A HYBRID METHOD FOR GENERAL THIRD ORDER ORDINARY DIFFERENTIAL EQUATIONS.

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DEDICATION

This work is dedicated to the almighty God, my late father, Chief Oluremi Joseph Olorunfemi, my mother Mrs.Lydia Olorunfemi and my children; Gift Olorunfemi, Temiloluwa Olorunfemi and Eyitayo Oluwaduyilemi Olorunfemi

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ABSTRACT

This work proposes numerical method for solving directly general third order ordinary differential equations by collocation at the grid points $x = x_{n+i}, i = o(1)k$ and at an off grid point $x = x_{n+u}$ and interpolation of the approximate solution at the grid points $x = x_{n+i}$, $i = o(1)\ k - 1$. k is the step-number of the method, u is an arbitrary rational number in (x_n, x_{n+k}).A predictor of order 2k-1 is also proposed to cater for y_{n+k} in the main method. Taylor series expansion is employed for the calculation of y_{n+1}, y_{n+2} and y_{n+u}, and their higher derivatives. Evaluation of the resulting method at $x = x_{n+k}$ for any value of u in the specified open interval yields a particular discrete scheme as a special case of the method. At u=3/2 the interval of absolute stability of the resulting discrete method is located at the origin which makes it to be of little or no practical applications. Finally the efficiency of the method is tested on some general initial value problems of third order ordinary differential equations.

CHAPTER ONE

INTRODUCTION

1.1 PROBLEM

The Mathematical formulation of most problems in science, engineering and management involves rate of change with respect to one or more independent variables. In every area of life, it is essential and necessary to develop a mathemetical model to represent problems. The models that involve the research for an unknown function which satisfies an equation in which the rate of change of the unknown functions plays a dominant role are called differential equations. Depending on the number of independent variables, differential equations can be categorized into two forms; ordinary and partial differential equations. If there is only one independent variable, the rate of change is known as derivative. In other word, any equation which connects the derivatives of a differetiable function of one independent variable with respect to itself is called ordinary differential equations. The general form of ordinary differential equation is

$$f(x, y, y', y'', \ldots\ldots\ldots\ldots y^{n}) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.1)$$

where x is the independent variable, y is the dependent variable and n is the order of the derivatives. If more than one independent variables exist, the equation will be called partial differential equation in the form;

$$\frac{a\partial\phi}{\partial x} + \frac{b\partial\phi}{\partial y} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 1.2$$

We note that, to obtain a solution to (1.1), n additional conditions will be required, which are called initial or boundary – conditions. If such information is provided at same values of the independent variables, we have the initial – value problem, for example

$$f\left(x, y, y', y'', \ldots\ldots\ldots\ldots y^{n}\right)=0, y^{i}\left(x_{0}\right)=\eta_{i,} i=0(1)n-1 \ldots\ldots\ldots\ldots\ldots(1.3)$$

If such information is provided at different values of the independent variable, we have a boundary value problem (BVP), for instance,

$$y'' = f\left(x, y, y'\right), y(a)=\alpha, y(b)=\beta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.4)$$

is a two – point boundary value problem.

System of initial value problems in ordinary differential equation also exist especially in stiff problems. A problem is stiff if its eigen values λ_j are widely seperated and non stiff otherwise. Stiffiness, as it is used in this context is a concept describing the nature of certain subsets of ordinary differential equations whose solution contain components with fast and slow responses. The fast responding components are called transients while the slow responding components are generally smooth and steady. For example

$$y' = f(x, y,), y(x_0) = \eta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.5)$$

The order of ordinary differential equations is the order of the highest derivatives that appears in the equation and its degree is the power to which the highest derivative is raised.

If no product of the dependent variable y(x) with itself or any of the devariative occur, then the equation is linear, otherwise, it is non – linear for example, $y^1 + y = 0$ is linear equation, whereas

$3y^1 {}_{-y}{}^2 = 0$ is non – linear.

1.2 METHODS OF SOLUTION

The fact that the propose schemes are based on linear multi – step method, it is necessary to discuss some of these techniques, particularly, the collocation hybrid method for the solution of initial value problems of ordinary differential equations. Linear multi – step method of step number k is a computational method for determining the sequence $\{y_n\}$of approximate solution which takes the form of a linear relationship between y_{n+j} and $f_{n+j}, j = 0,1,\ldots,k$

In general, linear multi – step method of step number k for third order initial value problems may thus be written in the form;

$$\sum_{j=0}^{k}\alpha_j y_{n+j} = h^3\sum_{j=0}^{k}\beta_j f_{n+j} \qquad (1.6)$$

Where α_j and β_j are constants and $\alpha_k \neq 0$, but assumed as $\alpha_k = 1$, y_{n+j} is an approximate to the theoretical solution $y(x_{n+j})$. if $\beta_k = 0$, method (1.6) is explicit and implicit if $\beta_k \neq 0$.

Hybrid methods are modified linear multi – step formulae which incorporate a function evaluation at an off-step point. They are among highly accurate numerical methods for solution of ordinary differential equations (ODEs). A k-step hybrid formula for third order ODEs is defined as

$$\sum_{j=0}^{k}\alpha_j y_{n+j} = h^3[\sum_{j=0}^{k}\beta_j f_{n+j} + \beta_u f_{n+u}] \qquad (1.7)$$

where $\alpha_k = +1$, α_0 and β_0 are not both zero, $u \notin \{0,1,\ldots\ldots, k\}$. In order to implement such a formula, even when it is explicit (that is, $\beta_k = 0$), predictor to estimate y_{n+k} and y_{n+u} are necessary. Examples of hybrid linear multi-step methods are as follows.

(i) Gragg and Stetter's formula (***Lambert, page 174***)

$$y_{n+1} - y_n = \tfrac{h}{6}\left(f_{n+1} + f_n + 4f_{n+\frac{1}{2}}\right) \qquad (1.8)$$

(ii) Butcher's formula (***lambert, page 167***)

$$y_{n+2}-\frac{1}{31}\left(32y_{n+1}-y_n\right)=\frac{h}{93}\left(15f_{n+2}+12f_{n+1}-f_n+54f_{n+\frac{1}{2}}\right) \quad \text{.............}(1.9)$$

where collocations at off grid points yeild the function $f_{n+1/2}$ in (1.8) and (1.9) respectively.

1.3 ORDER OF ACCURACY AND ERROR CONSTANT

The order of a linear-multistep method may be found by the use of Taylor expansion of the difference equation.

$$\sum_{j=0}^{k}\alpha_j y_{n+j}=h^3\sum_{j=0}^{k}\beta_j f_{n+j}+h^3\beta_u f_{n+u} \quad \text{..............................}(1.10)$$

and the associated linear difference operator L defined as

$$L\left[y_{(x_n)}:h\right]=\sum_{j=0}^{k}\left[\alpha_j y\left(x_n+jh\right)-h^3\beta_j y'''\left(x_n+jh\right)\right]-h^3\beta_u y'''\left(x_n+uh\right) \quad \text{.........}(1.11)$$

Where y(x) is an arbitrarry function which is continously differentiable on the interval [a,b], Expanding the test functioin y(x+jh) and deriavative $y^{III}(x_n + jh)$ and $y^{III}(x_n + uh)$ by Taylor series about x_n , and collecting terms in equal powers of h in (1.11) gives

$$L\left(y_{(x_n)}:h\right)=c_0 y_{(x)}+c_1 h y'(x)+\text{.......}+c_q h^q y^q(x)+ \quad \text{...................}(1.12)$$

Where the C_q are constants.

DEFINITION 1.1

The difference operator (1.11) and its associated linear multi – step method are said to be of order P if in (1.12)

$C_0 = C_1 = C_2 = \text{......} = C_P = 0, C_{P+1} \neq 0$ where

$C_0 = \alpha_0 + \alpha_1 + \alpha_2 + \text{......} + \alpha_K$

$C_1 = \alpha_1 + 2\alpha_2 + \text{......} + k\alpha_K - (\beta_0 + \beta_1 + \beta_2 + \text{.........} + \beta_K)$

.

.

.

$$C_q = \frac{1}{q}[(\alpha_1 + 2^q\alpha_2 + \ldots\ldots + k^q\alpha_k)] - \frac{1}{(q-1)}[(\beta_1 + 2^{q-1}\beta_2 + \ldots\ldots + k^{q-1}\beta_k) \quad \ldots\ldots\ldots (1.13)$$

q=2,3,…………(*see lambert, 1973, page 23*)

1.4 STABILITY AND CONSISTENCY

DEFINITION 1.2

A linear multi-step method (1.7) is said to be zero – stable, if no root of the first characteristic polynomial P(r) has modulus greater than one and if every root of modulus one has multiplicity not greater than three.

DEFINITION 1.3

A linear multi-step method is said to be absolutely table for a given $\bar{h}$, if for that $\bar{h}$, all the roots (r_s) of $\Pi(r,\bar{h}) = p(r) - \bar{h}\sigma(r) = 0$, where $\bar{h} = h^3\lambda^3$ ………….. (1.14)

[$\Pi(r,\bar{h})$] is the characteristic polynomial] satisfy $|r_s| < 1$, s = 1,2,………, k, and to be absolutely unstable for that $\bar{h}$ otherwise.

DEFINITION 1.4

An interval [α,β] of the real line is said to be an interval of absolute stability if the method (1.7) is absolutely stable for all $\bar{h} \in [\alpha, \beta]$.

DEFINITION 1.5

A numerical method (1.7) is said to be A – stable, if its region of absolute stability contains the whole of the left hand half plane Re ($h^3\lambda^3$)<0, of the complex plane.

DEFINITION 1.6

A linear multi-step (1.7) is said to be P – stable, if its region of absolute stabilility contains the whole of the right hand half plane. Re($h^1\lambda^1$) >0, of the complex plane.

The concept of consistency of LMM is very important in the sense that, it controls the magnitude of local truncation error(l.t.e) (***Fatunla 1988 page 113***).

DEFINITION 1.7

The linear multi-step method (1.7) is said to be consistent if it has order at least one and that the first and second characteristics polynomials fulfil the conditions

$p(1) = p^1(r) = 0$ and $p^{''}(r) = 3!\sigma(r)$, for r =1

1.5 SYMMETRIC PROPERTY

DEFINITION 1.8

A linear multi – step method (1.6) is symmetric if

$\alpha_j = -\alpha_{k-j}, \beta_j = \beta_{k-j}, j = 0,1,.............k$

1.6 CONVERGENCE OF THE LMM

A theorem which guarrantees the convergence of LMM is reproduced below without proof.

THEOREM (Henrici (1962))

The necessary and sufficient conditions for a linear multistep method to be convergent is for it to be consistent and zero stable.

1.7 HYBRID PREDICTOR

Before any of the hybrid formulae can be implemented, the value of f_{n+u} must be computed and may be obtained by the use of special predictor of the form.

$$y_{n+u} + \sum_{j=0}^{k-1} \bar{\alpha}_j y_{n+j} = h \sum_{j=0}^{k-1} \bar{\beta}_j f_{n+j} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.15)$$

The order, error constant, stability and consistency properties associated with LMM and hybrid methods are also applicable to the predictor.

1.8 ERRORS OF NUMERICAL APPROXIMATION TECHNIQUES.

A noticeable characteristic of numerical schemes is that errors are generated when they are adopted for aproximation of solution of ordinary differential equations.

The degree of accuracy of the schemes is determined by the magnitude of errors involved in the schemes. If the magnitude strictly tends to zero, we say, the scheme is convergent and accurate. Otherwise, it is non convergent and inaccurate.

Errors of numerical approximation techniques for ordinary differential equations arise from different causes, which can be classified as follows: truncation errors, round off errors, inherent errors and discretization errors (lambert(1973)).

(i) TRUNCATION ERRORS

Truncation errors are errors introduced as a result of ignoring some of the higher terms of the power series (e.g. in Taylor series expansion) during the development of the scheme.

Mathematically, truncation errors are defined as the amount by which the solution of the differential equation fails to satisfy the difference equation. That is, the truncation error of (1.6) or (1.7) can be defined as

$$y(x_{n+k}) - y_{n+k} = C_{p+1} h^{p+1} y_{(x_n)}^{p+1} + O(h^{p+2}) \ldots\ldots (1.16)$$

(lambert, 1973, page 28) where P is the order of the method, c_{p+1} is the error constant and $c_{p+1}h^{p+1}\, y^{p+1}{}_{(xn)}$ is the truncation error.

The effect of the truncation error may be reduced by making the step size h as small as possible.

(ii) ROUND-OFF,ERRORS

Round-off errors are errors introduced as a result of the computing devices. This is represented mathmatically as

$$T_{n+1} = y_{n+1} - p_{n+1} \qquad \ldots\ldots(1.17)$$

where y_{n+1} is the expected solution of the differences equation (1.7), p_{n+1} is computer output.

Storage and manipulation of numbers may lead to accumulation of these errors (round off errors). The effects of round off error could cause loss of accuracy, which may be disastrous. We can control these effects by employing double precision Arithemtic, if the numerical solution of the scheme involves iteration process.

(iii). INHERENT ERRORS.

Inherent errors are errors associated with the differential equation in the course of the model development. They are not easily traceable but they can be avoided if properly and envisaged earlier in the course of numerical schemes derivation and implementation.

(iv). DISCRETIZATION ERRORS.

Discretization errors are errors introduced as a result of transforming differential equations into difference equations.

Mathematically, the discretization error e_{n+1} associated with the formula (1.7) is the difference between the exact solution $y_{(xn)}$ and the numerical solution y_{n+1} generated by (1.7) at point x_{n+1}, that is

$$e_{n+1} = y_{n+1} - y(x_{n+1}) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.18)$$

1.9. EVALUATION OF METHOD.

Evaluation of method can be divided into three classes; Reliability (accuracy), efficiency and convenience (Gear 1971).

(i) RELIABILITY: The large number of different ways in which a user might interprete the accuracy of a method is refered to as local or Global, Relative or absolute and so on. But as long as a method is stable, its reliability in so far as accuracy is concerned depends almost entirely on how well it can estimate its local error.

(ii) EFFICIENCY: the efficient of a method is the course of solving the problem, or at least of solving some representative subset of the problems. Thus a method may be accurate but not efficient due to variuos involvement in the computation. The Runge Kutta method that can handle equation when discountiunity occurs involves a considerable number of function evaluation per iteration. It probably involves work to converge to solution and it is therefore, not efficient.

(iii) **CONVENIENCE:** This reveals the availability of suitable option about the ease with which they can be used and about the appropriateness of the documentation. A Runge kutta method with a fixed step size is one of the most convinient methods, but when the problem is linear and stiff with small step size then the trapezoidal rule can be effective. Thus, convenience must be added to other measures of evaluation (Gladwell, 1980).

1.10 MOTIVATION

The commonest method for solving an initial value problem of the form

$$y^n = f(x, y, y', \ldots\ldots y^{n-1}), y_{(a)} = y_{(0)}, y^i(a) = y_i, i = 1(1)_{n-1} \ldots\ldots\ldots\ldots\ldots\ldots (1.19)$$

as observed in literature is by reducing it into a system of first order equation. This approach is burdensome especially, when we consider time and cost implication involved as discussed in Awoyemi (1999,2001).

In this work, hybrid methods of high order which are consistent and zero stable are proposed. The discrete Schemes arising from this method for u =1/2, 3/2,and 5/2 are of order seven, six and six respectively. A predictor of order five which is p – stable is also proposed to calculate y_{n+k} in the main method. By p – stability,we mean the interval of absolute stability is infinite to the right of the x-axis, that is $(0,\infty)$ for details Fatunla (1988) and Awoyemi (2003). The time and cost of implementing the schemes will be highly minimised, since, the burden mentioned above are taking care of. The resulting algorithm is straight forward and could be easily put into a short program for the test problems of general third order ordinary differential equations.

1.11 AIMS AND OBJECTIVES

The purpose of this study is to:

(i) Develop hybrid collocation methods of high order for solving directly general third order initial value problems without reducing them to a system of first order equations. When the method is evaluated at $x = x_{n+3}$ for $u = {}^3/_2, {}^1/_2$ and ${}^5/_2$, schemes of order 7,6 and 6 are obtained as special cases of the method.

(ii) develop a predictor for calculating y_{n+k}

(iii) analyse the scheme with respect to zero stability, consistency, order, error constant, convergence and interval of absolute stability.

(iv) develop a computer program and apply the method to solve some numerical examples.

1.12 RESEARCH METHODOLOGY

To achieve the above objectives, we shall apply collocation and interpolation technique to generate our schemes (Chap. 3)

Numerical examples on third order problems are considered to test the efficiency of the schemes.

1.13 ORGANIZATION OF WORK

The remaining part of the thesis will be organised as described below.

In chapter two, we will discuss the literature reviewed of the method.

Chapter three deals with the development of the proposed schemes and its predictors. Also, their first and second derivatives.

Chapter four discusses the analysis of the methods with respect to order, error constant, consistency, zero stability, and interval of absolute stability and P-stability of the methods. Numerical computation, results and comparison of the results of the schemes with other schemes are also stated.

Finally, chapter five deals with the discussion of the results, conclusion and recommendation.

CHAPTER TWO

LITERATURE REVIEW.

The mathematical formulation of physical phenomenena in sciences, engineering, control theory and management often leads to an initial value problem of the form (1.19). It has been well known in literature that, analytic solution to such problem is of less value since a large number of the problems could not be solved by analytic approach. Thus, we have to resort to numerical methods of solution.

There have been methods used in solving (1.19). the commonest of which, according to Awoyemi (2000,2001), being reduction of the problems into first order system and then adopt any appropriate numerical method for first order equations to solve the system. Some of the disadvantages of this approach according to Awoyemi (2000,2001) are the lengthy and complicated computer programme requirement to test the accuracy of such method. As a result, computer time is wasted and cost implication is high.

Eminent scholars have attempted to solve the problem (1.19) directly without reduction to first order system of equation. Brown (1974,1977), lambert (1973), Enright (1974), Wanner etal (1978), Twizell and Khaliq (1981,1984) proposed independently a method called multi – derivative method to solve second order initial value problems. Bun and Vasil'yev (1990) proposed a numerical method, that does not require that the equation be reduced to a system of first – order equation, which is based on polynomial interpolation with multiple nodes for solving a system of non – linear differention equations with higher – order derivatives. Awoyemi (1996,1999,2000,2001,2002,2003) used multi – derivative collocation approach to solve directly higher order initial value problems of ordinary differential equations of the type (1.19) for n =2,3 and 4. Wright etal (1991) adopted a method called the mesh selection in collocation to solve second and fourth order boundary value problems. Shaw (1966) proposed some multi-step formulae for special order ordinary differential equations which are based on polynomial interpolation. Rusell and Shampine (1971) proposed a collocation method for boundary value problems which is based on collocation with piecewise polynomial functions. Twizell and Khaliq (1984) developed a class of p-stable two – step higher derivative formular for special second order initial value problems by adopting pade approximation technique. Henrici (1962) and Lambert (1973) discussed the theory of direct finite difference methods for the linear second order initial value problems.

In this thesis, a collocaion hybrid method for solving directly general third order initial value problems of ordinary differential equations of the form.

$$y^{III} = f(x, y, y^{I}, y^{II}), y(a) = y(0), y^{I}(a) = \eta_1, y^{II}(a) = \eta_2, a, yF \in R \ldots\ldots\ldots\ldots\ldots\ldots(2.1)$$

for a step number k = 3 is considered. In this problem, the minimum value of k according to Awoyemi (2003) is three.

It has been revealed in literature that, between the period 1964 – 1965 [Lambert (1973) page 162] linear multistep formulae which incorporated a function evaluation at an off – grid point or step point emerged. Such formulae simultaneously proposed by Gragg and Stetter (1964), Butcher (1965), Gear (1964), Ademiluyi (1987) were classified as "Hybrid".

The hybrid methods share with Runge Kutta methods the property of utilising data at points other than the step points $(x_n = a + nh)$ (lambert (1973))
Generally, a k – step hybrid formula is defined as:

$$\sum_{j=0}^{k} \alpha_j y_{n+1} = \sum_{j=0}^{k} \beta f_{n+1} + \beta_u f_{n+u} \qquad (2.2)$$

where $\alpha_k = 1, u \notin (0,1,2,\ldots\ldots,k)$.

CHAPTER THREE

THE PROPOSED SCHEMES

3.1 THE DERIVATION OF THE PROPOSED METHODS

In this chapter, we derived hybrid collocation methods for solving general third order initial value problems of ordinary differential equations of the form;

$$y''' = f(x, y, y', y''), y(a) = y(0) =, y'(a) = \eta_1, y''(a) = \eta_2 \text{.......................................}(3.1)$$

a, y, f∈R. For a step number k =3 we consider an approximate solution to (3.1) in the form

$$y(x) = \sum_{j=0}^{2k+1} a_j x^j \text{..}(3.2)$$

The first, second and third derivatives of (3.2) are given as follows.

$$y'(x) = \sum_{j=0}^{2k+1} j a_j x^{j-1} \text{............................}(3.3)$$

$$y''(x) = \sum_{j=1}^{2k+1} j(j-1) ajx^{j-2} \text{.......... ....}(3.4)$$

$$y'''(x) = \sum_{j=0}^{2k+1} j(j-1)(j-2) a_j x^{j-3} \text{.......... ....}(3.5a)$$

From (3.1) and (3.5a)

$$\sum_{j=0}^{2k+1} j(j-1)(j-2) a_j x^{j-3} = f(x, y, y', y'') \text{.......... ....}(3.5b)$$

Where a_j's are the parameter to be determined.

Thus, collocating (3.5b) at the grid point $x = x_{n+j}, j = 0(1)3$ and u and interpolate (3.2) at the grid point $x = x_{n+j}, j = 0,1,2. k = ,3$ give the following resulting linear system of equations

$$6a_3 + 24a_4 x_n + 60a_5 x_n^2 + 120a_6 x_n^3 210a_7 x_n^4 = fn$$

$$6a_3 + 24a_4x_{n+1} + 60a_5x^2{}_{n+1} + 120a_6x^3{}_{n+1} + 210a_7x^4{}_{n+1} = f_{n+1}$$

$$6a_3 + 24a_4x_{n+2} + 60a_5x^2{}_{n+2} + 120a_6x^3{}_{n+2} + 210a_7x^4{}_{n+2} = f_{n+2}$$

$$6a_3 + 24a_4x_{n+3} + 60a_5x^2{}_{n+3} + 120a_6x^3{}_{n+3} + 210a_7x^4{}_{n+3} = f_{n+3}$$

$$6a_3 + 24a_4x_{n+u} + 60a_5x^2{}_{n+u} + 120a_6x^3{}_{n+u} + 210a_7x^4{}_{n+u} = f_{n+u}$$

$$a_0 + a_1x_n + a_2x^2{}_n + a_3x^3{}_n + a_4x^4{}_n + a_5x^5{}_n + a_6x^6{}_n + a_7x^7{}_n = y_n$$

$$a_0 + a_1x_{n+1} + a_2x^2{}_{n+1} + a_3x^3{}_{n+1} + a_4x^4{}_{n+1} + a_5x^5{}_{n+1} + a_6x^6{}_{n+1} + a_7x^7{}_{n+1} = y_{n+1}$$

$$a_0 + a_1x_{n+2} + a_2x^2{}_{n+2} + a_3x^3{}_{n+2} + a_4x^4{}_{n+2} + a_5x^5{}_{n+2} + a_6x^6{}_{n+2} + a_7x^7{}_{n+2} = y_{n+2}$$

........................(3.6)

where $x_{n+i=}x_n + ih$

We solve (3.6) for a_j's, j = 0(1)7, by Gaussian Elimination method, we have

$$a_7 = \frac{1}{1260u(u-1)(u-2)(u-3)}\begin{bmatrix} 6f_{n+u} - u(u-1)(u-2)f_{n+3} + 3u(u-1)(u-3)f_{n+2} - 3u(u-2)(u-3)f_{n+1} + \\ (u-)(u-2)(u-3)f_n \end{bmatrix}$$

$$a_6 = \frac{1}{720h^3}\left[f_{n+3} - 3f_{n+2} - f_n - 1260\ a_7h^3(4x_n + 6h)\right]$$

$$a_5 = \frac{1}{120h^2}\left[f_{n+2-}2f_{n+1} + f_n - 240a_6(3x_n + 3h) - 420a_7\left(6x_n{}^2 + 12hx_n + 7h^2\right)\right]$$

$$a_4 = \frac{1}{24h}\left[f_{n+1-}f_{n-}60_{a_5}(2x_{n+h}) - 120a_6\left(3x_n{}^2 + 3hx_n + h^2\right) - 210a_7\left(4x_n{}^3 + 6hx_n{}^2 + 4h^2x_n + h^3\right)\right]$$

$$a_3 = \tfrac{1}{6}\left[f_n - 24a_4x_n - 60a_5x_n^2 - 120a_6x_n^3 - 210a_7x_n^4\right]$$

$$a_2 = \left(1/2h^2\right)\begin{bmatrix} y_{n+2} - 2y_{n+1} + y_{n-a_5}(3x_n + 3h) - a_4\left(6x_n^2 + 12hx_n + 7h^2\right) - a_5\left(10x_n^3 + 30hx_n^2 + 35h^2x_n + 15h^3\right) - \\ a_6\left(15x_n^4 + 60hx_n^3 + 105h^2x_n^2 + 90h^3x_n + 31h^4\right) - \\ a_7\left(21x_n^5 + 105hx_n^4 + 245h^2x_n^3 + 315h^3x_n^2 + 217h^4x_n + 63h^5\right) \end{bmatrix}$$

$$a_1 = y_{n+1} - y_n - a_2(2x_n + h) - a_3(3x_n^2 + 3hx_n + h^2) - a_4(4x_n^3 + 6hx_n^2 + 4h^2xn + h^3) -$$
$$a_5(5x_n^4 + 10hx_n^3 + 10h^2x_n^2 + 5h^3x_n + h^4) - a_6(6x_n^5 + 15hx_n^4 + 20h^2x_n^3 + 15h^3x_n^2 + 6h^4x_n + h^5) -$$
$$a_7(7x_n^6 + 21hx_n^5 + 35h^2x_n^4 + 35h^3x_n^3 + 21h^4x_n^2 + 7h^5x_n + h^6)$$

$$a_0 = y_n - a_1x_n - a_2x_n^2 - a_3x_n^3 - a_4x_n^4 - a_5x_n^5 - a_6x_n^6 - a_7x^7 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.7)$$

Substituting the values of aj's in the approximate solution (3.2), we obtain

$$y_k(x) = y_n + a_1(x - x_n) + a_2(x^2 - x_n^2) + a_3(x^3 - x_n^3) + a_4(x^4 - x_n^4) + a_5(x^5 - x_n^5) + a_6(x^6 - x_n^6) +$$
$$a_7(x^7 - x_n^7) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.8)$$

Evaluate (3.8) at $x = x_{n+3}$ where, $x_{n+3} = x_n + 3h$. After some algebraic manipulation,

we have a scheme in the form

$$y(x) = \sum_{j=0}^{k-1} \alpha_j(x) y_{n+j} + \sum_{j=0}^{k} \beta_j(x) f_{n+j} + \beta_u(x) f_{n+j} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.9)$$

Where $f_{n+j} = f(x_{n+j}, y_{n+j} y'_{n+j}, y''_{n+j}), \alpha_j(x)$ and $\beta_j(x)$ are function of x

We now let $t = (x - x_{n+2})/h$ ………………………… (3.10)

From (3.10), $\frac{dt}{dx} = \frac{1}{h}$ ………………………………….(3.11)

The following continuous coefficients in (3.9) are obtained as a function of t as follows:

$$\alpha_0(t) = \tfrac{1}{2}(t^2 + t)$$

$$\alpha_1(t) = -(t^2 + 2t)$$

$$\alpha_2(t) = \tfrac{1}{2}(t^2 + 3t + 2)$$

$$\beta_0(t) = \tfrac{h^3}{5040u}\left[4t^7 + (14 - 7u)t^6 - 14t^5 + (-70 + 35u)t^4 + (98 - 28u)t^2 + 52t\right]$$

$$\beta_1(t) = \tfrac{h^3}{5040u(u-1)}\left[-12t^7 + (21u - 63)t^6 + 42ut^5 + (-210u + 420)t^4 + (1449u - 1743)t^2 + (1218u - 1374)t\right]$$

$$\beta_2(t) = \tfrac{h^3}{5040(u-2)}\begin{bmatrix} 12t^7 + (-21u - 84)t^6 + (-84u + 126)t^5 + (105u + 420)t^4 + (840u - 1680)t^3 + \\ (1176u - 2058)t^2 + (504u - 852)t \end{bmatrix}$$

$$\beta_3(t)=\frac{h^3}{5040(u-3)}\begin{bmatrix}-4t^7+(7u-35)t^6+(42u+112)t^5+(105u+420)t^4+(840u-1680)t^3+\\(1176u-2058)t^2+(504u-852)t\end{bmatrix}$$

$$\beta_u(t)=\frac{h^3}{5040u(u-1)(u-2)(u-3)}\left[4t^7+14t^6-14t^5-70t^4+98t^2+52t\right] \qquad \text{..................(3.12)}$$

The first and second derivatives of (3.12) in terms of t are given below respectively

$$\alpha'_0(t)=\frac{1}{2h}(2t+1)$$

$$\alpha'_1(t)=\frac{-1}{h}(2t+2)$$

$$\alpha'_2(t)=\frac{1}{2h}(2t+3)$$

$$\beta'_0(t)=\frac{h^2}{2520u}\left[14t^6+(42-21u)t^5-35t^4+(-140+70u)t^3+(98-28u)t+26\right]$$

$$\beta'_1(t)=\frac{h^2}{2520(u-1)}\begin{bmatrix}-42t^6+(63u-189u)t^5+105ut^4+(-420u+840)t^3+(1449u-1743)t+\\(609u-687)\end{bmatrix}$$

$$\beta'_2(t)=\frac{h^2}{2520(u-2)}\begin{bmatrix}42t^6+(-63u+252)t^5+(-210u+315)t^4+(210u-840)t^3+\\(1260u-2520)t^2+(1176u-2058)t+(252u-426)\end{bmatrix}$$

$$\beta'_3(t)=\frac{h^2}{2520(u-3)}\begin{bmatrix}-14t^6+(21u-105u)t^5+(105u-280)t^4+(140u-280)t^3+\\(-77u+133)t+(-21u+37)\end{bmatrix}$$

$$\beta'_u(t)=\frac{h^2}{2520u(u-1)(u-2)(u-3)}\left[14t^6+42t^5-35t^4-140t^3+98t+26\right] \qquad \text{...............(3.13)}$$

$$\alpha_0''(t) = \frac{1}{2h^2}[2] = \frac{1}{h^2}$$

$$\alpha_1''(t) = \frac{-1}{h^2}[2] = \frac{-2}{h^2}$$

$$\alpha_{(2)}''(t) = \frac{1}{2h^2}[2] = \frac{1}{h^2}$$

$$\beta''_0(t) = \frac{h}{2520u}\left[84t^5 + (210 - 105u)t^4 - 140t^3 + (-420 + 210u)t^2 + (98 - 28u)\right]$$

$$\beta''_1(t) = \frac{h}{2520(u-1)}\begin{bmatrix} -252t^5 + (315u - 945)t^4 + 420ut^3 + (-1260u + 2520)t^2 \\ + (1449u - 1743) \end{bmatrix}$$

$$\beta''_2(t) = \frac{h}{2520(u-2)}\begin{bmatrix} 252t^5 + (-315u + 1260)t^4 + (840u + 1260)t^3 + (630u - 2520)t^2 \\ + (2520u - 5040)t + (1176u - 2058) \end{bmatrix}$$

$$\beta''_3(t) = \frac{h}{2520(u-3)}\begin{bmatrix} -84t^5 + (105u - 525)t^4 + (420u - 1120)t^3 + (420u - 1120)t^2 \\ + (-77u + 133) \end{bmatrix}$$

$$\beta''_u(t) = \frac{h}{2520u(u-1)(u-2)(u-3)}\begin{bmatrix} 84t^5 + 210t^4 - 140t^3 - 420t^2 + 98 \\ \end{bmatrix} \quad \ldots\ldots\ldots\ldots(3.14)$$

The discrete schemes and their first and second derivatives arising from (3.12), (3.13) and (3.14) respectively, at $t = 1 \Rightarrow x = x_{n+3}$ are given as follows:

$$y_{n+3} - 3y_{n+2} + 3y_{n+1} - y_n = \frac{h^3}{60u(u-1)(u-2)(u-3)}\begin{bmatrix} -u(u-1)(u-2)f_{n+3} + 3u(u-1)(u-3)(10u-19) \\ f_{n+2} + 3u(u-2)(u-3)(10u-11)f_{n+1} + (u-1) \\ (u-2)(u-3)f_n + 6f_{n+u} \end{bmatrix}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3.15)$$

$$y'_{n+3} = \frac{1}{2}h[5y_{n+2} - 8y_{n+1} + 3y_n] = \frac{h^2}{2520u(u-1)(u-2)(u-3)}\begin{bmatrix} u(u-1)(u-2)(168u-509)f_{n+3} + 15u(u-1)(u-3) \\ (175u-349)f_{n+2} + 3u(u-2)(u-3)(602u-607)f_{n+1} + \\ (u-1)(u-2)(u-3)(21u+5)f_n + 30f_{n+u} \end{bmatrix}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.16)$$

$$y''_{n+3} = \frac{1}{2}h\left[y_{n+2} - 2y_{n+1} + y_n\right] + \frac{h}{360u(u-1)(u-2)(u-3)}\begin{bmatrix} 4u(u-1)(u-2)(u-3)(31u-87)f_{n+3} + 3u(u-1)(u-3)(151u-326)f_{n+2} + \\ 12u(u-2)(u-3)(11u-5)f_{n+1} + (u-1)(u-2)(u-3)(11u-24)f_n - 144f_{n+u} \end{bmatrix}$$

....(3.17)

Now, we put u = $^3/_2$, the scheme and its first and second derivatives become;

$$y_{n+3} - 3y_{n+2} + 3y_{n+1} - y_n = \frac{h^3}{90}\left[f_{n+3} + 36f_{n+2} + 36f_{n+1} + f_n + 16f_{n+\frac{3}{2}}\right] \quad \text{......................(3.18)}$$

Order p = 7, $C_{p+2} = \frac{-57}{362880}$

$$y'_{n+3} = \frac{h}{2h}\left[5y_{n+2} - 8y_{n+1} + 3y_n\right] + \frac{h}{7560}^2\left[514f_{n+3} + 7785f_{n+2} + 160f_{n+\frac{3}{2}} + 5328f_n\right]$$

.... (3.19)

Order p = 6, $C_{p+2} = \frac{-243}{90720}$

$$y''_{n+3} = \left(\frac{1}{h^2}\right)\left[y_{n+2} - 2y_{n+1} + y_n\right] + \frac{h}{360}\left[108f_{n+3} + 597f_{n+2} - 256f_{n+\frac{3}{2}} + 276f_{n+1} - 5f_n\right]$$

..............................(3.20)

Order p = 6, $c_{p+2} = \frac{-294}{60480}$

U = $^5/_2$, we have,

$$y_{n+3} - 3y_{n+2} + 3y_{n+1} - y_n = \frac{h^3}{150}\left[5f_{n+3} - 16f_{n+\frac{5}{2}} + 90f_{n+2} + 70f_{n+1} + f_n\right]$$

...............(3.21)

Order p = 6, $c_{p+2} = \frac{-1}{1200}$

$$y'_{n+3} = \frac{1}{2h}\left(5y_{n+2} - 8y_{n+1} + 3y_n\right) + \frac{h^2}{2520}\left[178f_{n+3} - 32f_{n+\frac{5}{2}} + 2655f_{n+2} + 1796f_{n+1} + 23f_n\right]$$

...................(3.22)

Order p = 6, $Cp_{+2} = \frac{-522}{181440}$

$$y''_{n+3} = \frac{1}{h}\left[y_{n+2} - 2y_{n+1} + y_n\right] + \frac{h}{1800}\left[380f_{n+3} + 768f_{n+\frac{5}{2}} + 1545f_{n+2} + 900f_{n+1} + 7f_n\right]$$

..................(3.23)

Order p = 6, $c_{p+2} = \frac{-2385}{1512000}$

For u = ½, we have,

$$y_{n+3} - 3y_{n+2} + 3y_{n+1} - y_n = \tfrac{h^3}{150}\left[f_{n+3} + 70f_{n+2} + 90f_{n+1} - 16f_{n+\frac{1}{2}} + 5f_n\right] \qquad (3.24)$$

Order p = 6, $c_{p+2} = \frac{1}{1200}$

$$y'_{n+3} = \tfrac{1}{2h}\left[5y_{n+2} - 8y_{n+1} + 3y_n\right] + \tfrac{h^2}{2520}\left[170f_{n+3} + 2615f_{n+2} + 1836f_{n+1} - 32f_{n+\frac{1}{2}} + 31f_n\right] \qquad (3.25)$$

Order p = 6, $c_{p+2} = \frac{-45}{18144}$

$$y''_{n+3} = \tfrac{1}{h^2}\left[y_{n+2} - 2y_{n+1} + y_n\right] + \tfrac{h}{1800}\left[572f_{n+3} + 2505f_{n+2} - 60f_{n+1} + 768f_{n+\frac{1}{2}} - 185f_n\right] \qquad (3.26)$$

Order p = 6, $c_{p+2} = \frac{-12465}{1512000}$

3.2 DETERMINATION OF PREDICTOR FOR y_{n+3}

The same procedure described for the development of the main schemes is applicable here, but in this case, we consider a basis function in the form.

$$y(x) = \sum_{j=0}^{2k} a_j x^j \qquad (3.27)$$

to be an approximate solution to problem (3.1)

The first, second and third derivatives of (3.27) are as follows respectively;

$$y'(x) = \sum_{j=0}^{2k} j a_j x^{j-1} \qquad (3.27a)$$

$$y''(x) = \sum_{j=0}^{2k} j(j-1) a_j x^{j-2} \qquad (3.27b)$$

$$y'''(x) = \sum_{j=0}^{2k} j(j-1)(j-2) a_j x^{j-3} \qquad (3.27c)$$

where a_j's are the parameters to be determined.

We collocate (3.27c) at the grid point $x = x_{n+j},\ j = 0,1,2$ and u. that is,

$$y'''(x) = \sum_{j=0}^{2k} j(j-1)(j-2)a_j x^{j-3} = f_{n+j} \quad \text{.......................} (3.28)$$

$j = 0,1,2$ and u.

while in (3.27) interpolation is taken at $x = x_{n+j}$ $j = 0(1)2$ that is,

$$y\ (x) = \sum_{j=0}^{2k} a_j x^j = y_{n+j}; j = 0(1)2 \quad \text{..............................} (3.29)$$

solving (3.28) and (3.29) for aj's by Gaussian elimination method and after some algebraic manipulation, we obtain our predictor in the form.

$$y(x) = \sum_{j=0}^{k-1} \alpha_j(x) y_{n+j} + \sum_{j=0}^{k-1} \beta_j(x) f_{n+j} + \beta_u(x) f_{n+j} \quad \text{........................} (3.30)$$

from (3.10), we have that $t = (x - x_{n+2})/h$

The following continous coeficients in (3.30) are obtained as a function of t as follows.

$$\alpha_0(t) = \tfrac{1}{2}(t + t^2)$$
$$\alpha_1(t) = -(2t + t^2)$$
$$\alpha_2(t) = \tfrac{1}{2}(3t + t^2 + 2)$$
$$\beta_0(t) = \tfrac{h^3}{240u}\left[-t^6 - (2u-6)t^5 + (5u-10)t^4 + (5u+11)t^2 + (-2u+6)t\right]$$
$$\beta_1(t) = \tfrac{h^3}{240(u-1)}\left[2t^6 + (-4u+16)t^5 + (-20u+40)t^4 + (80u-102)t^2 + (64u-76)t\right]$$

$$\beta_2(t) = \tfrac{h^3}{240(u-2)}\left[-t^6 + (2u-10)t^5 + (15u-40)t^4 + (40u-80)t^3 + (45u-79)t^2 + (18u-30)t\right]$$

$$\beta_u(t) = \tfrac{h^3}{5040u(u-1)(u-2)(u-3)}\left[2t^6 + 12t^5 + 20t^4 - 22t^2 + 12t\right]$$

$$\text{..................................} (3.31)$$

The first and the second derivatives of (3.31) in terms of t are given below respectively;

$$\alpha_0^1(t) = \tfrac{1}{2h}(1 + 2t)$$

$$\alpha_1^1(t) = \tfrac{-1}{h}(2 + 2t)$$
$$\alpha_2^1(t) = \tfrac{1}{2h}(3 + 2t)$$

$\beta_0^{\prime}(t)=\frac{h^2}{120u}\left[-3t^5+(5u-15)t^4+(10u-20)t^3-(-5u+11)t+(-u+3)\right]$
$\beta_1^{\prime}(t)=\frac{h^2}{120(u-1)}\left[6t^5+(-10u+40)t^4+(-40u+80)t^3+(80u-102)t \;+(32u-38)\right]$

$\beta_2^{\prime}(t)=\frac{h^2}{120(u-2)}\left[-3t^5+(5u-25)t^4+(30u-80)t^3+(60u-102)t^2+(45u-79)t+(9u-15)\right]$
$\beta_u^{\prime}(t)=\frac{h^2}{120u(u-1)(u-2)}\left[6t^5+30t^4-22t \;+6\right]$

..................................(3.32)

$\alpha_0^{\prime\prime}(t)=\frac{1}{2h^2}[2]=\frac{1}{h^2}$
$\alpha_1^{\prime\prime}(t)=\frac{-2}{h^2}$
$\alpha_2^{\prime\prime}(t)=\frac{1}{h^2}$
$\beta_0^{\prime\prime}(t)=\frac{h}{120u}\left[-15t^4+(20u-60)t^3-(30u-60)t^2+(-5u+11)\right]$

$\beta_1^{\prime\prime}(t)=\frac{h}{120(u-1)}\left[30t^4+(-40u+160)t^3+(-120u+240)t^2+(80u-102)\right]$
$\beta_2^{\prime\prime}(t)=\frac{h}{120(u-2)}\left[-15t^4+(20u-100)t^3+(90u-240)t^2+(120u-240)t+(45u-79)\right]$

$\beta_u^{\prime\prime}(t)=\frac{h^2}{120u(u-1)(u-2)}\left[30t^4+120t^3+120t^2-22\right]$

....(3.33)

The discrete explicit scheme and its first and second derivatives arising from (3.31), (3.32) and (3.330 respectively at $t=1 \Rightarrow x=x_{n+3}$ are given as follows

$y_{n+3}-3y_{n+2}+3y_{n+1}-y_n=\frac{h^3}{2}\left[f_{n+2}+f_{n+1}\right]$.......................... .....(3.34)

order p = 5, $c_{p+2}={}^{1}/_{240}$, with interval of periodicity x(0) = (0,∞). Thus the schemes is p-stable (Fatunla 1988 and Awoyemi; 2003).

$y_{n+3}^{\prime}={}^{1}/_{2h}\left[5y_{n+2}-8y_{n+1}+3y_n\right]+\frac{h^2}{120u(u-1)(u-2)}\begin{bmatrix}u(u-1)(149u-322)f_{n+2}+u(u-2)(62u-14)f_{n+1}- \\ (u-1)(u-2)(9u-24)f_n+48f_{n+u}\end{bmatrix}$...(3.35)

$y_{n+3}^{\prime\prime}={}^{1}/_{h}\left[y_{n+2}-2y_{n+1}+y_n\right]+\frac{h}{120u(u-1)(u-2)}\begin{bmatrix}u(u-1)(275u-674)f_{n+2}+u(u-2)(80u-328)f_{n+1}+ \\ (u-1)(u-2)(45u-124)f_n+248f_{n+u}\end{bmatrix}$....(3.36)

put u=${}^{1}/_{2}$, the first and the second derivatives of the predictor become.

$$y'_{n+3} = \frac{1}{2h}\left[5y_{n+2} - 8y_{n+1} + 3y_n\right] + \frac{h^2}{120}\left[165f_{n+2} - 34f_{n+1} + 128f_{n+\frac{1}{2}} - 39f_n\right] \ldots\ldots\ldots (3.37)$$

Order p = 5, $c_{p+2} = \frac{85}{1008}$

$$y''_{n+3} = \frac{1}{2h}\left[5y_{n+2} - 8y_{n+1} + 3y_n\right] + \frac{h^2}{120}\left[197f_{n+2} - 128f_{n+\frac{1}{2}} + 158f_{n+1} - 7f_n\right] \ldots (3.38)$$

order p = 5, $c_{p+2} = \frac{1001}{5040}$ *0* for $u = \frac{3}{2}$, we have

$$y'_{n+3} = \frac{1}{2h}\left[5y_{n+2} - 8y_{n+1} + 3y_n\right] + \frac{h^2}{120}\left[197f_{n+2} - 128f_{n+\frac{1}{2}} + 158f_{n+1} - 7f_n\right] \ldots\ldots (3.39)$$

ordrer $p = 5$, $c_{p+2} = \frac{257}{5040}$

$$y''_{n+3} = \frac{1}{h^2}\left[y_{n+2} - 2y_{n+1} + y_n\right] + \frac{h}{360}\left[1569f_{n+2} - 1984f_{n+\frac{1}{2}} + 1248f_{n+1} - 133f_n\right] \ldots\ldots (3.40)$$

order $p = 5$, $c_{p+2} = \frac{567}{5040}$

for $u = \frac{5}{2}$, we have

$$y'_{n+3} = \frac{1}{2h}\left[5y_{n+2} - 8y_{n+1} + 3y_n\right] + \frac{h^2}{600}\left[128f_{n+\frac{1}{2}} + 505f_{n+2} + 470f_{n+1} - 3f_n\right] \ldots\ldots\ldots (3.41)$$

order $p = 5$, $c_{p+2} = \frac{89}{5040}$

$$y''_{n+3} = \frac{1}{h^2}\left[y_{n+2} - 2y_{n+1} + y_n\right] + \frac{h}{1800}\left[1984f_{n+\frac{1}{2}} + 405f_{n+2} + 1280f_{n+1} - 69f_n\right] \ldots (3.42)$$

order $p = 5$, $c_{p+2} = \frac{133}{5040}$

We use Taylor series expansion to calculate the value of y_{n+1}, y_{n+2}, y_{n+u} and their first and second order derivatives at $x = x_n$, in(3.9) and (3.30) respectively as follows:

$$y_{n+1} = y(x_{n+h}) = y_{(xn)} + hy^1_{(xn)} + \frac{h^2 y^{11}_{(xn)}}{2!} + \frac{h^3 f_n}{3!} + \frac{h^4 f_n}{4!} + \frac{h^5 f''_n}{5!} + \ldots\ldots\ldots$$

$$y^1_{n+1} = y^1_{(xn)} + hy^{11}_{(xn)} + \frac{h^2 f_n}{2!} + \frac{h^3 f_n}{3!} + \frac{h^4 f''_n}{4!} + \ldots\ldots\ldots$$

$$y^{11}_{n+1} = y^{11}_{(xn)} + hf_n + \frac{h^2 f_n}{2!} + \frac{h^3 f''_n}{3!} + \ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.43)$$

$$y_{n+2} = y(x_{n+2h}) = y_{(xn)} + 2hy^{1}_{(xn)} + \frac{(2h)^2 y^{11}_{(xn)}}{2!} + \frac{(2h)^3 f_n}{3!} + \frac{(2h)^4 f'_n}{4!} + \frac{(2h)^5 f''_n}{5!} + \ldots\ldots\ldots$$

$$y^{1}_{n+2} = y^{1}_{(xn)} + 2hy^{11}_{(xn)} + \frac{(2h)^2 f_n}{2!} + \frac{(2h)^3 f'_n}{3!} + \frac{(2h)^4 f''_n}{4!} + \ldots\ldots\ldots$$

$$y^{11}_{n+2} = y^{11}_{(xn)} + 2hf_n + \frac{(2h)^2 f'_n}{2!} + \frac{(2h)^3 f''_n}{3!} + \ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots(3.44)$$

$$y_{n+u} = y(x_{xn+uh}) = y_{(xn)} + uhy^{1}_{(xn)} + \frac{(uh)^2 y^{11}_{(xn)}}{2!} + \frac{(uh)^3 f_n}{3!} + \frac{(uh)^4 f'_n}{4!} + \frac{(uh)^5 f''_n}{5!} + \ldots\ldots\ldots$$

$$y^{1}_{n+u} = y^{1}_{(xn)} + uhy^{11}_{(xn)} + \frac{(uh)^2 f_n}{2!} + \frac{(uh)^3 f'_n}{3!} + \frac{(uh)^4 f''_n}{4!} + \ldots\ldots\ldots$$

$$y^{11}_{n+u} = y^{11}_{(xn)} + uhf_n + \frac{(2h)^2 f'_n}{2!} + \frac{(2h)^3 f''_n}{3!} + \ldots\ldots\ldots$$

Where $f_n = f(x_n, y_n, y^1_n, y^{11}_n), f^i_n = f^i(x_n, y_n, y^1_n, y^{11}_n)$, i = 1, 2

$$\ldots\ldots\ldots(3.45)$$

for convience, we put

$$f_{n+j} = f(x_{n+j}, y_{n+j}, y^1_{n+j}, y^{11}_{n+j}), \ j = 1,0,2,3 \ldots\ldots\ldots\ldots(3.46)$$

in the form.

$f = f\ (x\ , y\ , y^1, y^{11})$ and find f^1 and f^{11} by partial derivative technique as follows.

$$f' = \frac{df}{dx} = \frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} + y''\frac{\partial f}{\partial y} + y''\frac{\partial f}{\partial y'} + f\frac{\partial f}{\partial y''}, y^j = \frac{\partial^j y}{\partial x^j} \quad j = 1,2,\ldots\ldots\ldots\ldots(3.47)$$

$$f'' = \frac{d^2 f}{d\,x^2} = 2(Ay' + By'' + Cf) + D + E \ldots\ldots\ldots(3.48)$$

where ;

$$A = \frac{\partial^2 f}{\partial x \partial y} + y''\frac{\partial^2 f}{\partial x \partial y'} + f\frac{\partial^2 f}{\partial x \partial y''}$$

$$B = \frac{\partial f}{\partial x \partial y} + f \frac{\partial^2 f}{\partial y' \partial y''}$$

$$C = \frac{\partial^2 f}{\partial x \partial y''}$$

$$D = y'' \frac{\partial f}{\partial y} + f \frac{\partial f}{\partial y'} + f' \frac{\partial f}{\partial y''}$$

$$E = \frac{\partial^2 f}{\partial x^2} + y'^2 \frac{\partial^2 f}{\partial y^2} + y''^2 \frac{\partial^2 f}{\partial y'^2} + f^2 \frac{\partial^2 f}{\partial y''^2}$$

The next chapter gives the analysis of the proposed schemes and numerical computations and results.

CHAPTER FOUR

4.0 THE ANALYSIS OF THE PROPOSED SCHEMES

The following properties of our methods (3.18),(3.21) and (3.24) are examined.

4.1 ORDER AND ERROR CONSTANT.

The order and error constant of our methods may be found by defining a linear operator L associated with (3.18), (3.21) and (3.24) as

$$l[y_{(x)},h]=\sum_{j=0}^{k}\left[\alpha_j y(x_{n+j})-h^3\beta_j y(x_{n+j})\right] \quad (4.1)$$

k = 3,

$\alpha_3 = 1$, α_0 and β_0 are not both zero, $y_{(x)}$ is an arbituary function which continously differentiable on the interval [a,b], $y(x_{n+j})=y(x_n+jh)$.

If $y_{(x)}$ represents the true solution of (2.1) and we adopt Taylor series expansion of $y(x_{n+j})$ and $y^{111}(x_{n+j})$, $j=0(1)k$ about $x=x_n$, we have ;

$$y(x_{n+k})=\sum_{r=0}^{\infty}\frac{(kh)^r y^r(x_n)}{r!} \quad (4.2)$$

$$y^{111}(x_{n+k})=\sum_{r=3}^{\infty}\frac{(kh)^{r-3} y^r(x_n)}{(r-3)!} \quad (4.3)$$

Ademiluyi and kayode (2001).

using (4.2) and (4.3) in (3.18) for k=0(1)3, and collecting like terms, we obtain

$$l[y(x),h]=c_0y(x_n)+c_1hy^1(x_n)+c_2{}^2y^2(x_n)+.....c_qh^qy^q(x_n)+c_{q+1}h^{q+1}y^{q+1}(x_n)+c_{q+2}h^{q+2}y^{q+2}(x_n)$$
$$+c_{q+3}h^{q+3}y^{q+3}(x_n)+0(h^{q+4}) \quad (4.4)$$

Where

$c_0 = 1 - 3 + 3 - 1 = 0$

$c_1 = 3 - 6 + 3 = 0$

$c_2 = \frac{9}{2} - \frac{12}{2} + \frac{3}{2} = 0$

$c_3 = \frac{27}{6} - \frac{24}{6} + \frac{3}{6} - \frac{1}{90} - \frac{36}{90} - \frac{36}{90} - \frac{1}{90} - \frac{16}{90} = 0$

$c_4 = \frac{81}{24} - \frac{48}{24} + \frac{3}{24} - \frac{3}{90} - \frac{72}{90} - \frac{36}{90} - \frac{24}{90} = 0$

$c_5 = \frac{243}{120} - \frac{96}{120} + \frac{3}{120} - \frac{-9}{180} - \frac{144}{180} - \frac{36}{180} - \frac{36}{180} = 0$

$c_6 = \frac{729}{720} - \frac{192}{720} + \frac{3}{720} - \frac{27}{540} - \frac{288}{540} - \frac{36}{540} - \frac{54}{540} = 0$

$c_7 = \frac{2187}{5040} - \frac{3844}{5040} + \frac{3}{5040} - \frac{81}{2160} - \frac{576}{2160} - \frac{36}{2160} - \frac{81}{2160} = 0$

$c_8 = \frac{6561}{40320} - \frac{768}{40320} + \frac{3}{40320} - \frac{243}{10800} - \frac{1152}{10800} - \frac{36}{10800} - \frac{243}{21600} = 0$

$c_9 = \frac{19683}{362880} - \frac{1536}{362880} + \frac{3}{362880} - \frac{729}{64800} - \frac{2304}{64800} - \frac{729}{259200} \neq 0$

$= \frac{-57}{362880}$

$= -0.00016$

From the result above

$c_0 = c_1 = c_2 = ... = c_8 = 0, c_9 = c_{p+2} \neq 0$ It implies that the scheme (3.18) is of order

$p = 7$ and the error

Constant $c_{p+2} = \frac{-57}{362880}$

[Lambert {1973,1991}, Fatunla {1988}]

For (3.19-3.26) and (3.37-3.42) these are summarized in the tables below.

Table 4.1

SCHEMES	Value of U	Order P	Error Constant
$y_{n+3} - 3y_{n+2} + 3y_{n+1} - y_n = \frac{h^3}{90}\left[f_{n+3} + 36f_{n+2} + 16f_{n+\frac{3}{2}} + 36f_{n+1} + f_n\right]$	$^3/_2$	7	$\frac{-57}{362880}$
$y^{1}_{n+3} = \frac{1}{2h}\left[5y_{n+2} - 8y_{n+1} + 3y_n\right] + \frac{h^2}{7560}\left[514f_{n+3} + 7785f_{n+2} + 160f_{n+\frac{3}{2}} + 5328f_{n+2} + 73f_n\right]$	$^3/_2$	6	$\frac{-243}{90720}$
$y^{11}_{n+3} = \frac{1}{h^2}\left[y_{n+2} - 2y_{n+1} + y_n\right] + \frac{h}{360}\left[108f_{n+3} + 597f_{n+2} - 256f_{n+\frac{3}{2}} + 276f_{n+2} - 5f_n\right]$	$^3/_2$	6	$\frac{-294}{60480}$
$y_{n+3} - 3y_{n+2} + 3y_{n+1} - y_n = \frac{h^3}{150}\left[5f_{n+2} - 16f_{n+\frac{5}{2}} + 90f_{n+1} + 70f_{n+1} + f_n\right]$	$^5/_2$	6	$\frac{-1}{1200}$
$y^{1}_{n+3} = \frac{1}{2h}\left[5y_{n+2} - 8y_{n+1} + 3y_n\right] + \frac{h^2}{2520}\left[178f_{n+3} - 32f_{n+\frac{5}{2}} + 2655f_{n+2} + 1796f_{n+1} + 23f_n\right]$	$^5/_2$	6	$\frac{-522}{181440}$
$y^{11}_{n+3} = \frac{1}{h^2}\left[y_{n+2} - 2y_{n+1} + y_n\right] + \frac{h}{360}\left[380f_{n+3} + 768f_{n+\frac{5}{2}} + 1545f_{n+2} + 900f_{n+1} + 7f_n\right]$	$^5/_2$	6	$\frac{-2385}{1512000}$
$y_{n+3} - 3y_{n+2} + 3y_{n+1} - 3y_{n+1} - y_n = \frac{h^3}{150}\left[f_{n+3} + 70f_{n+2} + 90f_{n+1} - 16f_{n+\frac{1}{2}} + 5f_n\right]$	$^1/_2$	6	$\frac{1}{1200}$
$y^{1}_{n+3} = \frac{1}{2h}\left[5y_{n+2} - 8y_{n+1} + 3y_n\right] + \frac{h^2}{2520}\left[170f_{n+3} + 2615f_{n+2} + 1836f_{n+\frac{1}{2}} + 31f_n\right]$	$^1/_2$	6	$\frac{-45}{18144}$
$y^{11}_{n+3} = \frac{1}{h^2}\left[y_{n+2} - 2y_{n+1} + y_n\right] + \frac{h}{1800}\left[572f_{n+3} + 2505f_{n+2} - 60f_{n+1} + 768f_{n+\frac{1}{2}} - 185f_n\right]$	½	6	$\frac{-12465}{1512000}$

THE PREDICTOR AND ITS FIRST AND SECOND DERIVATIVES AT DIFFERENT VALUE OF U

Table 4.2

SCHEMES	Value of u	Order P	Error Constant
$y_{n+3}-3y_{n+2}+3y_{n+1}-y_n=\frac{h^3}{2}\left[f_{n+2}+f_{n+1}\right]$	-	5	$\frac{1}{240}$
$y'_{n+3}=\frac{1}{2h^2}\left[5y_{n+2}-8y_{n+1}+3y_n\right]+\frac{h^2}{120}\left[197f_{n+2}-128f_{n+\frac{1}{2}}+158f_{n+1}-7f_n\right]$	$^3/_2$	5	$\frac{257}{5040}$
$y''_{n+3}=\frac{1}{h^2}\left[5y_{n+2}-2y_{n+1}+y_n\right]+\frac{h}{360}\left[1569f_{n+2}-1984f_{n+\frac{3}{2}}+1248f_{n+1}-113f_n\right]$	$^5/_2$	5	$\frac{567}{5040}$
$y'_{n+3}=\frac{1}{2h}\left[5y_{n+2}-8y_{n+1}+3y_n\right]+\frac{h^2}{600}\left[128f_{n+\frac{1}{2}}+505f_{n+2}+470f_{n+1}-3f_n\right]$	$^3/_2$	5	$\frac{89}{5040}$
$y''_{n+3}=\frac{1}{h^2}\left[y_{n+2}-2y_{n+1}+y_n\right]+\frac{h}{1800}\left[1984f_{n+\frac{1}{2}}+405f_{n+2}+1280f_{n+1}-69f_n\right]$	$^3/_2$	5	$\frac{133}{5040}$
$y'_{n+3}=\frac{1}{2h}\left[5y_{n+2}-8y_{n+1}+3y_n\right]+\frac{h^2}{120}\left[165f_{n+3}-34f_{n+1}+128f_{n+\frac{1}{2}}-39f_n\right]$	$^1/_2$	5	$\frac{85}{1008}$
$y''_{n+3}=\frac{1}{h^2}\left[y_{n+2}-2y_{n+1}+y_n\right]+\frac{h}{360}\left[107f_{n+2}-1728f_{n+1}+1984f_{n+\frac{1}{2}}-609f_n\right]$	$^1/_2$	5	$\frac{1001}{5040}$

4.2. ZERO STABILITY OF THE SCHEMES

The scheme (3.18), (3.21) and (3.24 will be said to be zero stable, if no root of the first characteristic polynimial $p(r)$of either (3.18), or (3.21) or (3.24) has modulus greater than one with multiplicity not greater than three.

We obtain the first characteristic polynomial of either (3.18) or (3.24) as

$p(r)=r^3-3r^2+3r-1=(r-1)^3$ in which the roots are obtained as r = 1,1,1. Since, the roots satisfied the condition for zero stability stated above, the schemes (3.18), (3.21) and (3.24) are zero stable. Also, the predictor is zero stable.

4.3. **CONSISTENCY PROPERTY.**

The scheme (3.18), (3.21) and (3.24) are consistent, if they satisfy the conditions;

(i) order $p \geq 1$

(ii) $\sum_{j=0}^{k} \alpha_j = 0$

(iii) $p(r) = p'(r) = 0 and p'''(r) = 3!\sigma(1), for r =$ 4.5

where p and σ are first and second characteristics polynomials repectively

We observed from the analysis of the proposed schemes, that is (3.18), (3.21) and (3.24) have order $p = 7,6,6$ respectively. It means that, the first condition for consistencey is satisfied. Also, we found that the

$\sum_{j=0}^{k} \alpha_j = 0$ in each case.

Again,we consider the third condition for consistency of the schemes. The first characteristics polynamial of (3.18), (3.21) and (3.24) is the same as defined earlier and its first and third derivatives respectively as

i. $p(r) = r^3 - 3r^2 + 3r - 1,$

ii. $p^{1}(r) = 3r^2 - 6r + 3^3$

iii. $p^{111}(r) = 6$ 4.6

Also, we define the second characteristics polynomial of (3.18), (3.21) and (3.24) respectively as

i. $\sigma(r) = \frac{1}{90}\left(r^3 + 36r^2 + 16r^{3/2} + 36r + 1\right)$

ii. $\sigma(r) = \frac{1}{150}\left(5r^3 - 16r^{3/2} + 90r^2 + 70r + 1\right)$

iii. $\sigma(r) = \frac{1}{150}\left(r^3 + 70r^2 + 90r - 16r^{1/2} + 5\right)$...(4.7)

We evaluate (4.6) and (4.7) at r = 1, that is,

$p(1) = 0, p^{1}(1) = 0, p^{111}(1) = 6$ and $\sigma(1) = 1$.

The evaluation shown that $P(1) = P^{1}(1) = 0$ and $P^{111}(1) = 3!\, \sigma(1) = 6$. These imply that, the schemes are consistent, since the conditions for consistency of a numerical schemes are met. Also, the predictor satisfied the conditions for consistency, therefore, it is consistent.

4.3. DETERMINATION OF INTERVAL OF ABSOLUTE STABILITY.

We shall apply the boundary locus method (Lambert [1973], Fatunla [1988] to determine the interval of absolute stability of (3.18), (3.21), (3.24) and their first and second derivatives given in (3.19), (3.20), (3.22), (3,23), (3.25) and (3.26) respectively. Also, we determine the interval of absolute stability of the main scheme of the predictor and its first and second derivatives at three values of u i.e $u = \frac{1}{2}, \frac{3}{2}$,and $\frac{4}{2}$, in the method. we define:

$$h(\theta) = \frac{\rho\{\exp(i\theta)\}}{\sigma\{\exp(i\theta)\}} \qquad (4.8)$$

where ρ is the first characteristic polynomial and σ is the second characteristics polynomial for (3.18)

$$h(\theta) = \frac{90\{\exp 3i\theta - 3\exp 2i\theta + 3\exp i\theta - 1\}}{\exp 3i\theta + 36\exp 2i\theta + 16\exp 3i\theta/2 + 36\exp i\theta + 1} = x(\theta) + iy(\theta) \qquad (4.9)$$

applying Euler's formula [spiegel (1974)] to (4.4) yields

$$x(\theta) = \frac{0}{2304\cos\theta/2 + 2736\cos 3\theta/2 + 64\cos 3\theta/2 + 144\cos 2\theta + 2\cos 3\theta + 2850} \qquad (4.10)$$

and $y(\theta) = 0$

we now assumed the values of θ when $0^0 \le \theta \le 180^0$, at interval of 30^0 and obtained values for x(θ) as shown in the table 4.3 below,

TABLE 4.3

θ	0^o	30^0	60^0	90^0	120^0	150^0	180^0
X(θ)	0	0	0	0	0	0	0

The result shows from the table that the interval of absolute stability of the method (3.18) is (0,0).

For equation (3.21)

$$h(\theta)=\frac{15\left[\exp 3i\theta-3\exp 2i\theta+3\exp i\theta-1\right]}{5\exp 3i\theta-16\exp 5i\theta/2+90\exp 2i\theta+70\exp i\theta+1}=x(\theta)+iy(\theta)\ldots\ldots(4.11)$$

$$x(\theta)=\frac{15\left[-4\cos 3\theta+16\cos 5\theta/2-8\cos 2\theta-48\cos 3\theta/2+68\cos\theta+32\cos\theta/2-56\right]}{10\cos 3\theta-32\cos 5\theta/2+880\cos 2\theta-2240\cos 3\theta/2+13640\cos\theta-3040\cos\theta/2+13282}\ldots\ldots(4.12)$$

$$y(\theta)=0$$

Evaluation $x(\theta)$, $0^{o}\leq\theta\leq 180^{0}$ at interval of 30^{0} gives the following result s

Table 4.4

θ	0^{o}	30^{0}	60^{0}	90^{0}	120^{0}	150^{0}	180^{0}
$X(\theta)$	0	-0.000	-0.001	-0.035	-0.401	-3.432	-37.5

Thus the interval of absolute stability of (3.21) is (-3.75,0)

For equation (3.24)

$$h(\theta)=\frac{150\left[\exp 3i\theta-3\exp 2i\theta+3\exp i\theta-1\right]}{\exp 3i\theta+70\exp 2i\theta+90\exp i\theta-16\exp i\theta/2+5}=x(\theta)+iy(\theta)\ldots\ldots(4.13)$$

$$x(\theta)=\frac{15\left[4\cos 3\theta-16\cos 5\theta/2+8\cos 2\theta+48\cos 3\theta/2-68\cos\theta-32\cos\theta/2+56\right]}{10\cos 3\theta-32\cos 5\theta/2+880\cos 2\theta-2240\cos 3\theta/2+13640\cos\theta-3040\cos\theta/2+13282}\ldots\ldots(4.14)$$

Evaluation $x(\theta)$,$0^{o}\leq\theta\leq 180^{o}$ at interval of 30^{o} gives

Table 4.5

θ	0^{o}	30^{o}	60^{o}	90^{o}	120^{o}	150^{o}	180^{o}
$X(\theta)$	0	0.000	0.001	0.035	0.401	3.432	37.5

Thus the interval of absolute stability of (3.24) is (0,37.5).

Three different values of u are considered to get the first and second derivatives of the mehods (3.18), (3.21) and (3.24 at $t = 1$. Their interval of absolute stabilities are also obtained following the same procedure as stated for the main methods. They are given by the table 4.6 below

TABLE 4.6

	Value of U	Interval of absolute stability
$y^{1}_{n+3}=\frac{1}{2h}\left[5y_{n+2}-8y_{n+1}+3y_{n}\right]+\frac{h^{2}}{7560}\left[514f_{n+3}+7785f_{n+2}+160f_{n+\frac{1}{2}}+5328f_{n+2}+73f_{n}\right]$	$^{1}/_{2}$	(-59.62,0)
$y^{11}_{n+3}=\frac{1}{h^{2}}\left[y_{n+2}-2y_{n+1}+y_{n}\right]+\frac{h}{360}\left[108f_{n+3}+597f_{n+2}-256f_{n+\frac{1}{2}}+276f_{n+2}-5f_{n}\right]$	$^{3}/_{2}$	(-7.72,0)
$y^{1}_{n+3}=\frac{1}{2h}\left[5y_{n+2}-8y_{n+1}+3y_{n}\right]+\frac{h^{2}}{2520}\left[178f_{n+3}-32f_{n+\frac{5}{2}}+2655f_{n+2}+1796f_{n+1}+23f_{n}\right]$	$^{5}/_{2}$	(-57.52,0)
$y^{11}_{n+3}=\frac{1}{h^{2}}\left[y_{n+2}-2y_{n+1}+y_{n}\right]+\frac{h}{360}\left[380f_{n+3}+768f_{n+\frac{5}{2}}+1545f_{n+2}+900f_{n+1}+7f_{n}\right]$	$^{5}/_{2}$	(-2.95,0)
$y^{1}_{n+3}=\frac{1}{2h}\left[5y_{n+2}-8y_{n+1}+3y_{n}\right]+\frac{h^{2}}{2520}\left[170f_{n+3}+2615f_{n+2}+1836f_{n+\frac{1}{2}}+31f_{n}\right]$	$^{1}/_{2}$	(-62.84,0)
$y^{11}_{n+3}=\frac{1}{h^{2}}\left[y_{n+2}-2y_{n+1}+y_{n}\right]+\frac{h}{1800}\left[572f_{n+3}+2505f_{n+2}-60f_{n+1}+768f_{n+\frac{1}{2}}-185f_{n}\right]$	½	(-3.37,0)

Similarly, we determined the interval of absolute stability of the predictor scheme in (3.34). In this case,

$$h(\theta)=\frac{2[\exp 3i\theta-3\exp 2i\theta+3\exp i\theta-1]}{\exp 2i\theta+\exp i\theta}=x(\theta)+iy(\theta)\ldots\ldots(4.15)$$

$$x(\theta)=\frac{0}{2[1+\cos\theta]}\ldots\ldots(4.16)$$

$$y(\theta)=0$$

Evaluatio of x(θ), $0\leq\theta\leq180^0$ at interval of 30^0 gives the following result as shown below,

Table 4.7

θ	0°	30°	60°	90°	120°	150°	180°
X(θ)	0	0	0	0	0	0	$^0/_0=\infty$

The result shows from the table that the interval of absolute stability of the scheme (3.34) is (0, ∞) (see Awoyemi (2003)) and therefore the predictor is p-stable, since, the interval of absolute stablity is infinite to the right of the x-axis. The interval of abslute stability of the first and second derivatives of the predictor at different values of u are also determined and shown in the table below,

TABLE 4.8

	Value of u	Interval of absolute stability
$y'_{n+3}=\frac{1}{2h^2}\left[5y_{n+2}-8y_{n+1}+3y_n\right]+\frac{h^2}{120}\left[197f_{n+2}-128f_{n+\frac{3}{2}}+158f_{n+1}-7f_n\right]$	$^3/_2$	*(-1.76,0)*
$y''_{n+3}=\frac{1}{h^2}\left[5y_{n+2}-2y_{n+1}+y_n\right]+\frac{h}{360}\left[1569f_{n+2}-1984f_{n+3/2}+1248f_{n+1}-113f_n\right]$	$^5/_2$	*(-0.07,0)*
$y'_{n+3}=\frac{1}{2h}\left[5y_{n+2}-8y_{n+1}+3y_n\right]+\frac{h^2}{600}\left[128f_{n+\frac{5}{2}}+505f_{n+2}+470f_{n+1}-3f_n\right]$	$^5/_2$	*(- 13.05,0)*
$y''_{n+3}=\frac{1}{h^2}\left[y_{n+2}-2y_{n+1}+y_n\right]+\frac{h}{1800}\left[1984f_{n+\frac{5}{2}}+405f_{n+2}+1280f_{n+1}-69f_n\right]$	$^5/_2$	*(0,1.21)*
$y'_{n+3}=\frac{1}{2h}\left[5y_{n+2}-8y_{n+1}+3y_n\right]+\frac{h^2}{120}\left[165f_{n+3}-34f_{n+1}+128f_{n+\frac{1}{2}}-39f_n\right]$	$^1/_2$	*(-3.66,0)*
$y''_{n+3}=\frac{1}{h^2}\left[y_{n+2}-2y_{n+1}+y_n\right]+\frac{h}{360}\left[107f_{n+2}-1728f_{n+1}+1984f_{n+\frac{1}{2}}-609f_n\right]$	$^1/_2$	*(-0.61,0)*

4.5 NUMERICAL COMPUTATION AND RESULTS

In this section, we consider some sample problems to demonstrate the efficiency of the schemes.

TEST PROBLEMS

(i) $y^{111}-2y^{11}-3y^{1}+10y=34xe^{-2x}-16e^{-2x}-10x^{2}+6x+34$

$y(0)=3, y^{1}(0)=0, y^{11}(0)=0, 0\le x\le 1$

Exact solution $y(x)=x^{2}e^{-2x}-x^{2}+3$

(ii) $y^{111}+2y^{11}-9y^{1}-18y=18x^{2}-18x+22, 0\le x\le 1$

$y(0)=-2, y^{1}(0)=-8, y^{11}(0)=-12$

Exact solution $y(x)-2e^{3x}+e^{-2x}+x^{2}-1$

(iii) $\frac{1}{2}yy''+y'''=0$ $0\le x\le 1$ (Blasius'problem)

$y(0)=0, y^{1}(0)=0, y^{11}(0)=\alpha$

put $\alpha=1$

Note

YEX = THEORETICAL SOLUTION.

YC = COMPUTED SOLUTION OF Y

ER (error) = $|YC-YEX|$

Result for Problem 1, H = .1 , u = 3/2

Table: 4.9

X	YEX	YC	ER
0.0000	0.3000000000D+01	0.3000000000D+01	0.0000000000D+00
0.1000	0.2998187308D+01	0.2998186667D+01	0.6408641133D-06
0.2000	0.2986812802D+01	0.2986797686D+01	0.1511330039D-04
0.3000	0.2959393047D+01	0.2959329403D+01	0.6361113654D-04
0.4000	0.2911892634D+01	0.2911725068D+01	0.1675667217D-03
0.5000	0.2841969860D+01	0.2841619089D+01	0.3507709233D-03
0.6000	0.2748429916D+01	0.2747788829D+01	0.6410875485D-03
0.7000	0.2630832512D+01	0.2629760870D+01	0.1070642249D-02
0.8000	0.2489213772D+01	0.2487531558D+01	0.1682213540D-02
0.9000	0.2323892099D+01	0.2321371496D+01	0.2520603660D-06
1.0000	0.2135335283D+01	0.2131691269D+01	0.3644014162D-02

Result for Problem 1, H = 0.05, u = 3/2

Table: 4.10

	YEX	YC	ER
0.0000	0.3000000000D+01	0.3000000000D+01	0.0000000000D+00
0.1000	0.2998187070D+01	0.2998187070D+01	0.2374868373D-06
0.2000	0.2986812802D+01	0.2986810222D+01	0.2579619661D-05
0.3000	0.2959393047D+01	0.2959383527D+01	0.9520558463D-05
0.4000	0.2911892634D+01	0.2911868915D+01	0.2371928556D-04
0.5000	0.2841969860D+01	0.2841921684D+01	0.4870618445D-04
0.6000	0.2748429916D+01	0.2748343515D+01	0.8640177259D-04
0.7000	0.2630832512D+01	0.2630689929D+01	0.1425828393D-03
0.8000	0.2489213772D+01	0.2488992022D+01	0.2217492122D-03
0.9000	0.2323892099D+01	0.2323562150D+01	0.3299414751D-03
1.0000	0.2135335283D+01	0.2134860903D+01	0.4748006024D-03

Result for Problem 1, H = 0.025, u = 3/2

Table 4.11.

X	YEX	YC	ER
0.0000	0.3000000000D+01	0.3000000000D+01	0.0000000000D+00
0.1000	0.2998807308D+01	0.2998187267D+01	0.4036882562D-07
0.2000	0.2986812802D+01	0.2986812437D+01	0.3652348917D-06
0.3000	*0.2959393047D+01*	*0.2959391763D+01*	*0.1284028744D-05*
0.4000	0.2911892634D+01	0.2911882506D+01	0.3128282505D-05
0.5000	0.2841969860D+01	0.2841963587D+01	0.6273380743D-05
0.6000	0.2748429916D+01	0.2748418757D+01	0.1115939426D-04
0.7000	0.2630832512D+01	0.2630814201D+01	0.1831162672D-04
0.8000	0.2481213772D+01	0.2489185410D+01	0.2836124189D-04
0.9000	0.2323892099D+01	0.2323850033D+01	0.4206606715D-04
1.0000	0.2135335283D+01	0.2135274952D+01	0.6033136708D-04

Result for Problem 2, h = .1, u = 3/2

Table: 4.12

X	YEX	YC	ER
0.0000	-0.2000000000D+01	-0.2000000000D+01	0.0000000000D+00
0.1000	-0.2870986862D+01	-0.2870984833D+01	0.2028740692D-05
0.2000	-0.3933917555D+01	-0.3933871016D+01	0.4653839182D-04
0.3000	-0.5280394585D+01	-0.5280190157D+01	0.2044292055D-03
0.4000	-0.7030904881D+01	-0.7030332949D+01	0.5719319717D-03
0.5000	-0.9345498700D+01	-0.9344212790D+01	0.1289509677D-02
0.6000	-0.1243810072D+02	-0.1243555578D+02	0.2544941160D-02
0.7000	-0.1659574286D+02	-0.1659110400D+01	0.7990471230D-02
0.8000	-0.2978416456D+02	-0.2977095000D+02	0.1321455800D-01
0.9000	-0.2978416456D+02	-0.2977095000D+02	0.1321455800D-01
1.0000	-0.4003573856D+02	-0.4001453708D+02	0.2120147891D-01

Result for Problem 2, h = 0.05, u = 3/2

Table 4.13.

X	YEX	YC	ER
0.0000	-0.2000000000D+01	-0.2000000000D+01	0.0000000000D+00
0.1000	-0.2870986862D+01	-0.2870986153D+01	0.7089016307D-06
0.2000	-0.3933917555D+01	-0.3933909521D+01	0.8034012909D-05
0.3000	-0.5280394586D+01	-0.5280303081D+01	0.3150529745D-04
0.4000	-0.7030904881D+01	-0.7030820725D+01	0.8415642574D-04
0.5000	-0.9345498700D+01	-0.9345314110D+01	0.1845892112D-03
0.6000	-0.1243810072D+02	-0.1243774089D+02	0.3598253228D-03
0.7000	-0.1659574286D+02	-0.1659509355D+02	0.6493066609D-03
0.8000	-0.2220445624D+02	-0.2220334570D+02	0.1110545314D-02
0.9000	-0.2978416456D+02	-0.2977095000D+02	0.1321455800D-01
1.0000	-0.4003573856D+02	-0.4001453708D+02	0.2120147891D-01

Result for Problem 2, h = 0.025, u = 3/2

Table 4.14

X	YEX	YC	ER
0.0000	-0.2000000000D+01	-0.2000000000D+01	0.0000000000D+00
0.1000	-0.2870986862D+01	-0.2870986741D+01	0.1210826857D-06
0.2000	-0.3933917555D+01	-0.3933916396D+01	0.1158545738D-05
0.3000	-0.5280394586D+01	-0.5280390239D+01	0.4346790126D-05
0.4000	-0.7030904881D+01	-0.7030893501D+01	0.1138050818D-04
0.5000	-0.9345498700D+01	-0.9345474017D+01	0.2468256341D-04
0.6000	-0.1243810072D+02	-0.1243805294D+02	0.4777448835D-04
0.7000	-0.1659574286D+02	-0.1659565706D+02	0.4777448835D-04
0.8000	-0.2220445624D+02	-0.2220431000D+02	0.1462421914D-03
0.9000	-0.2978416456D+02	-0.2978392456D+02	0.2399983032D-03
1.0000	-0.4003573856D+02	-0.4003535575D+02	0.3828168567D-03

Result for Problem 1, h = .1 , u = 5/2

Table: 4.15

X	YEX	YC	ER
0.0000	0.3000000000D+01	0.3000000000D+01	0.0000000000D+00
0.1000	0.2998187308D+01	0.2998186667D+01	0.6408641133D-06
0.2000	0.2986812802D+01	0.2986797686D+01	0.1511330039D-04
0.3000	0.2959393047D+01	0.2959329403D+01	0.6361113654D-04
0.4000	0.2911892634D+01	0.2911725068D+01	0.1675667217D-03
0.5000	0.2841969860D+01	0.2841619089D+01	0.3507709233D-03
0.6000	0.2748429916D+01	0.2747788829D+01	0.6410875485D-03
0.7000	0.2630832512D+01	0.2629760870D+01	0.1070642249D-02
0.8000	0.2489213772D+01	0.2487531558D+01	0.1682213540D-02
0.9000	0.2323892099D+01	0.2321371496D+01	0.2520603660D-06
1.0000	0.2135335283D+01	0.2131691269D+01	0.3644014162D-02

Result for Problem 1, h = 0.05, u = 5/2

Table: 4.16

	YEX	YC	ER
0.0000	0.3000000000D+01	0.3000000000D+01	0.0000000000D+00
0.1000	0.2998187070D+01	0.2998187070D+01	0.2374868373D-06
0.2000	0.2986812802D+01	0.2986810222D+01	0.2579619661D-05
0.3000	0.2959393047D+01	0.2959383527D+01	0.9520558463D-05
0.4000	0.2911892634D+01	0.2911868915D+01	0.2371928556D-04
0.5000	0.2841969860D+01	0.2841921684D+01	0.4870618445D-04
0.6000	0.2748429916D+01	0.2748343515D+01	0.8640177259D-04
0.7000	0.2630832512D+01	0.2630689929D+01	0.1425828393D-03
0.8000	0.2489213772D+01	0.2488992022D+01	0.2217492122D-03
0.9000	0.2323892099D+01	0.2323562150D+01	0.3299414751D-03
1.0000	0.2135335283D+01	0.2134860903D+01	0.4748006024D-03

Result for Problem 1, h = 0.025, u = 5/2

Table 4.17

X	YEX	YC	ER
0.0000	0.3000000000D+01	0.3000000000D+01	0.0000000000D+00
0.1000	0.2998807308D+01	0.2998187267D+01	0.4036882562D-07
0.2000	0.2986812802D+01	0.2986812437D+01	0.3652348917D-06
0.3000	0.2959393047D+01	0.2959391763D+01	0.1284028744D-05
0.4000	0.2911892634D+01	0.2911882506D+01	0.3128282505D-05
0.5000	0.2841969860D+01	0.2841963587D+01	0.6273380743D-05
0.6000	0.2748429916D+01	0.2748418757D+01	0.1115939426D-04
0.7000	0.2630832512D+01	0.2630814201D+01	0.1831162672D-04
0.8000	0.2481213772D+01	0.2489185410D+01	0.2836124189D-04
0.9000	0.2323892099D+01	0.2323850033D+01	0.4206606715D-04
1.0000	0.2135335283D+01	0.2135274952D+01	0.6033136708D-04

Result for Problem 2, h = .1, u = 5/2

Table: 4.18

X	YEX	YC	ER
0.0000	-0.2000000000D+01	-0.2000000000D+01	0.0000000000D+00
0.1000	-0.2870986862D+01	-0.2870984833D+01	0.2028740692D-05
0.2000	-0.3933917555D+01	-0.3933871016D+01	0.4653839182D-04
0.3000	-0.5280394585D+01	-0.5280190157D+01	0.2044292055D-03
0.4000	-0.7030904881D+01	-0.7030332949D+01	0.5719319717D-03
0.5000	-0.9345498700D+01	-0.9344212790D+01	0.1289509677D-02
0.6000	-0.1243810072D+02	-0.1243555578D+02	0.2544941160D-02
0.7000	-0.1659574286D+02	-0.1659110400D+01	0.7990471230D-02
0.8000	-0.2978416456D+02	-0.2977095000D+02	0.1321455800D-01
0.9000	-0.2978416456D+02	-0.2977095000D+02	0.1321455800D-01
1.0000	-0.4003573856D+02	-0.4001453708D+02	0.2120147891D-01

Result for Problem 2, h = 0.05, u = 5/2

Table 4.19

X	YEX	YC	ER
0.0000	-0.2000000000D+01	-0.2000000000D+01	0.0000000000D+00
0.1000	-0.2870986862D+01	-0.2870986153D+01	0.7089016307D-06
0.2000	-0.3933917555D+01	-0.3933909521D+01	0.8034012909D-05
0.3000	-0.5280394586D+01	-0.5280303081D+01	0.3150529745D-04
0.4000	-0.7030904881D+01	-0.7030820725D+01	0.8415642574D-04
0.5000	-0.9345498700D+01	-0.9345314110D+01	0.1845892112D-03
0.6000	-0.1243810072D+02	-0.1243774089D+02	0.3598253228D-03
0.7000	-0.1659574286D+02	-0.1659509355D+02	0.6493066609D-03
0.8000	-0.2220445624D+02	-0.2220334570D+02	0.1110545314D-02
0.9000	-0.2978416456D+02	-0.2977095000D+02	0.1321455800D-01
1.0000	-0.4003573856D+02	-0.4001453708D+02	0.2120147891D-01

Result for Problem 2, h = 0.025, u = 5/2

Table 4.20

X	YEX	YC	ER
0.0000	-0.2000000000D+01	-0.2000000000D+01	0.0000000000D+00
0.1000	-0.2870986862D+01	-0.2870986741D+01	0.1210826857D-06
0.2000	-0.3933917555D+01	-0.3933916396D+01	0.1158545738D-05
0.3000	-0.5280394586D+01	-0.5280390239D+01	0.4346790126D-05
0.4000	-0.7030904881D+01	-0.7030893501D+01	0.1138050818D-04
0.5000	-0.9345498700D+01	-0.9345474017D+01	0.2468256341D-04
0.6000	-0.1243810072D+02	-0.1243805294D+02	0.4777448835D-04
0.7000	-0.1659574286D+02	-0.1659565706D+02	0.4777448835D-04
0.8000	-0.2220445624D+02	-0.2220431000D+02	0.1462421914D-03
0.9000	-0.2978416456D+02	-0.2978392456D+02	0.2399983032D-03
1.0000	-0.4003573856D+02	-0.4003535575D+02	0.3828168567D-03

Result for Problem 1, h = .1 , u = ½

Table 4.21

X	YEX	YC	ER
0.0000	0.3000000000D+01	0.3000000000D+01	0.0000000000D+00
0.1000	0.2998187308D+01	0.2998186667D+01	0.6408641133D-06
0.2000	0.2986812802D+01	0.2986797686D+01	0.1511330039D-04
0.3000	0.2959393047D+01	0.2959329403D+01	0.6361113654D-04
0.4000	0.2911892634D+01	0.2911725068D+01	0.1675667217D-03
0.5000	0.2841969860D+01	0.2841619089D+01	0.3507709233D-03
0.6000	0.2748429916D+01	0.2747788829D+01	0.6410875485D-03
0.7000	0.2630832512D+01	0.2629760870D+01	0.1070642249D-02
0.8000	0.2489213772D+01	0.2487531558D+01	0.1682213540D-02
0.9000	0.2323892099D+01	0.2321371496D+01	0.2520603660D-06
1.0000	0.2135335283D+01	0.2131691269D+01	0.3644014162D-02

Result for Problem 1, h = 0.05, u = ½

Table: 4.22

	YEX	YC	ER
0.0000	0.3000000000D+01	0.3000000000D+01	0.0000000000D+00
0.1000	0.2998187070D+01	0.2998187070D+01	0.2374868373D-06
0.2000	0.2986812802D+01	0.2986810222D+01	0.2579619661D-05
0.3000	0.2959393047D+01	0.2959383527D+01	0.9520558463D-05
0.4000	0.2911892634D+01	0.2911868915D+01	0.2371928556D-04
0.5000	0.2841969860D+01	0.2841921684D+01	0.4870618445D-04
0.6000	0.2748429916D+01	0.2748343515D+01	0.8640177259D-04
0.7000	0.2630832512D+01	0.2630689929D+01	0.1425828393D-03
0.8000	0.2489213772D+01	0.2488992022D+01	0.2217492122D-03
0.9000	0.2323892099D+01	0.2323562150D+01	0.3299414751D-03
1.0000	0.2135335283D+01	0.2134860903D+01	0.4748006024D-03

Result for Problem 1, h = 0.025, u = ½

Table: 4.23

X	YEX	YC	ER
0.0000	0.3000000000D+01	0.3000000000D+01	0.0000000000D+00
0.1000	0.2998807308D+01	0.2998187267D+01	0.4036882562D-07
0.2000	0.2986812802D+01	0.2986812437D+01	0.3652348917D-06
0.3000	0.2959393047D+01	0.2959391763D+01	0.1284028744D-05
0.4000	0.2911892634D+01	0.2911882506D+01	0.3128282505D-05
0.5000	0.2841969860D+01	0.2841963587D+01	0.6273380743D-05
0.6000	0.2748429916D+01	0.2748418757D+01	0.1115939426D-04
0.7000	0.2630832512D+01	0.2630814201D+01	0.1831162672D-04
0.8000	0.2481213772D+01	0.2489185410D+01	0.2836124189D-04
0.9000	0.2323892099D+01	0.2323850033D+01	0.4206606715D-04
1.0000	0.2135335283D+01	0.2135274952D+01	0.6033136708D-04

Result for Problem 2, h = .1, u = ½

Table 4.24

X	YEX	YC	ER
0.0000	-0.2000000000D+01	-0.2000000000D+01	0.0000000000D+00
0.1000	-0.2870986862D+01	-0.2870984833D+01	0.2028740692D-05
0.2000	-0.3933917555D+01	-0.3933871016D+01	0.4653839182D-04
0.3000	-0.5280394585D+01	-0.5280190157D+01	0.2044292055D-03
0.4000	-0.7030904881D+01	-0.7030332949D+01	0.5719319717D-03
0.5000	-0.9345498700D+01	-0.9344212790D+01	0.1289509677D-02
0.6000	-0.1243810072D+02	-0.1243555578D+02	0.2544941160D-02
0.7000	-0.1659574286D+02	-0.1659110400D+01	0.7990471230D-02
0.8000	-0.2978416456D+02	-0.2977095000D+02	0.1321455800D-01
0.9000	-0.2978416456D+02	-0.2977095000D+02	0.1321455800D-01
1.0000	-0.4003573856D+02	-0.4001453708D+02	0.2120147891D-01

Result for Problem 2, h = 0.05, u = ½

Table: 4.25

X	YEX	YC	ER
0.0000	-0.2000000000D+01	-0.2000000000D+01	0.0000000000D+00
0.1000	-0.2870986862D+01	-0.2870986153D+01	0.7089016307D-06
0.2000	-0.3933917555D+01	-0.3933909521D+01	0.8034012909D-05
0.3000	-0.5280394586D+01	-0.5280303081D+01	0.3150529745D-04
0.4000	-0.7030904881D+01	-0.7030820725D+01	0.8415642574D-04
0.5000	-0.9345498700D+01	-0.9345314110D+01	0.1845892112D-03
0.6000	-0.1243810072D+02	-0.1243774089D+02	0.3598253228D-03
0.7000	-0.1659574286D+02	-0.1659509355D+02	0.6493066609D-03
0.8000	-0.2220445624D+02	-0.2220334570D+02	0.1110545314D-02
0.9000	-0.2978416456D+02	-0.2977095000D+02	0.1321455800D-01
1.0000	-0.4003573856D+02	-0.4001453708D+02	0.2120147891D-01

Result for Problem 2, h = 0.025, u = ½

Table 4.26

X	YEX	YC	ER
0.0000	-0.2000000000D+01	-0.2000000000D+01	0.0000000000D+00
0.1000	-0.2870986862D+01	-0.2870986741D+01	0.1210826857D-06
0.2000	-0.3933917555D+01	-0.3933916396D+01	0.1158545738D-05
0.3000	-0.5280394586D+01	-0.5280390239D+01	0.4346790126D-05
0.4000	-0.7030904881D+01	-0.7030893501D+01	0.1138050818D-04
0.5000	-0.9345498700D+01	-0.9345474017D+01	0.2468256341D-04
0.6000	-0.1243810072D+02	-0.1243805294D+02	0.4777448835D-04
0.7000	-0.1659574286D+02	-0.1659565706D+02	0.4777448835D-04
0.8000	-0.2220445624D+02	-0.2220431000D+02	0.1462421914D-03
0.9000	-0.2978416456D+02	-0.2978392456D+02	0.2399983032D-03
1.0000	-0.4003573856D+02	-0.4003535575D+02	0.3828168567D-03

Table 4.27 PROBLEM 3, u =1/2

	h = 0.1	h=0.05	h = 0.025	h = 0.0125	h = 0.00625
X	Y1	Y2	Y3	Y4	Y5
0.1	.4999958333D-02	.4111958334D-02	.4999958334D-02	.4999958334D-02	.4999958334D-02
0.2	.1111866679D-01	.1999866682D-01	.1999866684D-01	.1999866684D-01	.1999866684D-01
0.3	.4498987869D-01	.4498987930D-01	.4498987944D-01	.4498987947D-01	.4498987947D-01
0.4	.7995737357D-01	.7995737715D-01	.7995737786D-01	.7995737797D-01	.7995737798D-01
0.5	.1248700417D+00	.1248700548D+00	.1248700571D+00	.1248700575D+00	.1248700575D+00
0.6	.1796770975D+00	.1796771342D+00	.1796771403D+00	.1796771412D+00	.1796771413D+00
0.7	.2443035157D+00	.2443033003D+00	.2443036149D+00	.2443036168D+00	.2443036171D+00
0.8	.3186458022D+00	.3186459781D+00	.3816460052D+00	.3186460089D+00	.3186460094D+00
0.9	.4025682355D+00	.4025685638D+00	.4025686130D+00	.4025686197D+00	.4025686206D+00
1.0	.4958997183D+00	.4959002866D+00	.4959003701D+00	.4959003814D+00	.4959003828D+00

Note for convergence

$$\underset{i \to \infty}{Lt}\, x_i = x$$

$$\underset{i \to \infty}{Lt}\, x_{i+1} = x$$

$$\underset{i \to \infty}{Lt}\, x_i - x_{i+1} = 0$$

FROM TABLE (4.27), WE HAVE

Table 4.28

X	Y1-Y2	Y2-Y3	Y3-Y4	Y4-Y5
0.1	1.0D-12	0	0	0
0.3	6.1D-10	1.4D-10	3.0D-11	0
0.5	1.3D-08	2.3D-09	4.0D-10	0
0.7	8.6D-08	1.4D-08	1.9D-09	3.0D-10
0.9	3.3D-07	4.9D-08	6.7D-09	9.0D-10
1.0	5.7D-07	5.4D-08	1.1D-08	1.4D-09

4.6 COMPARISON BETWEEN AWOYEMI (2003) AND THE NEW SCHEMES.

PROBLEM 1, ERROR VALUES, h= .1/16

Table 4.29

X	K = 4 AWOYEMI(2003)	K = 3, U = ½ NEW METHOD	K = 3, U = 3/2 NEW METHOD	K =3, U =5/2 NEW METHOD
0.2	.6198586533D-08	.6198580689D-08	.6198586089D-08	.6198586089D-08
0.4	.5071005038D-07	.5071011744D-07	.5071011744D-07	.5071011744D-07
0.6	.1783260193D-06	.1783260810D-06	.1783260810D-06	.1783260810D-06
0.8	.4501255768D-06	.4501256332D-06	.4501256332D-06	.4501256332D-06
1.0	.9537385033D-06	.9537384513D-06	.9537384513D-06	.9537384513D-06

PROBLEM 2, ERROR VALUES, h=.1/16

Table 4.30

X	k=4 AWOYEMI(2003)	k=3, U=1/2 NEW METHOD	K=3, U=3/2 NEW METHOD	K=3, U=5/2 NEW METHOD
0.2	.2003599242D-07	.2003599331D-07	.2003599331D-07	.2003599331D-07
0.4	.1884257728D-06	.1884259433D-06	.1884259433D-06	.1884259433D-06
0.6	.7806285645D-06	.7806288682D-06	.7806288682D-06	.7806288682D-06
0.8	.2375233962D-05	.2375234512D-05	.2375234512D-05	.2375234512D-05
1.0	.6197233333D-05	.6197232508D-05	.6197232508D-05	.6197232508D-05

CHAPTER FIVE

DISCUSSION OF RESULTS, CONCLUSION AND RECOMMENDATIONS

5.1 DISCUSSION

A hybrid collocation methods for general third order ordinary differential equations for step number k=3 have been developed and used to solve some initial value problems directly without reducing them into a system of first order equations. The methods have their continuous coefficients α_j, β_j and β_u as functions of t,t∈(0,1]. Thus, an infinite number of discrete schemes could be obtained in the interval t∈(0,1]. In this work, t=1 which implies $x = x_{n+3}$ to obtain the discrete schemes, their first and second order derivatives. Also, to obtain the predictor, their first and second derivatives, the same procedures are followed.

The analysis of the schemes in chapter four shows that the developed schemes (3.18), (3.21) and (3.24) have order p=7,6,6 with error constants; $\frac{-57}{362880}$, $\frac{-1}{1200}$ and $\frac{1}{1200}$ repectively. The order of acuracy of the first and the second derivatives and their error constants are given in table 4.1 to be.

Table 5.1

SCHEMES	Value of U	Order P	Error Constant
$y_{n+3}-3y_{n+2}+3y_{n+1}-y_n=\frac{h^3}{90}\left[f_{n+3}+36f_{n+2}+16f_{n+3/2}+36f_{n+1}+f_n\right]$	$3/2$	7	$\frac{-57}{362880}$
$y'_{n+3}=\frac{1}{2h}[5y_{n+2}-8y_{n+1}+3y_n]+\frac{h^2}{7560}\left[514f_{n+3}+7785f_{n+2}+160f_{n+3/2}+5328f_{n+2}+73f_n\right]$	$3/2$	6	$\frac{-243}{90720}$
$y''_{n+3}=\frac{1}{h^2}[y_{n+2}-2y_{n+1}+y_n]+\frac{h}{360}\left[108f_{n+3}+597f_{n+2}-256f_{n+3/2}+276f_{n+2}-5f_n\right]$	$3/2$	6	$\frac{-294}{60480}$
$y_{n+3}-3y_{n+2}+3y_{n+1}-y_n=\frac{h^3}{150}\left[5f_{n+2}-16f_{n+5/2}+90f_{n+1}+70f_{n+1}+f_n\right]$	$5/2$	6	$\frac{-1}{1200}$
$y'_{n+3}=\frac{1}{2h}[5y_{n+2}-8y_{n+1}+3y_n]+\frac{h^2}{2520}\left[178f_{n+3}-32f_{n+5/2}+2655f_{n+2}+1796f_{n+1}+23f_n\right]$	$3/2$	6	$\frac{-522}{181440}$
$y''_{n+3}=\frac{1}{h^2}[y_{n+2}-2y_{n+1}+y_n]+\frac{h}{360}\left[380f_{n+3}+768f_{n+5/2}+1545f_{n+2}+900f_{n+1}+7f_n\right]$	$5/2$	6	$\frac{-2385}{1512000}$
$y_{n+3}-3y_{n+2}+3y_{n+1}-3y_{n+1}-y_n=\frac{h^3}{150}\left[f_{n+3}+70f_{n+2}+90f_{n+1}-16f_{n+1/2}+5f_n\right]$	$1/2$	6	$\frac{1}{1200}$
$y'_{n+3}=\frac{1}{2h}[5y_{n+2}-8y_{n+1}+3y_n]+\frac{h^2}{2520}\left[170f_{n+3}+2615f_{n+2}+1836f_{n+1/2}+31f_n\right]$	$1/2$	6	$\frac{-45}{18144}$
$y''_{n+3}=\frac{1}{h^2}[y_{n+2}-2y_{n+1}+y_n]+\frac{h}{1800}\left[572f_{n+3}+2505f_{n+2}-60f_{n+1}+768f_{n+1/2}-185f_n\right]$	½	6	$\frac{-12465}{1512000}$

The predictor, their first and second derivatives all have order 5 with error constant given in table 4.2 to be.

Table 5.2

SCHEMES	Value of u	Order P	Error Constant
$y_{n+3}-3y_{n+2}+3y_{n+1}-y_n=\frac{h^3}{2}\left[f_{n+2}^{'}+f_{n+1}\right]$	-	5	$\frac{1}{240}$
$y^{1}_{n+3}=\frac{1}{2h^2}\left[5y_{n+2}-8y_{n+1}+3y_n\right]+\frac{h^2}{120}\left[197f_{n+2}-128f_{n+\frac{1}{2}}+158f_{n+1}-7f_n\right]$	$^3/_2$	5	$\frac{257}{5040}$
$y^{11}_{n+3}=\frac{1}{h^2}\left[5y_{n+2}-2y_{n+1}+y_n\right]+\frac{h}{360}\left[1569f_{n+2}-1984f_{n+3/2}+1248f_{n+1}-113f_n\right]$	$^3/_2$	5	$\frac{567}{5040}$
$y^{1}_{n+3}=\frac{1}{2h}\left[5y_{n+2}-8y_{n+1}+3y_n\right]+\frac{h^2}{600}\left[128f_{n+\frac{1}{2}}+505f_{n+2}+470f_{n+1}-3f_n\right]$	$^5/_2$	5	$\frac{89}{5040}$
$y^{11}_{n+3}=\frac{1}{h^2}\left[y_{n+2}-2y_{n+1}+y_n\right]+\frac{h}{1800}\left[1984f_{n+\frac{1}{2}}+405f_{n+2}+1280f_{n+1}-69f_n\right]$	$^5/_2$	5	$\frac{133}{5040}$
$y^{1}_{n+3}=\frac{1}{2h}\left[5y_{n+2}-8y_{n+1}+3y_n\right]+\frac{h^2}{120}\left[165f_{n+3}-34f_{n+1}+128f_{n+\frac{1}{2}}-39f_n\right]$	$^1/_2$	5	$\frac{85}{1008}$
$y^{11}_{n+3}=\frac{1}{h^2}\left[y_{n+2}-2y_{n+1}+y_n\right]+\frac{h}{360}\left[107f_{n+2}-1728f_{n+1}+1984f_{n+\frac{1}{2}}-609f_n\right]$	$^1/_2$	5	$\frac{1001}{5040}$

It is noticed from the tables 4.9 – 4.26 that any choice of $u\in(x_n\ x_{n+3})$ for rational u will produce identical results to 10 or more decimal places. As the decimal points are increasing, there may be changes in the values of the result. Thus, any value of u could be used to produce a schemes of identical results as shown in table 4.9 – 4.26 for problems 1and 2 and for any other problems for that matter. The schemes produced have good results and large stability intervals except when u is chosen at the centre of the interval, that is $u={}^3/_2$ For this value of u, even though the results are absolute stability is located at the origin just like Simpson's method for the first order ODEs. Such schemes are of little or no practical applications in real life.

Table (4.9 – 4.30) shows the result and error of the schemes (3.18), (3.21) and (3.24) they revealed that as step length h is decreasing, the acuracy of the methods is increasing.

Table 4.28 shows the result y_n, n =1 (1) 5 of the Blasius' problems in fluid mechanics which is non - linear and has no analytical solution. It is known in literature that if $\{x_n\}$ is a sequence of numerical solutions whose analytic solution is x then

$$\underset{n \to \infty}{Lt}\, x_n = x \quad \text{...5.1}$$

Also $\underset{n \to \infty}{Lt}\, x_{n+1} = x$ 5.2

Thus, from equations (5.1) and (5.2)

$$\underset{n \to \infty}{Lt}\, (x_n - x_{n+1}) = 0 \quad \text{..............................5.3}$$

Equation (5.3) is thus used to produce differences of the consecutive values of the results shown in table 4.28 for the non-linear problem 3. It could be observed in table 4.28 that each row is tending to zero as n is increasing for every value of x.

The results of the new methods are compared with the results obtained in Awoyemi (2003) with the step number K= 4. Although the results are found to be highly comparable, the new schemes are still more accurate than Awoyemi (2003).

The schemes are consistents and zero stable. The predictor for y_{n+3} in the schemes (3.18), (3.21) and (3.24) has interval of absolute stability as $(0, \infty)$ hence, it is P-stable(Fatunla (1988) and Awoyemi (2003))

5.2 CONCLUSION.

In this research work, a hybrid collocation methods for the solution of general third order ordinary differential equations have been developed. Three standard general problems including one non-linear of third order ODEs have been solved to test the efficiency of the methods.

Based on the analysis in chapter 4 and results in table 4.9-4.28, the developed methods are very adequate for solving linear and non-linear third order ODEs without reducing them to a system of first order ODEs. The effects of this are that;

(i) Computer time is not wasted

(ii) Cost implication is reduced.

(iii) A short computer program could be written to carter for the numerical aspect of the work.

An infinite numbers of discrete methods could be obtained by assigning different arbitary values of u at $x = x_{n+3}$. Each method obtained is independent of the others and could be used to solve any general third order ODEs.

The continuous coeficient of the methods allowed the first and second order derivatives of the methods to be determined.

5.3 RECOMMENDATIONS

The collocation approach of this work made use of polynomial as basis function. Further research could still be done using different function as basis or trial functions for possible computational advantages. It is also noted that the collocation points in the work involved only one off grid point. An additional off grid point could be added to see whether the scheme obtained will produce better results or not.

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```
 NAME OF FILE: IDOWU1
 K=3
 SOLUTION OF GENERAL THIRD ORDER INITIAL VALUE PROBLEMS
 OF THE FORM Y"'=F(X,Y,Y',Y")
 IMPLICIT DOUBLE PRECISION(A-H,O-Z)
 DIMENSION YN3C(81,10),YEX(81,10),ERC(81,10),YPN3C(81,10)
1,DYN3C(81,10)
 F(X,Y,Z,R)=2.D0*R+3.D0*Z-10.D0*Y+34.D0*X*DEXP(-2.D0*X)-16.D0*
1DEXP(-2.D0*X)-10.D0*X*X+6.D0*X+34.D0
 Y(X)=X*X*DEXP(-2.D0*X)-X*X+3.D0
 OPEN(6,FILE='IDOWU1.OUT')
 N=80
 NSTEP=10
 A=0.D0
 H=.0125D0
 B=A+H
 DX=H/FLOAT(N)
 D=1.D0
 U=1.5D0
 XN=A
 YN=3.D0
 YP=0.D0
 DY=0.D0
 XN1=XN+H
 XN2=XN+2.D0*H
 XN3=XN+3.D0*H
 XNU=XN+U*H
 WRITE(6,5)
 FORMAT(7X,'X',15X,'YEX',20X,'YC',20X,'ER'/)
 CALCULATE PREDICTORS AND THEIR DERIVATIVES
*K=0
 DO 1 I=1,81
 DO 2 J=1,NSTEP
 CALCULATE FP
 F0=F(XN,YN,YP,DY)
 DFX=(-68.D0*XN+66.D0)*DEXP(-2.D0*XN)-20.D0*XN+6.D0
 DFY=-10.D0
 DFYP=3.D0
 DFDY=2.D0
 FP=DFX+YP*DFY+DY*DFYP+F0*DFDY
 CACULATE FPP
 DFXX=(136.D0*XN-200.D0)*DEXP(-2.D0*XN)-20.D0
 DFXY=0.D0
 DFYYP=0.D0
 DFYDY=0.D0
 DFXYP=0.D0
 DFYPDY=0.D0
 DFXDY=0.D0
 DFYY=0.D0
 DFYPYP=0.D0
 DFDYDY=0.D0
 A=DFXY+DY*DFYYP+F0*DFYDY
 B=DFXYP+F0*DFYPDY
 C=DFXDY
 D=DY*DFY+F0*DFYP+FP*DFDY+DFXX
 E=YP*YP*DFYY+DY*DY*DFYPYP+F0*F0*DFDYDY
 FPP=2.D0*(A*YP+B*DY+C*F0)+D+E
 YN1=YN+H*YP+((H*H)/2.D0)*DY+((H**3)/6.D0)*F0+((H**4)/
124.D0)*FP+((H**5)/120.D0)*FPP
 YPN1=YP+H*DY+(H*H/2.D0)*F0+((H**3)/6.D0)*FP+((H**4)/
```

```
124.D0)*FPP
 DYN1=DY+H*F0+((H*H)/2.D0)*FP+((H**3)/6.D0)*FPP
 F1=F(XN1,YN1,YPN1,DYN1)
 YN2=YN+2.D0*H*YP+2.D0*H*H*DY+((8.D0*H**3)/6.D0)*F0+((
116.D0*H**4)/24.D0)*FP+((32.D0*H**5)/120.D0)*FPP
 YPN2=YP+2.D0*H*DY+2.D0*H*H*F0+(8.D0*H**3/6.D0)*FP+((16
1.D0*H**4)/24.D0)*FPP
 DYN2=DY+2.D0*H*F0+2.D0*H*H*FP+(8.D0*H**3/6.D0)*FPP
 F2=F(XN2,YN2,YPN2,DYN2)
 YNU=YN+U*H*YP+(((U*H)**2)/2.D0)*DY+(1.D0/6.D0)*(U*H)**3*F0
1+(1.D0/24.D0)*(U*H)**4*FP+(1.D0/120.D0)*(U*H)**5*FPP
 YPNU=YP+U*H*DY+(1.D0/2.D0)*(U*H)**2*F0+(1.D0/6.D0)*(U*H)**3
1*FP+(1.D0/24.D0)*(U*H)**4*FPP
 DYNU=DY+U*H*F0+(1.D0/2.D0)*(U*H)**2*FP+(1.D0/6.D0)*(U*H)**3
1*FPP
 FU=F(XNU,YNU,YPNU,DYNU)
 K=K+1
 IF(K.GE.2) THEN
 YN3=YC
 YPN3=YPC
 DYN3=DYC
 ELSE
 YN3=3.D0*YN2-3.D0*YN1+YN+(H**3)*(F2+F1)/2.D0
 YPN3=(1.D0/(2.D0*H))*(5.D0*YN2-8.D0*YN1+3.D0*YN)+(H*H/24.D0)*
1(28.D0*F2+16.D0*F1)
 DYN3=(1.D0/(H*H))*(YN2-2.D0*YN1+YN)+(H/24.D0)*(46.D0*F2+2.D0*F1)
 ENDIF
 F3=F(XN3,YN3,YPN3,DYN3)
 CALCULATE COEFFICIENT OF CONTINUOUS METHOD
 X=XN
 T=(X-XN2)/H
 A2T=(T*T+3.D0*T+2.D0)/2.D0
 A1T=-(T*T+2.D0*T)
 A0T=(T*T+T)/2.D0
 A1=(H**3)/5040.D0
 B0T=(A1/U)*(4.D0*T**7+(14.D0-7.D0*U)*T**6-14.D0*T**5+(-70.D0+
135.D0*U)*T**4+(98.D0-28.D0*U)*T*T+52.D0*T)
 B1T=(A1/(U-1.D0))*(-12.D0*T**7+(21.D0*U-63.D0)*T**6+42.D0*U*T*
1*5+(-210.D0*U+420.D0)*T**4+(1449.D0*U-1743.D0)*T*T+(1218.D0*U
2-1374.D0)*T)
 B2T=(A1/(U-2.D0))*(12.D0*T**7+(-21.D0*U+84.D0)*T**6+(-84.D0*U+
1126.D0)*T**5+(105.D0*U-420.D0)*T**4+(840.D0*U-1680.D0)*T**3+
2(1176.D0*U-2058.D0)*T*T+(504.D0*U-852.D0)*T)
 B3T=(A1/(U-3.D0))*(-4.D0*T**7+(7.D0*U-35.D0)*T**6+(42.D0*U-112
1.D0)*T**5+(70.D0*U-140.D0)*T**4+(-77.D0*U+133.D0)*T*T+
2(-42.D0*U+74.D0)*T)
 A2=U*(U-1.D0)*(U-2.D0)*(U-3.D0)
 BUT=(A1/A2)*(4.D0*T**7+14.D0*T**6-14.D0*T**5-70.D0*T**4+98.D0*
1T*T+52.D0*T)
 YN3C(I,J)=A0T*YN+A1T*YN1+A2T*YN2+B0T*F0+B1T*F1+B2T*F2+B3T*F3+
1BUT*FU
 YC=YN3C(I,J)
 AP2T=(2.D0*T+3.D0)/(2.D0*H)
 AP1T=-(2.D0*T+2.D0)/H
 AP0T=(2.D0*T+1.D0)/(2.D0*H)
 B1=H/2520.D0
 B2=14.D0*T**6+(42.D0-21.D0*U)*T**5-35.D0*T**4+(-140.D0+70.D0*U)*
1T**3+(98.D0-28.D0*U)*T+26.D0
 BP0T=H*B1*B2/U
 B3=-42.D0*T**6+(63.D0*U-189.D0)*T**5+105.D0*U*T**4+(-420.D0*U+840
```

```
     1.D0)*T**3+(1449.D0*U-1743.D0)*T+(609.D0*U-687.D0)
      BP1T=H*B1*B3/(U-1.D0)
      B4=42.D0*T**6+(-63.D0*U+252.D0)*T**5+(-210*U+315.D0)*T**4+(210.D0
     1*U-840.D0)*T**3+(1260.D0*U-2520.D0)*T*T+(1176.D0*U-2058.D0)*T
     2+252.D0*U-426.D0
      BP2T=H*B1*B4/(U-2.D0)
      B5=-14.D0*T**6+(21.D0*U-105.D0)*T**5+(105.D0*U-280.D0)*T**4+(140
     1.D0*U-280.D0)*T**3+(-77.D0*U+133.D0)*T-21.D0*U+37.D0
      BP3T=H*B1*B5/(U-3.D0)
      B6=U*(U-1.D0)*(U-2.D0)*(U-3.D0)
      B7=14.D0*T**6+42.D0*T**5-35.D0*T**4-140.D0*T**3+98.D0*T+26.D0
      BPUT=H*B1*B7/B6
      YPN3C(I,J)=AP0T*YN+AP1T*YN1+AP2T*YN2+BP1T*F1+BP2T*F2+BP3T*F3
     1+BPUT*FU+BP0T*F0 '
      YPC=YPN3C(I,J)
      ADY2T=1.D0/(H*H)
      ADY1T=-2.D0/(H*H)
      ADY0T=ADY2T
      C1=84.D0*T**5+(210.D0-105.D0*U)*T**4-140.D0*T**3+(210.D0*U-420
     1.D0)*T*T+98.D0-28.D0*U
      BDY0T=B1*C1/U
      C2=-252.D0*T**5+(315.D0*U-945.D0)*T**4+420.D0*U*T**3+(-1260.D0*
     1U+2520)*T*T+1449.D0*U-1743.D0
      BDY1T=B1*C2/(U-1.D0)
      C3=252.D0*T**5+(-315.D0*U+1260.D0)*T**4+(1260.D0-840.D0*U)*T**3
     1+(630.D0*U-2520.D0)*T*T+(2520.D0*U-5040)*T+1176.D0*U-2058.D0
      BDY2T=B1*C3/(U-2.D0)
      C4=-84.D0*T**5+(105.D0*U-525.D0)*T**4+(420.D0*U-1120.D0)*T**3+
     1(420.D0*U-1120.D0)*T*T-77.D0*U+133.D0
      BDY3T=B1*C4/(U-3.D0)
      C5=84.D0*T**5+210.D0*T**4-140.D0*T**3-420.D0*T*T+98.D0
      BDYUT=B1*C5/B6
      DYN3C(I,J)=ADY0T*YN+ADY1T*YN1+ADY2T*YN2+BDY1T*F1+BDY2T*F2+
     1BDY3T*F3+BDYUT*FU+BDY0T*F0
      DYC=DYN3C(I,J)
C     CALCULATE EXACT SOLUTION
      IF(X.GE.B) THEN
      YEX(I,J)=Y(X)
      ERC(I,J)=DABS(YN3C(I,J)-YEX(I,J))
      YE=YEX(I,J)
      ER=ERC(I,J)
      WRITE(6,10)X,YE,YC,ER
  10  FORMAT(2X,F7.4,3X,3D20.10)
C     CHANGE VARIABLE
      XN=XN1
      YN=YN1
      YP=YPN1
      DY=DYN1
      XN1=XN2
      YN1=YN2
      YPN1=YPN2
      DYN1=DYN2
      XN2=XN3
      YN2=YN3
      YPN2=YPN3
      DYN2=DYN3
      XN3=XN3+H
      ELSE
      X=X+DX
      GO TO 2
```

```
      ENDIF
      IF(B.GE.D) GO TO 1
      B=B+H
      GO TO 1
2     CONTINUE
1     CONTINUE
      STOP
      END
```