \sum_{i=1}^{2} V_i H_i$$

by putting the values of Ki and Hi into this, equation, we have:

$$y_{n+1} - y_n = h \sum_{i=1}^{2} W_i f(x_n + c_i h, y_n + \sum_{i=1}^{2} a_{ij} K_i) - y_n^2 h \sum_{i=1}^{2} V_i g(x_n + d_i h, z_n + b_{ij} H_j)$$

$$y_{n+1} - y_n = h \sum_{i=1}^{2} W_i f(x_n + c_i h, y_n + \sum_{i=1}^{2} a_{ij} K_i) - y_n^2 h \sum_{i=1}^{2} V_i g(x_n + d_i h, z_n + b_{ij} H_j)$$

$$\frac{y_{n+1} - y_n}{h} = \sum_{i=1}^{2} W_i f(x_n + c_i h, y_n + \sum_{i=1}^{2} a_{ij} K_i) - y_n^2 (-1/y_n^2) \sum_{i=1}^{2} V_i f(x_n + d_i h, z_n + \sum_{i=1}^{2} b_{ij} H_j)$$

$$= \sum_{i=1}^{2} W_i f(x_n + c_i h, y_n + \sum_{i=1}^{2} a_{ij} K_i) + \sum_{i=1}^{2} V_i f(x_n + d_i h, y_n + \sum_{i=1}^{2} b_{ij} H_j)$$

taking the limit of this as $h \to 0$, we have

$$\lim_{h \to 0} \left\{ \frac{y_{n+1} - y_n}{h} \right\} = \sum_{i=1}^{2} W_i f(x_n, y_n) + \sum_{i=1}^{2} V_i f(x_n + y_n)$$

$$= f(x_n, y_n) \left[\sum_{i=1}^{2} W_i + \sum_{i=1}^{2} V_i \right]$$

$$\text{but} \sum_{i=1}^{2} W_i + \sum_{i=1}^{2} V_i = W_1 + W_2 + V_1 + V_2$$

$$= W_1 + V_1 + W_2 + V_2$$

But from (3.62), $W_1 + V_1 + W_2 + V_2 = 1$

$$\lim_{h \to 0} \left\{ \frac{y_{n+1} - y_n}{h} \right\} \qquad = f(x_n, y_n)$$

Showing that the two stage method is consistent. According to Lambert

(1963). A consistent one step method is convergent. Hence, the new schemes is convergent. To see this, we consider the convergence of the method below:

4.4 THE CONVERGENCE PROPERTIES

The numerical scheme (3.1) for solving Ordinary Differential Equation (1.5) is said to be convergent if the numerical approximation y_{n+1} that is generated by it tends to the exact solution $y(x_{n+1})$ at $x = x_{n+1}$ as the step size tends to zero.

That is, $\text{Lim}\,(y(x_{n+1}) - y_{n+1}) = 0$

For analysis of the convergence of the scheme, we consider the following relevant standard theorems stated without proof.

THEOREM 1

Let e_j, $j = 0(1)n$ be set of real numbers, if there exist , finite constants R and S, such that

$$|e_j| \le R\,|e_{j-1}| + S \qquad j = 0(1)\,n-1 \quad \text{.................}(4.20)$$

$$\text{then } |e_j| \le \frac{R^j - 1}{R - 1}S + R^j|e_0| \text{ where } R < 1 \quad \text{.................}(4.21)$$

Let e_{n+1} and T_{n+1} denote the deseretization and truncation errors generated by (3.1) respectively.

By adopting binomial expansion and ignoring terms of order $0(h^2)$ in equation (3.2) and (4.2) , we obtain

$$y(x_{n+1}) = y(x_n) + h(\Psi_2(x_n, y(x_n), h) + h\varphi_1(x_n, y(x_n), h) + (\text{higher terms}) + T_{n+1} \text{..}(4.22)$$

Where φ_1, Ψ_2 and φ_2 are continuous functions in the domain $a \le x \le b$, $|y| < \infty$, $0 \le h \le h_0$ Define as

$$h\,\varphi_1(x_n, y(x_n), h) = \sum_{i=1}^{r} W_i K_i \text{ ....}(4.23)$$

$h\varphi_1(x_n, z(x_n); h) \quad \sum_{i=1}^{r} W_i H_i$

$$= -h/y^2(x_n)\Psi_2(x_n, y(x_n); h) \quad \text{.................(4.24)}$$

Where

$$\Psi_2(x_n, y(x_n); h) = (1 + y(x_n)\sum_{i=1}^{r} b_n H_i)\, O_2(x_n, y(x_n; h) \quad \text{........(4.25)}$$

From (4.2) equation (3.1) yields

$$y_{n+1} = y_n + h\Psi_2(x_n, y_n; h) + h\varphi_1(x_n, y_n; h) + (\text{higher terms}) \quad \text{.........(4.26)}$$

Subtracting (4.22) from (4.26) and using (4.1): We have

$$e_{n+1} = e_n + h[\Psi_2(x_n, y(x_n); h) - \Psi_2(x_n, y_n, h)] + h[\varphi_1(x_n, y(x_n); h] - \varphi_1(x_n\, y_n; h)] + T_{n+1} \quad \text{........(4.27)}$$

By mean value theorem,

$$e_{n+1} = e_n + \frac{h\partial\Psi_2}{\partial y}(y(x_n) - y_n) + \frac{h d\varphi_1}{\partial y}(y(x_n) - y_n) + T_{n+1} \quad \text{.........(4.28)}$$

$$= e_n + h\frac{\partial\Psi_2}{\partial y}e_n + h\frac{\partial\varphi_1}{\partial y}e_n + T_{n+1}$$

$$= e_n\left(1 + h\frac{\partial\Psi_2}{\partial y} + h\frac{\partial\varphi_1}{\partial y}\right) + T_{n+1}$$

taking absolute value on both sides of this, we obtain the inequality

$$|e_{n+1}| \leq |e_n| + \left|\frac{h\partial\Psi_2}{\partial y}(y(x_n) - y_n)\right| + \left|\frac{h\partial\varphi_1}{\partial y}(y(x_n) - y_n)\right| + |T_{n+1}| \quad \text{.........(4.29)}$$

By Lipstich condition, we have

Setting $\frac{\partial\Psi_2}{\partial y} = k$ and $\frac{\partial\varphi_1}{\partial y} = L$, we have

$$|e_{n+1}| \leq |e_n| + kh|e_n| + hL|e_n| + T \quad \text{........(4.30)}$$

where

$$k = \left|\frac{\partial\Psi}{\partial y}\right| \text{ and } L = \left|\frac{\partial\varphi_1}{\partial y}\right|$$

and k and L are the Lipschitz constants for $\varphi_1(x, y; h)$ and $\Psi_2(x, y, h)$respectively.

$$\text{Let } T = \sup_{a \leq x \leq b} |T_{n+1}| \quad \text{....... (4.31)}$$

Settin $N = L + k$ .... (4.32)

the inequality (4.30) becomes

$$/e_{n+1}/ \leq /e_n/ \; (1+hN) + T, \qquad n = 0,1 \quad \text{.....................(4.33)}$$

From theorem 1, expression (4.33) becomes

$$/e_{n+1}/ = \left(\frac{1+hN}{N}\right)^{n+1} + (1+hN)^n /e_o/ \quad \text{.............................(4.34)}$$

By Pade's approximation

$$(1+hN)^n \approx e^{nhN} \quad \text{..............................(4.35)}$$

also by equation (1.5),

$$h = \frac{b-a}{n} \quad \text{or } nh = b-a$$

thus,

$$(1+hN)^N = e^{N(b-a)} \quad \text{..............................(4.36)}$$

$$e^{nhN} \leq e^{N(b-a)} \quad \text{..............................(4.37)}$$

therefore, inequality (4.34) modifies into:

$$/e_n/ = \frac{(e^{N(b-a)} - 1)\,T}{hN} + e^{N(b-a)} /e_o/ \quad \text{.......................(4.38)}$$

adopting mean value theorem on

$$T_{n+1} = \Psi(x_n, y(x_n)\, h^{p+1} + 0h(^{p+2}), \text{ we have}$$

$$T_{n+1} = h\Psi_2\, [x+0h, y(x_n+0h) - \Psi_2\,(x_n, y(x_n)) + h[\varnothing_1(x_n+0h, y(x_n+0h),$$

$$-\varphi_1\,(x_n, y(x_n)]$$

$$- h[\Psi_2(x_n+0h, y(x_n+0h)), \quad \Psi_2(x_n+0h, y(x_n))] \; + \Psi_2\,(x_n+0h, y(x_n))$$

$$\Psi_2\,(x_n, y(x_n))] \quad + h[\varphi_1(x_n+0h, y(x_n+0h)) - \varphi_1\,(x_n+0h,$$

$$y(x_n)) + \varphi_1\,(x_n+0h, y(x_n)) - \varphi_1\,(x_n, y(x_n)) \quad \text{................(4.39)}$$

$$0 \leq 0 \leq 1$$

By taking equation (4.31) into consideration, and taking the absolute value of (4.39), we obtain

$$T = hL/Y(x_n+0h) - y(x_n)/ + jh^2 0 + hk/y(x_n+0h) - y(x_n) + mh^2 Q$$

$$= h^2 0\, N/y^1(\varepsilon_1)/ + (J+m)h^2 0, \qquad x_n \leq \varepsilon \leq x_{n+1} \quad \text{................(4.40)}$$

Where m and J are the partial derivative of φ_1 and Ψ_2 with respect to x

Setting Q = J + m

And $Y = \mathrm{Sup}\, /y^1(x)/$ .... (4.41)

$a \leq x \leq b$

$$T = h^2 0 [N/y^1/\varepsilon_1/ + (J+m)]$$

$$= h^2 0 [N/y^1(\varepsilon_1)/ + Q]$$

$$T = h^2 0[NY + Q) \quad \text{.......................................} (4.42)$$

Where $/y^1(\varepsilon_1)/ = y$

Putting (4.42) into (4.38), we have

$$/e_n/ \leq \frac{h^2 0(NY+Q)}{hN}(e^{N(b-a)} - 1) + e^{N(b-a)}/e_0/$$

$$/e_n/ \leq \frac{hQ(Ny+Q)}{N} e^{N(b-a)} - \frac{hQ(NY+Q)}{N} + e^{N(b-a)}/e_0/$$

$$/e_n/ \leq \frac{hQ}{N}[(NY+Q)e^{N(b-a)} - (NY+Q)] + e^{N(b-a)}/e_0/$$

$$/e_n/ = \frac{hQ}{N}\left[(e^{N(b-a)} - 1)(NY+Q)\right] + e^{N(b-a)} /e_0/ \quad \text{..................} (4.43)$$

Assuming there is no error in the input data, that is $e_0=0$, then taking the limit as $h \rightarrow 0$, we obtain,

$$\lim_{\substack{h\to 0 \\ N\to\infty}} /e_n/ = \lim_{\substack{h\to 0 \\ N\to\infty}} \left[\frac{hQ}{N}\left(e^{N(b-a)} - 1\right)(NY+Q)\right] + e^{N(b-a)}/0/$$

$$\lim_{\substack{h\to 0 \\ N\to\infty}} /e_n/ = 0$$

Which implies that $\lim_{h\to 0} y_n = y(x_n)$ ....(4.44)

That is, the convergence of the scheme (3.1) is established.

Having established the consistency and convergence of the scheme, we now consider the stability property of the scheme.

4.5 STABILITY PROPERTIES OF THE SCHEMES

Stability analysis of the semi-implicit schemes is important since it forms the basis by which suitability of the scheme is assessed. Here, Dalquist (1963) stability scalar test initial value problem

$y' = \lambda y \; ; \; y(x_n) = y_n$ ....(4.45)

becomes an important tool.

Consequently, scheme (3.1)was applied to the scalar initial value problem(4.45).

Under the assumption that

$Re(\lambda) < 0$ (λ being a complex constant with negative real part)

This leads to the following system of linear equations:

$$\left.\begin{array}{l} (1-\lambda ah_{11})k_1 + 0 + \qquad\qquad = \lambda hy_n \\ -\lambda hak_1 + (1-\lambda ha_{22})k_2 + 0 + \qquad = \lambda hy_n \\ \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \\ -\lambda ha_{n1}k_1 + \lambda ha_{r2}k_2 - \lambda ha_{r3}k_3 + \cdots + (1-\lambda ha_{rr})k_r = \lambda hy_n \end{array}\right\} \ldots\ldots\ldots(4.46)$$

putting these linear equations in matrix form, we have

$$\begin{pmatrix} 1-\lambda ha_{11}) & 0 & 0 \cdots & 0 \\ \lambda ha_{21} & (1-\lambda ha_{22}) - & 0 \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \lambda ha_{r1} & -\lambda ha_{r2} & \lambda ha_{r2} - \cdots & (1-\lambda ha_{rr}) \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_r \end{pmatrix} = \lambda hy \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \quad (4.47)$$

In compact form, equation (4.47) becomes

$Dk = Y$ ....(4.48)

Where

$Y = \lambda hy_n [1, 1, \ldots\ldots\ldots\ldots 1]^T$

And

$$D = \begin{pmatrix} (1-\lambda ha_{11}) & -0 & \cdots & 0 \\ -\lambda ha_{21} & (1-\lambda ha_{22}) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ -\lambda ha_{r1} & -\lambda ha_{r2} & \cdots & (1-\lambda ha_{rr}) \end{pmatrix} \ldots\ldots\ldots(4.49)$$

Assuming D is invertible, then

$$k = D^{-1} Y \quad \text{.................................}(4.50)$$

therefore, k_i 's are obtained from (4.50) as

$$k = (1-\Lambda\alpha)^{-1} e\alpha y_n \quad \text{..............................}(4.51)$$

where $\alpha = \lambda h$

$$H = -(1+\alpha\beta)^{-1} e\alpha z_n \quad \text{......................................}(4.52)$$

Where

$A = (a_{ij}) \qquad i,j = 1(1)r$

$B = (b_{ij}) \qquad i,j = 1(1)r$

$e = (1, \ 1 \ \text{......................} 1)^T$

and

$\alpha = \lambda h$

$$z = 1/y_n \quad \text{...}(4.53)$$

And e being r x r unit matrix

Putting (4.51) and (4.52) into (3.1) we obtain a difference equation (first order difference equation) of the form.

$$y_{n+1} = U(\alpha)y_n \qquad \alpha = \lambda h \quad \text{......................}(4.54)$$

Where

$$U(\alpha) = \frac{1+\alpha W^T(1-\alpha\Lambda)^{-1}e}{1-\alpha V^T(1+\alpha B)^{-1}e} \quad \text{......................}(4.55)$$

is the so called stability function

$$y_{n+1} = \left(\frac{1+\alpha W^T(1-\alpha\Lambda)^{-1}e}{1-\alpha V^T(1+\alpha\beta)^{-1}e}\right) y_n \quad \text{......................}(4.56)$$

Where

$W^T = (W_1, \ W_2 \ \text{..............................} W_r)$

$V^T = (V_1, \ V_2 \ \text{..............................} V_r)$

The parameters a_{ij}, b_{ij}, c_r, d_r, W_i and V_i in the scheme (3.1) are chosen to ensure that $U(\alpha)$ is a Pade's approximation to e^{α}. For better understanding of the Pade's approximation, the following definitions are stated:

DEFINITION 1

Let $Q_r(x)$ denote a polynomial of degree r in α specified as

$$Q_r(\alpha) = 1 + \frac{\alpha^{\prime}}{2r!} + \frac{r(r-1)}{2r(2r-1)}\frac{\alpha^2}{2!} + \frac{r(r-1)}{2r(2r-1)} \ldots\ldots \frac{1\alpha^{r}}{(r+1)!} \quad \ldots\ldots\ldots(4.57)$$

$$P(\alpha) = 1 + \frac{\alpha^{\prime}}{2r!} + \frac{r(r-1)}{2r(2r-1)}\frac{\alpha^2}{2!} + \frac{r(r-1)}{2r(2r-1)} \ldots\ldots \frac{1\alpha^{s}}{(s+1)!}$$

and

$$R_{rs}(\alpha) = \frac{Q_r(\alpha)}{P_s(\alpha)} \quad \ldots\ldots(4.58)$$

Where

$P_s(\alpha)$ denote a polynomial of degree s.

Then we say that $R_s(s3)$ is an (r,s) Pade's approximation of order r+s, to e^{α} if:

$$R_{r,s}(\alpha) = e^{\alpha} + 0(\alpha^{r+s+1}) \quad \ldots\ldots(4.59)$$

Which can be achieved by expressing e^{α} as a power series in α^{i} and equating the coefficient of α^{i}, $i = 1(1)\ r+s$ in equation

$$\sum_{i=1}^{n} a_i \alpha^{i} = \left(\sum_{i=0}^{s} b_i \alpha^{i}\right)\frac{(\alpha^{i})}{1!} \quad \ldots\ldots(4.60)$$

to define uniquely the coefficient a_i s, b_i s appearing in $R_{rs}(\alpha)$

DEFINITION 2

The scheme (3.1) is said to be absolutely stable at point (α, U (α)) in the complex plane if the stability function (4.56) satisfies

$$/U(\alpha)/ < 1 \quad \ldots\ldots(4.61)$$

The corresponding region R of absolute stability of the scheme is then defined as:

$R = \{\alpha : /U(\alpha)/ < 1\}$ ....(4.62)

DEFINITION 3

The integration scheme (3.1) is said to be A – stable if the Region of absolutely stability specified in (4.62) include the entire left half of the complex plane denoted by

$A_s = \{\alpha : a \in C \text{ and } Re(\alpha)/ < 0\}$ ....(4.63)

This A-stability property is one of the desirable properties for any numerical scheme as postulated by Dalquist (1963) for solving initial value problems that are stiff ordinary differential equations.

As earlier-mentioned, to relate Pade's approximation to e^{α} with definitions stated above which apparently highlight the adequacy of Pade's approximation to e^{α} in investigating the stability properties of the numerical schemes, the following theorems and definitions are considered.

DEFINITION 4

A Pade's approximation to e^{α} is said to be:

(1) A – acceptable if $/R_{r,s}(\alpha)/ < 1$ where $Re(\alpha) < 0$

(2) A (0) acceptable if $/R_{r,s}(\alpha)/ < 1$ whenever α is real and negative

(3) L – acceptable if it is A – acceptable and in addition satisfies

$/R_{r,s}$- $(\alpha)/$ tends to zero as

$Re(\alpha)$ tends to $-\infty$

Birkoff and Virga (1965) gave further theorems on $R_{r,s}(\alpha)$ as

(a) If r = s $R_{r,s}(\alpha)$ is A – acceptable

(b) If r = s+1 or r = s+2, $R_{r,s}(\alpha)$ is L-acceptable

Linger and Willoughby (1967) considered stages one and two of Pade's approximation to e^{α}

$$R_{1,1}(\alpha,\gamma) = \frac{1+\frac{1}{2}(1-\gamma)\alpha}{1-\frac{1}{2}(1+\gamma)\alpha} \quad \text{.......................................} (4.64)$$

and

$$R_{2,2}(\alpha,\gamma,\beta) = \frac{1+\frac{1}{2}(1-\gamma)\alpha+\frac{1}{4}(\beta-\gamma)\alpha^2}{1-\frac{1}{2}(1+\beta)\alpha+\frac{1}{4}(\beta+\gamma)\alpha^2} \text{..........} (4.65)$$

with the following conditions:

(i) $R_{1,1}(\alpha,\gamma)$ is A – acceptable if and only if $\gamma \geq 0$

L – acceptable if and only if $\gamma = 1$

(ii) $R_{2,2}(\alpha,\gamma,\beta)$ is A acceptable, if and only if $\alpha = \beta > 0$.

Any method whose stability function satisfies Pade approximation conditions, and its A-acceptable, is absolutely stable.

With the above definitions and theorems, we shall now analyze the stability properties of the family of one-stage and two stage schemes in (3·12) and (3·47)

4.5.1 ONE STAGE SCHEMES

For result-oriented analysis of the stability properties, the definitions and theorems stated above are adopted for the general one stage scheme.

$$y_{n+1} = \frac{y_n + W_1 k_1}{1 + y_n V_1 H_1} \text{..........................} (4.66)$$

Where

$k_1 = hf(x_n + c_1 h,\ y_n + a_{11} k_1)$

$H_1 = hg(x_n + d_1 h\ \ z_n + b_{11} H_1)$

$g(x_n, z_n) = -z_n^2 f(x_n, y_n)$

By applying (4.68) to the stability test equation (4.55), we obtain the recurrent relation.

$$y_{n+1} = \left(\frac{1+W_1\ \alpha(1-a_{11}\ \alpha)^{-1}}{1-V_1\ \alpha(1+b_{11}\ \lambda)^{-1}}\right) y_n \text{......................} (4.67)$$

For the approximation to the solution and for its convergence, we consider the function

$$U(\alpha) = \frac{1 + W_1 \alpha(1 - a_{11}\alpha)^{-1}}{1 - V_1 \alpha(1 + b_{11}\alpha)^{-1}} \quad \text{.........................(4.68)}$$

Which can be shown to satisfy Pade's approximation to e^{α}, that is, $U(\alpha)$ can be expressed in the form

$$U(\alpha) = \sum_{i=1}^{2} a_i \alpha^i + 0(\alpha^3) \quad \text{...(4.69)}$$

For example, the associated stability function $u(\alpha)$ in (4.56) is

$$U(\alpha) = \frac{1 + \frac{1}{2}\alpha}{1 - \frac{1}{2}\alpha} \quad \text{..(4.70)}$$

Which is (1.1) Pade's approximation to e^{α} (see equation (4.59) of definition 1)

Since $U(\alpha) = 1 + \alpha + \frac{1}{4}\alpha^2 + 0\,\alpha^3$ ....(4.71)

The stability function (4.70) satisfies (4.58) with $(-\infty, 0)$ as the corresponding interval of absolute stability. This implies that the schemes are A – stable (in line with definition 3)

This property stimulates the use of the schemes to solve Ordinary Differential Equations with Derivative Discontinuities and Stiff ODES. Also, according to Hinges and Willoughby (1976) one stage family of semi-implicit Rational Runge-Kutta Schemes as developed in (3.45) to (3.48) are A – acceptable since

$$/R_{1,1}(\alpha)/ \leq 1 \quad \text{..(4.72)}$$

With $Re(\alpha) < 0$ and A(0) acceptable since it satisfies (4.71) with negative real α.

4.5.2 TWO STAGE SCHEMES

The two stage scheme is of the form

$$y_{n+1} = \frac{y_n + W_1k_1 + W_2k_2}{1 + y_n(V_1H_1 + V_2H_2)} \quad \text{.......................(4.73)}$$

Where

$k_1 = hf(x_n + C_1h, y_n + a_{11}k_1 + a_{12}k_2)$

$k_2 = hf(x_n + C_2h, y_n + a_{21}k_1 + a_{22}k_2)$

$H_1 = hg(x_n + d_1h, z_n + b_{11}H_1 + b_{12}H_2)$

$H_2 = hg(x_n + d_2h, z_n + b_{21}H_1 + b_{22}H_2)$ ....(4.74)

Applying this formula to the stability test equation (4.55) we obtain a system of linear equations for K_i's that can be written in the matrix form

$$\begin{pmatrix} 1-\lambda h a_{11} & 0 \\ 1 - a_{21}\lambda h & 1-\lambda h a_{22} \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \lambda h y_n \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{....................(4.75)}$$

that can be re-written in compact form

$Dk = Y$ ....(4.76)

Where

$$D = \begin{pmatrix} 1-\lambda h\, a_{11} & 0 \\ 1\ \lambda h\, a_{21} & 1-\lambda h\, a_{22} \end{pmatrix} \quad \text{..........................(4.77)}$$

and $Y = \lambda h y_n [1, 1]^t$ ....(4.78)

Equation (4.78) can be written as

$k = D^{-1} Y$ ....(4.79)

If the inverse of D, that is, D^{-1} exist,

k can be obtained from (4.76) as

$k = \alpha (I - A\alpha)^{-1} e y_n$ ....(4.80)

Similarly, H_1 and H_2 yields

$H = (I + B(\alpha)^{-1} e\alpha_n$ ....(4.81)

$e = (1,1)^T_j \quad \alpha = \lambda h$

$A = (a_{ij}), \quad B = (b_{ij}) \qquad\qquad i, j = 1(1)2$

The I appearing in (4.80) is a (2x2) unit matrix.

Putting (4.78) and (4.79) into (4.73), we obtain

$$y_{n+1} = \frac{1 + \alpha W^1 (1 - \alpha A)^{-1} e}{1 - \alpha V^1 (1 + \alpha B)^{-1} e} y_n \quad \text{.......................................}(4.82)$$

Which is a difference equation with stability vector function

$$U(\lambda) = \frac{1 + \alpha W^1 (1 - \alpha A)^{-1} e}{1 - \alpha V^1 (1 + \alpha B)^{-1} e} \quad \text{.......................................}(4.83)$$

Where

$W^1 = [\ W_1,\ \ W_2]$

And $V^1 = [\ V_1,\ \ V_2]$

This can be shown to satisfy a 2 by 2 Pade's approximation to e^{α} since it can be expressed as

$$R_{(2,2)}(\alpha) = 1 + \alpha + \tfrac{3}{4}\alpha^2 + \tfrac{1}{4}\alpha^3 + 5/144\,\alpha^4 + 0(\alpha^5) \text{-------}(4.84)$$

This scheme is A – stable with $(-\infty, 0)$ as corresponding interval of absolute stability, since it satisfies

$$\lim_{\alpha \to \infty} /U(\alpha)/ < 1 \quad \text{.......................................}(4.85)$$

The scheme is also A – acceptable since

$$/R_{2,2}(\alpha)/ < 1 \quad \text{.......................................}(4.86)$$

COMPUTER IMPLEMENTATION AND NUMERICAL RESULTS

It is important to translate the new numerical scheme (3.1) into computer codes so as to be able to demonstrate its applicability and suitability for solving problems of our interest (initial value problem of type (1.1))

There are several computer programming languages such as FORTRAN, BASIC, PASCAL that are suitable for implementation of computational formulae, however, in this work, we consider the use of the Fortran programming language as the mode of implementation of our scheme, because its implementation involves the following structure that makes it convenient to use:

I. Re-writing of the scheme in an algorithmic form.

ii. Translating the algorithm into a computer flow chart.

iii. Conversion of flow chart into computer code.

iv. Implementation of the code with sample problems on a digital computer.

v. Discussion the results.

5.1 COMPUTER ALGORITHM.

An algorithm of a problem can be defined as a set of steps taken towards obtaining the solution of the given problem.

In this section, we develop the Numerical Algorithm for implementing the semi-implicit, Rational Runge-Kutta methods described in chapter three and adopt the errror estimation techniques discussed in chapter four, using the approriate step size control measures.

The algorithm is

Step 1: Declaration of variables.

Step 2: Define function f(x,y) exact

Step 3: Select input values x_0, x_{last}, y_0, h, tol.

Step 4: Initialise variables by setting

$i = 0$ (counter)

$x = x_0$

$y = y_0$

$H = h_{old}$

Step 5: Compute the appropriate values of y(x)

for i = 0 to N

$Gk(i)_{11} = x_i + (0.211325 * H_i)$

$Gk(i)_{12} = y_i + (0.211325 * Rk_1(x_i))$

$$Rk_1(x_{i+1}) = Rk_1(x_i) - \frac{Gk_1(x_i)}{G_2k_1(x_i)}$$

$$G_1k_2(x_i) = 1 + \frac{Gk_2(x_i)}{G_1k_2(x_i)}$$

$$Rk_2(x_{i+1}) = Rk_2(x_i) - \frac{Gk_2(x_i)}{G_1k_2(x_i)}$$

$y_{i+1} = y_i + 0.5 * (Rk_1(x_{i+1}) + Rk_2(x_{i+1}))$

$x_{i+1} = x_i + 0.1 * H_i$

$H_{i+1} = 0.5 * H_i$

Step 6:

Estimate the values of absolute error.

Set $E_{i+1} = ABS((y_{i+1}) - YE_{i+1})$

Estimate the local truncation error (L. T. E)

using subroutine ADAPT ($x, y, h, tol, lte, H_{new}, y_{n2}$)

Set $y_{new} = y_{old} + hf(x, y)$

$D_{new} = Abs(tol * y_{new})$

Call IMPRRK (x_0, y_0, h, tol, y_{n1})

Set $h_2 = 0.5 \times h_{old}$

Call IMPRRK ($x_0, y_0, h_2, tol, y_{n2}$)

Set $x_1 = x_0 + h_2$

Call IMPRRK ($x_1, y_{n2}, h_2, tol, y_{n3}$)

Set L.T.E $= (1 - 2^{(r)})^{-1} (y_{n1} - y_{n})$

$D_{old} = Abs (y_{n1} - y_{n})$

$D_{np} = (D_{new} / D_{old})$

While $(D_{old} < D_{new})$

then set $h_{new} = h_{old} \times (D_{np})^{0.2}$

Else

set $h_{new} = h_{old} \times (D_{np})^{0.25}$

Step 7:

while (L. T. E < tol)

Then Return the results

Else

Step 8:

Adjust the step size and replace h_{old} by h_{new}

$h_{old} = h_{new}$

and repeat step 6 and 7.

Step 9:

Output the results

Step 10:

Stopping criterion

while $(x_n < x_{last})$ Then set

$x_n = x_{n+1}$

$y_n = y_{n+1}$

$h_{old} = h_{new}$

$n = n + 1$

and repeat step 5 - 10

Else

Step 11:

Stop.

PROGRAM FLOW CHART

A computer flow chart is the representation of the alogrithm or the plan of solution to a given problem in form of diagram. It is the basis by which the process of solution, the relevant operations to be performed, the computation, the point of decision and other information at the point of the solution are indicated.

As a result of its documenting features, flow chart are of significant interest. It is constructed by using selected geometrical symbols such as squares, rectangles, diamonds shape or circles. Each of the geometrical symbols used represent some activities which could be input / output of data, taking a decision, terminating the process of solution and so on.

The direction of flow in the chart is indicated by the joining of the symbols by directed lines segment called arrow. The fig. below shows the flow chart of the above numerical algorithm for implemetation of our method.

FLOW CHART OF THE IMPLEMENTATION.

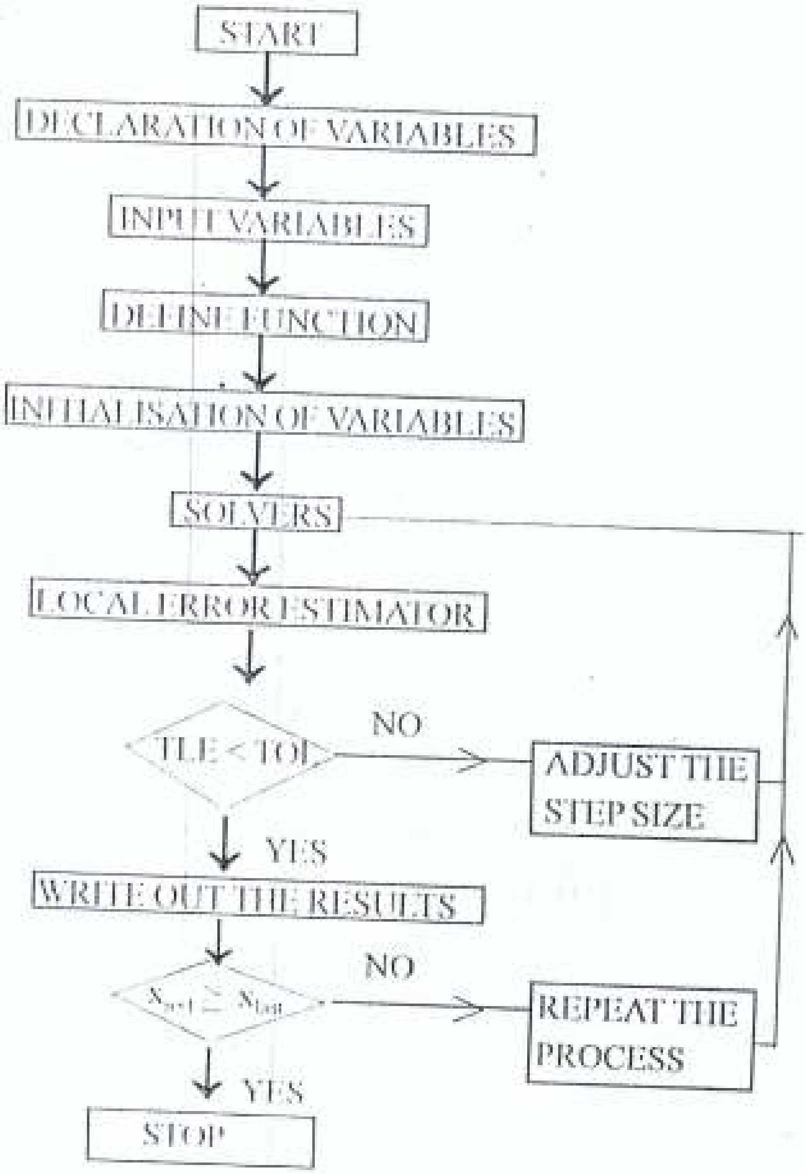

FIG 5.1

5.3 PROGRAMMING IMPLEMENTATION

This section, consider, the computer implementation of the above alogrithm. The method adopted is that of variable step size fixed order. The program comprises three modules called FUNCS, SIMPRRK and ADAPT respectively.

FUNC - Subroutine that evaluate functions f and g

SIMPRRK - Main solvers (Semi-implicit Rational Runge-Kutta)

ADAPT - Estimates the descritization error.

The program starts by declaring the values of variables in double precision mode in order to reduce round off error. After this, the program chooses initial estimate for the variables and function FUNC (was defined as a function sub-program to) eveluates f(x,y) and g(x,y)

Next, is the adoption of the solver called IMPRRK which call on subroutine FUNC to supply the slopes of the integral curve of the solution. On the receipt of the estimates, IMPRRK generates the approxmates solution y_{n+1} to y(x) at x_{n+1} and call on ADAPT to estimate the values of the error (LTE) associated with the computation so as to choose the new step size which will increase the rate of convergence.

On the receipt of the error estimate from the subrotine ADAPT, IMPRRK then test for the convergence of the solution by comparing the magnitude of the error with allowable error tolerance. As soon as the condition /LTE/ < Tol is met, the program will ask whether the upper point has been reached, if yes, it will output result. Othewrwise IMPRRK will go to the next step to generate the next round of approximation to the solution. This process is repeated and continued until the upper end point is reached. When the program reached the end point (x_{n+1}) it will stop the afore metioned process.

5.4 NUMERICAL COMPUTATIONS AND RESULTS.

To demonstrate the applicability and suitability of the method for solving differential equation with derivative discontinuities ,some sample problems were considered. These sample problems were solved with the new scheme and the results are shown in tables 1 - 6. In order to asses the perfromance of our scheme, the result of our scheme was compared with the result obtained with the other existing method, that is, the modified Euler scheme of the same order , and the result are as shown on page 71.

PROBLEM 1

We consider the initial value problem

$y^1 = -x/y$; $x_0 = -1, y_0 = 1$

$-1 \leq x < 1$

Whose theoretical solution

$y(x) = \sqrt{2 - x^2}$

which is a family of circle centre (0,0). The differential equation is with derivative discontinuities at ($\sqrt{2}$, 0)

This problem is solved numerical using the one-stage formular and the result is as shown in table 1.The Same problem was solved using two stage scheme (3.68) and the result is as shown in table 4.

PROBLEM 2:

An initial value problem:

$y^1 = y/x$ $x_0 = 1\ y_0 = -1$

Whose theoretical solution is

$y = -x$

The equation has derivative discontinuity at x = 0. It was solved with the one stage fomular (3.43). The result is as shown in the table 2. The same problem was solved using two stage formula (3.68) and the result is as shown in table 5.

Problem 3:

An initial value problem

$y^1 = 1/x$ $x_0 = 1, y_0 = -1$

Whose theoretical solution is

$y = \ln x$

The differential equation was considered, derivative discontinuities at x = 0

The numerical solution of this problem is shown in table 3 and 6 for the one stage and two-stage scheme respectively.

5.5 DISCUSSION OF NUMERICAL RESULTS

From the results of solutions to problems 1 to 3 as displayed in tables 1,2,3,4, 5 and 6, it was observed that the discretisation error obtained from the solution are sufficiently small, showing that the schemes were accurate, stable and convergent.

Considering tables 6a and 6b, where results obtained from our method was compared against modified Euler method of the same order; It can be seen that it compared favourably well with the said existing method of solution.

Table 1

Result of One stage Scheme for problem 1

Mesh Size .10000000D+00

X_n	YEXACT	Y_n	ERROR
-0.10000000D+01	.10000000D+01	.10000000D + 01	.00000000D+07
-0.90000000D+00	.10908710D+01	.10883430D + 01	.41826020D-03
-0.80000000D+00	.11661900D+01	.10552338D + 01	.11095620D-02
-0.69999990D+00	.12288210D+01	.10782578D + 01	.15056320D-02
-0.59999990D+00	.12806250D+01	.12630379D + 01	.17587010D-02
-0.49999990D+00	.13228760D+01	.13036342D + 01	.19241740D-02
-0.39999990D+00	.13564660D+01	.13361820D + 01	.20283990D-02
-0.29999990D+00	.13820280D+01	.13609691D + 01	.21058860D-02
-0.19999990D+00	.14000000D+01	.13785968D + 01	.21403148D-02
-0.99999930D-01	.14106740D+01	.13889798D + 01	.21694160D-02
0.74505810D-07	.14142140D+01	.13925157D + 01	.21698240D-02
0.10000010D+00	.14106740D+01	.13891141D + 01	.21559860D-02
0.20000010D+00	.14000000D+01	.13787042D + 01	.21295800D-02
0.30000010D+00	.13820280D+01	.13611331D + 01	.20894820D-02
0.40000010D+00	.13564660D+01	.13362975D + 01	.01684410D-02
0.50000010D+00	.13228760D+01	.13037601D + 01	.19115860D-02
0.60000010D+00	.12806250D+01	.12630035D + 01	.17621410D-02
0.70000010D+00	.12288210D+01	.12139158D + 01	.14905110D-02
0.80000010D+00	.11661900D+01	.11541358D + 01	.12054160D-02
0.90000000D+00	.10908710D+01	.10839561D + 01	.69148960D-03
0.10000000D+01	.99999990D+00	.999803410D + 01	.19644814D-03
0.11000000D+01	.88881930D+00	.887002351D + 01	.18169410D-02
0.12000000D+01	.74833120D+00	.74301429D + 01	.53169020D-02
0.13000000D+01	.55677590D+00	.54215079D + 01	.14625110D-01

Table 2

Result of one Stage Scheme for problem 2

Mesh Size .10000000D+00

X_n	YEXACT	Y_n	ERROR
.10000000D+01	-.10000000D+01	-.10000000D+ 01	.00000000D+00
.11000000D+01	-.11000000D+01	-.86712549D+ 00	.21112590D+ 00
.12000000D+01	-.12000000D+01	-.72823888D+ 00	.45038830D+ 00
.13000000D+01	-.13000000D+01	-.56249500D+ 00	.73749500D+ 00
.14000000D+01	-.14000000D+01	.15446630D + 01	.11446630D+ 01
.15000000D+01	-.15000000D+01	.15426815D + 01	.39268150D+ 01
.16000000D+01	-.16000000D+01	.15611934D + 01	.40119340D+ 01
.17000000D+01	-.17000000D+01	.15743918D + 01	.40439118D+ 01
.18000000D+01	-.18000000D+01	.15869345D + 01	.40693455D+ 01
.19000000D+01	-.19000000D+01	.15986931D + 01	.40869315D+ 01
.20000000D+01	-.20000000D+01	.26092830D + 01	.40928300D+ 01
.21000000D+01	-.21000000D+01	.26197108D + 01	.40971080D+ 01
.22000000D+01	-.22000000D+01	.26285906D + 01	.40859060D+ 01
.23000000D+01	-.23000000D+01	.26361110D + 01	.40631110D+ 01
.24000000D+01	-.24000000D+01	.26423921D + 01	.40239210D+ 01
.25000000D+01	-.25000000D+01	.26464513D + 01	.39645130D+ 01
.26000000D+01	-.26000000D+01	.26483591D + 01	.38835910D+ 01
.27000000D+01	-.27000000D+01	.26483021D + 01	.37630210D+ 01
.27999990D+01	-.27999990D+01	.26376414D + 01	.35764150D+ 01
.28999990D+01	-.28999990D+01	.24135220D + 01	.32352215D+ 01
.29999990D+01	-.29999990D+01	.33654520D + 01	.17056860D+ 01
.30999990D+01	-.30999990D+01	.20068304D + 02	.16877690D+ 02
.31999990D+01	-.31999990D+01	-.19961450D + 02	.16648560D+ 02
.32999990D+01	-.32999990D+01	-.19846560D + 02	.16528790D+ 02

Table 3

Result of One stage Scheme for problem 3

Mesh Size .10000000D+00

X_n	YEXACT	Y_n	ERROR
.10000000D+01	.10000000D+01	.10000000D + 01	.00000000D + 00
.11000000D+01	.10953100D+01	.98976450D + 01	.10538060D + 00
.12000000D+01	.11823220D+01	.97857260D + 00	.20348260D + 00
.13000000D+01	.12623640D+01	.96628230D + 00	.29605220D + 00
.14000000D+01	.13364720D+01	.93785430D + 00	.38354030D + 00
.15000000D+01	.14054650D+01	.92159070D + 00	.46749210D + 00
.16000000D+01	.14700040D+01	.90406850D + 00	.54826310D + 00
.17000000D+01	.15306280D+01	.88420560D + 00	.62640110D + 00
.18000000D+01	.15877870D+01	.86439450D + 00	.70249520D + 00
.19000000D+01	.16418540D+01	.84176580D + 00	.77726610D + 00
.20000000D+01	.16931470D+01	.81797650D + 00	.85069410D + 00
.21000000D+01	.17419370D+01	.79086850D + 00	.92365160D + 00
.22000000D+01	.17884570D+01	.76358740D + 00	.99637620D + 00
.23000000D+01	.18329090D+01	.73286670D + 00	.10681650D + 01
.24000000D+01	.18754690D+01	.69835240D + 00	.11414560D + 01
.25000000D+01	.19162910D+01	.66268570D + 00	.12156810D + 01
.26000000D+01	.19555110D+01	.62225860D + 00	.12916720D + 01
.27000000D+01	.19932520D+01	.57713320D + 00	.13708640D + 01
.27999990D+01	.20296190D+01	.52589570D + 00	.14516820D + 01
.28999990D+01	.20647110D+01	.46890460D + 00	.15368850D + 01
.29999990D+01	.20986120D+01	.40078660D + 00	.16288160D + 01
.30999990D+01	.21314020D+01	.31625430D + 00	.17303910D + 01
.31999990D+01	.21631510D+01	.19936850D + 00	.18459280D + 01
.32999990D+01	.21939220D+01	.41572860D – 01	.19936526D + 01

Table 4

Result of Two stage Scheme for problem 1

Mesh Size .10000000D+00

X_n	YEXACT	Y_n	ERROR
-0.10000000D+01	.10000000D+01	.10000000D+01	.00000000D+00
-0.90000000D+00	.10908710D+01	.10904530D+01	.41830540D-03
-0.80000000D+00	.11661900D+01	.11650800D+01	.11106730D-02
-0.69999990D+00	.12288210D+01	.12273130D+01	.15075210D-02
-0.59999990D+00	.12806250D+01	.12788640D+01	.17606020D-02
-0.49999990D+00	.13228760D+01	.13609500D+01	.19261840D-02
-0.39999990D+00	.13564660D+01	.13544310D+01	.20354990D-02
-0.29999990D+00	.13820280D+01	.13799210D+01	.21066670D-02
-0.19999990D+00	.14000000D+01	.13978500D+01	.21504160D-02
-0.99999930D-01	.14106740D+01	.14085000D+01	.21731850D-02
0.74505810D-07	.14142140D+01	.14120350D+01	.21781920D-02
0.10000010D+00	.14106740D+01	.14085070D+01	.21668670D-02
0.20000010D+00	.14000000D+01	.13978610D+01	.21384950D-02
0.30000010D+00	.13820280D+01	.13799370D+01	.20906930D-02
0.40000010D+00	.13564660D+01	.13544470D+01	.20184520D-02
0.50000010D+00	.13228760D+01	.13209620D+01	.19136670D-02
0.60000010D+00	.12806250D+01	.12788630D+01	.17621520D-02
0.70000010D+00	.12288210D+01	.12272800D+01	.15404220D-02
0.80000010D+00	.11661900D+01	.11649840D+01	.12065170D-02
0.90000000D+00	.10908710D+01	.10901090D+01	.68175790D-03
0.10000000D+01	.99999990D+00	.10001970D+01	.19729140D-03
0.11000000D+01	.88881930D+00	.89063800D+00	.18187170D-02
0.12000000D+01	.74833120D+00	.75364960D+00	.53184030D-02
0.13000000D+01	.55677590D+00	.57631010D+00	.15534220D-01
0.14000000D+01	.19999840D+00	.29434340D+00	.94345030D-01

Table 5

Result of Two stage Scheme for problem 2

Mesh Size .10000000D+00

X_n	YEXACT	Y_n	ERROR
.10000000D+01	-.10000000D+01	-.10000000D+01	.00000000D+00
.11000000D+01	-.11000000D+01	-.88887330D+01	.21112670D+00
.12000000D+01	-.12000000D+01	-.74959070D+01	.45040930D+00
.13000000D+01	-.13000000D+01	-.56246610D+01	.73753400D+00
.14000000D+01	-.14000000D+01	-.25752660D+01	.11424730D+01
.15000000D+01	-.15000000D+01	.24274170D+01	.39274170D+01
.16000000D+01	-.16000000D+01	.24121690D+01	.40121690D+01
.17000000D+01	-.17000000D+01	.23441890D+01	.40441890D+01
.18000000D+01	-.18000000D+01	.22685520D+01	.40685520D+01
.19000000D+01	-.19000000D+01	.21857400D+01	.40857410D+01
.20000000D+01	-.20000000D+01	.20949490D+01	.40949500D+01
.21000000D+01	-.21000000D+01	.19951090D+01	.40951090D+01
.22000000D+01	-.22000000D+01	.18848090D+01	.40848080D+01
.23000000D+01	-.23000000D+01	.17621320D+01	.40621320D+01
.24000000D+01	-.24000000D+01	.16243610D+01	.40243610D+01
.25000000D+01	-.25000000D+01	.14674190D+01	.39674190D+01
.26000000D+01	-.26000000D+01	.12846820D+01	.38846820D+01
.27000000D+01	-.27000000D+01	.10640440D+01	.37640430D+01
.27999990D+01	-.27999990D+01	.77852770D+01	.35785270D+01
.28999990D+01	-.28999990D+01	.33744410D+01	.32374430D+01
.29999990D+01	-.29999990D+01	-.20077970D+02	.17077970D+02
.30999990D+01	-.30999990D+01	-.19988440D+02	.16888440D+02
.31999990D+01	-.31999990D+01	-.19972960D+02	.16772960D+02
.32999990D+01	-.32999990D+01	-.19956670D+02	.16656680D+02
.33999990D+01	-.33999990D+01	-.19939880D+02	.16539880D+02

Table 6

Result of Two stage Scheme for problem 3

Mesh Size .10000000D+00

X_n	YEXACT	Y_n	ERROR
.10000000D+01	.10000000D+01	.10000000D+01	.00000000D+00
.11000000D+01	.10953100D+01	.98989950D+00	.10541070D+00
.12000000D+01	.11823220D+01	.97868190D+00	.20363970D+00
.13000000D+01	.12623640D+01	.96630110D+00	.29606330D+00
.14000000D+01	.13364720D+01	.95271190D+00	.38376040D+01
.15000000D+01	.14054650D+01	.93786200D+00	.46760310D+00
.16000000D+01	.14700040D+01	.92169060D+00	.54831310D+00
.17000000D+01	.15306280D+01	.90412720D+00	.62659110D+00
.18000000D+01	.15877870D+01	.88508930D+00	.70269750D+00
.19000000D+01	.16418540D+01	.86447980D+00	.77737420D+00
.20000000D+01	.16931470D+01	.84218420D+00	.85096310D+00
.21000000D+01	.17419370D+01	.81806520D+00	.92387220D+00
.22000000D+01	.17884570D+01	.79195760D+00	.99649980D+00
.23000000D+01	.18329090D+01	.76365890D+00	.10692500D+01
.24000000D+01	.18754690D+01	.73291780D+00	.11425510D+01
.25000000D+01	.19162910D+01	.69941560D+00	.12168750D+01
.26000000D+01	.19555110D+01	.66273930D+00	.12927720D+01
.27000000D+01	.19932520D+01	.62233670D+00	.13709150D+01
.27999990D+01	.20296190D+01	.57744200D+00	.14521770D+01
.28999990D+01	.20647110D+01	.52693890D+00	.15377720D+01
.29999990D+01	.20986120D+01	.46908590D+00	.16295260D+01
.30999990D+01	.21314020D+01	.40088750D+00	.17305150D+01
.31999990D+01	.21631510D+01	.31633280D+00	.18468180D+01
.32999990D+01	.21939220D+01	.19941330D+00	.19945090D+01
.33999990D+01	.22237750D+01	-.41582550D-01	.22653580D+01

THE RESULTS OF THE COMPARISION OF THE NEW SCHEME WITH MODIFIED EULER'S METHOD

Results of one stage scheme for problem 2

Mesh Size .10000000D+00

X_n	YEXACT	Y_n	ERROR	e Y_n	E_2
.10000000D−01	−.10000000D−01	−.10000000D+ 01	.00000000D−00	−.10000000D+ 01	.00000000D+ 00
.11000000D−01	−.11000000D−01	−.86712549D− 00	.21112590D+ 00	−.86640000D+ 01	.20834680D+ 00
.12000000D−01	−.12000000D−01	−.72823888D− 00	.45038830D− 00	−.74213480D+ 01	.44926570D+ 00
.13000000D−01	−.13000000D−01	−.56249500D− 00	.73749500D+ 00	−.56243790D+ 01	.73728640D+ 00
.14000000D+01	−.14000000D−01	.15446630D − 01	.11446630D− 01	.25438750D− 01	.11429560D+ 01
.15000000D−01	−.15000000D−01	.15426815D − 01	.39268150D− 01	.24417680D− 01	.39156081D− 01
.16000000D−01	−.16000000D−01	.15611934D − 01	.40119340D− 01	.24108870D− 01	.40113450D− 01
.17000000D−01	−.17000000D−01	.15743918D − 01	.40439118D− 01	.23434860D− 01	.40436710D+ 01
.18000000D−01	−.18000000D−01	.15869345D − 01	.40693455D+ 01	.22658350D− 01	.40674320D− 01
.19000000D−01	−.19000000D−01	.15986931D − 01	.40869315D− 01	.21875860D− 01	.40846520D+ 01
.20000000D−01	−.20000000D−01	.26092830D − 01	.40928300D+ 01	.20986432D− 01	.40937640D+ 01
.21000000D−01	−.21000000D−01	.26197108D − 01	.40971080D− 01	.19688743D− 01	.40946220D+ 01
.22000000D+01	−.22000000D−01	.26285906D − 01	.40859060D− 01	.18665742D− 01	.40836070D+ 01
.23000000D−01	−.23000000D−01	.26361110D − 01	.40631110D− 01	.17662142D+ 01	.40614230D+ 01
.24000000D−01	−.24000000D−01	.26423921D − 01	.40239210D− 01	.16224882D− 01	.40232560D+ 01
.25000000D−01	−.25000000D−01	.26464513D − 01	.39645130D− 01	.14654620D+ 01	.39583210D+ 01
.26000000D−01	−.26000000D−01	.26483591D − 01	.38835910D− 01	.12842430D− 01	.38835240D− 01
.27000000D−01	−.27000000D−01	.26483021D − 01	.37630210D− 01	.10652631D− 01	.37630350D− 01
.27999990D−01	−.27999990D−01	.26376414D − 01	.35764150D− 01	.74135200D− 01	.35667380D− 01
.28999990D−01	−.28999990D−01	.24135220D − 01	.32352215D− 01	.33646610D− 00	.32363420D− 01
.29999990D−01	−.29999990D−01	.33654520D − 01	.17056860D− 01	−.20057720D− 02	.17068820D− 01
.30999990D−01	−.30999990D−01	.20868304D − 02	.16877698D− 02	−.19886250D− 02	.16878340D+ 01
.31999990D−01	−.31999990D−01	−.19961450D − 02	.16648560D− 02	−.19950940D− 02	.16761830D− 01
.32999990D−01	−.32999990D−01	−.19846560D − 02	.16528790D− 02	−.19835730D− 02	.16645610D− 01

THE RESULTS OF ONE STAGE SCHEME ADOPTING RICHARDSON METHOD TO ESTIMATE THE LOCAL TRUNCATION ERROR FOR PROBLEM 2

Mesh Size .1000000D+00

x_n	YEXACT	Y_n	ERROR	L.T.E.
.1000000D+01	-.1000000D+01	-.1000000D+01	.0000000D+00	.98656720D-02
.1100000D+01	-.1100000D+01	-.8671259D+00	.2111259D+00	.97862520D-02
.1200000D+01	-.1200000D+01	-.7282388D+00	.1503883D+00	.93245320D-02
.1300000D+01	-.1300000D+01	-.5624950D+00	.7374950D+00	.86425230D-01
.1400000D+01	-.1400000D+01	.1544663D+01	.1144663D+01	.16746320D-02
.1500000D+01	-.1500000D+01	.1542681D+01	.3926815D+01	.14536210D-02
.1600000D+01	-.1600000D+01	.1561193D+01	.4011934D+01	.73214529D-03
.1700000D+01	-.1700000D+01	.1574391D+01	.4043911D+01	.48674920D-03
.1800000D+01	-.1800000D+01	.1586934D+01	.4069345D+01	.36724610D-03
.1900000D+01	-.1900000D+01	.1598693D+01	.4086931D+01	.22657320D-03
.2000000D+01	-.2000000D+01	.2609283D+01	.4092830D+01	.17948610D-03
.2100000D+01	-.2100000D+01	.2619710D+01	.4097108D+01	.11096540D-04
.2200000D+01	-.2200000D+01	.2628590D+01	.4085906D+01	.74869230D-04
.2300000D+01	-.2300000D+01	.2636111D+01	.4063111D+01	.48946520D-04
.2400000D+01	-.2400000D+01	.2642392D+01	.4023921D+01	.32165710D-04
.2500000D+01	-.2500000D+01	.2646451D+01	.3964513D+01	.18795462D-04
.2600000D+01	-.2600000D+01	.2648359D+01	.3883591D+01	.47925341D-04
.2700000D+01	-.2700000D+01	.2648302D+01	.3763021D+01	.32565820D-04
.2799990D+01	-.2799990D+01	.2637641D+01	.3576415D+01	.11278560D-04
.2899990D+01	-.2899990D+01	.2413522D+01	.3235221D+01	.28922368D-04
.2999990D+01	-.2999990D+01	.3365452D+01	.1705686D+01	.16946871D-04
.3099990D+01	-.3099990D+01	.2006830D+02	.1687769D+02	.21367920D-04
.3199990D+01	-.3199990D+01	-.1996145D+02	.1664856D+02	.25069216D-04
.3299990D+01	-.3299990D+01	-.1984656D+02	.1652879D+02	.23681924D-04

THE RESULTS OF TWO STAGE ADOPING RICHARDSON METHOD TO ESTIMATE THE LOCAL TRUNCATION ERROR ASSOCIATED WITH PROBLEM I

Mesh Size .10000000D+00

X_n	YEXACT	Y_n	ERROR	L.T.E
-0.10000000D+01	.10000000D+01	.10000000D + 01	.00000000D+07	-1.10000000
-0.90000000D+00	.10908710D+01	.10883430D + 01	.41826020D-03	-1.10904543
-0.80000000D+00	.11661900D+01	.10552338D + 01	.11095620D-02	-1.11650850
-0.69999990D+00	.12288210D+01	.10782578D + 01	.15056320D-02	-1.12273120
-0.59999990D+00	.12806250D+01	.12630379D + 01	.17587010D-02	-1.12788550
-0.49999990D+00	.13228760D+01	.13036342D + 01	.19241740D-02	-1.13209460
-0.39999990D+00	.13564660D+01	.13361820D + 01	.20283990D-02	-1.13544280
-0.29999990D+00	.13820280D+01	.13609691D + 01	.21058860D-02	-1.13799110
-0.19999990D+00	.14000000D+01	.13785968D + 01	.21403148D-02	-1.13978400
-0.99999930D-01	.14106740D+01	.13889798D + 01	.21694160D-02	-1.14085000
0.74505810D-07	.14142140D+01	.13925157D + 01	.21698240D-02	-1.14120349
0.10000010D+00	.14106740D+01	.13891141D + 01	.21559860D-02	-1.14085075
0.20000010D+00	.14000000D+01	.13787042D + 01	.21295800D-02	-1.13978530
0.30000010D+00	.13820280D+01	.13611331D + 01	.20894820D-02	-1.13799365
0.40000010D+00	.13564660D+01	.13362975D + 01	.01684410D-02	-1.13544462
0.50000010D+00	.13228760D+01	.13037601D + 01	.19115860D-02	-1.13208570
0.60000010D+00	.12806250D+01	.12630035D + 01	.17621410D-02	-1.12768910
0.70000010D+00	.12288210D+01	.12139158D + 01	.14905110D-02	-1.12256820
0.80000010D+00	.11661900D+01	.11541358D + 01	.12051160D-02	-1.11648560
0.90000000D+00	.10908710D+01	.10839561D + 01	.69148960D-03	-1.10901680
0.10000000D+01	.99999990D+00	.99980341D + 01	.19644814D-03	-1.10001685
0.11000000D+01	.88881930D+00	.88700235D + 01	.18169410D-02	-1.89056840
0.12000000D+01	.74833120D+00	.74301429D + 01	.53169020D-02	-1.75354750
0.13000000D+01	.55677590D+00	.54215079D + 01	.14625110D-01	-1.57231012

GENERAL CONCLUSION

6.1 SUMMARY

In this project, we have developed a class of semi-implicit Rational Runge-Kutta method for solving ordinary differential equations with derivatives discontinuities.

This is motivated by Rational Runge - Kutta Scheme proposed by Hong Yuan Fu (1982) and variety of application areas of ordinary differential equations and the need to cater for the deficiencies identified in the adoption of the existing methods of solving this class of differential equations.

The method was derived using power sderies expansion technique, analysed, computerised and implemented with sample problems on a micro-computer. The results showed that the schemes is absolutely stable, convergent, efficient and effective towards solving ordinary differential equations with derivative discontinuities.

6.2 LIMITATIONS

Due to the fact that, in the development of the scheme, we adopted the power series (Taylor and binominal) expansion, it is, subject to point to point error and possible error propagation. Besides these, other limitations to the work are as listed below:

i. There is difficulty in the selection of starting step size (that is, the step size that will over step the points of discontinuities in the problem) that will balance the: restriction by accuracy in the neighbourhood of discontinuities and

ii. Step size restriction dictated by stability.

6.3 RECOMMENDATION

Based on the aforementioned limitations, their is need to adopt some strategies which can be the basis for finding appropriate balance between the step size (h), the order of accuracy and stability of the method. in order to achieve a better, accurate, efficient and effective scheme.

Thus strategy includes the choice of step size (h) such that

$$h >> \frac{1}{y_n - g(x_n, z_n)}$$

This will ensure that the point of discontinuities can be stepped over.

the use of double precision arithemetic to minimise the effects of round off error.

6.4 CONTRIBUTION TO KNOWLEDGE

The new scheme will lead to a better general purpose computer alogrithm for solution of differential equations of electrical networks, control problems and National economy affected by inflation from which differential equations with derivative discontinuities arise. By its stability properties, the schemes will be capable of solving stiff and stiff oscillatory differential equations as well.

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