AXIAL FORCE INFLUENCE ON THE RESPONSE TO MOVING CONCENTRATED MASSES OF RECTANGULAR PLATES INCORPORATING ROTATORY INERTIA CORRECTION FACTOR.

BY

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CERTIFICATION

(A) **BY THE STUDENT:** This work has not been presented for the award of a degree, or any other purpose.

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DEDICATION

This work is dedicated to Almighty God for his loving kindness towards me

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ABSTRACT

This thesis studies the influence of axial force on the dynamic response to moving concentrated masses of rectangular plate incorporating rotatory inertia correction factor. The responses of the elastic structures to moving concentrated forces are special cases of such dynamical problems.

The governing equation of this problem is a fourth order partial differential equation. The solution technique is based on the use of the property of the Dirac-delta function as an even function to express it in series form, the versatile two-dimensional generalized integral transform with the normal modes of the plate as the kernel of transformation and a modification of the Struble's asymptotic technique. By the use of this technique, one is able to obtain closed form solutions for all variants of classical end conditions for this class of problems. The closed form solutions are analysed and numerical analyses in plotted curves are presented.

The results show that as the axial force (prestress), N_x and N_y, foundation moduli K and rotatory inertia Ro increase, the response amplitudes of the dynamical system decrease for both illustrative examples. However, higher values of N_x, N_y, K and Ro are required for a more noticeable effect in the case of simple-clamped boundary conditions than those of simply supported boundary conditions. It is also found that for both illustrative examples, the moving force solution is not an upper bound for the accurate solution of the moving mass problem of a

rectangular plate under the action of a concentrated moving load. This important result also agrees with the result of similar problems in literature.

Finally, in all the illustrative examples considered, for the same natural frequency, the critical speed for the moving mass problem is smaller than that of the moving force problem. Hence resonance is reached earlier in moving mass problem.

NOMENCLATURE

R_0 is the measure of rotatory inertia effect

L_x is the height of the rectangular plate

L_y is the length of the rectangular plate

N_x is the the axial force along x-direction

N_y is the the axial force along y-direction

K is the foundation stiffness of the rectangular plate

W(x,y,t) is the transverse displacement rectangular plate

D is the bending rigidity

P(x,y,t) is the concentrated load

M is the the mass moving load

v is the Poisson's ratio

$\delta(\ )$ is the Dirac delta function

E is the Young's Modulus

μ is the mass per unit area

c is the constant velocity

$W_j(x)$ is the j^{th} normal mode in the direction of x-axis of the plate

α_j, β_j are mode frequencies the plate

$\Omega_{j,k}$ is the natural circular frequency

$\beta_{j,k}$ is the modified frequency of the plate traversed by moving force

$\omega_{j,k}$ is the modified frequency of the plate traversed by moving mass

$\alpha^{*}_{j,k}$ is the modified frequency of simply supported of the plate traversed by moving force

$\gamma_{j,k}$ is the modified frequency of simply supported of the plate traversed by moving mass

$\alpha^{iv}_{r,j,k}$ is the modified frequency of simple-clamped of the plate traversed by moving force

$\alpha^{iv}_{p,j,k}$ is the modified frequency of simple-clamped of the plate traversed by moving mass

∇ is the Laplase operator.

CHAPTER ONE

1.0 INTRODUCTION

For more than a century the analysis of continuous elastic systems subjected to moving subsystems has been a subject of interest in many fields, from structural to mechanical to aerospace engineering [13]. However, it is especially in bridge engineering that this problem finds its widest field of application. Aside from the presence of structural damage due to environmental loads such as corrosion, material loss and support determination, the dynamic excitation caused by moving loads dramatically reduce the useful life of the bridge. In order to recognize when the structure is approaching an overstressed condition, it is necessary to understand the complexity of the dynamic interaction between the continuous system (bridge) and the subsystem (vehicle) moving on it.

Examples of structural configurations which may be elastic, inelastic or viscoelastic, on which these loads travel includes beams, plates, shells etc. The transverse motions of these structural configurations under moving loads are governed by the fourth order non-homogenous partial differential equations whose coefficients may be constant variable or singular. The vibrations of these structures are examined only during the period of the load traverse. Once the load departs from it, the structure begins to vibrate in free vibration, and this process no longer falls within the scope of our discussion. The attenuation of the whole phenomenon is greatly affected by the damping characteristics of both the structure and the materials.

In the analysis of the effects of vehicles moving over large-span bridges, Inglis[12] introduced a theory according to which the gravitational effects of the moving load may be separated from the inertia ones. In the calculation, the force is considered as moving along the beam while the mass of the vehicle acts at a definite, constant point x_0. The argument had been whether or not the second part of the assumption is justified. The inertia action of the mass on the deformed structure is described by [5] the D'Alembert's principle as the product of mass and acceleration. When the inertia effect of the moving load is considered, the governing partial differential equation of motion becomes complex and cumbersome and no longer has constant coefficients. In particular, the coefficients become variable and singular. If the inertia effect of the moving load is neglected in the governing equation, the problem is termed ***moving force problem*** and when it is retained it is termed ***moving mass problem***. Though, the problem, when the inertia effect of the moving load is considered negligible and has been greatly simplified, the following question arises: how safe is a design based on this assumption? The justification of this assumption would have been established had the solution of this approximate model been proved to be an upper bound for the actual deflection of the elastic system. Thus has not been so

6. **Resonance:** Is a phenomenon that occurs during vibration when the frequency of the moving load equals the natural frequency of the elastic structure. Practically speaking this means the state at which the deflection of the elastic structúres increases beyond bounds.

7. **Critical Speed:** This is defined as the speed of the load, which brings about resonance effects in the system.

8. **Deflection:** This is the vertical sag of structural members under load. The deflections of slender members are generally higher than that of thicker ones.

9. **Amplitude:** This is the maximum displacement in a structure subjected to dynamic loading or vibration.

BOUNDARY CONDITIONS OF STRUCTURAL MEMBERS

Aside the problem arising from the inclusion of the inertia terms in moving mass problems, difficulties often arise from the type of specified end-conditions. These end-conditions can be classified [3, 9] into two, viz:

(a) geometric boundary conditions,

(b) dynamic/force boundary conditions.

the geometric conditions relate to the deflection, say, $v(x,t)$ and slope $\frac{\partial v(x,t)}{\partial x}$, (where x is the spatial coordinate and t the time) while the dynamic or force boundary conditions relate to bending moments $\frac{\partial^2 v(x,t)}{\partial x^2}$, and shear force $\frac{\partial^3 v(x,t)}{\partial x^3}$, .

There are five end conditions that are of practical interest to an applied mathematician or an engineer. These are [3, 9, 4, 6, 31].

(i) clamped end conditions

(ii) simply supported or pinned end conditions

(iii) free end conditions

(iv) sliding end condition and

(v) non-classical end conditions.

The first four are known as classical end conditions.

At a clamped end, both deflection and slope vanish and at pinned or simply supported end, deflection and bending moment vanish. Furthermore, bending moment and shear force vanish at free ends and while slope and shear force vanish at sliding end. Thus, in particular, the first four boundary conditions may be written mathematically, in case of a uniform beam, as follows,

$$v(x,t) = \frac{\partial v(x,t)}{\partial x} = 0 \text{ at a clamped end} \tag{1.1}$$

$$v(x,t) = \frac{\partial^2 v(x,t)}{\partial x^2} = 0 \text{ at a pinned end.} \tag{1.2}$$

$$\frac{\partial^2 v(x,t)}{\partial x^2} = \frac{\partial^3 v(x,t)}{\partial x^3} = 0 \text{ at a free end} \tag{1.3}$$

$$\frac{\partial v(x,t)}{\partial x} = \frac{\partial^3 v(x,t)}{\partial x^3} = 0 \text{ at a sliding end} \tag{1.4}$$

As an example of non-classical end conditions, we briefly discuss the elastically supported end condition. Suppose a beam is hinged or pinned at one of its ends and supported by an elastic spring, with modulus k at the other end, the magnitude of the shearing force must be k times the displacement, i.e.

$$\frac{\partial^3 v(x,t)}{\partial x^3} = \pm kv \tag{1.5}$$

Thus, for such elastically supported end, the boundary conditions are

$$\frac{\partial^2 v(x,t)}{\partial x^2} = 0, \quad \frac{\partial^3 v(x,t)}{\partial x^3} - k_1 v\,(x,t) = 0 \tag{1.6}$$

where k_1 is an arbitrary spring modulus.

Other examples of non-classical end conditions include excited or time dependent boundary conditions.

In most of the work done on problems involving moving loads, methods of solution had been suitable only for simply supported end conditions. Thus, there is a need for method which is suitable, with or without consideration for inertia effects of the moving load, for all boundary conditions and structures that are of practical interest.

1.2 REVIEW OF RELATED LITERATURE

The problem of the response of an elastic system (beam or plate) to a moving load (moving force or moving mass) has been the objective of numerous investigations in Engineering, Mathematical Physics and Applied Mathematics for many years [6,31]. In particular, the dynamic response of a simply supported beam, traversed by a constant force moving at a uniform speed was first studied by Krylov [15]. His results were obtained by using the method of expansion of eigenfunctions. He assumed that the mass of the load is smaller than that of the beam. Later, Timoshenko [29] used energy methods to obtain solutions in series form for simply supported finite beam on an elastic foundation subjected to time dependent point loads moving with uniform velocities across the beam. Kenny [14] similarly investigated the dynamic response of infinite elastic beams on elastic foundation under the influence of load moving at constant speeds. He included the effects of viscous damping in the governing differential equation. Steel [28] also investigated the response of a finite simply supported Bernoulli-Euler beam to a unit force moving at a uniform velocity. He analysed the effects of this moving force on beams with and without an elastic foundation. Using a considerably simpler vector formulation with a Laplace rather than Fourier transformation, Steel[27] presented a review of the transient response of the Euler-Bernoulli beam and the Timoshenko beam on elastic foundation due to moving loads. The problem of a cylindrical shell with an engulfing axisymmetric pressure wave is shown to be generally quite analogous to Timoshenko beam. In a much later development Oni [20] considered the problem of a harmonic time-variable concentrated force moving at a uniform velocity over a finite deep beam. The methods of integral transformations are used. Series solution which converges is obtained for the deflection of simply supported beams and analysed for various speeds of the load. Just as for elastic beams, the problem of dynamic response of elastic plates to moving loads when the mass effect of the moving loads is considered have only attracted the attentions of few researchers. Among the earliest researchers into this subject was Holl [10] who solved the problem of a rectangular plate carrying uniformly moving loads. He concluded that a critical velocity existed for each mode of vibration. Livesly [16] on the other hand, considered the problem of a uniformly traveling load on an infinite plate and showed that there exists a certain critical velocity, beyond which stresses and deflections become infinite. However, in these studies, the plates considered were idealized by one where mass is approximately neglected.

Much later, Stanisic et al [26] made landmark feet when they studied the two-dimensional problems of flexural vibration of plates under the actions of loads, paying more attention to moving mass. Only the inertia term that measures the effect of local acceleration in the direction of the deflection was considered. The method of solution was based on the Fourier sine transform technique suitable only for simply-supported boundary conditions. The solutions so obtained were shown to converge very rapidly. For a plate structure, without an elastic foundation, Wu et al [32] used the finite element method to study the dynamic response under moving loads. He examined the effects of eccentricity, span length, acceleration and initial velocity of the moving load. The dynamics of a plate executing small motions relative to a reference frame undergoing overall rigid-body motion was presented by Banerjee and Kane [22]

Also, Aiyesimi [1] studied the dynamic response of an elastic, isotropic rectangular non-Mindlin plate resting on a visco-elastic foundation and under the action of a force moving with variable velocity. His method was based on the finite integral Fourier transform suitable only for simply – supported end conditions. it was shown that there is a slight drop in the maximum amplitude for the static load case before a steady state was attained. The work of Stanisic et al (1968) was taken up much later by Gbadeyan and Oni [7] who studied the dynamic analyses of an elastic plate continuously supported by an elastic Pasternak foundation traversed by an arbitrary number of concentrated masses. All the components of the inertia terms were considered and the rectangular plate was assumed to be simply supported, the deflection of the plate was calculated for several values of the foundation moduli and shown graphically as a function of time. As in the previous paper, the method of solution is suitable only for simply-supported boundary conditions. More recently, Oni [19] developed a versatile solution technique for solving two-dimensional moving load problems for all variants of classical boundary conditions. The technique involves the use of the modified generalized two-dimensional integral transform to reduce the fourth order differential equation governing the motion of the plate to second order ordinary differential equation which is then treated using the modified asymptotic method of Struble, Nayfey [18]. The elegant method in Oni [19] was extended by Oni [21] to investigate the dynamic behaviour under several masses of rectangular plates resting on a Pasternak elastic foundation and having an arbitrary end supports. The solution method was based on the modified two-dimensional generalized transform and a modification of Struble's asymptotic method. It was found that the critical speed for the system consisting of a rectangular plate resting on Pasternak's subgrade and traversed by a moving mass is reached prior that traversed by a moving force. Also, a two-dimensional theory, on the correction for rotatory inertia, on flexural motions of isotropic, elastic plate under moving load was studied by Oni [8]. The generalized two-dimensional integral transform with the normal modes of the plate as the kernel of transformation is used for the solution of the problem. The results show that the moving force solution is not always an upper bound for the accurate solution for the plate problem. In a more recent article, Huang and Thambiratnam [11] studied isotropic homogeneous elastic rectangular plate resting on an elastic Winkler foundation under a single concentrated load. Finite strip method is employed. Numerical examples show that when the load moves with zero or a small initial velocity, the dynamic response of the structure is steady and unlike the response due to the sudden application of a load. Worthy of note, also, is the work of Shadnam et al. [24] who investigated the dynamics of plates under the influence of relatively large masses, moving along an arbitrary trajectory on the plate surface. As an example, the dynamic response of a rectangular plate, simply supported on all its edges, under a mass moving parallel to one of its sides and also traveling along circular trajectory is presented by means of operational calculus. Analysis shows that the response of structures due to moving mass, which has often been neglected in the past must be properly taken into account because if often differs significantly from the moving force model. Shadnam [25] in a similar manner, worked on periodicity on the response of non-linear plate under moving mass. More recently, Oni and Omolofe [29] worked on the flexural vibration of prestressed Bernoulli – Euler beam resting on elastic foundation and traversed by masses traveling at varying speeds. Closed forms solutions were obtained and analyzed for both the problems of uniform and non uniform – Euler beams. However, no attempt was made to extend the theory developed in this study to solve the problem of flexural motions of prestressed rectangular plate under moving loads.

The plate model under consideration gives consideration to the effects of cross-sectional dimension on the response of the plates. This is so because, it is well known that a typical element of an elastic system performs not only a translatory motion but also rotates [30].

For simplicity in analysis two opposite sides of the plate are simply supported and other (two opposite edges) supported at will. Infact, plate structures of bridges are known usually to have two opposite edges simply supported and the other edges are free [22]

1.2 OBJECTIVES OF THE RESEARCH

This study concerns the influence of axial force on the dynamic response to moving concentrated masses of rectangular plates incorporating rotatory inertial correction factor.

The specific objectives of the study are to:

(a) obtain the analytical solution of the fourth order partial differential equation with variable and singular coefficients for all variants of boundary conditions.

(b) investigate and classify the influence of axial force on the response to moving concentrated masses of rectangular plates.

(c) classify the effects of rotatory inertia and foundation`s stiffness on the response amplitudes of the rectangular plates under the action of moving loads.

(d) examine the reliability of the moving force solution as a safe approximation to the moving mass solution.

(e) establish the resonance conditions for both moving force and moving mass problems and the effect axial force, rotatory inertia and foundation moduli on the resonance conditions

1.3 GOVERNING DIFFERENTIAL EQUATION

A single differential equation governing the deflections w of a rectangular plate under the action of a moving load is obtained from a two-dimensional theory of flexural motions of isotropic, elastic plates deduced from the three-dimensional equations of elasticity. The theory includes the effects rotatory inertia and shear deformation in the same manner as Timoshenko's one dimensional theory of bars. The single equation is given by [17]

$$\left(\nabla^2 - \frac{\rho}{G}\frac{\partial^2}{\partial t^2}\right)\left(D\nabla^2 - \frac{\rho h^2}{12}\frac{\partial^2}{\partial t^2}\right)w(x,y,t) + \rho h\frac{\partial^2 w(x,y,t)}{\partial t^2} =$$

$$\left(1 - \frac{D\nabla^2}{Gh} + \frac{\rho h}{12G}\frac{\partial^2}{\partial t^2}\right)P(x,y,t) \qquad (1.7)$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the two-dimensional Laplace operator, ρ is the density G is the modulus of elasticity in shear, D is the bending rigidity of the plate, h is the plate's thickness, x is the position co-ordinate in x-direction, y is the position co-ordinate in y-direction, t is the time, w(x,y,t) is the traverse displacement and P(x,y,t) is the moving load.

Equation (1.7) is the two-dimensional analogue of Timoshenko's beam equation. If the shear deformation terms are omitted from equation (1.7), it reduces to

$$\left(D\nabla^2 - \frac{\rho h^2}{12}\frac{\partial^2}{\partial t^2}\right)\nabla^2 w(x,y,t) + \rho h\frac{\partial^2 w(x,y,t)}{\partial t^2} = \mathrm{P(x,y,t)} \qquad (1.8)$$

Similarly, if rotatory inertia terms are neglected, equation (1.7) becomes

$$D\nabla^4 w(x,y,t) + \rho h\frac{\partial^2 w(x,y,t)}{\partial t^2} = \mathrm{P(x,y,t)} \qquad (1.9)$$

If this rectangular plate is prestressed, two additional terms given by

$$-\left(N_x \frac{\partial^2 w(x,y,t)}{\partial x^2} + N_y \frac{\partial^2 w(x,y,t)}{\partial y^2}\right) \quad (1.10)$$

where N_x is the axial prestress in the x-direction and N_y is the axial prestress in the y-direction are added to the left hand side of the equation

Equation (1.7) is the two-dimentional analogue of the Rayleigh beam equation in the one-dimensional theory of flextural motions of elastic beams carrying moving loads.

1.4 FEATURES OF THE THESIS

The procedure in the other sections of this study is as follows:

The initial-boundary value problem of axial force influence on the dynamic response to moving concentrated masses of rectangular plates on a Winkler elastic foundation is solved in general form in chapter two. Illustrative examples involving particular boundary conditions, numerical calculations and discussions of results are presented in chapter three

Finally, chapter four of the study contains summary of research work, contributions to knowledge, limitation of study and suggestion for further work

CHAPTER TWO

2.1 GOVERNING EQUATION

A rectangular plate of thickness h and lateral dimension L_x and L_y (Respectively in the x and y direction in the rectangular axis) under the actions of a concentrated load P(x, y, t) of mass M traveling from point $y = y_1$ on the plate along a straight line parallel to the x – axis with constant velocity c is considered in this thesis.

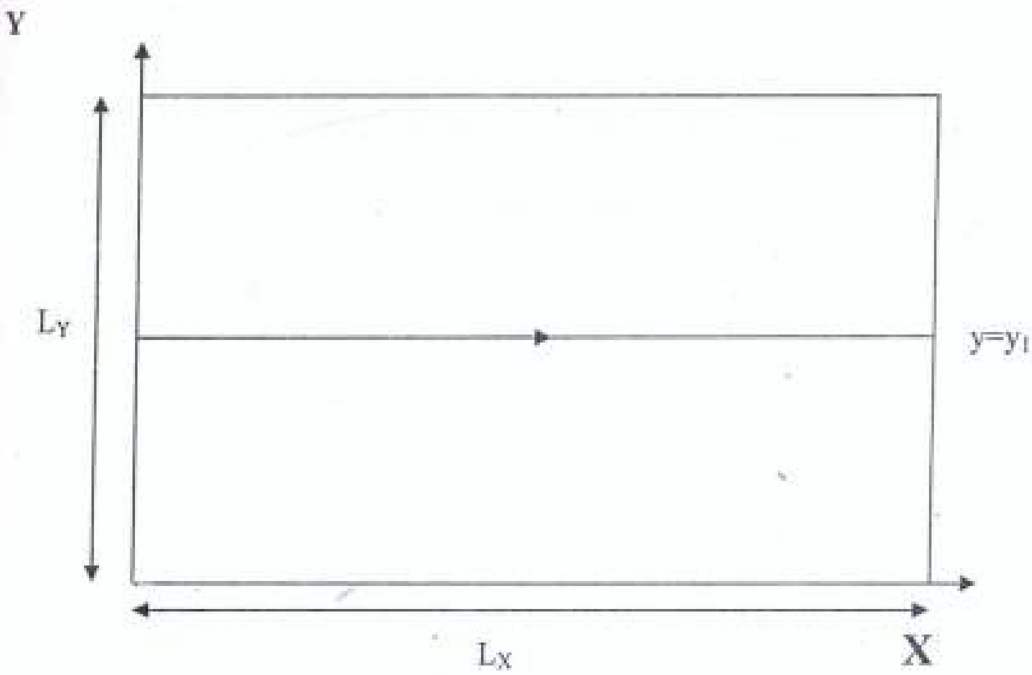

Fig 2.0: Plate structure carrying an arbitrary number N of the concentrated masses M, moving at a straight line parallel to x – axis.

Neglecting damping and the effects of shear deformation, according to the two-dimensional theory of flexural motions of isotropic elastic rectangular plate, the transverse displacement w (x, y, t), of the mid-surface of such rectangular plate exhibiting anisotropic prestress is found by solving [17]

$$\left(D\nabla^2 - \mu R_o \frac{\partial^2}{\partial t^2}\right)\nabla^2 w(x,y,t) - \left(N_x \frac{\partial^2 w(x,y,t)}{\partial x^2} + N_y \frac{\partial^2 w(x,y,t)}{\partial y^2}\right) +$$

$$\mu \frac{\partial^2 w(x,y,t)}{\partial t^2} + Kw(x,y,t) = P(x,y,t) \tag{2.1}$$

Where

$$D = \frac{Eh^3}{12(1-\nu)} \quad , \quad \nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \tag{2.2}$$

E is the Young's modulus of the plate, ν is the Poisson's ratio, t is time, x is the position coordinate in x – direction, y is the position coordinate in y direction, μ is the mass per unit area of the plate and R_o is the measure of rotatory inertia effect. In this system, when the inertia effect of the moving load on the transverse displacement of the rectangular plate is taken into consideration, the load P (x, y, t) takes the form

$$P(x,y,t) = P_f(x,y,t)\left[1 - \frac{1}{g}\left(\frac{\partial^2}{\partial t^2} + 2c\frac{\partial^2}{\partial x \partial t} + c^2\frac{\partial^2}{\partial x^2}\right)\right] \tag{2.3}$$

Where

$$P_f(x, y, t) = Mg\,\delta\,(x - ct)\,\delta\,(y - y_1) \tag{2.4}$$

M is the mass of moving load and δ () is the Dirac delta function defined as

$$\delta(x-ct) = \begin{cases} 0, x \neq ct \\ \infty, x = ct \end{cases} \tag{2.5}$$

With the properties

$$\text{(i)} \quad \int_{-\infty}^{\infty} \delta(x) = 1 \tag{2.6}$$

(ii) $\delta(-x)=\delta(x)$ (2.7)

(iii) $$\int_a^b \delta(t-k)f(t)=\begin{cases}0, k<a<b\\ f(k), a<k<b\\ 0, a<b<k\end{cases}$$ (2.8)

Furthermore, two opposite sides of the plate are simply supported and the other two opposite edges are taken to be arbitrary.

Infact, plate structures of bridges are known usually to have two opposite edges simply supported and the other edges free

Thus, at edges y =0 and y = L_y the following conditions pertain

$$w(x, o, t) = 0 = w(x, L_y, t) \text{ and}$$

$$\frac{\partial^2}{\partial y^2} w(x, o, t) = 0 = \frac{\partial^2}{\partial y^2} w(x, L_y, t) \quad (2.9)$$

For simplicity, the associated initial conditions are

$$w(x, y, t)/_{t=0} = 0 = \frac{\partial w}{\partial t}(x, y, t)/_{t=0} = 0 \quad (2.10)$$

Using equation (2.2), (2.3), (2.4), in equation (2.1), the governing equation of motion, after some rearrangements takes the firm

$$D_o\left[\frac{\partial^4}{\partial x^4}w(x,y,t)+2\frac{\partial^4}{\partial x^2\partial y^2}w(x,y,t)+\frac{\partial^4}{\partial y^4}w(x,y,t)\right]-N_x^o\frac{\partial^2}{\partial x^2}w(x,y,t)-$$
$$N_y^o\frac{\partial^2}{\partial y^2}w(x,y,t)+K_o w(x,y,t)-R_o\frac{\partial^2}{\partial t^2}\nabla^2 w(x,y,t)+\frac{\partial^2}{\partial t^2}w(x,y,t)+\frac{Mg}{\mu}\delta(x-ct)\delta(y-y_1)\bullet$$
$$\left(\frac{\partial^2}{\partial t^2}+2c\frac{\partial^2}{\partial x\partial t}+c^2\frac{\partial^2}{\partial x^2}\right)w(x,y,t)=\frac{Mg}{\mu}\delta(x-ct)\delta(y-y_1)$$

(2.11)

Where

$$D_m = \frac{D}{\mu},\ N_x^u = \frac{N_x}{\mu},\ N_y^u = \frac{N_y}{\mu},\quad K_0 = \frac{K}{\mu} \qquad (2.12)$$

2.2 ANALYTICAL SOLUTION PROCEDURE

In order to solve equation (2.1) subject to the end conditions (2.9), in the first instance, a transformation techniques based on the two dimensional Fourier sine integral transformations is employed. This technique is termed Generalized two-dimensional integral transformation and it is defined by

$$u(j,k,t) = \int_0^{L_x}\int_0^{L_y} w(x,y,t) Sin\frac{k\pi y}{L_y} W_j(x)\,dxdy \qquad (2.13)$$

with the inverse

$$w(x,y,t) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{2}{L_y}\frac{\mu}{w_j}u(j,k,t)\sin\frac{k\pi y}{L_y}W_j(x) \qquad (2.14)$$

Where

$$w_j = \int_0^{L_x} \mu W_j^2(x)\,dx \qquad (2.15)$$

and $W_j(x)$ is the J^{th} normal mode in the direction of x-axis vibration of the plate. This is obtained from the equation governing the free vibration of plate. It is given by

$$W_j(x) = Sin\frac{\alpha_j x}{L_x} + A_j Cos\frac{\alpha_j x}{L_x} + B_j Sinh\frac{\beta_j x}{L_x} + C_j Cosh\frac{\beta_j x}{L_x} \qquad (2.16)$$

Where

$$\alpha_j = L_x\left[\frac{k^2\pi^2}{L_y^2} - \frac{(\Omega_{jk}^2)^{1/2}}{D_m}\right]^{1/2} \qquad (2.17)$$

And

$$\beta_j = L_x\left[\frac{k^2\pi^2}{L_y^2} + \frac{(\Omega_{jk}^2)^{1/2}}{D_m}\right]^{1/2} \tag{2.18}$$

Are mode frequencies and A_j, B_j and C_j are constants and $D_m = D/\mu$. The parameter $\Omega_{j,k}$ is the natural circular frequency defined by

$$\Omega_{j,k}^2 = D_m\left(\frac{j^4\pi^4}{L_x^4} + 2\frac{j^2k^2\pi^4}{L_x^2L_y^2} + \frac{k^4\pi^4}{L_y^4}\right) \tag{2.19}$$

The function (2.16) satisfies all classical boundary conditions for this class of plate problem in the x direction.

Applying the generalized integral transformation (2.13) to (2.11), equation (2.11) can be written as.

$$\begin{aligned} &D_m Z_0(0,L_x,L_y) + D_m H_1(j,k,L_x,L_y) + U_n(j,k,t) - N_x^o H_2(j,k,L_x,L_y) - \\ &N_y^o H_3(j,k,L_x,L_y) - R_o H_4(j,k,L_x,L_y) - R_o H_5(j,k,L_x,L_y) \\ &\frac{M}{\mu}\left[G_1(p,k,t) + G_2(p,k,t) + G_3(p,k,t)\right] = \frac{Mg}{\mu} Sin\frac{k\Pi y_1}{L_y} W_j(ct) \end{aligned} \tag{2.20}$$

Where

$$\begin{aligned} Z_0(0,L_x,L_y) = \int_0^{L_y}\Big[&W_j(x)\frac{\partial^3 w}{\partial x^3} - W'_j(x)\frac{\partial^2 w}{\partial x^2} + W_j'''(x)\frac{\partial w}{\partial x} - wW_j'''(x) - \\ &\frac{k^2\pi^2}{L_y^2}\frac{\partial w}{\partial x}W_j(x) + \frac{k^2\pi^2}{L_y^2}W_j'(x)w\Big] Sin\frac{k\pi y}{L_y} \end{aligned} \tag{2.21}$$

$$H_1(j,k,L_x,L_y) = \int_0^{L_x}\int_0^{L_y} wW_j^{iv}(x) Sin\frac{k\pi y}{L_y}\,dxdy - \frac{k^2\pi^2}{L_y}\int_0^{L_x}\int_0^{L_y} wW_j''(x) Sin\frac{k\pi y}{L_y}\,dxdy +$$

$$+\frac{k^4\pi^4}{L_y^4}\int_0^{L_x}\int_0^{L_y} wW_j(x)Sin\frac{k\pi y}{L_y}dxdy \quad (2.22)$$

$$H_2(j,k,L_x,L_y)=\int_0^{L_x}\int_0^{L_y}\frac{\partial^2 w(x,y,t)}{\partial x^2}Sin\frac{k\pi y}{L_y}W_j(x)dxdy \quad (2.23)$$

$$H_3(j,k,L_x,L_y)=\int_0^{L_x}\int_0^{L_y}\frac{\partial^2 w(x,y,t)}{\partial y^2}Sin\frac{k\pi y}{L_y}W_j(x)dxdy \quad (2.24)$$

$$H_4(j,k,L_x,L_y)=\int_0^{L_x}\int_0^{L_y}\frac{\partial^4 w(x,y,t)}{\partial t^2\partial x^2}Sin\frac{k\pi y}{L_y}W_j(x)dxdy \quad (2.25)$$

$$H_5(j,k,L_x,L_y)=\int_0^{L_x}\int_0^{L_y}\frac{\partial^4 w(x,y,t)}{\partial t^2\partial y^2}Sin\frac{k\pi y}{L_y}W_j(x)dxdy \quad (2.26)$$

$$G_1(p,k,t)=\int_0^{L_x}\int_0^{L_y}\delta(x-ct)\delta(y-y_1)\frac{\partial^2 w(x,y,t)}{\partial t^2}Sin\frac{k\pi y}{L_y}W_j(x)dxdy \quad (2.27)$$

$$G_2(p,k,t)=2c\int_0^{L_x}\int_0^{L_y}\delta(x-ct)\delta(y-y_1)\frac{\partial^2 w(x,y,t)}{\partial x\partial t}Sin\frac{k\pi y}{L_y}W_j(x)dxdy \quad (2.28)$$

$$G_3(p,k,t)=c^2\int_0^{L_x}\int_0^{L_y}\delta(x-ct)\delta(y-y_1)\frac{\partial^2 w(x,y,t)}{\partial x^2}Sin\frac{k\pi y}{L_y}W_j(x)dxdy \quad (2.29)$$

it is remarked at this juncture that when

$$w(x,y,t)=Sin\frac{k\pi y}{L_y}W_j(x)Sin\Omega_{j,k}t \quad (2.30)$$

Where

$$\Omega_{j,k}=\left(\frac{D}{\mu}\left(\frac{j^4\pi^4}{L_x^4}+2\frac{j^2k^2\pi^4}{L_x^2L_y^2}+\frac{k^4\pi^4}{L_y^4}\right)\right)^{1/2} \quad (2.31)$$

is the natural circular frequency of a rectangular plate is substituted into the equation of free vibration of plate namely

$$D\left[\frac{\partial^4 w(x,y,t)}{\partial x^4}+2\frac{\partial^4 w(x,y,t)}{\partial x^2\partial y^2}+\frac{\partial^4 w(x,y,t)}{\partial y^4}\right]+\mu\frac{\partial^2 w(x,y,t)}{\partial t^2}=0 \qquad (2.32)$$

One obtains

$$D\left[W_j^{iv}(x)-2\frac{k^2\pi^2}{L_y^2}W_j^{''}(x)+\frac{k^4\pi^2}{L_y^2}W_j^{''}(x)+\frac{k^4\pi^4}{L_y^4}W_j(x)\right]=\Omega_{j,k}^2W_j(x) \qquad (2.33)$$

On multiplying (2.33) by $w(x,y,t)Sin\frac{k\pi y}{L_y}$

And integrating with respect to x and y between the limits o, L_x and o, Ly respectively, clearly

$$D_m\left[\int_0^{L_x}\int_0^{L_y}wW_j^{iv}(x)Sin\frac{k\pi y}{L_y}dxdy-2\frac{k^2\pi^2}{L_y^2}\int_0^{L_x}\int_0^{L_y}wW_j^{''}(x)Sin\frac{k\pi y}{L_y}dxdy+\right.$$

$$\left.+\frac{k^4\pi^4}{L_y^4}\int_0^{L_x}\int_0^{L_y}wW_j(x)Sin\frac{k\pi y}{L_y}dxdy\right]=\Omega_{j,k}^2u(j,k,t) \qquad (2.34)$$

Consequently,

$$H_1(j,k,L_xL_y)=\frac{\mu}{D}\Omega_{j,k}^2u(j,k,t) \qquad (2.35)$$

In order to evaluate H_2 (j,k,L_x, L_y), it is noted that for any arbitrary subscripts j = p , k = q, equation (2.14) can be written as

$$\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\frac{2}{L_x}\frac{\mu}{w_p}U(p,q,t)Sin\frac{q\pi y}{L_y}W_p(x) \qquad (2.36a)$$

It follows that

To evaluate integral (2.20) , use is made of the property of the Dirac Delta function as an function to express it as a Fourier Cosine Series, namely

$$\delta(x-ct)=\frac{1}{L_x}+\frac{2}{L_x}\sum Cos\frac{n\pi ct}{L_x}Cos\frac{n\pi x}{L_x} \tag{2.44}$$

Similarly,

$$\delta(y-y_1)=\frac{1}{L_y}+\frac{2}{L_y}\sum_{n=1}^{\infty}Cos\frac{n\pi y_1}{L_y}Cos\frac{n\pi y}{L_y} \tag{2.45}$$

It follows from (2.14), that

$$w_n(x,y,t)=\frac{2}{L_y}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\frac{\mu}{w_p}u_n(p,q,t)Sin\frac{q\pi y}{L_y}W_p(x) \tag{2.46}$$

Thus, using (2.44), (2.45) and (2.46)

$$\delta(x-ct)\delta(y-y_1)w_n(x,y,t)$$

$$=\frac{8}{L_xL_y^2}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\mu}{w_p}u_n(p,q,t)\cos\frac{n\pi ct}{L_x}\cos\frac{m\pi y_1}{L_y}\left[\cos\frac{n\pi x}{L_x}W_p(x)\right]\left[\cos\frac{m\pi y}{L_y}\sin\frac{q\pi y}{L_y}\right]$$

$$+\frac{4}{L_xL_y^2}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{n=1}^{\infty}\frac{\mu}{w_p}u_n(p,q,t)\cos\frac{n\pi ct}{L_x}\left[\cos\frac{\eta\pi x}{L_x}W_p(x)\right]\sin\frac{q\pi y}{L_y}$$

$$+\frac{4}{L_xL_y^2}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=1}^{\infty}\frac{\mu}{W_p}u_n(p,q,t)\left[\cos\frac{m\pi y}{L_y}\sin\frac{q\pi y}{L_y}\right]W_p(x)$$

$$+\frac{4}{L_xL_y^2}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\frac{\mu}{w_p}u_n(p,q,t)\sin\frac{q\pi y}{L_y}W_p(x) \tag{2.47}$$

$$G_1(p,k,t)=\frac{4}{L_xL_y^2}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\mu}{w_p}u_n(p,q,t)\cos\frac{n\pi ct}{L_x}\cos\frac{m\pi y_1}{L_y}$$

$$\left[\int_0^{L_x} \cos\frac{n\pi x}{L_x} W_j(x) W_p(x) dx\right]\left[\int_0^{L_y} \sin\frac{(m+q)\pi y}{L_y} \sin\frac{k\pi y}{L_y} dy\right.$$

$$\left.-\int_0^{L_y} \sin\frac{(m-q)\pi y}{L_y} \sin\frac{k\pi y}{L_y} dy\right] + \frac{4}{L_x L_y^2}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{n=1}^{\infty}\frac{\mu}{w_p} u_{tt}(p,q,t)\cos\frac{n\pi ct}{L_x}$$

$$\left[\int_0^{L_x} \cos\frac{n\pi x}{L_x} W_j(x) W_p(x) dx\right]\left[\int_0^{L_y} \sin\frac{q\pi y}{L_y} \sin\frac{k\pi y}{L_y} dy\right]$$

$$+\frac{2}{L_x L_y^2}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=1}^{\infty}\frac{\mu}{w_p} U_{tt}(p,q,t)\cos\frac{m\pi y_1}{L_y}\left[\int_0^{L_y} \sin\frac{(m+q)\pi y}{L_y} \sin\frac{k\pi y}{L_y} dy\right.$$

$$\left.-\int_0^{L_y} \sin\frac{(m-q)\pi y}{L_y} \sin\frac{k\pi y}{L_y} dy\right]\left[\int_0^{L_x} W_j(x) W_p(x)\right]$$

$$+\frac{2}{L_x L_y^2}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\frac{\mu}{w_p} u_{tt}(p,q,t)\left[\int_0^{L_y} \sin\frac{q\pi y}{L_y} \sin\frac{k\pi y}{L_y} dy\right]\left[\int_0^{L_x} W_j(x) W_p(x) dx\right] \qquad (2.48a)$$

Which when simplified and rearranged becomes

$$G_1(p,k,t) = \frac{4}{L_x L_y}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{n=1}^{\infty}\frac{\mu}{w_p} u_{tt}(p,q,t)\cos\frac{n\pi ct}{L_x}\sin\frac{k\pi y_1}{L_y}$$

$$\sin\frac{q\pi y_1}{L_y}\Lambda(n,j,p) + \frac{2}{L_x L_y}\sum_{p=1}^{\infty}\sum_{n=1}^{\infty}\frac{\mu}{w_p} u_{tt}(p,q,t)\cos\frac{n\pi ct}{L_x}\Lambda(n,j,p)$$

$$+\frac{2}{L_x L_y}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\frac{\mu}{w_p} u_{tt}(p,q,t)\sin\frac{q\pi y_0}{L_y}\Lambda(j,p) + \frac{1}{L_x L_y}\sum_{p=1}^{\infty}\frac{\mu}{w_p} u_{tt}(p,q,t)\Lambda(j,p) \qquad (2.48b)$$

Where $\Lambda(j,p)$ is as defined in (2.40) and

$$\Lambda(n,j,p) = \int_0^{L_x} \cos\frac{n\pi x}{L_x} W_j(x) W_p(x) dx \qquad (2.49)$$

Using similar argument in the evaluation of $G_1(p,k,t)$, the following results are obtained from $G_2(p,k,t)$ and $G_3(p,k,t)$, namely,

$$G_2(p,k,t)=\frac{4}{L_xL_y}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{n=1}^{\infty}\frac{\mu}{w_p}U_t(p,q,t)\cos\frac{n\pi ct}{L_x}\sin\frac{k\pi y_1}{L_y}\sin\frac{q\pi y_1}{L_y}$$

$$\Lambda^1(n,j,p)+\frac{2}{L_xL_y}\sum_{p=1}^{\infty}\sum_{n=1}^{\infty}\frac{\mu}{w_p}U_t(p,k,t)\cos\frac{n\pi ct}{L_x}\Lambda^1(n,j,p)$$

$$+\frac{2}{L_xL_y}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\frac{\mu}{w_p}u_t(p,q,t)\sin\frac{k\pi y_1}{L_y}\sin\frac{q\pi y_1}{L_y}\Lambda^1(j,p)+\frac{1}{L_xL_y}\sum_{p=1}^{\infty}\frac{\mu}{w_p}u_t(p,k,t)\Lambda^1(j,p)$$

(2.50)

where

$$\Lambda^1(j,p)=\int_0^{L_x}W_j(x)\frac{d}{dx}W_p(x)dx \qquad (2.51)$$

$$\Lambda^1(n,j,p)=\int_0^{L_x}\cos\frac{n\pi x}{L_x}W_j(x)\frac{d}{dx}W_p(x)dx \qquad (2.52)$$

And

$$G_3(p,k,t)=\frac{4}{L_xL_y}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{n=1}^{\infty}\frac{\mu}{w_p}u(p,q,t)\cos\frac{n\pi ct}{L_x}\sin\frac{k\pi y_1}{L_y}\sin\frac{q\pi y_1}{L_y}\Lambda^2(n,j,p)$$

$$+\frac{2}{L_xL_y}\sum_{p=1}^{\infty}\sum_{n=1}^{\infty}\frac{\mu}{w_p}u(p,k,t)\cos\frac{n\pi ct}{L_x}\Lambda^2(n,j,p)+\frac{2}{L_xL_y}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\frac{\mu}{w_p}u(p,q,t)\sin\frac{k\pi y_1}{L_y}\bullet$$

$$\sin\frac{q\pi y_1}{L_y}\Lambda^2(j,p)+\frac{1}{L_xL_y}\sum_{p=1}^{\infty}\frac{\mu}{w_p}u(p,k,t)\Lambda^2(j,p) \qquad (2.53)$$

where

$$\Lambda^2(j,p)=\int W_j(x)\frac{d^2}{dx^2}W_p(x)dx \qquad (2.54)$$

$$\Lambda^2(n,j,p)=\int_0 \cos\frac{n\pi x}{L_x}W_j(x)\frac{d^2}{dx^2}W_p(x)dx \tag{2.55}$$

Substituting (2.22), (2.23), (2.24), (2.25), (2.26),(2.27),(2.28) and (2.29) into equation (2.20) after same simplification and rearrangements yields

$$u_{tt}(j,k,t)+\left(\Omega^2_{j,k}+K\right)u(j,k,t)+D_n Z_n(0,L_x,L_y)$$

$$-\sum_{p=1}^{\infty}\frac{\mu}{w_p}u(p,k,t)\left[N_x^0\Lambda^2(p,j)-N_y^0\frac{k^2\pi^2}{L_y^2}\Lambda(p,j)\right]$$

$$-R_o\sum_{p=1}^{\infty}\frac{\mu}{w_p}u_{tt}(p,k,t)\left[\Lambda^2(p,j)-\frac{k^2\pi^2}{L_y^2}\Lambda(p,j)\right]$$

$$+\frac{M}{\mu L_x L_y}\left\{4\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{n=1}^{\infty}\frac{\mu}{w_p}u_{tt}(p,q,t)\cos\frac{n\pi ct}{L_x}\sin\frac{k\pi y_1}{L_y}\sin\frac{q\pi y_1}{L_y}\Lambda(n,j,p)\right.$$

$$+2\sum_{p=1}^{\infty}\sum_{n=1}^{\infty}\frac{\mu}{w_p}u_{tt}(p,k,t)\cos\frac{n\pi ct}{L_x}\Lambda(n,j,p)$$

$$+2\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\frac{\mu}{w_p}u_{tt}(p,q,t)\sin\frac{k\pi y_1}{L_y}\sin\frac{q\pi y_1}{L_y}\Lambda(j,p)+\sum_{p=1}^{\infty}\frac{\mu}{w_p}u_{tt}(p,k,t)\Lambda(j,p)$$

$$+2c\left[4\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{n=1}^{\infty}\frac{\mu}{w_p}u_t(p,q,t)\cos\frac{n\pi ct}{L_x}\sin\frac{k\pi y_1}{L_y}\sin\frac{q\pi y_1}{L_y}\Lambda^1(n,j,p)\right.$$

$$+2\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\frac{\mu}{w_p}u_{tt}(p,q,t)\cos\frac{n\pi ct}{L_x}\Lambda^1(n,j,p)$$

$$\left.+2\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\frac{\mu}{w_p}u_t(p,q,t)\sin\frac{k\pi y_1}{L_y}\sin\frac{q\pi y_1}{L_y}\Lambda(j,p)+\sum_{p=1}^{\infty}\frac{\mu}{w_p}u_t(p,k,t)\Lambda^1(j,p)\right]$$

$$+c^2\left[4\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{n=1}^{\infty}\frac{\mu}{w_p}U(p,q,t)\cos\frac{n\pi ct}{L_x}\sin\frac{k\pi y_1}{L_y}\sin\frac{q\pi y_1}{L_y}\Lambda(n,j,p)\right.$$

$$+2\sum_{p=1}^{\infty}\sum_{n=1}^{\infty}\frac{\mu}{w_p}u(p,k,t)\cos\frac{n\pi ct}{L_x}\Lambda^2(n,j,p)$$

$$+2\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\frac{\mu}{w_p}u(p,q,t)\sin\frac{k\pi y_1}{L_y}\sin\frac{q\pi y_1}{L_y}\Lambda^2(j,p)+\sum_{p=1}^{\infty}\frac{\mu}{w_p}u(p,k,t)\Lambda^2(j,p)\Bigg]\Bigg\}$$

$$=\frac{Mg}{\mu}\sin\frac{k\pi y_1}{L_y}W_j(ct) \tag{2.56}$$

It is remarked at this juncture this $Z_n(0,L_x,L_y)=0$ for all classical boundary conditions.

It can be shown that p^th^ normal mode in the direction of x – axis of vibration of a plate is

$$W_p(x)=\sin\alpha_p\frac{x}{L_x}+A_p\cos\alpha_p\frac{x}{L_x}+B_p\sinh\beta_p\frac{x}{L_x}+C_p\cosh\beta_p\frac{x}{L_x} \tag{2.57}$$

Thus,

$$\frac{d}{dx}W_p(x)=\frac{\alpha_p}{L_x}\left[\cos\alpha_p\frac{x}{L_x}-A_p\sin\alpha_p\frac{x}{L_x}+B_p\frac{\beta_p}{\alpha_p}\cosh\beta_p\frac{x}{L_x}+C_p\frac{\beta_p}{\alpha_p}\sinh\beta_p\frac{x}{L_x}\right] \tag{2.58}$$

$$\frac{d^2}{dx^2}W_p(x)=\frac{\alpha_p^2}{L_x^2}\left[-\sin\alpha_p\frac{x}{L_x}-A_p\cos\alpha_p\frac{x}{L_x}+B_p\frac{\beta_p^2}{\alpha_p^2}\sinh\beta_p\frac{x}{L_x}+C_p\frac{\beta_p^2}{\alpha_p^2}\cosh\beta_p\frac{x}{L_x}\right] \tag{2.59}$$

Consequently,

$$\Lambda(j,p)=\big[I_1+A_pI_2+B_pI_3+C_pI_4+A_jI_5+A_jA_pI_6+A_jB_pI_7$$

$$+A_jC_pI_8+B_jI_9+B_jA_pI_{10}+B_jB_pI_{11}+B_jC_pI_{12}$$

$$+C_j I_{13} + C_j A_p I_{14} + C_j B_p I_{15} + C_j C_p I_{16}\Big] \tag{2.60}$$

$$\Lambda^1(j,p) = \frac{\alpha_p}{L_x}\Big[I_2 - A_p I_1 + B_p I_4 + C_p I_3 + A_j I_6 - A_j A_p I_5 + A_j B_p I_8$$

$$+ A_j C_p I_7 + B_j I_{10} - B_j A_p I_9 + B_j B_p I_{12} + B_j C_p I_{11}$$

$$+ C_j I_{14} - C_j A_p I_{13} + C_j B_p I_{16} + C_j C_p I_{15}\Big] \tag{2.61}$$

$$\Lambda^2(j,p) = \frac{\alpha_p^2}{L_x^2}\Big[- I_1 - A_p I_2 + B_p I_3 + C_p I_4 - A_j I_5 - A_j A_p I_6 + A_j B_p I_7$$

$$+ A_j C_p I_8 - B_j I_9 - B_j A_p I_{10} + B_j B_p I_{11} + B_j C_p I_{12}$$

$$- C_j I_{13} - C_j A_p I_{14} + C_j B_p I_{15} + C_j C_p I_{16}\Big] \tag{2.62}$$

$$\Lambda(n,j,p) = \Big[I_{17} + A_p I_{18} + B_p I_{19} + C_p I_{20} + A_j I_{21} + A_j A_p I_{22} + A_j B_p I_{23}$$

$$+ A_j C_p I_{24} + B_j I_{25} + B_j A_p I_{26} + B_j B_p I_{27} + B_j C_p I_{28}$$

$$+ C_j I_{29} + C_j A_p I_{30} + C_j B_p I_{31} + C_j C_p I_{32}\Big] \tag{2.63}$$

$$\Lambda^1(n,j,p) = \frac{\alpha_p}{L_x}\Big[I_{18} - A_p I_{17} + B_p I_{20} + C_p I_{19} + A_j I_{22} - A_j A_p I_{21} + A_j B_p I_{24}$$

$$+ A_j C_p I_{23} + B_j I_{26} - B_j A_p I_{25} + B_j B_p I_{28} + B_j C_p I_{27}$$

$$+ C_j I_{30} - C_j A_p I_{29} + C_j B_p I_{32} + C_j C_p I_{31}\Big] \tag{2.64}$$

And

$$\Lambda^2(n,j,p) = \frac{\alpha_p^2}{L_x^2}\Big[- I_{17} - A_p I_{18} + B_p I_{19} + C_p I_{20} - A_j I_{21} - A_j A_p I_{22} + A_j B_p I_{23}$$

$$+ A_j C_p I_{24} - B_j I_{25} - B_j A_p I_{26} + B_j B_p I_{27} + B_j C_p I_{28}$$

$$- C_j I_{29} - C_j A_p I_{30} + C_j B_p I_{31} + C_j C_p I_{32}\Big] \tag{2.65}$$

Where

$$I_1 = \int_0^{L_x} Sin\frac{\alpha_j x}{L_x} Sin\frac{\alpha_p x}{L_x}dx, \qquad I_2 = \int_0^{L_x} Sin\frac{\alpha_j x}{L_x} Cos\frac{\beta_p x}{L_x}dx,$$

$$I_3 = \int_0^{L_x} Sin\frac{\alpha_j x}{L_x} Sinh\frac{\beta_p x}{L_x}dx, \qquad I_4 = \int_0^{L_x} Sin\frac{\alpha_j x}{L_x} Cosh\frac{\beta_p x}{L_x}dx,$$

$$I_5 = \int_0^{L_x} Cos\frac{\alpha_j x}{L_x} Sin\frac{\alpha_p x}{L_x}dx, \qquad I_6 = \int_0^{L_x} Cos\frac{\alpha_j x}{L_x} Cos\frac{\alpha_p x}{L_x}dx,$$

$$I_7 = \int_0^{L_x} Cos\frac{\alpha_j x}{L_x} Sinh\frac{\beta_p x}{L_x}dx, \qquad I_8 = \int_0^{L_x} Cos\frac{\alpha_j x}{L_x} Cosh\frac{\beta_p x}{L_x}dx,$$

$$I_9 = \int_0^{L_x} Sinh\frac{\beta_j x}{L_x} Sin\frac{\alpha_p x}{L_x}dx, \qquad I_{10} = \int_0^{L_x} Sinh\frac{\beta_j x}{L_x} Cos\frac{\alpha_p x}{L_x}dx,$$

$$I_{11} = \int_0^{L_x} Sinh\frac{\beta_j x}{L_x} Sinh\frac{\beta_p x}{L_x}dx, \qquad I_{12} = \int_0^{L_x} Sinh\frac{\beta_j x}{L_x} Cosh\frac{\beta_p x}{L_x}dx,$$

$$I_{13} = \int_0^{L_x} Cosh\frac{\beta_j x}{L_x} Sin\frac{\alpha_p x}{L_x}dx, \qquad I_{14} = \int_0^{L_x} Cosh\frac{\beta_j x}{L_x} Cos\frac{\alpha_p x}{L_x}dx,$$

$$I_{15} = \int_0^{L_x} Cosh\frac{\beta_j x}{L_x} Sinh\frac{\beta_p x}{L_x}dx, \qquad I_{16} = \int_0^{L_x} Cosh\frac{\beta_j x}{L_x} Cosh\frac{\beta_p x}{L_x}dx,$$

$$I_{171} = \int_0^{L_x} Cos\frac{n\pi x}{L_x} Sin\frac{\alpha_j x}{L_x} Sin\frac{\alpha_p x}{L_x}dx, \quad I_{18} = \int_0^{L_x} Cos\frac{n\pi x}{L_x} Sin\frac{\alpha_j x}{L_x} Cos\frac{\beta_p x}{L_x}dx,$$

$$I_{19} = \int_0^{L_x} Cos\frac{n\pi x}{L_x} Sin\frac{\alpha_j x}{L_x} Sinh\frac{\beta_p x}{L_x}dx, \qquad I_{20} = \int_0^{L_x} Cos\frac{n\pi x}{L_x} Sin\frac{\alpha_j x}{L_x} Cosh\frac{\beta_p x}{L_x}dx,$$

$$I_{21} = \int_0^{L_x} Cos\frac{n\pi x}{L_x} Cos\frac{\alpha_j x}{L_x} Sin\frac{\alpha_p x}{L_x}dx, \quad I_{22} = \int_0^{L_x} Cos\frac{n\pi x}{L_x} Cos\frac{\alpha_j x}{L_x} Cos\frac{\alpha_p x}{L_x}dx,$$

$$I_{23} = \int_0^{L_x} Cos\frac{n\pi x}{L_x} Cos\frac{\alpha_j x}{L_x} Sinh\frac{\beta_p x}{L_x}dx, \qquad I_{24} = \int_0^{L_x} Cos\frac{n\pi x}{L_x} Cos\frac{\alpha_j x}{L_x} Cosh\frac{\beta_p x}{L_x}dx,$$

$$I_{25}=\int_0^{L_x} Cos\frac{n\pi x}{L_x}Sinh\frac{\beta_j x}{L_x}Sin\frac{\alpha_p x}{L_x}dx,\quad I_{26}=\int_0^{L_x} Cos\frac{n\pi x}{L_x}Sinh\frac{\beta_j x}{L_x}Cos\frac{\alpha_p x}{L_x}dx,$$

$$I_{27}=\int_0^{L_x} Cos\frac{n\pi x}{L_x}Sinh\frac{\beta_j x}{L_x}Sinh\frac{\beta_p x}{L_x}dx,\quad I_{28}=\int_0^{L_x} Cos\frac{n\pi x}{L_x}Sinh\frac{\beta_j x}{L_x}Cosh\frac{\beta_p x}{L_x}dx,$$

$$I_{29}=\int_0^{L_x} Cos\frac{n\pi x}{L_x}Cosh\frac{\beta_j x}{L_x}Sin\frac{\alpha_p x}{L_x}dx,\quad I_{30}=\int_0^{L_x} Cos\frac{n\pi x}{L_x}Cosh\frac{\beta_j x}{L_x}Cos\frac{\alpha_p x}{L_x}dx,$$

$$I_{31}=\int_0^{L_x} Cos\frac{n\pi x}{L_x}Cosh\frac{\beta_j x}{L_x}Sinh\frac{\beta_p x}{L_x}dx,\quad I_{32}=\int_0^{L_x} Cos\frac{n\pi x}{L_x}Cosh\frac{\beta_j x}{L_x}Cosh\frac{\beta_p x}{L_x}dx,$$

In order to solve equation (2.56), in view of [5], it is arranged to take the form

$$u_{tt}(j,k,t)+\alpha_{j,k}^2 u(j,k,t)-R_o\sum_{p=1}^{\infty}\frac{\mu}{w_p}u_{tt}(p,k,t)\left[\Lambda^2(p,j)-\frac{k^2\pi^2}{L_y^2}\Lambda(p,j)\right]$$

$$-\sum_{p=1}^{\infty}\frac{\mu}{w_p}u(p,k,t)\left[N_x^o\Lambda^2(p,j)-\ N_y^o\frac{k^2\pi^2}{L_y^2}\Lambda(p,j)\right]$$

$$+\Gamma_o\left\{4\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{n=1}^{\infty}\frac{\mu}{w_p}u_{tt}(p,q,t)\cos\frac{n\pi ct}{L_x}\sin\frac{k\pi y_1}{L_y}\sin\frac{q\pi y_1}{L_y}\Lambda(n,j,p)\right.$$

$$+2\sum_{p=1}^{\infty}\sum_{n=1}^{\infty}\frac{\mu}{w_p}u_{tt}(p,k,t)\cos\frac{n\pi ct}{L_x}\Lambda(n,j,p)+2\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\frac{\mu}{w_p}u_{tt}(p,q,t)\sin\frac{k\pi y_1}{L_y}\sin\frac{q\pi y_1}{L_y}\Lambda(j,p)$$

$$+\sum_{p=1}^{\infty}\frac{\mu}{w_p}u_{tt}(p,k,t)\Lambda(p,j)+2c\left[4\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{n=1}^{\infty}\frac{\mu}{w_p}u_t(p,q,t)\cos\frac{n\pi ct}{L_x}\sin\frac{k\pi y_1}{L_y}\sin\frac{q\pi y_1}{L_y}\Lambda^1(n,j,p)\right.$$

$$+2\sum_{p=1}^{\infty}\sum_{n=1}^{\infty}\frac{\mu}{w_p}u_t(p,k,t)\cos\frac{n\pi ct}{L_x}\Lambda(n,j,p)$$

$$\left.+2\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\frac{\mu}{w_p}u_t(p,q,t)\sin\frac{k\pi y_1}{L_y}\sin\frac{q\pi y_1}{L_y}\Lambda^1(j,p)+\sum_{p=1}^{\infty}\frac{\mu}{w_p}u_t(p,k,t)\Lambda^1(p,j)\right]$$

$$+c^2\left\{4\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{n=1}^{\infty}\frac{\mu}{w_p}u(p,q,t)\cos\frac{n\pi ct}{L_x}\sin\frac{k\pi y_1}{L_y}\sin\frac{q\pi y_1}{L_y}\Lambda^2(n,j,p)\right.$$

$$+2\sum_{p=1}^{\infty}\sum_{n=1}^{\infty}\frac{\mu}{w_p}u(p,k,t)\cos\frac{n\pi ct}{L_x}\Lambda^2(n,j,p)+2\sum_{p=1}^{\infty}\sum_{n=1}^{\infty}\frac{\mu}{w_p}u(p,k,t)\sin\frac{k\pi y_1}{L_y}\sin\frac{q\pi y_1}{L_y}\Lambda^2(j,p)$$

$$+\sum_{p=1}^{\infty}\frac{\mu}{w_p}u(p,k,t)\Lambda^2(p,j)=\frac{Mg}{\mu}\sin\frac{k\pi y_1}{L_y}W_j(c,t) \quad (2.66)$$

Where

$$\Gamma_o=\frac{M}{\mu L_x L_y} \quad (2.67)$$

And

$$P_o=\frac{Mg}{\mu} \quad (2.68)$$

Equation (2.66) is now the fundamental transformed equation of our problem, when the presressed plate has arbitrary end support conditions along x – direction.

2.3 SOLUTION OF THE TRANSFORMED EQUATION

In this section, two special cases of equation (2.66) are discussed. They are termed moving force and moving mass problems.

2.3.1 RECTANGULAR PLATE TRAVERSED BY A MOVING FORCE PROBLEM

In this section an approximate model of the differential equation describing the response of a prestressed rectangular plate with rotatory inertia effect and traversed by a moving force may be obtained from equation (2.66) by setting $\Gamma_{n}=0$

Thus, setting, $\Gamma_{n}=0$ equation (2.66) reduces to

$$u_{tt}(j,k,t)+\alpha_{j,k}^{2}u\,(j,k,t)-R_{n}\sum_{p=1}^{\infty}\frac{\mu}{w_{p}}u_{tt}(p,k,t)\left[\Lambda^{2}(p,j)-\frac{k^{2}\pi^{2}}{L_{y}^{2}}\Lambda(p,j)_{\cdot}\right]-$$

$$\sum_{p=1}^{p}\frac{\mu}{w_{p}}u\,(p,k,t)\left[N_{x}^{0}\Lambda^{2}(p,j)-N_{y}^{0}\frac{k^{2}\pi^{2}}{L_{y}^{2}}\Lambda(p,j)\right]=\frac{Mg}{\mu}Sin\frac{k\pi y_{1}}{L_{y}}W_{j}(c,t) \qquad (2.69)$$

The equation represents the classical case of a moving force problem of the system. This is an approximate model, where the inertia effect of the moving mass is assumed to be negligible. Obviously, an exact analytical solution of this equation is not possible. Consequently, an approximate analytical solution technique which is a modification of the asymptotic method of Struble discussed in [7] and [23] shall be used.

Incorporating the rotatory inertia term and rearrange equation (2.69).

$$u_{tt}(j,k,t)+\alpha_{j,k}^{2}u_{t}(j,k,t)+Dh\sum_{p=1}^{\infty}\frac{\mu}{w_{p}}u_{tt}(p,k,t)\left[\frac{k^{2}\pi^{2}}{L_{y}^{2}}\Lambda(p,j)-\Lambda^{2}(p,j)\right]$$

$$-\sum_{p=1}^{\infty}\frac{\mu}{w_{p}}u\,(p,k,t)\left[N_{x}\Lambda^{2}(p,j)-N_{y}\frac{k^{2}\pi^{2}}{L_{y}^{2}}\Lambda(p,j)\right]=\sum_{j=1}^{N}P_{L}Sin\frac{k\pi y_{1}}{L_{y}}W_{j}(c,t) \qquad (2.70)$$

Set

$$P_{L}=\frac{Mg}{\mu} \qquad (2.71)$$

$$\Gamma=Dh,\qquad p=j \qquad (2.72)$$

$$-\left[1+\Gamma\frac{\mu}{w_p}\left(\frac{k^2\pi^2}{L_y^2}\Lambda(j,j)-\Lambda^2(j,j)\right)\right]u_u(j,k,t)+\left[\alpha_{j,k}^2-\frac{\mu}{w_p}\left(N_x^0\Lambda^2(j,j)-N_y^0\frac{k^2\pi^2}{L_y^2}\Lambda(j,j)\right)\right]\bullet$$

$$u(j,k,t)+\Gamma\sum_{\substack{p=1\\p\neq 1}}^{\infty}\frac{\mu}{w_p}\left\{\left(\frac{k^2\pi^2}{L_y^2}\Lambda(p,j)-\Lambda^2(p,j)\right)u_u(p,k,t)-\right.$$

$$\left.\frac{1}{Dh}u(p,k,t)\left(N_x^0\Lambda^2(p,j)-N_y^0\frac{k^2\pi^2}{L_y^2}\Lambda(p,j)\right)\right\}=P_z Sin\frac{K\pi y_1}{L_y}W_j(ct) \qquad (2.73)$$

By means of this technique, we seek the modified frequency corresponding to the frequency of the free system due to the presence of the effect of rotatory inertia. An equivalent free system operator defined by the modified frequency then replaces equation (2.69). To this end, we set the right hand side of (2.73) to zero and consider a parameter $\lambda^* < 1$ for any arbitrary ratio $\Gamma*$ defined as

$$\lambda^* = \frac{\Gamma*}{1+\Gamma*} \qquad (2.74)$$

So that

$$\Gamma* = \lambda^* + 0(\lambda)^2 \qquad (2.75)$$

Substituting equation (2.74) into homogenous part, of equation (2.72) yields.

$$\left[1+\lambda^*\frac{\mu}{w_p}\left(\frac{k^2\pi^2}{\lambda^2 y}\Lambda(j,j)-\Lambda^2(j,j)\right)\right]u_u(j,k,t)+\left[\alpha_{j,k}^2-\frac{\mu}{w_p Dh}\left(N_x^0\Lambda^2(j,j)-N_y^0\frac{k^2\pi^2}{L_y^2}\Lambda(j,j)\right)\right]\bullet$$

$$u(j,k,t)+\lambda*\sum_{\substack{p=1\\p\neq j}}^{\infty}\frac{\mu}{w_p}\left\{u_u(p,k,t)\left(\frac{k^2\pi^2}{L_y^2}\Lambda(p,j)-\Lambda^2(p,j)\right)\right.$$

$$\left.-\frac{1}{Dh}u(p,k,t)\left[N_x^0\Lambda^2(p,j)-N_y^0\frac{k^2\pi^2}{L_y^2}\Lambda(p,j)\right]\right\}=0 \qquad (2.76)$$

Approximate Analytical Method of Solution

When λ^* is set to zero in equation (2.76), we obtain a situation corresponding to the case in which effect of the cross sectional dimensions of the plate is regarded as negligible. In such a case, the solution is of the form

$$u_{st}(j,k,t) = C_{sf}\,Cos\,(\alpha_{jk}t - \phi_{sf}) \qquad (2.77)$$

Where

$C_{st}, \alpha_{jk} and \phi_{sf}$ are constants.

Furthermore, as $\lambda^* < 1$, stubble's techniques require that the solution of equation (2.76) can be of the form.

$$u(j,k,t) = A(j,k,t)Cos\,[\alpha_{jk}t - \phi\,(j,k,t) + \lambda^* u\,(j,k,t) + O(\lambda^*) \qquad (2.78)$$

Where

$A(j,k,t)$ and $\phi(j,k,t)$ are slowly varying functions of time or equivalently

$$\frac{dA(i,j,t)}{dt} \approx O(\lambda^*) \text{ and } \frac{d^2A(j,k,t)}{dt^2} \approx O(\lambda^*)^2 \qquad (2.79)$$

$$\frac{d\phi(j,k,t)}{dt} \approx O(\lambda^*) \text{ and } \frac{d^2\phi(j,k,t)}{dt^2} \approx O(\lambda^*) \qquad (2.80)$$

Differentiating equation (2.77) first and second time one obtains.

$$\ddot{u}(j,k,t) = \ddot{A}(j,k,t)Cos\left((\alpha_{jk}t - \phi(j,k,t) + A(j,k,t)(\ddot{\phi}(j,k,t) - \alpha_{jk})\right) \bullet$$

$$Sin(\alpha_{jk}t - \phi(j,k,t)) + \lambda^* \ddot{u}_1(j,k,t) \qquad (2.81)$$

And

$$\overset{00}{u}(j,k,t)=\left[2\overset{0}{A}(j,k,t)+\overset{0}{\phi}(j,k,t)+A(j,k,t)\overset{00}{\phi}(j,k,t)-2d_{jk}\overset{0}{A}(j,k,t)\right]Sin(\alpha_{jk}t-\phi(j,k,t))+$$

$$+2A(j,k,t)\overset{0}{\phi}(j,k,t)\alpha_{jk}-A(j,k,t)-\overset{2}{\alpha}_{jk}A(j,k,t)+\overset{00}{A}(j,k,t)\Big]Cos\left(\alpha_{jk}t-\phi(j,k,t)\right)$$

$$+\lambda^{*}\overset{00}{u}_{1}(j,k,t)+O(\lambda^{*})^{2} \tag{2.82}$$

In order to obtain the modified frequency equation (2.78), (2.81) and (2.82) are substituted into the equation (2.76) and one obtains.

$$2\alpha_{jk}\overset{0}{A}(j,k,t)Sin(\alpha_{jk}t-\phi(j,k,t))+2A(j,k,t)\overset{0}{\phi}(j,k,t)\alpha_{jk}Cos(\alpha_{jk}t-\phi(j,k,t))+$$

$$+\lambda^{*}\overset{00}{u}(j,k,t)+\alpha_{jk}^{2}\lambda^{*}u_{1}(j,k,t)-\frac{\lambda^{*}\mu}{w_{p}Dh}\left[N_{s}^{o}\Lambda^{2}(j,k,t)-N_{s}\frac{k^{2}\pi^{2}}{L_{y}^{2}}\Lambda(j,k,t)\right]-$$

$$\Gamma\sum_{p=1}^{r}\frac{\mu}{w_{p}}\left\{\alpha_{jk}^{2}\left(\frac{k^{2}\pi^{2}}{L_{y}^{2}}\Lambda(p,j)-\Lambda^{2}(p,j)\right)-\frac{1}{Dh}\left[N_{s}^{o}\Lambda^{2}(p,j)-N_{s}^{o}\frac{k^{2}\pi^{2}}{L_{y}^{2}}\Lambda(p,j)\right]\right\}\bullet$$

$$A(j,k,t)Cos(\alpha_{jk}t-\phi(j,k,t))=0 \tag{2.83}$$

where terms higher than (λ^{*}) are neglected. Now, since only the terms involving

$Sin(\alpha_{jk}t-\phi(j,k,t))$ and $Cos(\alpha_{jk}t-\phi(j,k,t))$

contribute to the variational equations describing the behavior of A (j, k, t) and $\phi(j,k,t)$, equation (2.83) reduces to

$$2\alpha_{jk}\overset{0}{A}(j,k,t)Sin(\alpha_{jk}t-\phi(j,k,t))+2A(j,k,t)\overset{0}{\phi}(j,k,t)\ \alpha_{jk}Cos\ (\alpha_{jk}t-\phi(j,k,t))-$$

$$\Gamma^{*}\sum_{p=1}^{r}\frac{\mu}{w_{p}}\left\{\alpha_{jk}^{2}\left(\frac{k^{2}\pi^{2}}{L_{y}^{2}}\Lambda(p,j)-\Lambda^{2}(p,j)\right)-\frac{1}{Dh}\left[N_{s}^{o}\Lambda^{2}(p,j)-N_{s}^{o}\frac{k^{2}\pi^{2}}{L_{y}^{2}}\Lambda(p,j)\right]\right\}ACos(\alpha_{jk}t-\phi(j,k,t))=0$$

(2 .84)

The so-called variation equations of this equation (2.84) are obtained by setting the coefficients Cos $\left[\alpha_{jk}t - \phi(j,k,t)\right]$ and Sin $\left[\alpha_{jk}t - \phi(j,k,t)\right]$ to zero

Thus, one obtains

$$2\ \alpha_{jk}A^{o}(j,k,t) = O \tag{2.85}$$

And

$$2A(j,k,t)\ \phi^{o}(j,k,t)\alpha_{jk} - A(j,k,t)\Gamma^{*}\frac{\mu}{w_{p}}\left[\alpha_{jk}^{2}Q_{0}(t) - \frac{1}{Dh}Q_{1}(t)\right] = O \tag{2.86}$$

Where

$$Q_{0}(t) = \frac{k^{2}\pi^{2}}{L_{y}^{2}}\Lambda(j,j) - \Lambda^{2}(j,j) \tag{2.87}$$

$$Q_{1}(t) = N_{x}^{o}\Lambda^{2}(j,j) - N_{y}^{o}\frac{k^{2}\pi^{2}}{L_{y}^{2}}\Lambda(j,j) \tag{2.88}$$

Solving equation (2.85), one obtain

$$A(j,k,t) = C_{st} \tag{2.89}$$

where C_{st} is a constant

The first order differential equation (2.86) describing the behavior of $\phi(j,k,t)$ implies.

$$\frac{d\phi(j,k,t)}{dt} = \frac{\lambda^{*}\mu}{2w_{p}}\left\{\alpha_{jk}Q_{0}(t) - \frac{1}{Dh\alpha_{jk}}Q_{1}(t)\right\} \tag{2.90}$$

hence,

$$\phi(j,k,t) = \frac{\lambda^{*}\mu}{2w_{p}}\left\{\alpha_{jk}Q_{0}(t) - \frac{1}{Dh\alpha_{jk}}Q_{1}(t)\right\}t + \phi_{a} \tag{2.91}$$

Therefore when cross section dimension is considered, the first approximate to the homogenous system is

$$u(j,k,t) = C_{af}Cos(\beta_{jk}t - \phi_{a}) \tag{2.92}$$

where

$$\beta_{jk} = \alpha_{jk}\left(1 - \frac{\lambda^{*}\mu}{2w_{p}}\left(Q_{0}(t) - \frac{1}{Dh\alpha_{jk}^{2}}Q_{1}(t)\right)\right) \tag{2.93}$$

represents the modified frequency due to the effect of rotatory inertia of the plate. It is observed that when $\lambda^{*} = 0$, we recover the frequency of the moving force problem when rotatory inertia or cross-sectional dimension (csd) effect is neglected. In order to solve the non homogeneous equation (2.69), the differential operator which acts on $u(j,k,t)$ and $u(p,k,t)$ is replaced by the equivalent free system operator defined by the modified frequency β_{jk} i.e.

$$u_{tt}(j,k,t) + \beta_{jk}^{2}u(j,k,t) = P_{L}\,V_{k}(y_{1})W_{j}(c,t) \tag{2.94}$$

Where

$$V_{k}(y_{1}) = Sin\frac{k\pi y_{1}}{L_{y}} \tag{2.95a}$$

$$W_{j}(c,t) = Sin\alpha_{gf}t + A_{j}Cos\alpha_{gf}t + B_{j}Sinh\beta_{gf}t + C_{j}Cosh\beta_{gf}t \tag{2.95b}$$

And

$$\alpha_{gf} = \frac{\alpha_{j}c}{L_{x}}, \quad \beta_{gf} = \frac{\beta_{j}c}{L_{x}} \tag{2.96}$$

Therefore, equation (2.94) becomes

$$u_{tt}(j,k,t)+\beta_{jk}^2 u(j,k,t)=P_L V_k(y_1)\left[Sin\alpha_{ij}t+A_j Cos\alpha_{ij}t+B_j Sinh\beta_{ij}t+C_j Cosh\beta_{ij}t\right]$$

(2.97)

Where

In order to obtain the solution of equation (2.97)

$$(\sim)=\int_0^\infty(\bullet)e^{-st}dt \tag{2.98}$$

Where S is the laplace parameter in conjunction with the transformed initial conditions

$$u(j,k,t)=O=u_t(j,k,o) \tag{2.99}$$

Thus, one obtains the simple algebraic equation given by

$$u(j,k,t)=\frac{P_L V_k(y_1)}{S^2+\beta_{jk}^2}\left[\frac{\alpha_j}{S^2+\alpha_{ij}^2}+\frac{A_j S}{S^2+\alpha_{ij}^2}+\beta_j\frac{\beta_j}{S^2-\beta_{ij}^2}+C_j\frac{S}{S^2-\beta_{ij}^2}\right] \tag{2.100}$$

By further re-arrangement of 2.100 we have

$$u(j,k,t)=\frac{P_L V_k(y_1)}{S^2+\beta_{jk}^2}\frac{\alpha_j}{S^2+\alpha_j^2}+\frac{PV_k(y_1)A_j}{S^2+\beta_{jk}^2}\frac{S}{S^2+\alpha_j^2}$$
$$+\frac{P_2 V_k(y_1)B_j}{S^2+\beta_{jk}^2}\frac{\beta_j}{S^2-\beta_{ij}^2}+\frac{P_L V_k C_j}{S^2+\beta_{jk}^2}\frac{S}{S^2-\beta_{ij}^2}\Bigg] \tag{2.101}$$

Next, is to find the laplace inversion of 2.101. This can be done by adopting the following representations:

$$g_1(s)=\frac{\alpha_j}{S^2+\alpha_{ij}^2};\ f_1(s)=\frac{P_L V_k(y_1)}{S^2+\beta_{jk}^2} \tag{2.102a}$$

$$\hat{g}_2(s) = \frac{s}{S^2 + \alpha_{jj}^2}; \hat{f}_2(s) = \frac{P_L V_k(y_1) A_j}{S^2 + \beta_{jk}^2} \tag{2.102b}$$

$$\hat{g}_3(s) = \frac{\beta_{jj}}{S^2 - \beta_{jj}^2}; \hat{f}_3(s) = \frac{P_L V_k(y_1) B_j}{S^2 + \beta_{jk}^2} \tag{2.102c}$$

$$\hat{g}_4(s) = \frac{s}{S^2 + \beta_{jj}^2}; \hat{f}_4(s) = \frac{P_L V_k(y_1) C_j}{S^2 + \beta_{jk}^2} \tag{2.102d}$$

The Laplace inversion of $u(j,k,t)$ is the convolution of $\hat{f}(s)$ and $\hat{g}(s)$ defined as

$$f(s) * g(s) = \int_0^t f_i(t-r) g_j(r) dr \tag{2.103}$$

for all i = 1, 2, 3, 4 and j = 1, 2, 3, 4

Now,

$$f_1(s) * g_1(s) = \frac{P_L V_k(y_1)}{\beta_{j,k}} \int_0^t Sin\beta_{j,k}(t-r) Sin\alpha_{jj} r dr \tag{2.104}$$

$$f_2(s) * g_2(s) = \frac{P_L V_k(y_1)}{\beta_{j,k}} \int_0^t Sin\beta_{j,k}(t-r) Cos\alpha_{jj} r dr \tag{2.105}$$

$$f_3(s) * g_3(s) = \frac{P_L V_k(y_1)}{\beta_{j,k}} \int_0^t Sin\beta_{j,k}(t-r) Sinh\beta_{jj} r dr \tag{2.106}$$

$$f_4(s) * g_4(s) = \frac{P_L V_k(y_1)}{\beta_{j,k}} \int_0^t Sin\beta_{j,k}(t-r) Cosh\beta_{jj} r dr \tag{2.107}$$

Thus, $u(j,k,t)$ is easily expressed in the form below

$$u(j,k,t) = u_1(s) + u_2(s) + u_3(s) + u_4(s) \tag{2.108}$$

Where

$$u_1(s) = f_1(s) * g_1(s) \tag{2.109a}$$

$$u_2(s) = f_2(s) * g_2(s) \tag{2.109b}$$

$$u_3(s) = f_3(s) * g_3(s) \tag{2.109c}$$

$$u_4(s) = f_4(s) * g_4(s) \tag{2.109d}$$

Consequently,

$$u_1(s) = -p_L V_k(y_1) Sin \frac{\beta_{j,k} t}{2\beta_{j,k}} \left[\frac{Cos(\beta_{j,k} + \alpha_{ij})t}{\beta_{j,k} + \alpha_j} - \frac{Cos(\beta_{j,k} - \alpha)t}{\beta_{j,k}} - \frac{2\alpha_{ij}}{\alpha^2_{ij} - \beta^2_{j,k}} \right]$$

$$-P_L V_4(y_1) \frac{Cos\beta_{j,k} t}{2\beta_{j,k}} \left[\frac{Sin(\beta_{ij} + \alpha_{ij})t -}{\beta_{j,k} + \alpha_{ij}} - \frac{Sin(\beta_{ij} - \alpha_{ij})t}{\beta_{ij} - \alpha_{ij}} \right] \tag{2.110}$$

$$u_2(s) = \frac{P_L V_k(y_1) Sin\beta_{j,k} t}{2\beta_{j,k}} \left[\frac{Sin(\beta_{j,k} + \alpha_{ij})t}{\beta_{j,k} + \alpha_{ij}} + \frac{Sin(\beta_{j,k} - \alpha_{ij})t}{\beta_{j,k} - \alpha_{ij}} + \frac{Cos\beta_{j,k} t}{2} \right.$$

$$\left(\frac{Cos(\beta_{ij} + \alpha_{ij})t}{\beta_{ij} + \alpha_{ij}} + \frac{Cos(\beta_{ij} - \alpha_{ij})t}{\beta_{ij} - \alpha_{ij}} - \frac{2\beta_{ij}}{\beta^2_{ij} - \alpha^2_{ij}} \right) \tag{2.111}$$

$$u_3(s) = \frac{\beta_{ij}}{\beta_{j,k}(\beta^2_{ij} + \beta^2_{j,k})} \left[\frac{\beta_{j,k}}{\beta_{ij}} Sinh\beta_{ij} t - Sin\beta_{j,k} t \right] \tag{2.112}$$

$$u_4(s) = \frac{1}{(\beta^2_{ij} + \beta^2_{j,k})} \left[Cosh\beta_{ij} t - Cos\beta_{j,k} t \right] \tag{2.113}$$

Subsisting equation 2.110 – 2.113 in equation (2.108) after some rearrangement one obtains

$$u(j,k,t) = \frac{P_L V_k(y_1)}{2(\beta^2_{j,k} - \alpha^2_{ij})} \left(Sin\alpha_{ij} t - \frac{\alpha_{ij}}{\beta_{j,k}} Sin\beta_{ij} t + A_j Cos\alpha_{ij} t - A_j Cos\beta_{j,k} t \right) + \frac{P_L V_k(y_1) \beta_{ij}}{2\beta_{j,k}(\beta^2_{ij} + \beta^2_{j,k})} \bullet$$

$$\left[\frac{B_j \beta_{j,k}}{\beta_j} Sinh\beta_{j,k} t - B_j Sin\beta_{j,k} t + C_j \frac{1}{\beta_j} \beta_{jk} Cosh\beta_j t - C_j \frac{\beta_{j,k}}{\beta_{ij}} Cos\beta_{j,k} t \right] \tag{2.114}$$

Which on inversion becomes

$$w(x,y,t)=\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{2}{L_y}\frac{\mu}{W_j}\left\{\frac{P_i V_k(y_1)}{2(\beta_{j,k}^2-\alpha_{ij}^2)}\left(Sin\alpha_{ij}t-\frac{\alpha_{ij}}{\beta_{j,k}}Sin\beta_{ij}t+A_j Cos\alpha t-A_j Cos\beta_{jk}t\right)\right\}$$

$$+\frac{P_i V_k(y_1)\beta_{ij}}{2(\beta_{j,k}(\beta_{ij}^2+\beta_{j,k}^2)}\left[\frac{B_{ij}\beta_{j,k}}{\beta_j}Sinh\beta_{ij}t-B_j Sin\beta_{j,k}t+\frac{C_j\beta_{jk}}{\beta_j}Cosh\beta_j t-\frac{C_j\beta_{jk}}{\beta_j}Cos\beta_{jk}t\right]\Bigg\}$$

$$Sin\frac{k\pi y_1}{L_y}\frac{Sink\pi y}{L_y}W_j(x) \qquad (2.115)$$

Equation (2.115) above represents the transverse displacement response of a rectangular plate having arbitrary edge supports and traversed by a moving force.

2.3.2 RECTANGULAR PLATE TRAVERSED BY A MOVING MASS PROBLEM

In this section we seek the solution to entire equation (2.66} when no term of the coupled differential equation is neglected.

Clearly, an exact analytical solution to equation (2.66) does not exist. An analytical approximate method is desirable as solution so obtained often show vital information about the vibrating system. Thus, the approximate analytical solution method of Struble technique that has been used in solving coupled differential equation in the moving force problem is employed to obtain its closed form solution. When the terms representing the inertia effect of the moving mass, is neglected, one obtains equation (2.69). The homogeneous part of this equation can be replaced by a free system operator defined by the modified frequency $\beta_{j,k}$

due to the presence of the effect of rotatory inertia and the shear modulus. Therefore equation 2.69 can be rewritten in the form.

$$u_{tt}(j,k,t)+\beta_{j,k}^{2}u\ (j,k,t)+\Gamma_{o}\Bigg\{4\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{n=1}^{\infty}\frac{\mu}{w_p}u_{tt}(p,q,t)Cos\frac{n\pi ct}{Lx}$$

$$\frac{Sink\pi y_1}{Ly}\frac{Sinq\pi y_1}{Ly}\Lambda\ (n,j,p)+2\sum_{p=1}^{\infty}\sum_{n=1}^{\infty}\frac{\mu}{w_p}u_{tt}(p,k,t)Cos\frac{n\pi ct}{Lx}$$

$$\Lambda\ (n,j,p)+2\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\frac{\mu}{w_p}u_{tt}(p,q,t)Sin\frac{k\pi y_1}{Ly}Sin\frac{q\pi y_1}{Ly}\Lambda\ (j,p)+\sum_{p=1}^{\infty}\frac{\mu}{w_p}u_{tt}(p,k,t)\Lambda\ (j,p)$$

$$2c\Bigg[4\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{n=1}^{\infty}\frac{\mu}{w_p}u_t(p,q,t)Cos\frac{n\pi ct}{Lx}Sin\frac{k\pi y_1}{Ly}Sin\frac{q\pi y_1}{Ly}\Lambda^1\ (n,j,p)$$

$$+2\sum_{p=1}^{\infty}\sum_{n=1}^{\infty}\frac{\mu}{w_p}u_t(p,k,t)Cos\frac{n\pi ct}{Lx}\Lambda^1\ (n,j,p)+2\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\frac{\mu}{w_p}u_t(p,q,t)Sin\frac{k\pi y_1}{Ly}Sin\frac{q\pi y_1}{Ly}\Lambda\ (j,p)$$

$$+\sum_{p=1}^{\infty}\frac{\mu}{w_p}u_t(p,k,t)\Lambda^1\ (j,p)\Bigg]+c^2\Bigg\{4\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{n=1}^{\infty}\frac{\mu}{w_p}u\ (p,q,t)Cos\frac{n\pi ct}{Lx}Sin\frac{k\pi y_1}{Ly}Sin\frac{q\pi y_1}{L_y}\Lambda^2\ (n,j,p)$$

$$+2\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\frac{\mu}{w_p}u(p,q,t)Cos\frac{n\pi ct}{Lx}\Lambda^2(j,p)+\ \sum_{p=1}^{\infty}\frac{\mu}{w_p}u\ (p,k,t)\Lambda^2\ (j,p)=P_LSin\frac{k\pi y_1}{L_y}$$

(2.116)

Which can be further rearranged to take the form

$$u_{tt}(j,k,t)+\frac{2\Gamma_oR_2}{1+\Gamma_oR_1}u_t(j,k,t)+\frac{(\beta_{j,k}+\Gamma_oR_3)}{1+\Gamma_oR_1}u(j,k,t)+\frac{\Gamma_o}{1+\Gamma_oR_1}\bullet$$

$$\Bigg\{4\sum_{\substack{p=1\\p\neq j}}^{\infty}\sum_{\substack{q=1\\q\neq k}}^{\infty}\sum_{n=1}^{\infty}\frac{\mu}{w_p}u_{tt}(p,q,t)Cos\frac{n\pi ct}{Lx}Sin\frac{k\pi y_1}{Ly}Sin\frac{q\pi y_1}{Ly}\Lambda\ (n,j,p)$$

$$+2\sum_{\substack{p=1\\p\neq j}}^{\infty}\sum_{n=1}^{\infty}\frac{\mu}{w_p}u_{tt}(p,k,t)Cos\frac{n\pi ct}{Lx}\Lambda\ (n,j,p)+2\sum_{\substack{p=1\\p\neq j}}^{\infty}\sum_{\substack{q=1\\q\neq k}}^{\infty}\frac{\mu}{w_p}u_{tt}(p,kt)Sin\frac{k\pi y_1}{Ly}\bullet$$

$$Sin\frac{q\pi y_1}{Ly}\Lambda(j,p)+\sum_{\substack{p=1\\p\neq j}}^{\infty}\frac{\mu}{w_p}u_{tt}(p,k,t)\Lambda(j,p)+2c\left[4\sum_{\substack{p=1\\p\neq j}}^{\infty}\sum_{\substack{q=1\\q\neq k}}^{\infty}\sum_{n=1}^{\infty}\frac{\mu}{w_p}u_{tt}(p,q,t)Cos\frac{n\pi ct}{L_x}\bullet\right.$$

$$Sin\frac{k\pi y_1}{L_y}Sin\frac{q\pi y_1}{Ly}\Lambda^1(n,j,p)+2\sum_{\substack{p=1\\p\neq j}}^{\infty}\sum_{n=1}^{\infty}\frac{\mu}{w_p}U_t(p,k,t)Cos\frac{n\pi ct}{Lx}\Lambda^1(n,j,p)$$

$$+2\sum_{\substack{p=1\\p\neq j}}^{\infty}\sum_{\substack{q=1\\q\neq k}}^{\infty}\frac{\mu}{w_p}u_t(p,q,t)Sin\frac{k\pi y_1}{Ly}Sin\frac{q\pi y_1}{Ly}\Lambda^1(j,p)+\sum_{\substack{p=1\\p\neq j}}^{\infty}\frac{\mu}{w_p}u_t(p,k,t)\Lambda^1(j,p)$$

$$+c^2\left[4\sum_{\substack{p=1\\p\neq j}}^{\infty}\sum_{\substack{q=1\\q\neq k}}^{\infty}\sum_{n=1}^{\infty}\frac{\mu}{w_p}u(p,q,t)Cos\frac{n\pi ct}{Lx}Sin\frac{k\pi y_1}{Ly}Sin\frac{q\pi y_1}{Ly}\Lambda^2(n,j,p)\right.$$

$$\bullet\Lambda^2(n,j,p)+2\sum_{\substack{p=1\\p\neq j}}^{\infty}\sum_{n=1}^{\infty}\frac{\mu}{w_p}Cos\frac{n\pi ct}{L_x}\Lambda^2(n,j,p)u(p,k,t)+2\sum_{\substack{p=1\\p\neq j}}^{\infty}\sum_{\substack{q=1\\q\neq k}}^{\infty}\frac{\mu}{w_p}u(p,q,t)Sin\frac{k\pi y_1}{L_y}\bullet$$

$$\left.Sin\frac{q\pi y}{L_y}\Lambda^2(j,p)+\sum_{\substack{p=1\\p\neq j}}^{\infty}\frac{\mu}{w_p}u(p,k,t)\Lambda^2(j,p)\right]=\frac{\Gamma_a L_x L_y g}{1+\Gamma_a R_1}Sin\frac{k\pi y_1}{L_y}W_j(c,t) \tag{2.117}$$

Equation (2.117) can be rewritten as

$$\frac{d^2u(j,k,t)}{dt^2}+\frac{2\Gamma_a R_2}{1+\Gamma_a R_1}\frac{du(j,k,t)}{dt}+\frac{\beta_{jk}^2+\Gamma_a R_3}{1+\Gamma_a R_1}u(j,k,t)+\frac{\Gamma_a}{1+\Gamma_a R_1}\sum_{\substack{p=1\\p\neq j}}^{\infty}\sum_{\substack{q=1\\q\neq k}}^{\infty}$$

$$\left[R_1(p,q,j,k)u_{tt}(p,q,t)+R_2(p,q,j,k)u_t(p,q,t+R_3(p,q,j,k)u(p,q,t)\right]=$$

$$\frac{\Gamma_a LxLyg}{1+\Gamma_a R_1}Sin\frac{k\pi y_1}{Ly}W_j(c,t) \tag{2.118}$$

Where

$$R_3 = c^2\left[4\sum_{n=1}^{\infty}\frac{\mu}{w_j}Cos\frac{n\pi ct}{L_x}Sin\frac{k\pi y_1}{L_y}\Lambda^2(n,j,j)+2\sum_{n=1}^{\infty}\frac{\mu}{w_j}Cos\frac{n\pi ct}{L_x}\Lambda^2(n,j,j)\right.$$

$$\left.+\frac{2\mu}{w_j}Sin^2\frac{k\pi y_1}{L_y}\Lambda^2(j,j)+\frac{\mu}{w_j}\Lambda^2(j,j)\right] \tag{2.119}$$

$$R_2 = 2c\left[4\sum_{n=1}^{\infty}\frac{\mu}{w_j}Cos\frac{n\pi ct}{L_x}Sin^2\frac{k\pi y_1}{L_y}\Lambda^1(n,j,j)+2\sum_{n=1}^{\infty}\frac{\mu}{w_j}Cos\frac{n\pi ct}{L_x}\Lambda^1(n,j,j)\right.$$

$$\left.+\frac{2\mu}{w_j}Sin^2\frac{k\pi y_1}{L_y}\Lambda^1(j,j)+\frac{\mu}{w_j}\Lambda^1(j,j)\right] \tag{2.120}$$

$$R_1 = 4\sum_{n=1}^{\infty}\frac{\mu}{w_j}Sin^2\frac{k\pi y_1}{L_y}\Lambda(j,j)+\frac{\mu}{w_j}\Lambda(j,j) \tag{2.121}$$

$$\Gamma_a = \frac{M}{\mu L_x L_y} \tag{2.122}$$

$$R_1(p,q,j,k)=4\sum_{n=1}^{\infty}\frac{\mu}{w_p}Cos\frac{n\pi ct}{L_x}Sin\frac{k\pi y_1}{L_y}Sin\frac{q\pi y_1}{L_y}\Lambda(n,j,p)+2\sum_{n=1}^{\infty}\frac{\mu}{w_p}Cos\frac{n\pi ct}{L_x}\Lambda(n,j,p)$$

$$+2\frac{\mu}{w_p}Sin\frac{k\pi y_1}{L_y}Sin\frac{q\pi y_1}{L_y}\Lambda(j,p)+\frac{\mu}{w_p}\Lambda(j,p) \tag{2.123}$$

$$R_2(p,q,j,k)=2c\left[4\sum_{n=1}^{\infty}\frac{\mu}{w_p}CCos\frac{n\pi ct}{L_x}Sin\frac{k\pi y_1}{L_y}Sin\frac{q\pi y_1}{L_y}\Lambda^1(n,j,p)+2\sum_{n=1}^{\infty}\frac{\mu}{w_p}Cos\frac{n\pi ct}{L_x}\bullet\right.$$

$$\Lambda^1(n,j,p)+2\frac{\mu}{w_p}Sin\frac{k\pi y_1}{L_y}Sin\frac{q\pi y_1}{L_y}\Lambda^1(j,p)+\frac{\mu}{w_p}\Lambda^1(j,p) \tag{2.124}$$

$$R_3(p,q,j,k)=c^2\left[4\sum_{n=1}^{\infty}\frac{\mu}{w_p}Cos\frac{n\pi ct}{L_x}Sin\frac{k\pi y_1}{L_y}Sin\frac{q\pi y_1}{L_y}\Lambda^2(n,j,p)+2\sum_{n=1}^{\infty}\frac{\mu}{w_p}Cos\frac{n\pi ct}{L_x}\bullet\right.$$

$$\Lambda^2(n,j,p)+\frac{2\mu}{w_p}Sin\frac{k\pi y_1}{L_y}Sin\frac{q\pi y_1}{L_y}\Lambda^2(n,j,p)+\frac{\mu}{w_p}\Lambda^2(n,j,p) \tag{2.125}$$

Now in order to obtain a modified frequency corresponding to the frequency of the free system due to the presence of the moving mass M, the homogeneous part of equation (2.118) is considered. An equivalent free system operator defined by the modified frequency then replaces equation (2.118).

To this end, the right hand side of equation (2.118) to zero and a parameter $\lambda \angle 1$ is considered for any arbitrary mass ratio Γ_n defined as

$$\lambda = \frac{\Gamma_n}{1+\Gamma_n} \qquad (2.126)$$

it can be shown that

$$\Gamma_n = \lambda + o(\lambda)^2 \qquad (2.127)$$

Thus

$$\frac{1}{1+\Gamma_n R_1} = 1 - \lambda R_1 + o(\lambda)^2 \qquad (2.128)$$

Where

$$/\ \lambda R_1 / \angle 1 \qquad (2:129)$$

Setting $\lambda = o$, case corresponding to the case when the inertia effect of the mass of the moving system is neglected, the solution of equation (2.78) can be written as

$$u\left(j,k,t\right) = C_\beta Cos(\beta_{j,k}^2 t - \phi\left(j,k,t\right)) \qquad (2.130)$$

Where C_β and ϕ_β are constants

Since $\lambda \angle 1$, Struble's technique requires that the solution of equation (2.118) be of the form

$$u(j,k,t) = A(j,k,t)Cos(\beta_{jk}t - \phi(j,k,t) + \lambda u_1(j,k,t) + o(\lambda)^2) \qquad (2.131)$$

Where $A(j,k,t)$ and $\phi(j,k,t)$ are slowly varying functions of time.

The main objective is to obtain the modified frequency. To this end, equation (2.131) and its derivatives are substituted into the homogenous part of equation (2.118). The resulting variational equations describing the behaviour of $A(j,k,t)$ and $\phi(j,k,t)$ during the motion of the mass determine the modified frequency.

Thus, substituting equation (2.131) and its derivatives into the homogenous part of equation (2.118) in conjunction with the expanded expression in equation (2.127), and (2.128) one obtains

$$\left[2\overset{o}{A}(j,k,t)\overset{o}{\phi}(j,k,t) + A(j,k,t)\overset{oo}{\phi}(j,k,t) - 2\beta_{jk}\overset{o}{A}(j,k,t)\right]Sin(\beta_{jk}t - \phi(j,k,t)) +$$

$$+\left[2A(j,k,t)\overset{o}{\phi}(j,k,t)\beta_{jk} - A(j,k,t)\overset{o}{\phi}{}^2(j,k,t) - \beta_{jk}^2 A(j,k,t) + \overset{oo}{A}(j,k,t)\right] \bullet$$

$$Cos(\beta_{jk}t - \phi(j,k,t)) + \lambda\overset{oo}{u}(j,k,t) + 2\lambda R_2\Big\{\overset{o}{A}(j,k,t)Cos(\beta_{j,k}t - \phi(j,k,t)) +$$

$$A(j,k,t)(\overset{o}{\phi}(j,k,t) - \beta_{j,k})Sin(\beta_{j,k}^2 t - \phi(j,k,t) + \lambda\overset{o}{u}_1(j,k,t)\Big\}$$

$$+(\beta_{j,k}^2 + \lambda R_1 - \lambda\beta_{j,k}^2 R_1)\{A(j,k,t)Cos(\beta_{j,k}t - \phi(j,k,t)) + \lambda u_1(j,k,t) +$$

$$\lambda\sum_{\substack{p=1\\p\neq j}}^{\infty}\sum_{\substack{q=1\\q\neq k}}^{\infty}\Big\{R_1(p,q,j,k)\Big\}\left[\left[2\overset{o}{A}(p,q,t) + A(p,q,t)\overset{oo}{\phi}(p,q,t)2\beta_{pq}\overset{o}{A}(p,q,t)\right]\right] \bullet$$

$$Sin(\beta_{pq}t - \phi\,(p,q,t)) +$$

$$\left[2A_{(p,q,t)}\overset{o}{\phi}_{(p,q,t)}\beta_{pq} + A_{(p,q,t)}\overset{o^2}{\phi}\,(p,q,t) - \beta_{pq}^2 A\,(p,q,t) + \overset{oo}{A}\,(p,q,t)\right] \bullet$$

$$Cos(\beta_{pq}t - \phi\,(p,q,t) + \lambda \overset{oo}{U}_1(p,q,t)\Big\} + 2CR_2(p,q,j,k)\Big\{\overset{o}{A}\,(p,q,t)Cos(\beta_{pq}t - \phi_{(p,q,t)}) +$$

$$A\,(p,q,t)\left[\overset{o}{\phi}\,(p,q,t) - \beta_{pq}\right]Sin(\beta_{pq}t - \phi\,(p,q,t)) + \lambda \overset{o}{u}\,(p,q,t)\Big\}$$

$$+ C^2 R_3(p,q,j,k)\{A\,((p,q,t))Cos(\beta_{pq}t - \phi\,(p,q,t) + \lambda u_1(p,q,t)\} = 0 \qquad (2.132)$$

Further, simplification and rearrangement yield

$$-2\beta_{jk}\overset{o}{A}\,(j,k,t)Sin(\beta_{jk}t - \phi\,(j,k,t)) + 2A\,(j,k,t)\overset{o}{\phi}\,(j,k,t)\beta_{jk}Cos(\beta_{jk}t - \phi\,(j,k,t)) + \lambda \overset{oo}{U}\,(j,k,t)$$

$$+\lambda\beta_{jk}^2 U_1(j,k,t) + \lambda\sum_{p=1}^{n}\sum_{q=1}^{m}\{-\beta_{pq}^2 A\,(p,q,t)Cos(\beta_{pq}t - \phi\,(p,q,t))R_1(p,q,j,k)$$

$$+ 2CR_2(p,q,j,k)\beta_{jk}A\,(p,q,t)Sin(\beta_{jk}t - \phi^{\cdot}(p,q,t))$$

$$+ C^2 R_3(p,q,j,k)A\,(p,q,t)Cos(\beta_{jk}t - \phi\,(p,q,t))\} = 0 \qquad (2.133)$$

At this junction, it is noted that

$$Cos\frac{n\pi ct}{L_x}Sin(\beta_{pq}t - \phi\,(p,q,t))$$

$$= \frac{1}{2}\left\{Sin\left(\beta_{pq}t - \phi\,(p,q,t) + \frac{n\pi ct}{L_x}\right) + Sin\left(\beta_{pq}t - \phi\,(p,q,t) - \frac{n\pi ct}{L_x}\right)\right\} \qquad (2.134)$$

And

$$Cos\frac{n\pi ct}{L_x}Cos(\beta_{pq}t - \phi\,(p,q,t))$$

$$= \frac{1}{2}\left\{Cos\left[\beta_{pq}t - \phi\,(p,q,t) + \frac{n\pi ct}{L_x}\right] + Cos\left[\beta_{pq}t - \phi\,(p,q,t) - \frac{n\pi ct}{L_x}\right]\right\} \qquad (2.135)$$

It should be noted that only the terms involving $Sin[\beta_{jk}t - \phi\,(j,k,t)]$ and $Cos[\beta_{jk}t - \phi\,(j,kt)]$ contribute to the variational equation describing the behavior of $A(j,k,t)$ and $\phi(j,k,t)$, in view of equation (2.134) and (2.135). equation (2.133) reduces to

$$\begin{aligned}
&-2\beta_{jk}\overset{n}{A}\,(j,k,t)Sin[\beta_{jk}t - \phi\,(j,k,t)] + 2A\,(j;k,t)\overset{o}{\phi}\,(j,k,t)\times\beta_{jk}Cos(\beta_{jk}t - \phi\,(j,k,t)) + \lambda\overset{\prime\prime\prime}{u}\,(j,k,t) \\
&+\lambda\beta_{jk}^2U\,(j,k,t) + S_a(j,k)A\,(j,k,t)Cos(\beta_{jk}t - \phi\,(j,k,t)) + \\
&+S_b(j,k)A\,(j,k,t)Sin(\beta_{jk}t - \phi\,(j,k,t)) \\
&+S_c(j,k)A\,(j,k,t)Cos(\beta_{jk}t - \phi\,(j,k,t)) = o \qquad (2.136)
\end{aligned}$$

Where

$$S_a(j,k) = -\lambda\beta_{jk}^2\left(\frac{2\mu}{w_j}Sin^2\frac{k\pi y_1}{L_y}\Lambda\,(j,j) + \frac{\mu}{w_j}\Lambda\,(j,j)\right) \qquad (2.137)$$

$$S_b(j,k) = 2c\lambda\frac{\beta_{jk}\mu}{w_j}\left(2Sin^2\frac{k\pi y_1}{L_y}\Lambda^1\,(j,j) + \Lambda^1\,(j,j)\right) \qquad (2.138)$$

$$S_c(j,k) = c^2\frac{\mu}{w_j}\left(2Sin^2\frac{k\pi y_1}{L_y}\Lambda^2\,(j,j) + \Lambda^2\,(j,j)\right) \qquad (2.139)$$

The variational equations of this problem one obtained by setting the co-efficient of $Sin[\beta_{jk}t - \phi_{(j,k,t)}]$ and $Cos[\beta_{jk}t - \phi\,(j,k,t)]$ in equation (2.133) to zero respectively.,

Thus, we have

$$-2\beta_{jk}\overset{\circ}{A}(j,k,t)+S_b(j,k)A(j,k,t)=0 \qquad (2.140)$$

And

$$-2\beta_{jk}A(j,k,t)\overset{\circ}{\phi}(j,k,t)+(S_a(j,k)+S_c(j,k))A(j,k,t)=0 \qquad (2.141)$$

Solving equation (2.140), one obtains

$$A(j,k,t)=C^{o}e^{\varepsilon(j,k)t} \qquad (2.142)$$

Where C^{o} is a constant and

$$\varepsilon=\frac{S_b}{2\beta} \qquad (2.143)$$

The second differential equation in $\phi(j,k,t)$ implies

$$\frac{d\phi(j,k,t)}{dt}=\frac{-(S_a+S_c)}{2\beta} \qquad (2.144)$$

Hence

$$\phi(j,k,t)=\frac{-(S_a(j,k)+S_c(j,k))t}{2\beta_{jk}}+\phi_{jk} \qquad (2.145)$$

Thus, when the effect of the mass of the particle is considered the first approximation to the homogeneous system is

$$u(j,k,t)=C^{o}\lambda^{\varepsilon(j,k)t}Cos\{\omega_{jk}t-\phi_{jk}] \qquad (2.146)$$

Where

$$\omega_{jk}=\beta_{jk}\left[\frac{1+(S_a(j,k)+S_c(j,k))}{2\beta_{jk}^2}\right] \qquad (2.147)$$

Is the modified frequency corresponding to the frequency of the free system due to the presence of moving mass. It is remarked at this juncture, that this modified frequency has in it the effects of the foundation modulus, rotatory inertia and foundation stiffness.

In order to solve the non-homogeneous equation (2.118) the differential operator which acts on $u(j,k,t)$ and $u(p,q,t)$ is replaced by the equivalent free system operator defined by the modified frequency ω_{jk} i.e.

$$u_{tt}(j,k,t)+\omega_{jk}^2 u\,(j,k,t)=\lambda gLxLyV_k(y_1)W_j(c,t) \tag{2.148}$$

$V_k(y_1)$ is defined in equation (2.95a) and $W_j(c,t)$ in equation (2.95b)

Therefore, equation (2.148) becomes

$$U_{tt}(j,k,t)+\omega_{jk}^2(j,k,t)=\lambda gLxLyV_k(y_1)[Sin\alpha_j t+A_jCos\alpha_j t+B_jSin\beta_j t+C_jCosh\beta_j t] \tag{2.149}$$

From equation above, it is noticed that equation (2.149) is analogous to equation (2.97) with ω_{jk} and $\lambda gLxLy$ replacing β_{jk} and P_L respectively

Therefore the solution to (2.149) is given by

$$u\,(j,k,t)=\frac{\lambda\, gL_xL_yV_k(y_1)}{2(\omega_{jk}^2-\alpha_{ij}^2)}\left(Sin\alpha_{ij}t-\frac{\alpha_{ij}}{\omega_{jk}}Sin\omega_{jk}t+A_jCos\omega_{jk}t-A_jCos\alpha_{ij}t\right)+$$

$$\frac{\lambda\, gL_xL_yV_k(y_1)}{2\omega_{j,k}(\beta_{ij}^2+\omega_{jk}^2)}\left[B_j\frac{\omega_{j,k}}{\beta_{ij}}Sinh\beta_{ij}t-B_jSin\omega_{j,k}t+C_j\frac{\omega_{j,k}}{\beta_{ijj}}Cooh\beta_{ij}t-C_j\frac{\omega_{j,k}}{\beta_{ij}}Cos\omega_{j,k}t\right] \tag{2.150}$$

which on inversion becomes

$$w(x,y,t)=\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{2}{L_y}\frac{\mu}{W_j}\frac{\lambda\, gL_yL_xV_k(y_1)}{2(\omega_{jk}^2-\alpha_{jf}^2)}\left(Sin\alpha_{jf}t-\frac{\alpha_{jf}}{\omega_{jk}}Sin\omega_{jk}t+A_jCos\omega_{jk}t-A_jCos\alpha_{jf}t\right)+$$

$$\frac{\lambda\, gL_xL_y\beta_{jf}}{2\omega_{jk}(\beta_{jf}^2+\omega_{jk}^2)}\left[B_j\frac{\omega_{jk}}{\beta_{jf}}Sinh\beta_{jf}t-B_jSin\omega_{jk}t+C_j\frac{\omega_{jk}}{\beta_{jf}}Cosh\beta_{jf}t-C_j\frac{\omega_{jk}}{\beta_{jf}}Cos\omega_{jk}t\right]\bullet$$

$$Sin\frac{k\pi y_1}{L_y}Sin\frac{k\pi y}{L_y}W_j(x) \qquad (2.151)$$

Equation (2.151) is the transverse displacement response of a prestress rectangular plate (incorporating rotatory inertia effects) and having arbitrary edge support and under the action of a moving mass.

The constants A_j, B_j, and C_j are to be determined from the choice of any of the classical supports conditions.

CHAPTER THREE

3.0 ILLUSTRATIVE EXAMPLES, NUMERICAL CALCULATIONS AND DISCUSSION OF RESULTS

3.1 ILLUSTRATIVE EXAMPLES

The foregoing analysis is illustrated by various practical examples in this section. The classical boundary conditions such as simply supported boundary conditions and simple clamped end conditions are taken into consideration.

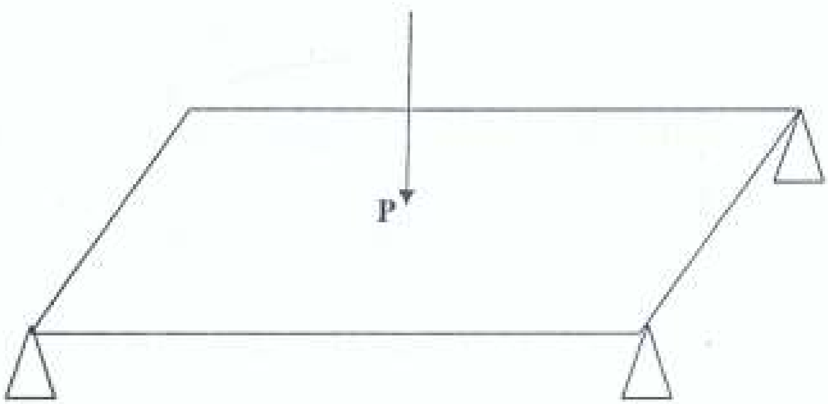

Fig 3.0(a) Plate under the action of moving concentrated masses having simple supports at all its edges. P denotes the load on the structure.

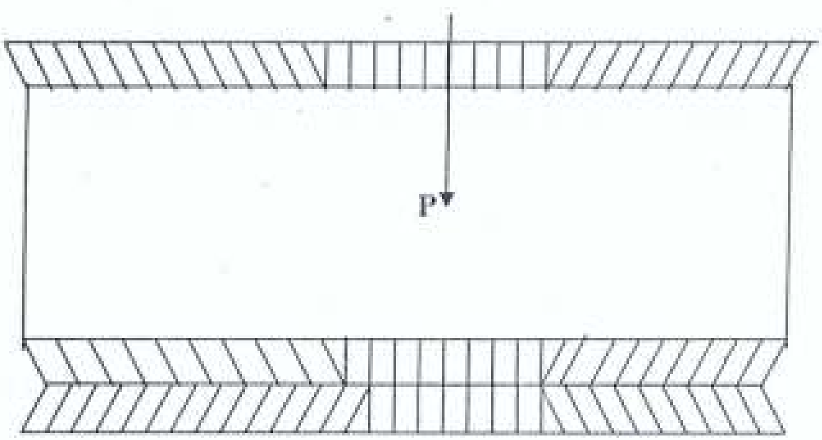

Fig 3.0(b): Plate clamped at edges x=0, x=L_x with simple supports at edges y=0, y=L_y A typical example is bridge

3.2 SIMPLE SUPPORT AT ALL EDGES

A prestressed rectangular plate under the action of moving mass having simple supports at all its edges has boundary conditions given by:

$$w(o,y,t)=0, \quad w(L_x,y,t)=0 \tag{3.1a}$$

$$w(x,o,t)=0, \quad w(x,L_y,t)=0 \tag{3.1b}$$

$$\frac{\partial^2}{\partial x^2}w(o,y,t)=0, \quad \frac{\partial^2}{\partial x^2}w(L_x,y,t)=0 \tag{3.2a}$$

$$\frac{\partial^2}{\partial y^2}w(x,o,t)=0, \quad \frac{\partial^2}{\partial y^2}w(x,L_y,t)=0 \tag{3.2b}$$

Hence for the normal modes

$$w_j(0)=0\ , \quad w_j(L_x)=0 \tag{3.3a}$$

$$w_k(0)=0\ , w_k(L_y)=0 \tag{3.3b}$$

$$\frac{\partial^2}{\partial x^2}w_j(0)=0\ , \frac{\partial^2}{\partial x^2}w_j(Lx)=0 \tag{3.4a}$$

$$\frac{\partial^2}{\partial y^2}w_k(0)=0, \frac{\partial^2}{\partial y^2}w_k(L_y)=0 \tag{3.4b}$$

Now we make use of boundary condition given by equation (2.98)

$$A_j=0, \beta_j=0 \quad C_j=0 \text{ and } \alpha_j=j\pi \tag{3.5a}$$

$$A_k=0, \beta_k=0, \text{ and } C_j=0 \text{ and } \alpha_k=k\pi \tag{3.5b}$$

Similarly,

$$A_p=0, \quad \beta_l=0 \quad C_p=0 \text{ and } \alpha_p=p\pi \tag{3.6a}$$

$$A_q = 0, \quad \beta_q = 0 \quad C_q = 0 \text{ and } \alpha_q = q\pi \tag{3.6b}$$

Also

$$w_j = w_p = \frac{\mu L_x}{2} \tag{3.7}$$

When we substitute equations (3.5a), (3.5b), (3.6) and (3.7) into the transformed equation (2.56) the transformed equation for a prestressed rectangular plate, having simple supports at all its edges is obtained

Namely,

$$u_{tt}(j,k,t) + \alpha_n^2 U\,(j,k,t) + \sum_{p=1}^{\infty} \frac{\mu}{w_p} u\,(j,k,t) X_1 + R_o \sum_{p=1}^{\infty} \frac{\mu}{w_p} u_{tt} X_2(p,k,t) +$$

$$\Gamma\left\{\left(4\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{n=1}^{\infty} \frac{\mu}{w_p} u_{tt}(p,q,t) Sin\frac{k\pi y_1}{L_y} Sin\frac{q\pi y_1}{L_y} X_3 + \right.\right.$$

$$\left. 2\sum_{p=1}^{\infty}\sum_{n=1}^{\infty} \frac{\mu}{w_p} u_{tt}(p,q,t) X_3 + \sum_{p=1}^{\infty} \frac{\mu}{w_p} u_{tt}(p,k,t) I_1 + 2\sum_{p=1}^{\infty}\sum_{q=1}^{\infty} \frac{\mu}{w_p} u_{tt}(p,q,t) Sin\frac{k\pi y_1}{L_y} Sin\frac{q\pi y_1 I_1}{L_y} \right)$$

$$+2c\left(4\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{n=1}^{\infty} \frac{\mu}{w_p} u_t(p,q,t) Cos\frac{n\pi ct}{L_x} Sin\frac{k\pi y_1}{L_y} Sin\frac{q\pi y_1}{L_y} X_4 + 2\sum_{p=1}^{\infty}\sum_{n=1}^{\infty} \frac{\mu}{w_p} u_t(p,k,t) Cos\frac{n\pi ct}{L_x} X_4 + \right.$$

$$\left. 2\sum_{p=1}^{\infty}\sum_{q=1}^{\infty} \frac{\mu}{w_p} u_t(p,q,t) Sin\frac{k\pi y_1}{L_y} Sin\frac{q\pi y_1}{L_y} X_5 + \sum_{p=1}^{\infty} \frac{\mu}{w_p} u_t(p,k,t) X_5 \right)$$

$$-c^2\left(4\sum_{p=1}^{\infty}\sum_{p=1}^{\infty}\sum_{n=1}^{\infty} \frac{\mu}{w_p} u(p,q,t) Sin\frac{k\pi y_1}{L_y} Sin\frac{q\pi y_1}{L_1} X_6 + \right.$$

$$2\sum_{p=1}^{\infty}\sum_{n=1}^{\infty} \frac{\mu}{w_p} u(p,k,t) X_6 + 2\sum_{p=1}^{\infty}\sum_{q=1}^{\infty} \frac{\mu}{w_p} u\,(p,q,t) Sin\frac{k\pi y_1}{L_y} Sin\frac{q\pi y_1}{L_y} X_7 +$$

$$\sum_{p=1}^{\infty}\frac{\mu}{w_p}u\,(p,k,t)X_7\Big) = \sum_{i=1}^{N}\frac{M\ g}{\mu}Sin\frac{k\pi y_1}{L_y}W_j(c,t) \tag{3.8}$$

$$X_1 = \left(N_x\frac{p^2\pi^2}{L_x^2}I_1 + N_y\frac{k^2\pi^2}{L_y^2}I_1\right) \tag{3.9a}$$

$$X_2 = \left(\frac{p^2\pi^2}{L_x^2}I_1 + \frac{k^2\pi^2}{L_y^2}I_1\right) \tag{3.9b}$$

$$X_3 = Cos\frac{n\pi ct}{L_x}I_{17} \tag{3.9c}$$

$$X_4 = \frac{p^2\pi^2}{L_x^2}I_{18} \tag{3.9d}$$

$$X_5 = \frac{p\pi}{L_x}I_2 \tag{3.9e}$$

$$X_6 = Cos\frac{n\pi ct}{L_x}\frac{p^2\pi^2}{L_x^2}I_{17} \tag{3.9f}$$

$$X_7 = \frac{p^2\pi^2}{L_x^2}I_1 \tag{3.9g}$$

Using (3.5a) and (3.6), it is easily shown that

$$I_1 = \frac{L_x}{2}, p=j \qquad (3.10$$

$$I_2 = \begin{cases} 0, & if \quad p \pm j = even \\ \frac{Lxbo}{2}, & if \quad P \pm j = odd \end{cases} \tag{3.11}$$

$$I_{18} = \begin{cases} 0, \ if \quad p \pm j = even \\ \frac{L_y a_0}{2}, if \quad P \pm j = odd \end{cases} \tag{3.12}$$

$$a_0(n,p,j) = \left(\frac{-4j\left(n^2 + p^2 - j^2\right)}{\pi\left[(n+p)^2 - j^2\right]\left[(n-p)^2 - j^2\right]} \right) \tag{3.13a}$$

$$b_0(p,j) = \left(\frac{-4j\pi}{p^2\pi^2 - j^2\pi^2} \right) \tag{3.13b}$$

Substituting 3.10 – 3.12 into 3.8 after some rearrangements one obtains.

$$u_n(j,k,t) + \alpha_{\mu}^2 u\,(p,k,t) + u\,(j,k,t)\left(N_x \frac{j^2\pi^2}{L_x^2} + N_y \frac{k^2\pi^2}{L_y^2} \right) + R_0 u_n(j,k,t)\left(\frac{j^2\pi^2}{L_x^2} + \frac{k^2\pi^2}{L_y^2} \right) +$$

$$\Gamma_u \left\{ \left(4\sum_{p=1}^{\infty}\sum_{q=1} u_n(p,k,t) Sin\frac{k\pi y_1}{L_y} Sin\frac{q\pi y_1}{L_y} Sin\frac{p\pi ct}{L_x} Sin\frac{j\pi ct}{L_x} \right.\right.$$

$$+ 2\sum_{p=1}^{\infty} u_n(p,k,t) Sin\frac{p\pi ct}{L_x} Sin\frac{j\pi ct}{L_x} + 2\sum_{q=1}^{\infty} u_n(p,q,t) Sin\frac{k\pi y_1}{L_y} Sin\frac{q\pi y_1}{L_y} + u_n(p,k,t)$$

$$+ 8c\left(\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\ \sum_{n=1}^{\infty} 2b_j u_t(p,k,t) Cos\frac{n\pi ct}{L_x} Sin\frac{q\pi y_1}{L_y} Sin\frac{k\pi y_1}{L_y} + \sum_{p=1}^{\infty}\sum_{n=1}^{\infty} b_j u_t(p,k,t) Cos\frac{n\pi ct}{L_x} + \right.$$

$$\left. \sum_{p=1}^{\infty}\sum_{q=1}^{\infty} 2a_j u_t(p,q,t) Sin\frac{q\pi y_1}{L_y} Sin\frac{k\pi y_1}{L_y} + \sum_{p=1}^{\infty} a_j u_t(p,k,t) \right) -$$

$$4c^2\left(\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{n=1}^{\infty} 2S_p u(p,q,t) Sin\frac{j\pi ct}{L_x} Sin\frac{p\pi ct}{L_x} Sin\frac{q\pi y_1}{L_y} Sin\frac{k\pi y_1}{L_y} + \sum_{p=1}^{\infty}\sum_{n=1}^{\infty} S_p u\,(p,k,t) Sin\frac{j\pi ct}{L_x} Sin\frac{p\pi ct}{L_x} + \right.$$

$$\sum_{p=1}^{\infty}\sum_{q=1}^{\infty} u(p,q,t) S_p Sin\frac{q\pi y}{L_y} Sin\frac{k\pi y}{L_y} + \frac{1}{2}\sum_{p} u\,(p,k,t) S_p = 2\Gamma_u Sin\frac{k\pi y}{L_y} Sin\frac{j\pi ct}{L_x}$$

(3.14)

where

$$b_j(n,p,j) = b_0 \frac{p\pi}{L_x}, \tag{3.15a}$$

$$a_j(p,j) = a_0 \frac{p\pi}{L_x} \tag{3.15b}$$

$$b_0(n,p,j) = \left(\frac{2_j\left(n^2 + p^2 - j^2\right)}{\pi\left[(n+p)^2 - j^2\right]\left[(n-p)^2 - j^2\right]} \right) \tag{3.16}$$

$$a_0(p,j) = \left(\frac{j\pi}{p^2\pi^2 - j^2\pi^2} \right) \tag{3.17}$$

$$S_j = \frac{j^2\pi^2}{2L_x^2}, \quad S_p = \frac{p^2\pi^2}{2L_x^2} \tag{3.18}$$

$$\Gamma_w = \frac{M_g}{2\mu} \qquad \Gamma_0 = \frac{M}{\mu L_x L_y} \tag{3.19}$$

Equation (3.14) is now the fundamental equation of our problem when the prestressed rectangular plate has, simple support at all its edges. In what follows, we shall discuss two cases of the equation.

3.21 SIMPLY SUPPORTED RECTANGULAR PLATE TRAVERSED BY MOVING FORCE

An approximate model of the system when the inertia effect of the moving mass M is neglected, i.e when $\Gamma_0 = 0$ is the moving force problem associated with the system. Thus, the differential equation reduces to

$$u_{tt}(j,k,t) + \alpha_{jk}^* u\ (j,k,t) = P_L^* Sin\frac{k\pi y}{L_y} Sin\frac{j\pi ct}{L_x} \tag{3.20}$$

where

$$\alpha_{jk}^{*2} = \frac{\alpha_{jk}^2 + \left(N_x \frac{j^2\pi^2}{L_x^2} + N_y \frac{k^2\pi^2}{L_y^2} \right)}{1 + R_0\left(\frac{j^2\pi^2}{L_x^2} + \frac{k^2\pi^2}{L_y^2} \right)} \tag{3.21}$$

$$P_L^* = \frac{2\Gamma_w}{1 + R_0\left(\frac{j^2\pi^2}{L_x^2} + \frac{k^2\pi^2}{L_y^2} \right)} \tag{3.22}$$

In order to solve equation (3.20), the method of Laplace transform defined by

$$(\sim) = \int_0^\infty (\bullet) e^{-st} dt \tag{3.23}$$

Where s is the Laplace parameter. Using (3.23) in conjunction with initial condition

$$u(j,k0) = 0 = u_t(j,k,0) \tag{3.24}$$

Thus, we obtain the simple algebraic equation given by.

$$\bar{u}_{(j,k,s)} = \frac{P_L^* V_k(y_1)}{S^2 + \alpha_{jk}^*} \bullet \frac{\alpha_{jf}}{S^2 + \alpha_{jf}^2} \tag{3.25}$$

The problem reduces to finding the Laplace inversion of (3.25). In order to do this, we adopt the following representation.

$$g(s) = \frac{\alpha_{jf}}{S^2 + \alpha_{jf}^2}, \quad f(s) = \frac{P_L^* V_k y_1}{S^2 + \alpha_{jk}^{*2}} \tag{3.26}$$

So that the Laplace inversion of $U(j,k,s)$ is the convolution of $f(s)$ and $g(s)$ defined as

$$f(s) * g(s) = \int_0^\infty f(t-r) g(r) dr \tag{3.27}$$

Consequents,

$$u(j,k,t) = \frac{P_i^* V_k(y_1)}{\alpha_{jk}^*} \int_0^{\infty} Sin\alpha_{jk}^* r Sin(\alpha_{jf} t - \alpha_{jf} r) dr \tag{3.28}$$

Carrying out the integration, it is easily shown that

$$u(j,k,t) = P_i^* V_k(y_1) \frac{1}{\left(\alpha_{jk}^{*2} - \alpha_{jf}^2\right)} \left[Sin\alpha_{jf} t - \frac{\alpha_{jf}}{\alpha_{jk}^*} Sin\alpha_{jk}^* t \right] \tag{3.29}$$

This inversion becomes

$$w(x,y,t) = \frac{4P_i^* V_k(y_1)}{L_x L_y} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{\left(\alpha_{jk}^{*2} - \alpha_{jf}^2\right)} \left[Sin\alpha_{jf} t - \frac{\alpha_{jf}}{\alpha_{jk}^*} Sin\alpha_{jk}^* t \right] Sin\frac{k\pi y}{L_y} Sin\frac{j\pi x}{L_x} \tag{3.30}$$

where

$$\alpha_{jf} = \frac{j\pi c}{L_x} \tag{3.31}$$

Equation (3.30) agrees with results previously obtained in literature when $\Gamma_0 = 0$ is set to zero and it represents the displacement response of the plate model due to moving force.

3.2.2 SIMPLY SUPPORTED RECTANGULAR PLATE TRAVERSED BY A MOVING MASS.

This section is concerned with finding the solution to the entire equation (3.14) when no term of the equation is neglected. In order to solve the boundary-value problem, we use the modified asymptotic method of Struble already alluded to. To this end equation (3.14) is arranged to take the form

$$\frac{d^2}{dt^2}u\left(j,k,t\right)-\frac{\Gamma_0 H_b(t)}{1+\Gamma_0 H_c(t)}\frac{du}{dt}(j,k,t)+\frac{\alpha_{\mu}^{*^2}-\Gamma_0 H_a(t)}{1+\Gamma_0 H_c(t)}u\left(j,k,t\right)+$$

$$\frac{\Gamma_0}{1+\Gamma_0 H_c(t)}\left\{4\sum_{\substack{p=1\\p\neq j}}^{\infty}\sum_{\substack{p=1\\p\neq k}}^{\infty}u_{tt}(p,q,t)Sin\frac{j\pi ct}{L_x}Sin\frac{p\pi ct}{L_x}Sin\frac{q\pi y_1}{L_y}Sin\frac{k\pi y_1}{L_y}+\right.$$

$$2\sum_{\substack{p=1\\p\neq 1}}^{\infty}u_{tt}(p,k,t)Sin\frac{j\pi ct}{L_x}Sin\frac{p\pi ct}{L_x}+2\sum_{q=1}^{p}{}_{tt}(j,q,t)Sin\frac{q\pi y_1}{L_y}Sin\frac{k\pi y_1}{L_y}-$$

$$\frac{8cp\pi}{L_x}\left\{\sum_{\substack{p=1\\p\neq j}}^{\infty}\sum_{\substack{q=1\\q\neq k}}^{\infty}\sum_{n=1}^{\infty}2b_j(n,p,j)u_t(p,q,t)Cos\frac{n\pi ct}{L_x}Sin\frac{q\pi y_1}{L_y}Sin\frac{k\pi y_1}{L_y}+\right.$$

$$\sum_{\substack{p=1\\p\neq j}}^{\infty}\sum_{n=1}^{\infty}b_j(n,p,j)u_t(p,k,t)Cos\frac{n\pi ct}{L_x}+$$

$$\left\{\sum_{\substack{p=1\\p\neq j}}^{\infty}\sum_{\substack{q=1\\q\neq k}}^{\infty}2a_j(p,j)u_t(p,q,t)Sin\frac{q\pi y_1}{L_y}Sin\frac{k\pi y_1}{L_y}+\sum_{\substack{p=1\\p\neq j}}^{\infty}a_j(p,j)u_t(p,k,t)\right]-$$

$$\frac{4c^2\pi^2}{L_x^2}\left\{\sum_{\substack{q=1\\q\neq k}}^{\infty}j^2u\left(j,q,t\right)Sin\frac{k\pi y_1}{L_y}Sin\frac{q\pi y_1}{L_y}+\sum_{\substack{p=1\\p\neq j}}^{\infty}P^2u\left(p,k,t\right)Sin\frac{j\pi ct}{L_x}\bullet\right.$$

$$Sin\frac{p\pi ct}{L_x}+2\sum_{\substack{p=1\\p\neq j}}^{\infty}\sum_{\substack{q=1\\q\neq k}}^{\infty}P^2u\left(p,q,t\right)Sin\frac{j\pi ct}{L_x}Sin\frac{p\pi ct}{L_x}Sin\frac{q\pi y_1}{L_y}Sin\frac{k\pi y_1}{L_y}$$

$$=P_0 Sin\frac{k\pi y_1}{L_y}Sin\frac{j\pi ct}{L_x} \qquad (3.32)$$

where

$$P_0=\frac{Mg}{\mu}=\Gamma_0 L_x L_y g \qquad (3.33)$$

$$H_a(t)=\frac{4c^2j^2\pi^2}{L_x^2}\left\{Sin^2\frac{k\pi y_1}{L_y}+Sin^2\frac{j\pi ct}{L_x}+2Sin^2\frac{j\pi ct}{L_x}Sin^2\frac{k\pi y_1}{L_y}+\frac{1}{2}\right\} \tag{3.34}$$

$$H_b(t)=\frac{8cj\pi}{L_x}\left\{2\sum_{n=1}^{\infty}b_j(n,j,j)Cos\frac{n\pi ct}{L_x}Sin^2\frac{k\pi y_1}{L_y}+\sum_{n=1}^{\infty}b_j(n,j,j)Cos\frac{n\pi ct}{L_x}\right\} \tag{3.35}$$

$$H_c(t)=4\left[Sin^2\frac{j\pi ct}{L_x}Sin^2\frac{k\pi y_1}{L_y}+\frac{1}{2}Sin^2\frac{j\pi ct}{L_x}+\frac{1}{2}Sin^2\frac{k\pi y_1}{L_y}+\frac{1}{4}\right] \tag{3.36}$$

$$b_i(n,j,j)=\frac{2jn^2}{\pi\left[n^2+2jn\right]\left[n^2-2jn\right]} \tag{3.37}$$

$$a_j(j,j)=0 \tag{3.38}$$

Thus, considering the homogenous part of the equation (3.32) and going through the same arguments and analysis as in the previous section (2.1), the modified frequency corresponding to the frequency of the free system due to the presence of the moving mass is

$$\gamma_{jk}=\alpha^*_{jk}\left\{1-\Gamma_n Sin^2\frac{k\pi y_1}{L_y}\left(1+\frac{2c^2\pi^2j^2}{\alpha^*_{jk}L_x^2}\right)\right\} \tag{3.39}$$

In order to solve the non-homogenous equation (3.32) the differential operator which acts on $u(j,k,t)$ and $u(p,q,t)$ is replaced by the equivalent free system operator defined by the modified frequency, γ_{jk}

$$\frac{d^2}{dt^2}u(j,k,t)+\gamma_{jk}^2u(j,k,t)=\epsilon L_xL_ygSin\frac{k\pi y_1}{L_y}\frac{Sinj\pi ct}{L_x} \tag{3.40}$$

Obviously, we can deduce that equation (3.40) is directly analogous to equation (3.18). Hence we have

$$u(j,k,t)=\frac{\in L_x L_y g V_k}{\left(\gamma_{jk}^2-\alpha_{jf}^2\right)}\left[Sin\alpha_{jf}t-\frac{\alpha_{jf}}{\gamma_{jk}}Sin\gamma_{jk}t\right] \tag{3.41}$$

which on inversion takes the form

$$w(x,y,t)=\frac{\in L_x L_y g V_k}{\left(\gamma_{jk}^2-\alpha_{jf}^2\right)}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\left[Sin\alpha_{jf}t-\frac{\alpha_{jf}}{\gamma_{jk}}Sin\gamma_{jk}t\right]\frac{Sink\pi y}{L_y}\frac{Sinj\pi x}{L_x} \tag{3.42}$$

3.3 RECTANGULAR PLATE CLAMPED AT EDGES x=0, x=L_x WITH SIMPLY SUPPORT AT EDGES $y=0$, $y=L_y$

For a rectangular plate clamped at edges $x=0$, $x=L_x$ with simply supports at edges $y=0$, $y=L_y$, the boundary conditions at such opposite edges are

$$\begin{aligned} w(o,y,t)&=0, \quad w(l_x,y,t)=0\\ w(x,o,t)&=0, \quad w(x,l_y,t)=0 \end{aligned} \tag{3.43a}$$

$$\begin{aligned} \frac{\partial w(o,y,t)}{\partial x}&=0, \quad \frac{\partial w(l_x,y,t)}{\partial x}=0\\ \frac{\partial^2 w(x,o,t)}{\partial y^2}&=0, \quad \frac{\partial^2 w(x,l_y,t)}{\partial y^2}=0 \end{aligned} \tag{3.43b}$$

Hence, for have the normal modes we have

$$\begin{aligned} w_j(o)&=0, \quad w_j(l_x)=0\\ w_k(o)&=0, \quad w_k(l_y)=0 \end{aligned} \tag{3.44a}$$

$$\frac{\partial w_j(o)}{\partial x} = 0, \qquad \frac{\partial w_j(l_x)}{\partial x} = 0$$
$$\frac{\partial^2 w_k(o)}{\partial y^2} = 0, \qquad \frac{\partial^2 w_k(l_x)}{\partial y^2} = 0 \tag{3.44b}$$

Our initial conditions remain as

$$w(x, y, o) = 0 = \frac{\partial w(x, y, o)}{\partial t} \tag{3.45}$$

Using the boundary condition 3.43a and 3.43b in equation (2.16) the following values of the constants and frequency equations are obtained for the clamped edges

$$A_j = \frac{\frac{\alpha_j}{\beta_j}\sinh\beta_j - \sin\alpha_j}{\cos\alpha_j - \cosh\beta_j} \Rightarrow A_p = \frac{\frac{\alpha_p}{\beta_p}\sinh\beta_p - \sin\alpha_p}{\cos\alpha_p - \cosh\beta_p} \tag{3.46}$$

$$B_j = -\frac{\alpha_j}{\beta_j} \qquad B_p = -\frac{\alpha_p}{\beta_p} \tag{3.47}$$

$$A_j = -C_j \qquad A_p = -C_p \tag{3.48}$$

The frequency equation of the clamped edges

$$2 - 2\cos\alpha_j\cosh\beta_j + \left(\frac{\alpha_j}{\beta_j} - \frac{\beta_i}{\alpha_j}\right)\sin\alpha_j\sinh\beta_j = 0 \tag{3.49}$$

From simple support edges, it is readily shown that

$$A_j = 0 \qquad A_p = 0$$
$$C_j = 0 \qquad C_p = 0 \tag{3.50}$$
$$B_j = 0 \qquad B_p = 0$$

With the corresponding frequency equation given by

$$\alpha_j = j\pi \qquad \alpha_p = p\pi \tag{3.51}$$

$$w_j = w_p = \frac{\mu L_x}{2} \tag{3.52}$$

For the chosen boundary conditions, it is required that the entire equation (2.56) is solved and considers only one mass M traveling with velocity c. First, the case of moving force is considered.

3.3.1 SIMPLE-CLAMPED PRESTRESSED RECTANGULAR PLATE UNDER THE ACTION OF A MOVING FORCE

The differential equation describing the response of a simple-clamped rectangular plate under the action of moving force may be obtained

Thus, setting $\varepsilon^0 = 0$, equation (2.56) reduces to

$$u_{tt}(j,k,t) + \theta_e^2 u\,(j,k,t) - \varepsilon^0 \sum_{p=1}^{\infty} \frac{\mu}{w_p} u_{tt}(p,k,t) \left[\Lambda^2(p,j) - \frac{k^2\pi^2}{L_y^2} \Lambda(p,j)_1 \right] -$$

$$\sum_{p=1}^{\infty} \frac{\mu}{w_p} u\,(p,k,t) \left[N_x^0 \Lambda^2(p,j) - N_y^0 \frac{k^2\pi^2}{L_y^2} \Lambda(p,j) \right] = P_L Sin \frac{k\pi y_1}{L_y} W_j(c,t) \tag{3.53}$$

where

$$\varepsilon^0 = \frac{M}{\mu L_x L_y} \tag{3.54}$$

$$\theta_e^2 = \Omega_{jk}^2 + K \tag{3.55}$$

$$P_L = \frac{Mg}{\mu} \tag{3.56}$$

$$W_j(c,t) = \sin\frac{\alpha_j ct}{L_x} + A_j\cos\frac{\alpha_j ct}{L_x} + B_j\sinh\frac{\beta_j ct}{L_x} + C_j\cosh\frac{\beta_j ct}{L_x} \quad (3.57)$$

Equation (3.53) is clearly analogous to equation (2.69) in section 2.1. Thus, similar arguments in this section, the modified frequency corresponding to the frequency of the free system due to the presence of the rotatory inertia factor z'', becomes

$$\alpha''_{c,j,k} = \theta_c\left[1 - \frac{\Gamma_{a0}\mu}{2w_j}\left\{\left(\frac{k^2\pi^2}{L_y^2}\Lambda(j,j) - \Lambda^2(j,j)\right) + \frac{1}{z''\theta_c^2}\left[N_r^0\Lambda^2(p,j) - N_r^0\frac{k^2\pi^2}{L_y^2}\Lambda(p,j)\right]\right\}\right]$$

(3.58)

Thus, in order to solve equation (3.53), the differential operator which acts on $u(j,k,t)$ and $u(j,p,t)$ can be replaced by the equivalent free system operator defined by the modified $\alpha''_{c,j,k}$ i.e

$$\frac{d^2}{dt^2}u(j,k,t) + \alpha''_{c,j,k}u(j,k,t) = P_i V_k(y_1)W_j(ct) \quad (3.59)$$

$$V_k(y_1) = \sin\frac{k\kappa y_1}{L_y} \quad (3.60)$$

Obviously, equation (3.59) is directly analogous to equation (2.97). Hence, solving (3.59) in conjunction with the conditions, one obtains

$$u(j,k,t) = \frac{P_i V_k(y_1)}{2\left(\alpha''^2_{c,j,k} - \alpha^2_{jj}\right)}\left(-A\cos\alpha_{jj}t + A_j\cos\alpha''_{c,j,k}t + \sin\alpha_{jj}t - \frac{\alpha_{jj}}{\alpha''_{c,j,k}}\sin\alpha''_{c,j,k}t\right)$$

$$+\frac{P_i\beta_j}{2\alpha''_{c,j,k}\left(\beta^2_{jj} + \alpha''^2_{c,j,k}\right)}\left(B_j\frac{\alpha''_{c,j,k}}{\beta_j}\sinh\beta_{jj}t - B_j\sin\alpha''_{c,j,k}t\right.$$

$$+C_j \frac{\alpha^{sv}_{c,j,k}}{\beta_{jf}} \cosh \beta_{jf} t - C_j \frac{\alpha^{sv}_{c,j,k}}{\beta_{jf}} \cos \alpha^{sv}_{c,j,k} t \Bigg) \tag{3.61}$$

Where

$$W_j(ct) = \sin \alpha_{jf} t + A_j \cos \alpha_{jf} + B_j \sinh \beta_{jf} t + C_j \cosh \beta_{jf} t \tag{3.62}$$

$$\alpha_{jf} = \frac{\alpha_j c}{L_x} \tag{3.63}$$

$$\beta_{jf} = \frac{\beta_j c}{L_x} \tag{3.64}$$

Which on inversion gives

$$w(x,y,t) = \frac{4P_1 V_k(y_1)}{L_x L_y} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \left(\frac{1}{2\left(\alpha^{sv^2}_{c,j,k} - \alpha^2_{jf}\right)} \Big(-A \cos \alpha_{jf} t + A_j \cos \alpha^{sv}_{c,j,k} t \right.$$

$$\left. + \sin \alpha_{jf} t - \frac{\alpha_{jf}}{\alpha^{sv}_{c,j,k}} \sin \alpha^{sv}_{c,j,k} t \Big) + \frac{\beta_{jf}}{2\alpha^{sv}_{c,j,k}\left(\beta^2_{jf} + \alpha^{sv^2}_{c,j,k}\right)} \Bigg(B_j \frac{\alpha^{sv}_{c,j,k}}{\beta_{jf}} \sinh \beta_{jf} t - B_j \sin \alpha^{sv}_{c,j,k} t \right.$$

$$\left. + C_j \frac{\alpha^{sv}_{c,j,k}}{\beta_{jf}} \cosh \beta_{jf} t - C_j \frac{\alpha^{sv}_{c,j,k}}{\beta_{jf}} \cos \alpha^{sv}_{c,j,k} t \Bigg) \right) \bullet$$

$$\sin \frac{k\pi y}{L_y} \left(\sin \frac{\alpha_j x}{L_x} + A_j \frac{\alpha_j x}{L_x} + B_j \sinh \frac{\beta_j x}{L_x} + C_j \cosh \frac{\beta_j x}{L_x} \right) \tag{3.65}$$

The equation above, (3.65) represents the transverse displacement response of our plate model when it is under the action of a moving force.

3.3.2 SIMPLE CLAMPED PRESTRESS RECTANGULAR PLATE UNDER THE ACTION OF A MOVING MASS

The term moving mass problem refers to when $\varepsilon^{o} \neq 0$ i.e when the inertial term is retained. In this case, the solution to the entire equation (2.66) is required. Here we apply the same argument in the previous section when we neglect the inertia term, the homogenous part of this equation can be replace by a free system operator defined by the modified frequency $\alpha''_{c,j,k}$, corresponding to the frequency of the free system due to the presence of the effect rotatory correction factor. Thus, equation (2.66) can be rearrange to take the form

$$u_{tt}(j,k,t)+\alpha^{rr^2}_{c,j,k}u\,(j,k,t)+\varepsilon^{o}\left\{4\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{n=1}^{\infty}\frac{\mu}{w_p}u_{tt}(p,q,t)Cos\frac{n\pi ct}{Lx}\frac{Sin k\pi y_1}{Ly}\frac{Sin q\pi y_1}{Ly}\Lambda\,(n,j,p)\right.$$

$$+2\sum_{p=1}^{\infty}\sum_{n=1}^{\infty}\frac{\mu}{w_p}u_{tt}(p,k,t)Cos\frac{n\pi ct}{Lx}\Lambda\,(n,j,p)$$

$$+2\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\frac{\mu}{w_p}u_{tt}(p,q,t)Sin\frac{k\pi y_1}{Ly}Sin\frac{q\pi y_1}{Ly}\Lambda\,(j,p)+\sum_{p=1}^{\infty}\frac{\mu}{w_p}u_{tt}(p,k,t)\Lambda\,(j,p)$$

$$-2c\left[4\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{n=1}^{\infty}\frac{\mu}{w_p}u_t(p,q,t)\,Cos\frac{n\pi ct}{Lx}Sin\frac{k\pi y_1}{Ly}Sin\frac{q\pi y_1}{Ly}\Lambda^1\,(n,j,p)+2\sum_{p=1}^{\infty}\sum_{n=1}^{\infty}\frac{\mu}{w_p}u_t(p,k,t)\right.$$

$$\left.Cos\frac{n\pi ct}{Lx}\Lambda^1\,(n,j,p)+2\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\frac{\mu}{w_p}u_t(p,q,t)Sin\frac{k\pi y_1}{Ly}Sin\frac{q\pi y_1}{Ly}\Lambda\,(j,p)+\sum_{p=1}^{\infty}\frac{\mu}{w_p}u_t(p,k,t)\Lambda^1\,(j,p)\right]$$

$$+c^2\left\{4\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{n=1}^{\infty}\frac{\mu}{w_p}u\,(p,q,t)Cos\frac{n\pi ct}{Lx}Sin\frac{k\pi y_1}{Ly}\,Sin\frac{q\pi y_1}{Ly}\Lambda^2\,(n,j,p)+\right.$$

$$2\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\frac{\mu}{w_p}u\,(p,q,t)Cos\frac{n\pi ct}{Lx}\Lambda^2(j,p)+\sum_{p=1}^{\infty}\frac{\mu}{w_p}u\,(p,k,t)\Lambda^2\,(j,p)=P_i Sin\frac{k\pi y_1}{L_y}W_j(c,t)$$

(3.66)

When we consider the homogenous part of equation (3.66) and going through the same arguments and analysis as in section 2.1, the modified frequency corresponding to the frequency of the free system due to the presence of moving mass is

$$\beta_{p,j,k}^{\prime\prime} = \alpha_{c,j,k}^{\prime\prime}\left[1+\left(\frac{S_a(j,k)+S_c(j,k)}{2\alpha_{c,j,k}^{\prime\prime 2}}\right)\right] \tag{3.67}$$

Where

$$S_a(j,k) = -\lambda\beta_{jk}^2\left(\frac{2\mu}{w_j}Sin^2\frac{k\pi y_1}{L_y}\Lambda\,(j,j)+\frac{\mu}{w_j}\Lambda\,(j,j)\right) \tag{3.68}$$

And

$$S_c(j,k) = c^2\frac{\mu}{w_j}\left(2Sin^2\frac{k\pi y_1}{L_y}\Lambda^2(j,j)+\Lambda^2(j,j)\right) \tag{3.69}$$

To solve the non-homogenous equation (3.66), the differential operator which act on $u(j,k,t)$ and $u(p,k,t)$ is replaced by the equivalent free system operator defined by the modified frequency $\alpha_{p,j,k}^{\prime\prime}$ i.e.

$$\frac{d^2}{dt^2}u(j,k,t)+\alpha_{p,j,k}^{\prime\prime 2}u(j,k,t) = \varepsilon L_x L_y g V_k(y_1)W_j(c,t) \tag{3.70}$$

Obviously, equation (3.70) is directly analogous to equation (2.117). Hence, we have

$$u(j,k,t) = \frac{L_x L_y g V_k(y_1)}{2\left(\alpha_{p,j,k}^{\prime\prime 2}-\alpha_{jf}^2\right)}\left(-A_j\cos\alpha_{jf}t + A_j\cos\alpha_{p,j,k}^{\prime\prime}t\sin\alpha_{jf}t - \frac{\alpha_{jf}}{\alpha_{p,j,k}^{\prime\prime}}\sin\alpha_{p,j,k}^{\prime\prime}t\right)$$

$$+\frac{\varepsilon L_x L_y g\beta_{jj} V_k(y_1)}{2\alpha''_{p,j,k}\left(\beta_{jj}^2+\alpha''^2_{p,j,k}\right)}\left(B_j\frac{\alpha''_{p,j,k}}{\beta_{jj}}\sinh\beta_{jj}t-B_j\sin\alpha''_{c,j,k}t\right.$$

$$\left.+C_j\frac{\alpha''_{c,j,k}}{\beta_{jj}}\cosh\beta_{jj}t-C_j\frac{\alpha''_{c,j,k}}{\beta_{jj}}\cos\alpha''_{p,j,k}t\right) \tag{3.71}$$

Which on inversion gives

$$w(x,y,t)=4\varepsilon V_k(y_1)\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\left(\frac{1}{2\left(\alpha''^2_{p,j,k}-\alpha_{jj}^2\right)}\left(-A_j\cos\alpha_{jj}t+A_j\cos\alpha''_{p,j,k}t\right.\right.$$

$$\left.+\sin\alpha_{jj}t-\frac{\alpha_{jj}}{\alpha''_{p,j,k}}\sin\alpha''_{p,j,k}t\right)+\frac{\beta_{jj}}{2\alpha''_{p,j,k}\left(\beta_{jj}^2+\alpha''^2_{p,j,k}\right)}\left(B_j\frac{\alpha''_{p,j,k}}{\beta_{jj}}\sinh\beta_{jj}t-B_j\sin\alpha''_{c,j,k}t\right.$$

$$\left.\left.+C_j\frac{\alpha''_{c,j,k}}{\beta_{jj}}\cosh\beta_{jj}t-C_j\frac{\alpha''_{c,j,k}}{\beta_{jj}}\cos\alpha''_{p,j,k}t\right)\right)$$

$$\sin\frac{k\pi y}{L_y}\left(\sin\frac{\alpha_j x}{L_x}+A_j\cos\frac{\alpha_j x}{L_x}+B_j\sinh\frac{\beta_j x}{L_x}+C_j\cosh\frac{\beta_j x}{L_x}\right) \tag{3.72}$$

Equation (3.85) represents the response displacement of a simple – clamped rectangular plate traversed by a moving mass.

3.4 DISCUSSION OF THE ANALYTICAL SOLUTION

In an undamped system as this, it is necessary to examine the phenomenon of resonance. Equation (3.30) clearly shows that the simply supported elastic rectangular plate traversed by a moving force will be in state of resonance whenever

$$\alpha^*_{jk}=\frac{j\pi c}{L_x} \tag{3.73}$$

While equation (3.42) shows that the plate under the action of a moving mass encounters a resonance effects at

$$\gamma_{jk} = \frac{j\pi c}{L_x} \tag{3.74}$$

Where

$$\gamma_{jk} = \alpha^{*}_{jk}\left\{1-\Gamma_{n} Sin^{2}\frac{k\pi y_{1}}{L_{y}}\left(1+\frac{2c^{2}\pi^{2}j^{2}}{\alpha^{*}_{jk}L_{x}^{2}}\right)\right\} \tag{3.75}$$

It is obvious from equation (3.73) and (3.75) that in the same natural frequency, the critical speed for the system of simply supported elastic rectangular plate with rotatory inertia effect and traversed by a moving force is greater than that traversed by a moving mass. Thus, resonance is reached earlier than in the moving force system.

Next, the phenomenon of the resonance for the simple-clamped plate under the action of a moving force and the moving mass is investigated. It is obvious from equation (3.65) that the simple-clamped elastic rectangular plate traversed by a moving force reaches a state of resonance whenever

$$\alpha^{sv}_{e,j,k} = \frac{j\pi c}{L_x} \tag{3.76}$$

While equation (3.72) shows that the same plate under the action of a moving mass experience resonance effect whenever

$$\beta^{sv}_{p,j,k} = \frac{j\pi c}{L_x} \tag{3.77}$$

Where

$$\beta_{e,j,k}^{vv} = \alpha_{e,j,k}^{vv}\left[1+\left(\frac{S_a(j,k)+S_c(j,k)}{2\alpha_{e,j,k}^{vv^2}}\right)\right] \tag{3.78}$$

Thus, from equation (3.76) and equation (3.78) it is evident that the same results and analysis similar to those of the simply supported plate are also obtained for simple-clamped plate.

3.5 NUMERICAL RESULTS AND DISCUSSION OF RESULTS

In order to illustrate the analytical results, the rectangular plate is taken to be of length $L_y = 0.914$m, and height L_x=0.457m.It is assumed that the mass travels at the constant velocity 1.5m/s. Furthermore, E, y and λ are chosen to be 2.109×10^9 kg/m^2, 0.4m and 0.2 respectively. The transverse deflection of the rectangular plate are calculated and plotted against time for values of rotatory inertia R_0, axial force along x-axis N_x axial force along y-axis N_y, foundation stiffness K. The results are shown on the various graphs for the two classes of boundary conditions so far considered

3.5.1 SIMPLY SUPPORTED RECTANGULAR PLATE

Fig3.1 and Fig3.2 display the effect of Rotatory inertia R_o on the transverse deflection of the simply supported plate for both cases of moving force and moving mass respectively for fixed values of K, N_x and N_y (K=2 x10^{6}N/m^3 and axial forces N_x=2 x10^{6}N and N_y=2.5 x10^6 N). The graphs show that the response amplitude decreases as R_o increases. The values of R_o used are 10, 20 and 30.

Fig3.3 and Fig3.4 depict the transverse displacement response of the simply supported plate in both cases of moving force and moving mass respectively for fixed values of N_x, N_y and R_o for various values of foundation stiffness K. It is evident that as K increases, the response amplitude decreases.

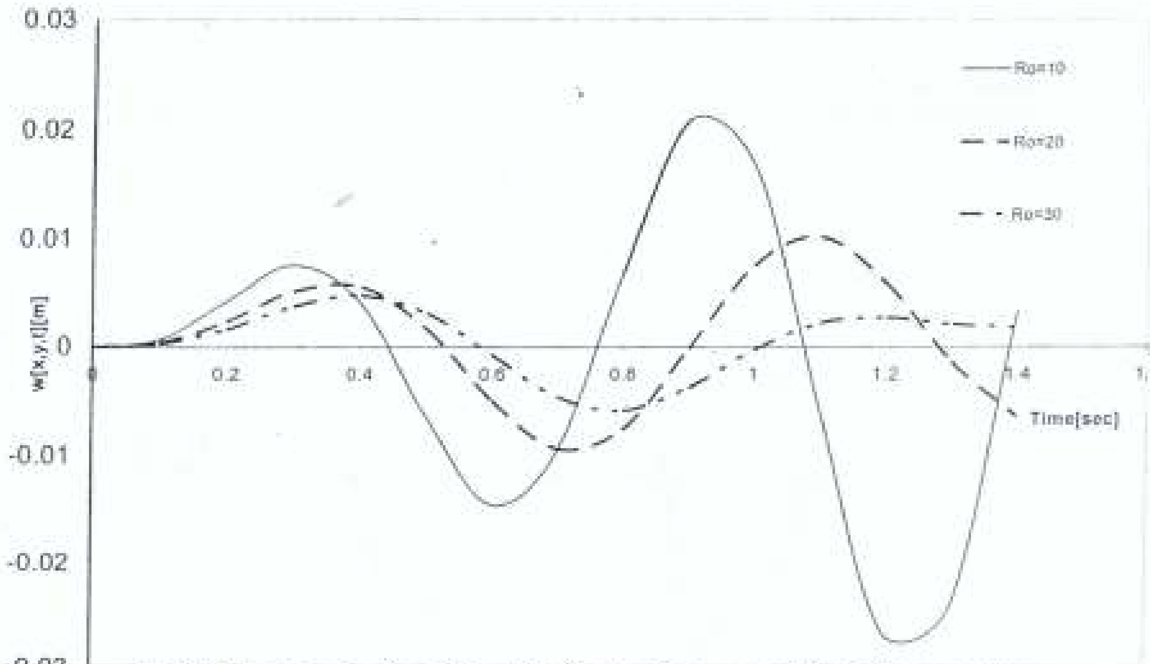

Fig3.1:Deflection Profile of Simply Supported Rectangular Plate Trasversed by Moving Force for fixed K=2x10^6 N/m^3,Nx=2x10^6 N,Ny=2.5x10^6 N for various values of Ro

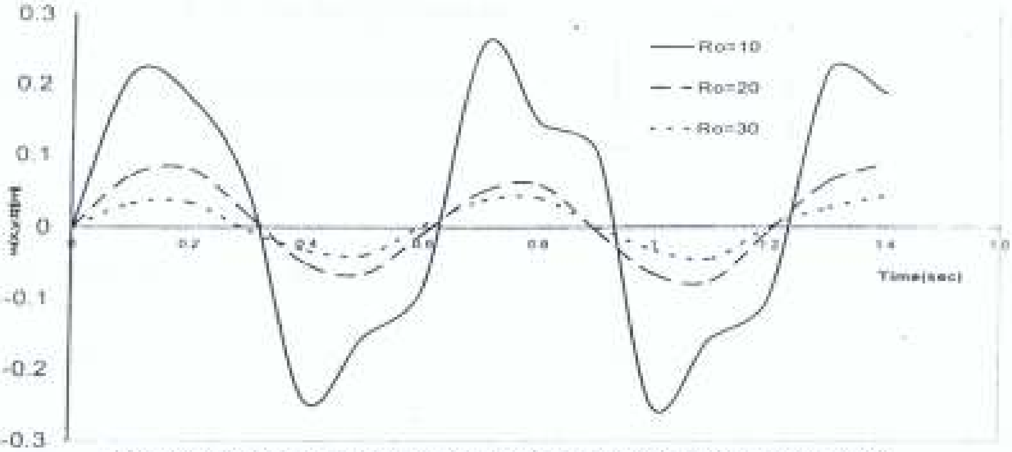

Fig3.2:Deflection Profile of Simply-Suppoted Rectangular Plate Trasversed by Moving Mass for fixed Nx=2x10^4N, Ny=2.5x10^4N, K=2x10^5N/m^3 for various values of Ro

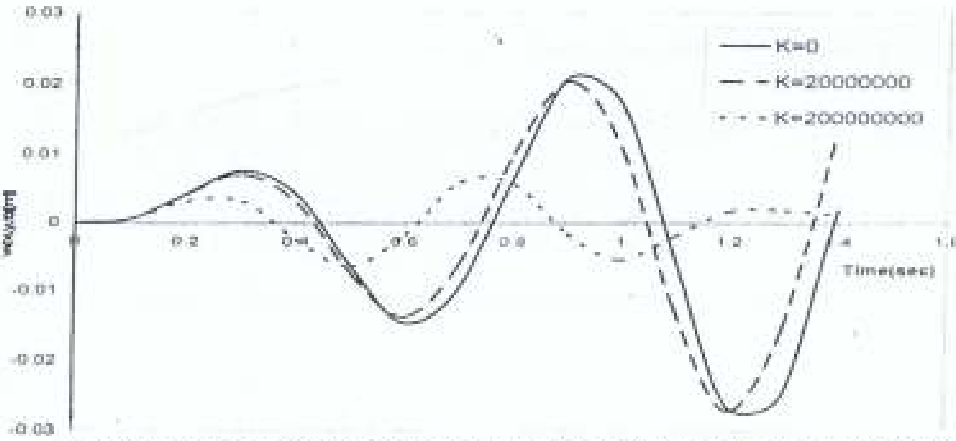

Fig3.3: Deflection Profile of Simple-Supported Rectangular Plate Trasversed by Moving Force for fixed Ro=10,Nx=2x10^4N,Ny=2.5x10^4N for various values of K

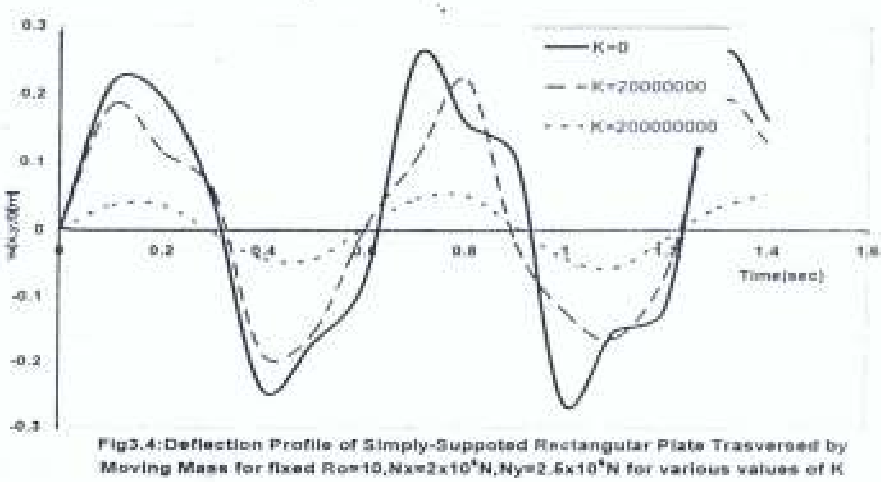

Fig3.4:Deflection Profile of Simply-Suppoted Rectangular Plate Trasversed by Moving Mass for fixed Ro=10,Nx=2x10^4N,Ny=2.5x10^4N for various values of K

Fig3.5, shows the deflection profile of the simply supported plate under moving force for fixed value of K, R_a and N_y (K=2 x10^6 N/m^3, R_a=10 and N_y=2.5 x10^6 N) for various values of axial force along x-axis N_x. This is repeated for R_a = 20 and 30 respectively in fig3.6 and fig3.7. Fig3.8, 3.9and 3.10 show the corresponding graphs for moving mass. Similar results obtained in fig3.5, 3.6 and 3.7 are obtained. The analyses show that as N_x increases, response amplitudes decreases.

Fig3.11, displays the deflection profile of the plate under moving force for fixed values of K and N_x (K=2 x10^6 N/m^3 and N_x=2 x10^6 N) for various values of axial force along y-axis N_y. The analyses show that as N_y increases, response amplitude decreases. This analysis is repeated for R_a=20 and 30 in fig3.12 and 3.13. Similar results obtained in fig3.11, 3.12 and 3.13 are obtained in fig3.14, 3.15 and 3.16

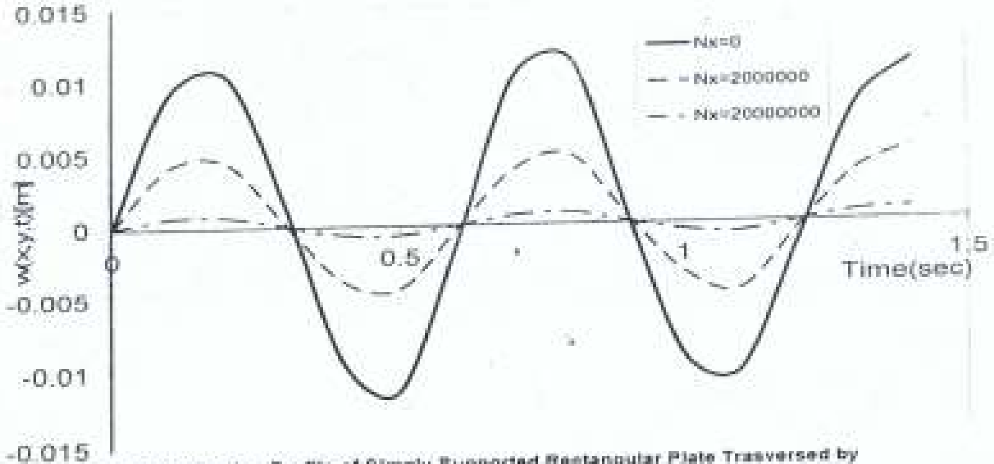

Fig3.5: Deflection Profile of Simply-Supported Rectangular Plate Trasversed by Moving Force for fixed Ro=10, $K=2x10^6 N/m^3$, Ny=2.5x106N for various values of Nx

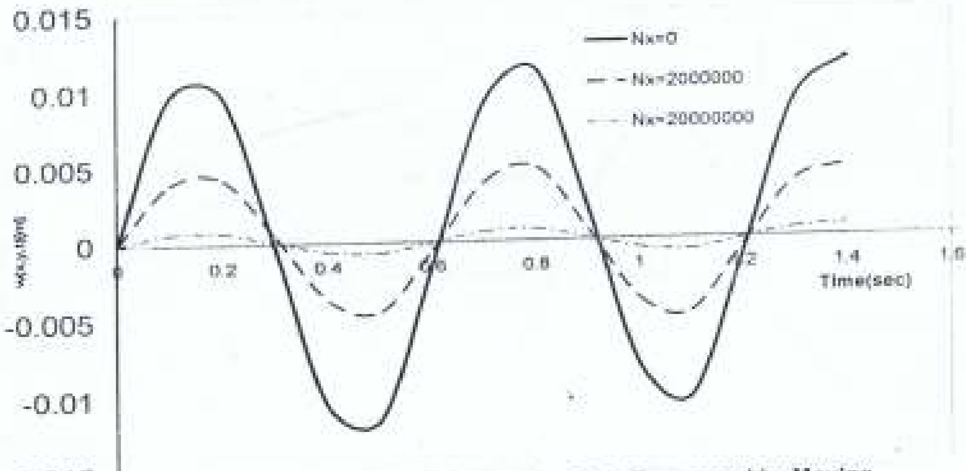

Fig3.6: Deflection Profile of Rectangular Plate Trasversed by Moving Force for fixed Ro=20, $K=2x10^6 N/m^3$, $Ny=2.5x10^6 N$ for various values of Nx

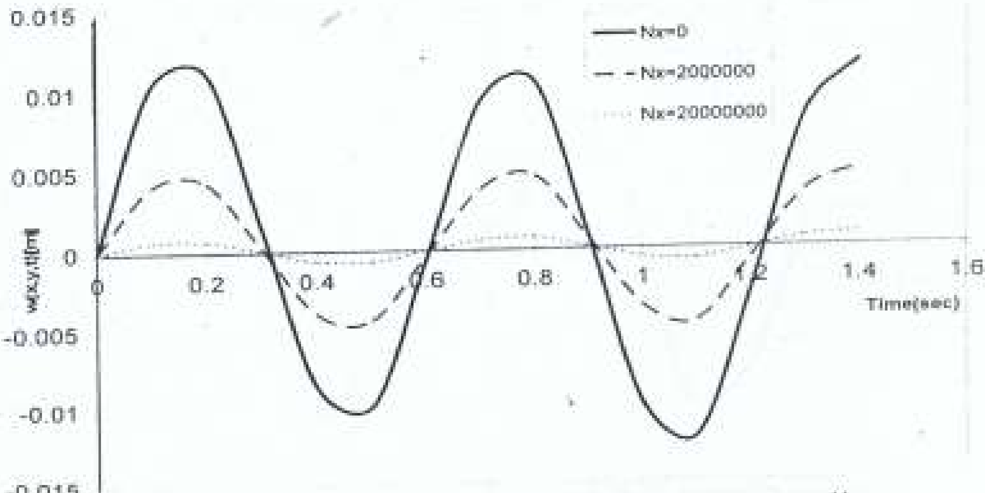

Fig3.7: Deflection Profile of Simply-Supported Rectangular Plate Trasversed by Moving Force for fixed Ro=30, $K=2x10^6 N/m^3$, $Ny=2.5x10^6 N$ for various values of Nx

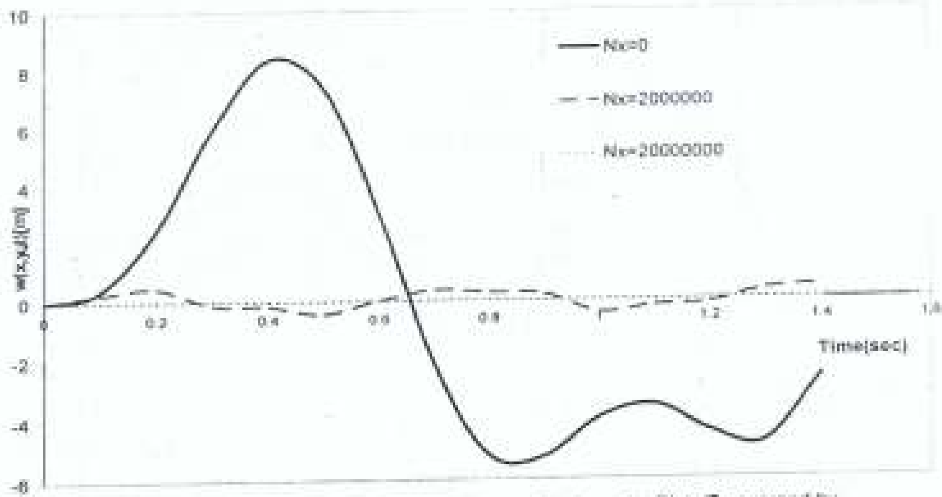

Fig3.8:Deflection Profile of Simply-Suppoted Rectangular Plate Trasversed by Moving Mass for fixed Ro=10,Ny=2.5x10^{6}N,K=2x10^{6}N/m^{3} for various values of Nx

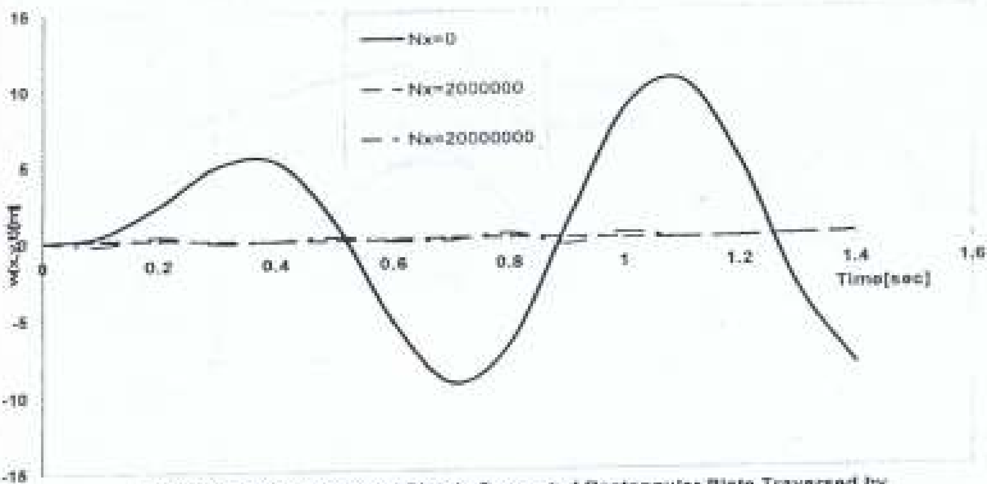

Fig3.9:Deflection Profile of Simply-Supported Rectangular Plate Traversed by Moving Mass for fixed Ro=20,Ny=2.5x10^{6}N,K=2x10^{6}N/m^{3} for various values of Nx

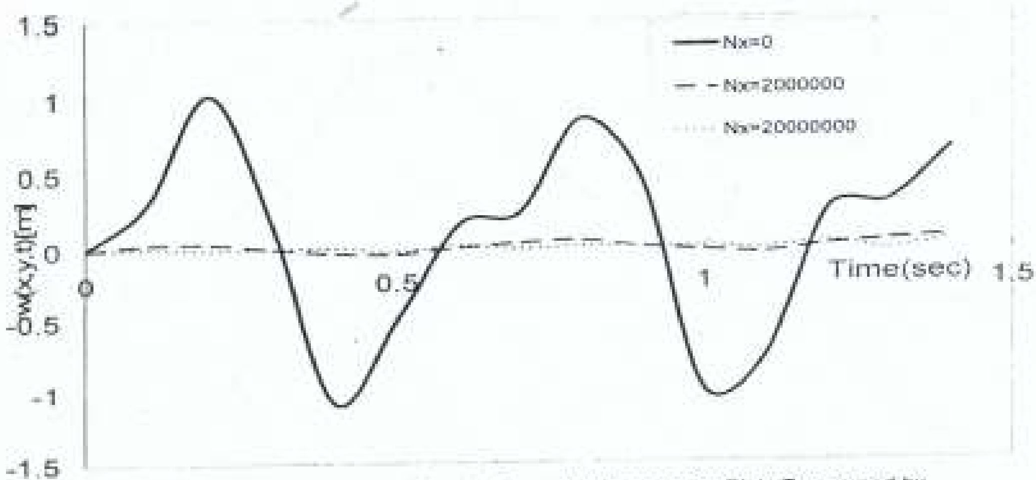

Fig3.10:Deflection Profile of Simply-Suppoted Rectangular Plate Trasversed by Moving Mass for fixed RO=30,Ny=2.5x10^{6}N,K=2x10N/m^{6} for various values of Nx

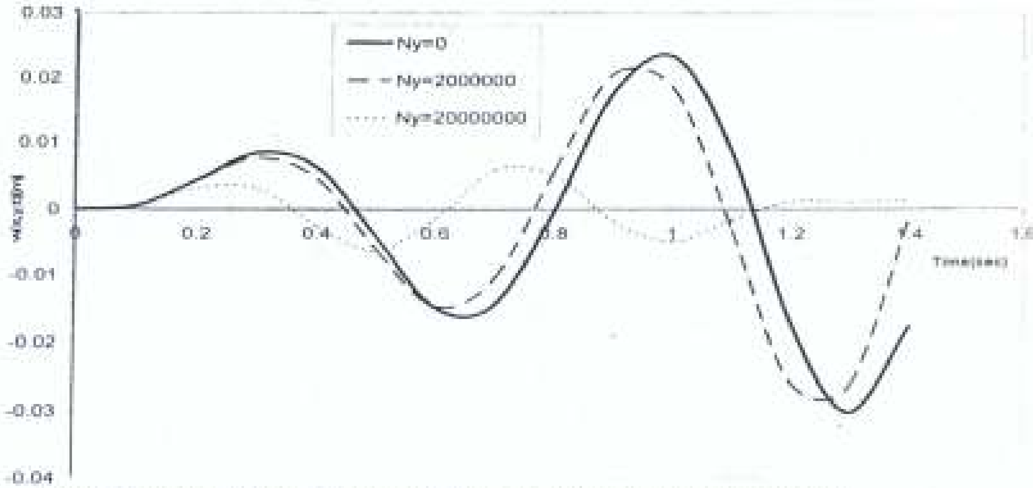

Fig3.11:Deflection Profile of Simply-Suppoted Rectangular Plate Trasversed by Moving Force for fixed Ro=10,Nx=2x10^{6}N,K=2x10^{6}N/m^{3}for various values of Ny

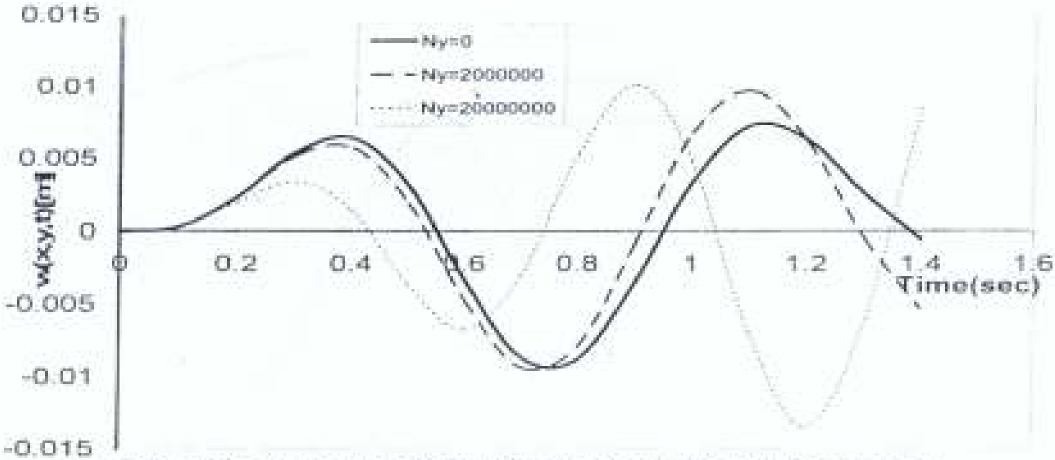

Fig3.12:Deflection Profile of Simply-Suppoted Rectangular Plate Trasversed by Moving Force for fixed Ro=20,Nx=2x10^{6}N,K=2x10^{6}N/m^{3} for various values of Ny

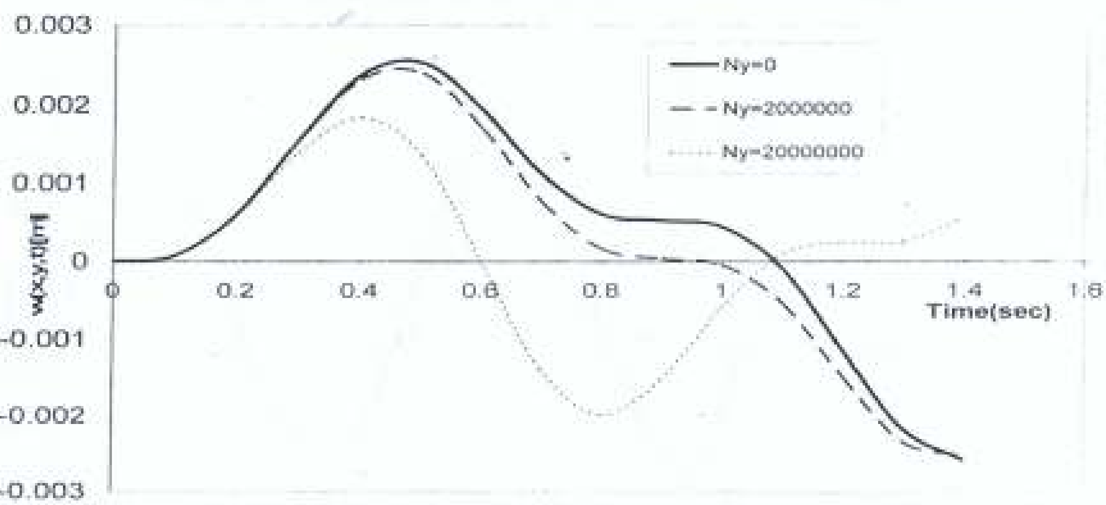

Fig3.13:Deflection Profile of Simply-Suppoted Rectangular Plate Trasversed by Moving Force for fixed Ro=30,Nx=2x10^{6}N,K=2x10^{6}N/m^{3} for various values of Ny

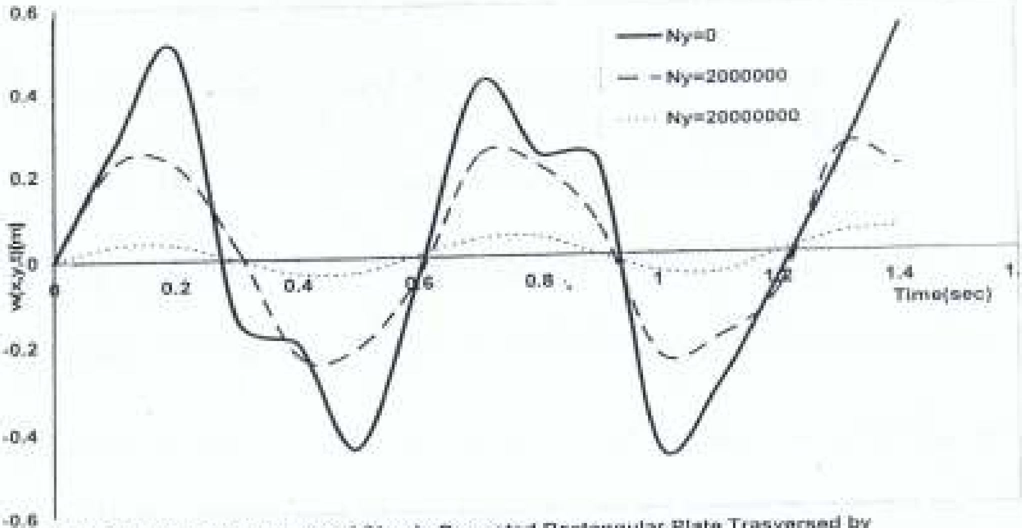

Fig3.14:Deflection Profile of Simply-Suppoted Rectangular Plate Trasversed by Moving Mass for fixed Ro=10,Nx=2×10^6N,K=2×10^6N/m^3 for various values of Ny

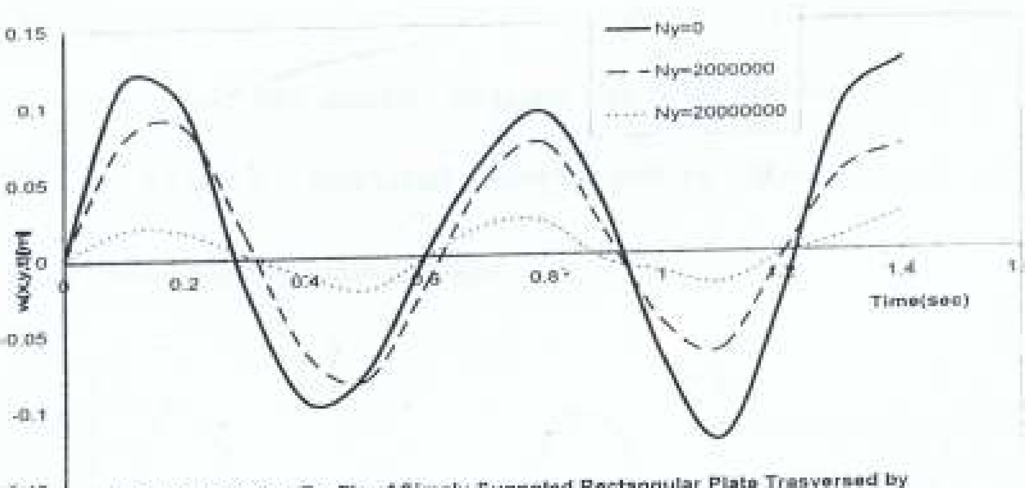

Fig3.15:Deflection Profile of Simply-Suppoted Rectangular Plate Trasversed by Moving Mass for fixed Ro=20,K=2×10^6N/m^3,Nx=2×10^6M for various values of Ny

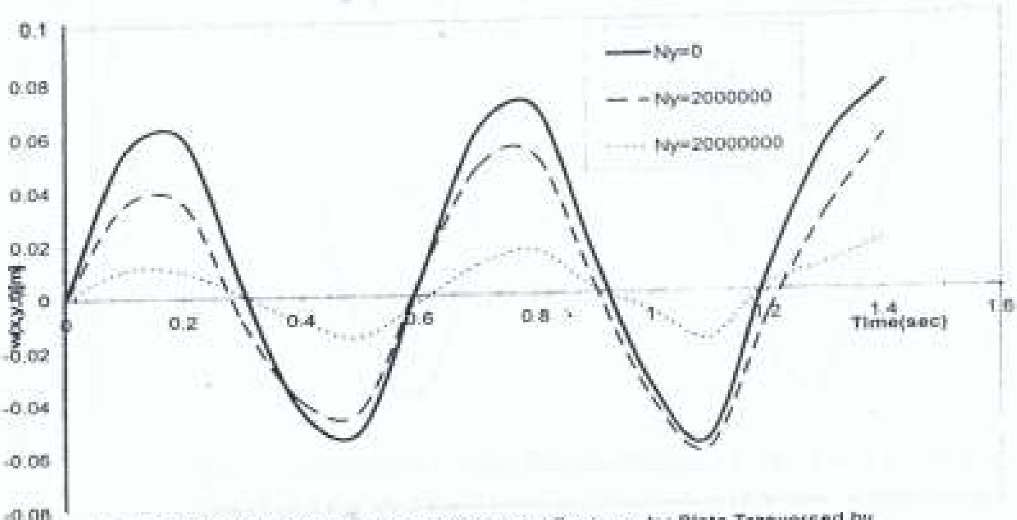

Fig3.16:Deflection Profile of Simply-Suppoted Rectangular Plate Trasversed by Moving Mass for fixed Ro=30,Nx=2×10^6N,K=2×10^6N/m^3 for various values of Ny

Fig3.17 and 3.18 display the deflection profile of the plate under moving load when N_x and N_y are increase simultaneously while other parameters K and R_o (K=2 x10^6 N/m^3 and R_o=10) are fixed for both moving force and moving mass of the rectangular plate respectively. For both cases, response amplitude decreases as N_x and N_y are increased simultaneously Also Fig3.19 compares the displacement curves of the moving force and moving mass for the plate for fixed R_o=20, K=2 x10^6 N/m^3, N_x=2 x10^6 N, and N_y=2.5 x10^6 N. Obviously, the response amplitude of moving mass is greater than that of moving force problem. In like manner, the result also holds for other choice of values of R_o, K, N_x and N_y. This result shows the moving force solution is not always an upper bound moving mass solution.

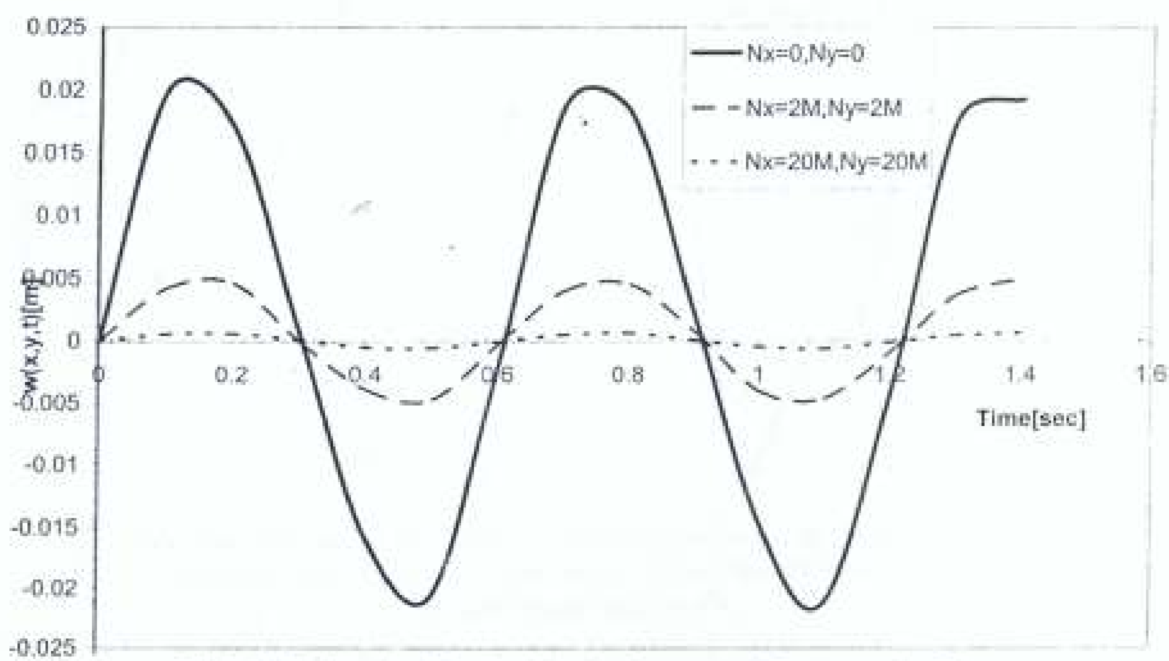

Fig3.17:Deflection Profile Simply-Supported Rectangular Plate Trasversed by Moving Force for fixed Ro=10,K=2x10^{6}N/m^3 when values of Nx and Ny are increase simultaneously

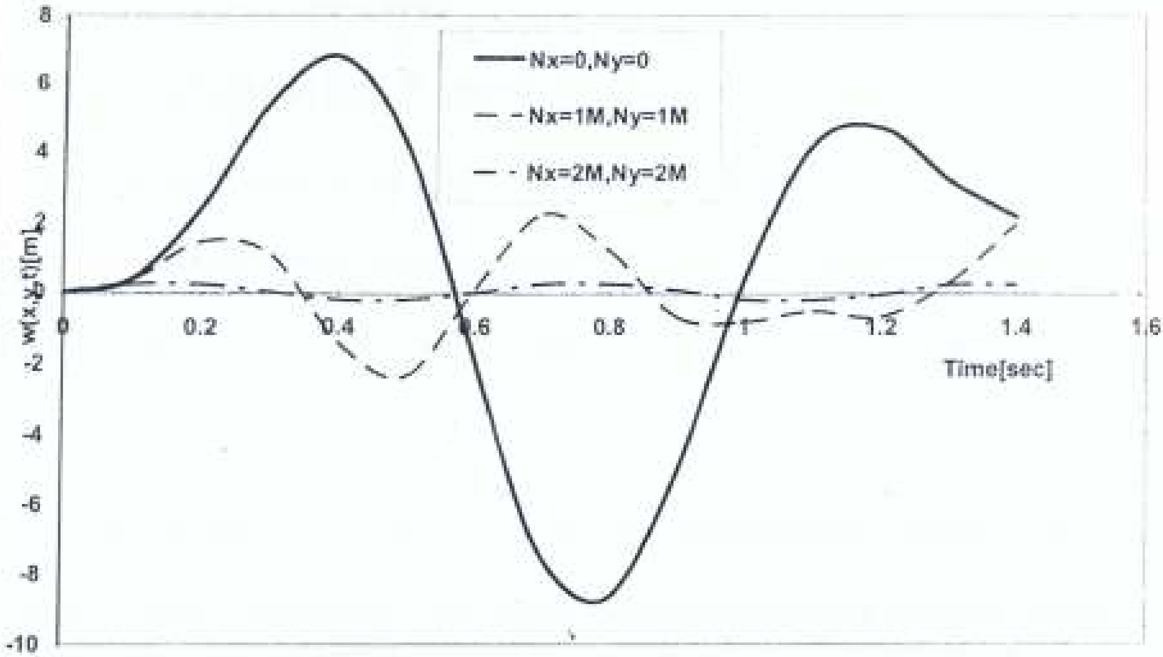

Fig3.18:Deflection Profile of Simply-Supported Rectangular Plate Trasversed by Moving Mass for fixed Ro=10,K=$2x10^6 N/m^3$ when values of Nx and Ny are increase simultaneously

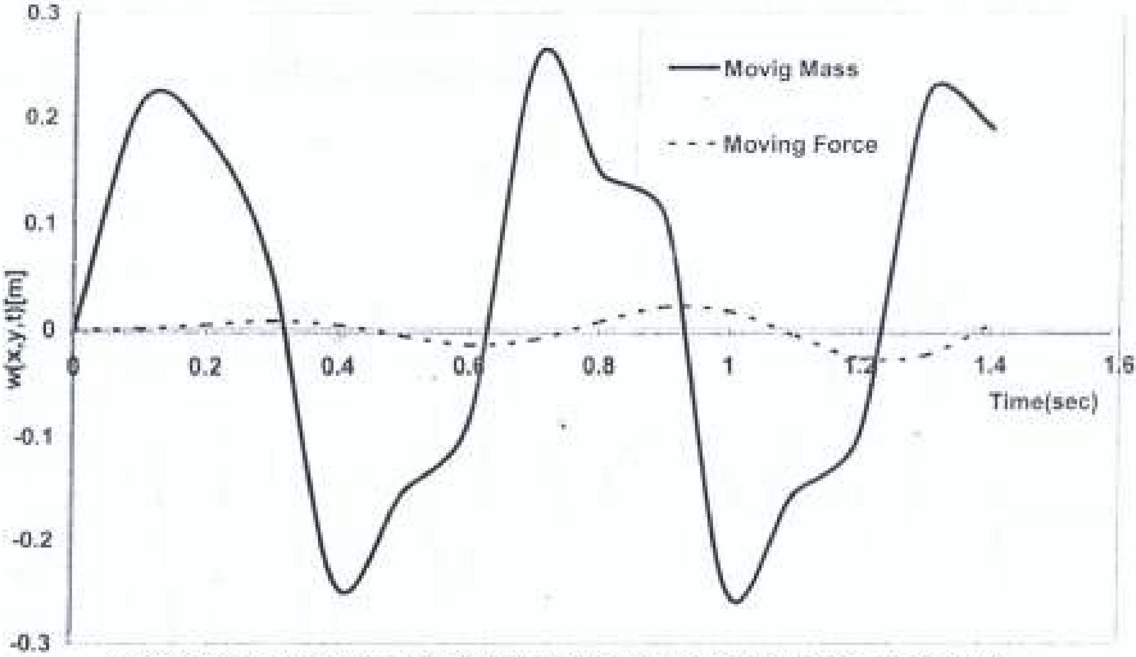

Fig3.19:Comparison of The Deflection of Moving Force and Moving Mass cases for Simple Supported Rectangular Plate for fixed Ro=10, K=$2x10^6 N/m^3$, Nx=$2x10^6$N and Ny=$2.5x10^6$N

3.5.2 SIMPLE-CLAMPED ENDS

Fig3.20 and Fig3.21 display the effect of Rotatory inertia R_o on the transverse deflection of the simple-clamped plate under the action of moving force and moving mass respectively for fixed values of K, N_x and N_y ($K=2\text{x}10^6 N/m^3$, $N_x=2 \text{ x}10^6$ N and $N_y=2.5 \text{ x}10^6$ N). The graphs show that the response amplitude decreases as the R_o increases. The values of R_o which are used are 10, 20 and 30.

Fig3.22 and Fig3.23 depict the transverse displacement response of the simple-clamped rectangular plate under the action of moving force and moving mass respectively for fixed values of N_x, N_y and R_o ($N_x=2 \text{ x}10^6$ N, $N_y=2.5 \text{ x}10^6$N and $R_o=10$) for various values of foundation stiffness K. The graph shows that as K increases, the response amplitude decreases.

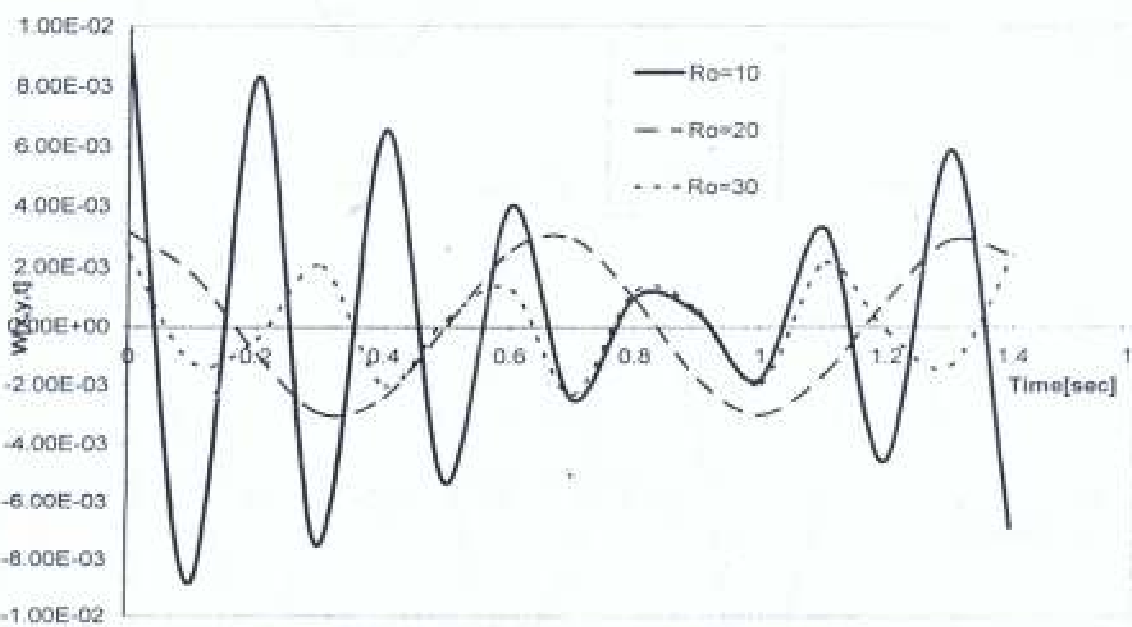

Fig3.20:Deflection Profile of Simple-Clamped Plate Trasversed by Moving Force for fixed $K=2\text{x}10^6 N/m^3$,$Nx=2\text{x}10^6N$,$Ny=2.5\text{x}10^6N$ for various values of Ro

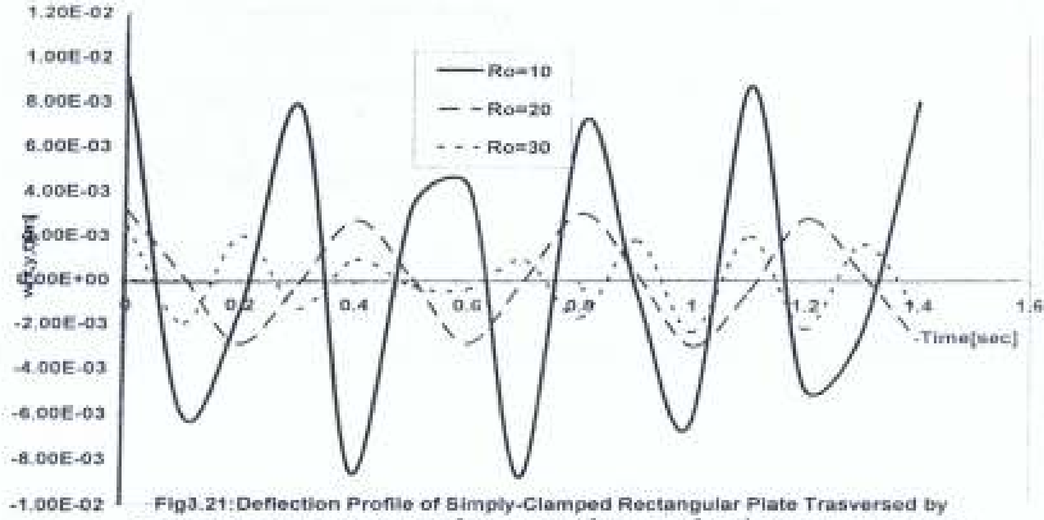

Fig3.21:Deflection Profile of Simply-Clamped Rectangular Plate Trasversed by Moving Mass for fixed Nx=2x10⁶N,Ny=2.5x10⁶N,K=2x10⁴N/m³ for various values of Ro

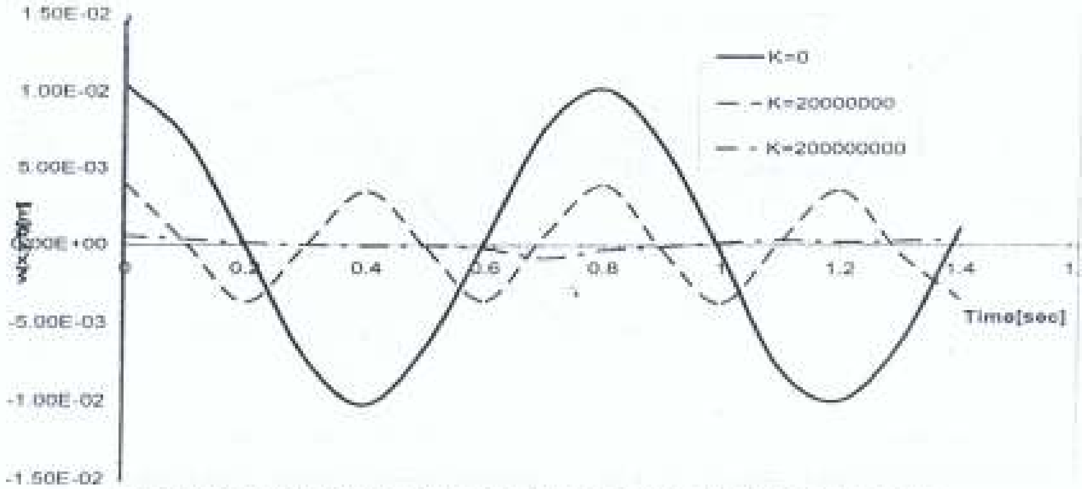

Fig3.22:Deflection Profile of Simple-Clamped Rectangular Plate Traversed by Moving Force for fixed Ro=10,Nx=2x10⁶N,Ny=2.5x10⁶N, for various values of K

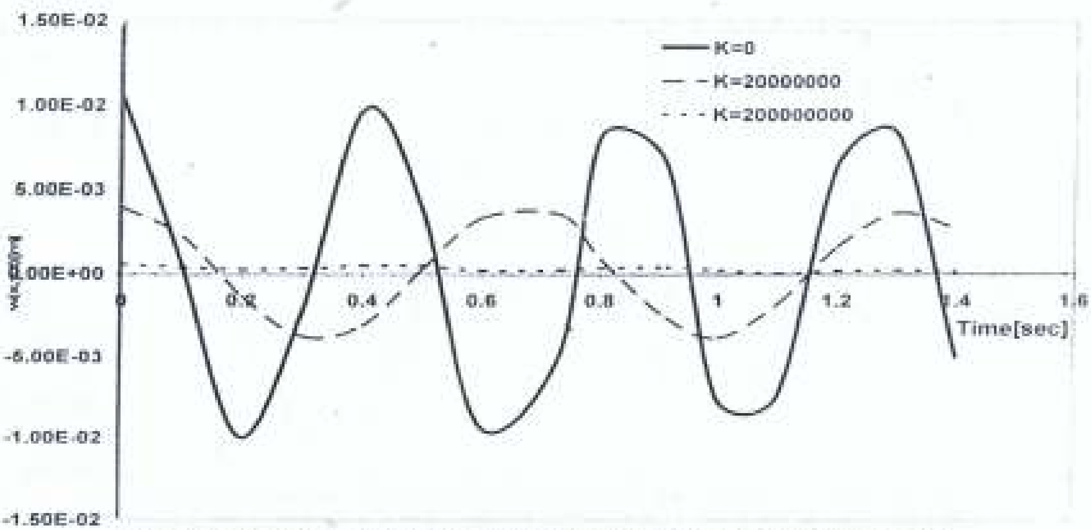

Fig3.23:Deflection Profile of Simple-Clamped Rectangular Plate Traversed by Moving Mass for fixed Ro=10,Nx=2x10⁶N,Ny=2.5x10⁶N for various values of K

Also, for various time t, fig3.24 and fig3.25 show deflection profile of rectangular plate under the action of moving force and moving mass respectively for various values of axial force along x-axis, N_x and for fixed values of K, N_y and R_o (K=2x10^6 N/m^3, N_y=2.5x10^6 N and R_o=10).It shows that higher values of axial force along x-axis, N_x reduce the deflection profiles of the plate in both cases

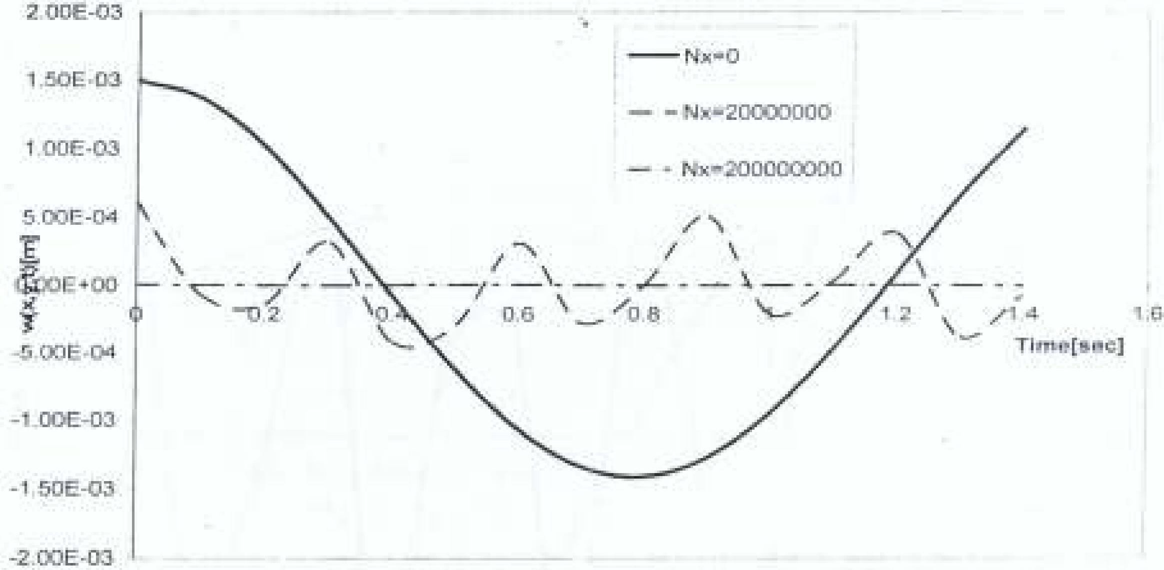

Fig3.24:Deflection Profile of Simiply-Clamped Rectangular Plate Traversed by Moving Force for fixed Ro=10,K=2x10^{6}N/m^3,Ny=2.5x10^{6}N for various values of Nx

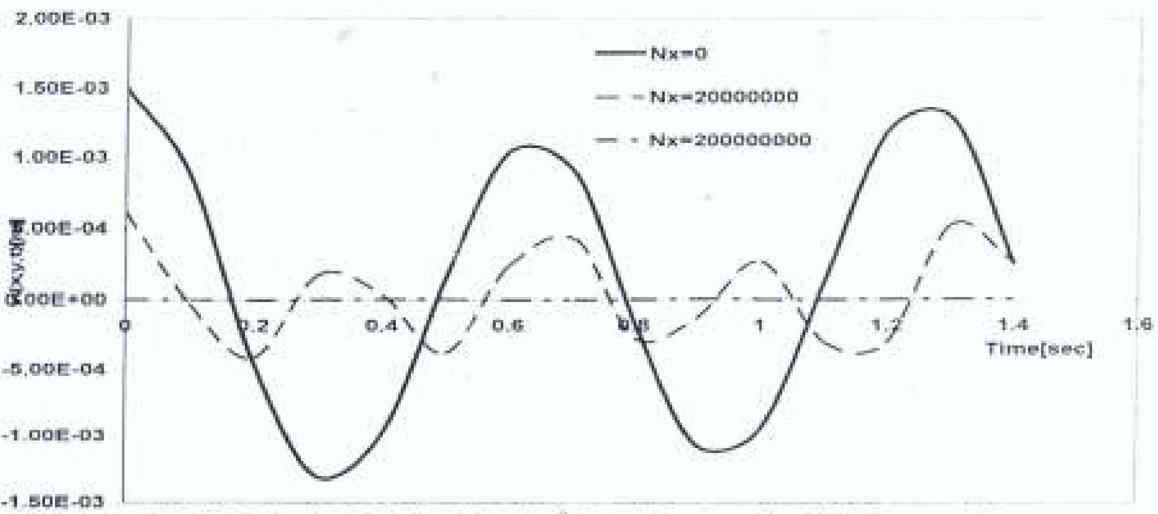

Fig3.25:Deflection Profile of Simply-Clamped Rectangular Plate Traversed by Moving Mass for fixed Ro=10,Ny=2.5x10^{6}N,K=2x10^{6}N/m^3 for various values of Nx

Fig3.26, 3.27 and 3.28 depict the transverse deflection of simple-clamped plate under moving force for various values of N_y for fixed values K, R_o and N_x (K=2x10^6 N/m ^{3}and N_x=2 x10^6 N). This analysis is carried out for various values of rotatory inertia (R_o=10, 20 and 30). The corresponding behaviors when the plate is traversed by concentrated masses are shown in fig3.29, 3.30 and 3.31 respectively. As N_y increases, the maximum amplitude of the plate decreases.

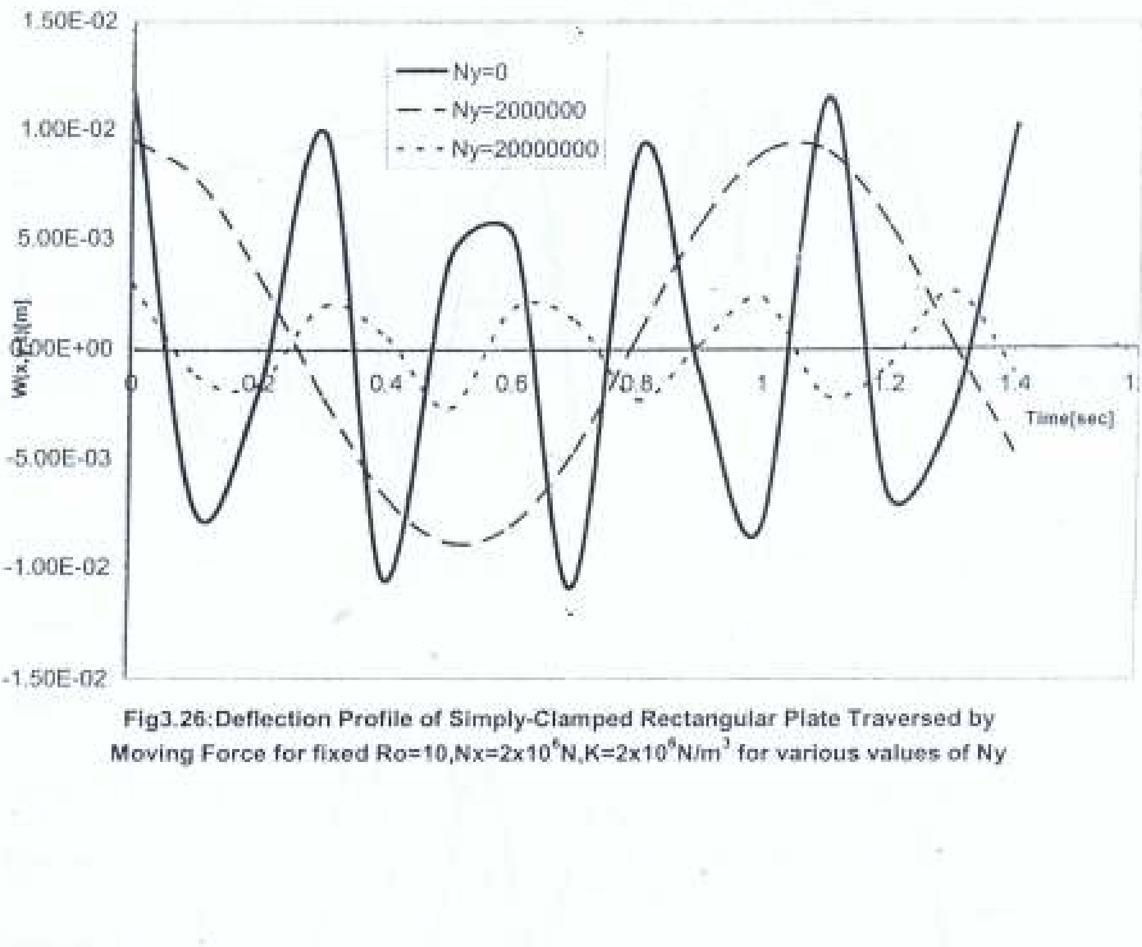

Fig3.26:Deflection Profile of Simply-Clamped Rectangular Plate Traversed by Moving Force for fixed Ro=10,Nx=2x10^6N,K=2x10^6N/m^3 for various values of Ny

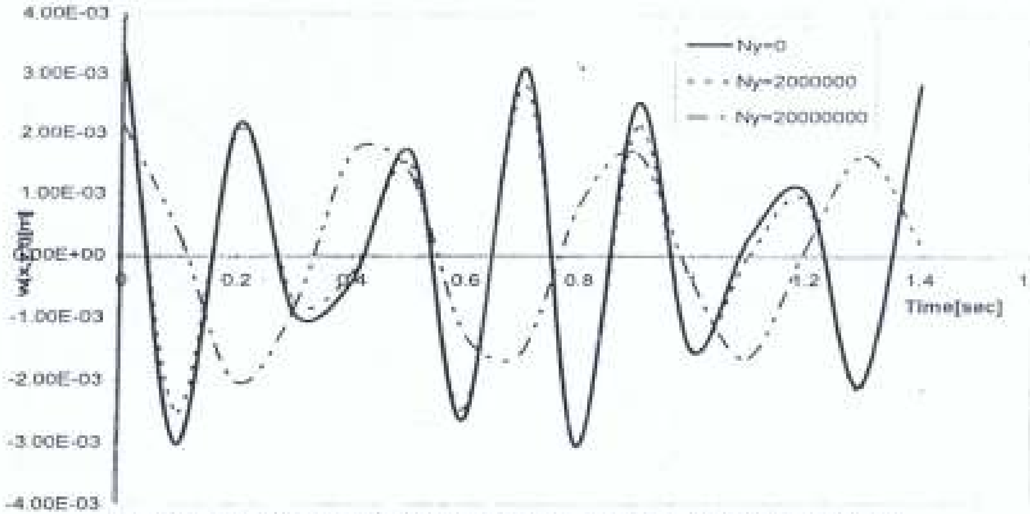

Fig3.27:Deflection Profile of Simply-Clamped Rectangular Plate Traversed by Moving Force for fixed Ro=20,Nx=2x10^{6}N,K=2x10^{6}N/m^{2} for various values of Ny

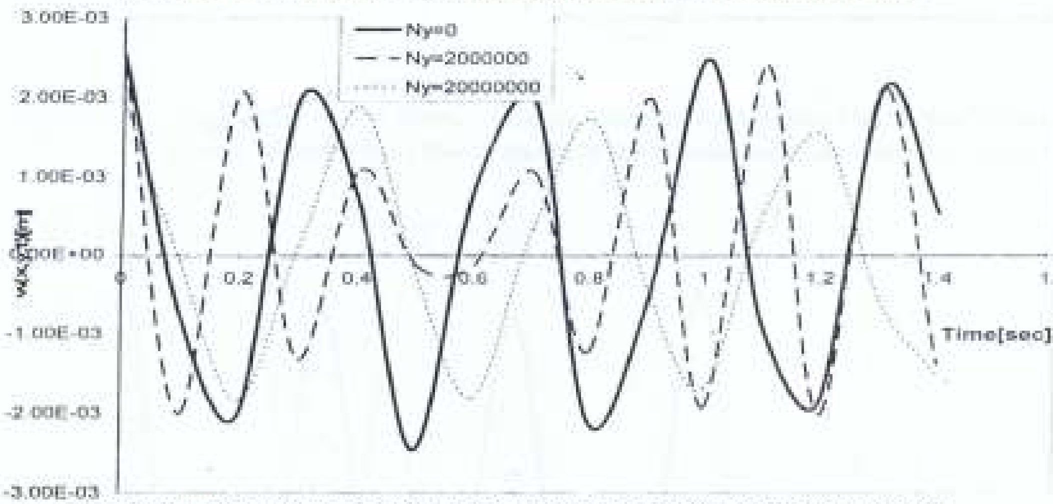

Fig3.28:Deflection Profile of Simly-Clamped Rectangular Plate Traversed by Moving Force for fixed Ro=30,Nx=2x10^{6}N,K=2x106N/m^{2} for various values of Ny

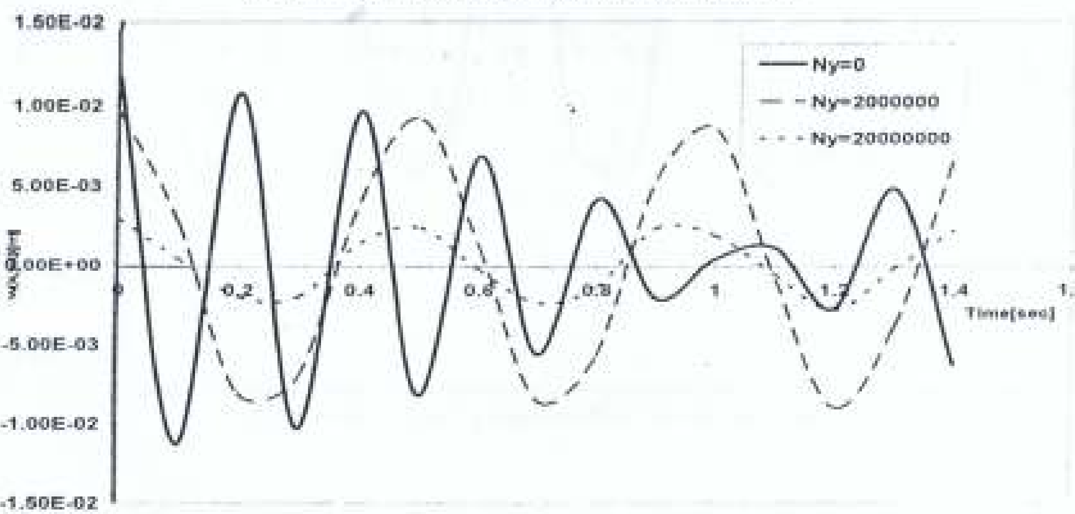

Fig3.29:Deflection Profile of Simply-Clamped Rectangular Plate Traversed by Moving Mass for fixed Ro=10,Nx=2x10^{6}N,K=2x10^{6}N/m^{2} for various values of Ny

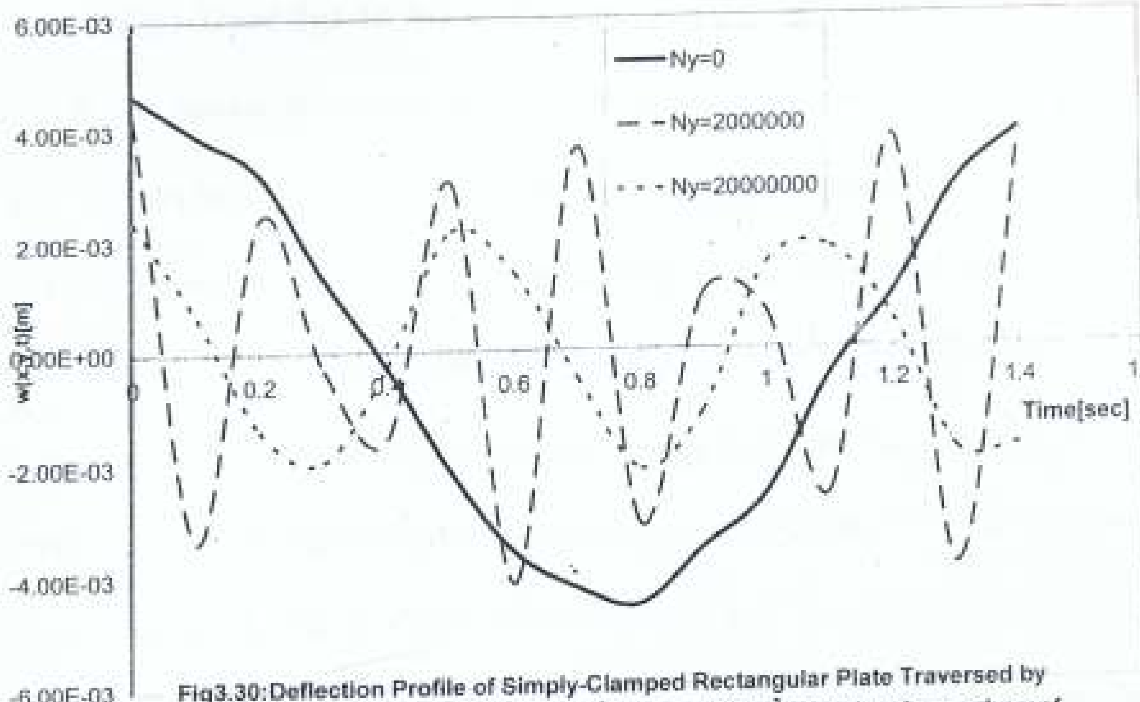

Fig3.30:Deflection Profile of Simply-Clamped Rectangular Plate Traversed by Moving Mass for fixed Ro=20,Nx=2x10^{6}N,K=2x106N/m^3 forand various values of Ny

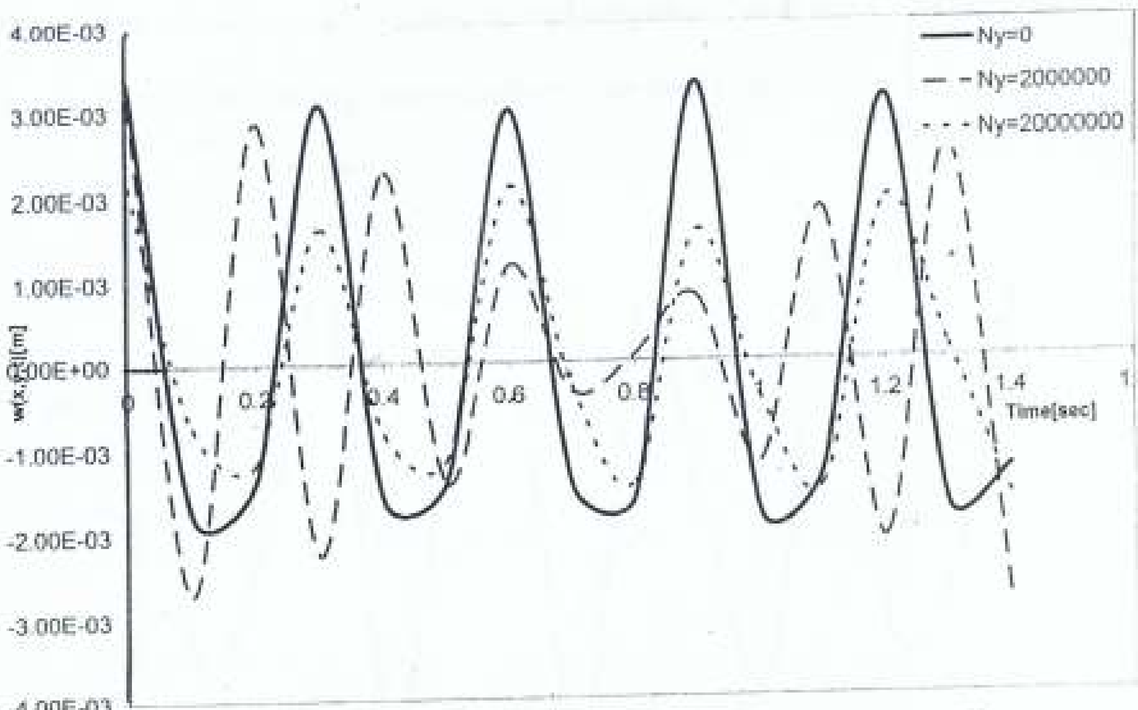

Fig3.31:Deflection Profile of Simply-Clamped Rectangular Plate Traversed by Moving Mass for fixed Ro=30,Nx=2x10^{6}N,K=2x10^{6}N/m^3 for various values of Ny

Fig3.32and fig3.33 display the response amplitude when axial forces N_x and N_y are increased simultaneously for fixed values of K and R_o (K=2 x10^6 N/m^3 and R_o=10) for simple-clamped rectangular plate under the action of moving force and moving mass respectively. Evidently, the response amplitudes decrease when values of N_x and N_y are increases simultaneously. Finally, fig3.34 depicts the comparison of transverse displacement of moving force and moving mass cases for simple-clamped rectangular plate traversed by a moving load for fixed values of R_o, N_y, N_x and K (R_o=10, N_y=2.5 x10^6 N and N_x=2 x10^6 N and K=2 x10^6 N/m^3).

Clearly, the response amplitude of moving mass is higher than that of the moving force. This important result show that relying on moving force solution as an approximation to moving mass solution is seriously misleading

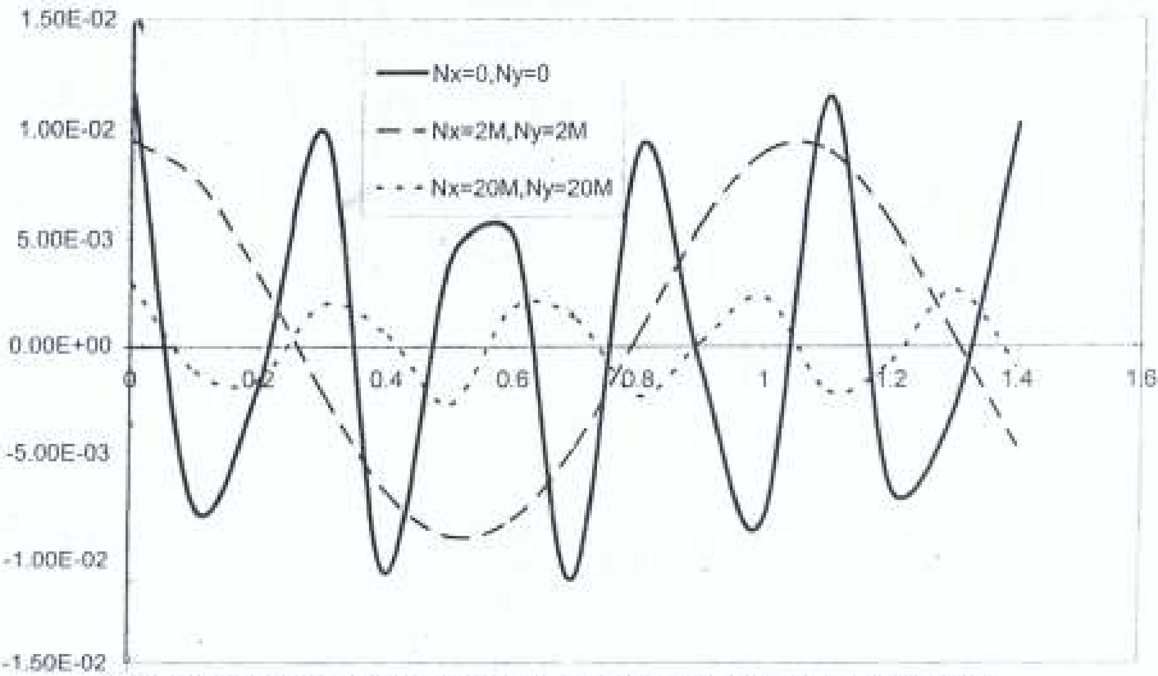

Fig3.32:Deflection Profile Simple-Clamped Rectangular Plate Traversed Moving Force for fixed Ro=10 and K=2x10^{6}N/m^3 when the values of Nx and Ny are increase simultaneously

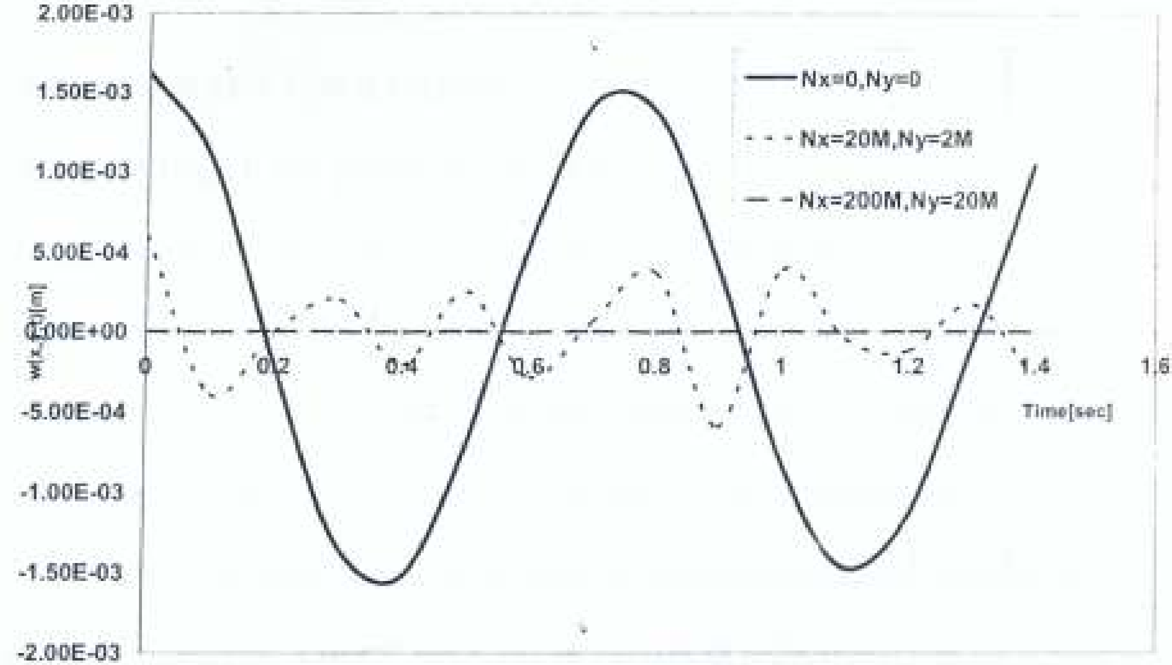

Fig3.33:Deflection Profile of Simple-Clamped Rectangular Plate Traversed by Moving Mass for fixed Ro=10 and K=2x10^{6}N/m^{3} when values of Nx and Ny are increase simultaneously

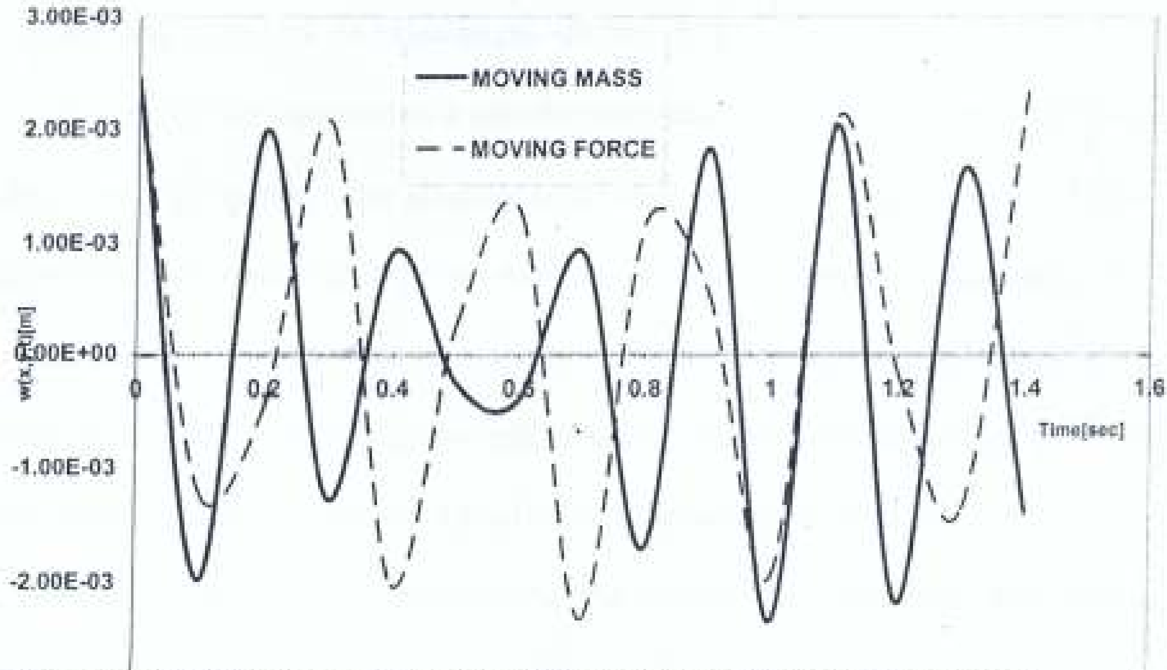

Fig3.34:Comparison of the Deflection of Moving force and Moving mass cases for Simple-Clamped Plate for fixed Ro=30,Ny=2.5x10^{6}N, Nx=2x10^{6}N and K=2x106N/m^{3}

CHAPTER 4

4.0 GENERAL CONCLUTION

4.1 SUMMARY OF RESEARCH WORK

The problem of assessing the dynamic behaviour of prestressed finite rectangular plate under the action of transverse traveling loads is investigated in this thesis. This plate model takes into consideration the effect of the rotatory inertia correction factor which was neglected in the non-Mindlin plate model. In addition, two opposite edges of the plate are simply supported and the other two opposite edges could take the form of any classical boundary conditions. This is so because, plate structures of bridges are known usually to have two opposite edges simply supported and the other edges are free [17].

The governing equation is a non-homogeneous fourth order partial differential equation with variable and singular coefficient. The main objective is to obtain a closed form solution valid for all variants of classical boundary conditions at the ends having arbitrary support condition. Unlike the method of Oni [21], the generalized two-dimensional integral transform with the normal modes of plate as the kernel of transformation is used for the solution of the problem.

Firstly, a closed form solution to the equation is obtained. The solution technique is based on

(i) the modified generalized two-dimensional integral transform with plate function (normal modes) as the kernel of transformation.

(ii) the modified asymptotic method of Struble.

(iii) integral transformation techniques and convolution theory.

The theory is then illustrated using some examples of classical boundary conditions commonly encountered in Engineering practice. They are

(i) simply supported end conditions.

(ii) simple-clamped end conditions.

Analysis of the closed form solutions obtained is carried out for the illustrative examples. The resonance conditions for the various end conditions are obtained.

The influence of the prestress (axial force), rotatory inertia and foundation moduli on the dynamic response to moving force and moving mass of prestressed finite rectangular plate under the action of moving concentrated masses is investigated. The transverse displacements for all the illustrative examples are calculated and presented in plotted curves.

This study exhibits the following interesting features:

(i) The dynamic response amplitudes of rectangular plates incorporating rotatory inertia correction factor decrease with an increase in the values of axial forces N_x and N_y when one is varied while the other is fixed for fixed foundation reaction modulus K and rotatory inertia Ro fixed

(ii) When the axial forces N_x and N_y (prestress values) are increased simultaneously, the response amplitude of the plate decreases

(iii) For both illustrative examples considered, the moving force solution is not always an upper bound for the accurate solution of the moving mass problem. Hence the non-reliability of moving force solution as a safe approximation to the moving mass problem is confirmed.

(iv) Higher values of axial forces N_x and N_y, foundation reaction modulus K and rotatory inertia R_o are required for a more noticeable effect in the case of simple-clamped end conditions than for the case of simply supported end conditions for both moving force and moving mass problems.

(v) For fixed values of axial forces, N_x and N_y, and rotatory inertia R_o, the dynamic response amplitudes of the rectangular plate decreases with an increase in values foundation reaction modulus K. similarly, as rotatory inertia R_o increases while other parameters are fixed, the response amplitudes decrease.

(vi) For fixed axial forces N_x and N_y, foundation reaction modulus K and rotatory inertia R_o, the dynamic response amplitudes for the moving mass problem is greater than that of the moving force problem for the two illustrative examples considered.

(vii) In the two illustrative examples considered, for the same natural frequency, the critical speed for moving mass problem is smaller than that of the moving force problem. Hence resonance is reached earlier in the former.

4.2 CONTRIBUTIONS TO KNOWLEDGE

(a) The study has provided analytical solution for the problem of the dynamic behaviors under moving concentrated masses of rectangular plate incorporating rotatory inertia correction factor for all variants of classical boundary conditions.

(b) It also provided vital information on the effect or influence of axial forces on the transverse deflection of the rectangular plate under the action of moving concentrated masses.

(c) Useful information has been provided on the effect of the foundation reaction modulus K and rotatory inertia R_o on the response displacement of rectangular plates

(d) Through the study, the non-reliability of the moving force solution as a safe approximation to the moving mass problem was confirmed.

(c) The study gave useful information on the resonance conditions for both moving force and moving mass problems for various boundary conditions.

4.3 LIMITATIONS TO STUDY AND RECOMMENDATIONS FOR FURTHER RESEARCH

The axial force influence on the dynamic response to moving concentrated masses of rectangular plates incorporating rotatory inertia correction factor is the main objective of the study

Illustrative examples have been limited to classical boundary conditions only. Non classical boundary conditions such as

(i) Elastically supported end conditions

(ii) Time dependent boundary conditions are not taken into consideration and such are suggested for future research.

Structures (plates or beam) on other uniform foundation models are left for further research. Also plate models resting on variable elastic non-Winkler foundation and visco-elastic foundation are not taken care of in this work.

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APPENDIX

SOLUTION OF THE INTEGRALS

$$I_1 = \begin{cases} \dfrac{L_x}{2}\left[\dfrac{Sin(\alpha_j - \alpha_p)}{\alpha_j - \alpha_p} - \dfrac{Sin(\alpha_j + \alpha_p)}{\alpha_j + \alpha_p}\right] & ,\alpha_j \neq \alpha_p \\ \dfrac{L}{2}\left[1 - \dfrac{Sin2\alpha_p}{2\alpha_p}\right] & ,\alpha_j = \alpha_p \end{cases}$$

$$I_2 = \begin{cases} \dfrac{-L_x}{2}\left[\dfrac{(\alpha_j - \alpha_p)(Cos(\alpha_j + \alpha_p) - 1) + (\alpha_j + \alpha_p)(Cos(\alpha_j - \alpha_p) - 1)}{\alpha_j^2 - \alpha_p^2}\right] & ,\alpha_j \neq \alpha_p \\ \dfrac{-L_x}{2}\left[\dfrac{Cos2\alpha_p - 1}{2\alpha_p}\right] & ,\alpha_j = \alpha_p \end{cases}$$

$$I_3 = \frac{\alpha_p L_x}{\alpha_p^2 + \alpha_j^2}\left[Sin\alpha_j Cosh\beta_p - \frac{\alpha_j}{\beta_p} Cos\alpha_j Sinh\beta_p\right]$$

$$I_4 = \frac{\beta_p L_x}{\beta_p^2 + \alpha_j^2}\left[Sin\alpha_j Sinh\beta_p - \frac{\alpha_j}{\beta_p}(Cos\alpha_j Cosh\beta_p - 1)\right]$$

$$I_5 = \begin{cases} \dfrac{L_x}{2}\left[\dfrac{(\alpha_j + \alpha_p)(Cos(\alpha_j - \alpha_p) - 1) + (\alpha_j - \alpha_p)(1 - Cos(\alpha_j + \alpha_p))}{\alpha_j^2 - \alpha_p^2}\right] & ,\alpha_j \neq \alpha_p \\ \dfrac{-L_x}{2}\left[\dfrac{Cos2\alpha_p - 1}{2\alpha_p}\right] & ,\alpha_j = \alpha_p \end{cases}$$

$$I_6 = \begin{cases} \dfrac{L_x}{2}\left[\dfrac{Sin(\alpha_j + \alpha_p)}{\alpha_j + \alpha_p} + \dfrac{Sin(\alpha_j - \alpha_p)}{\alpha_j - \alpha_p}\right] & ,\alpha_j \neq \alpha_p \\ \dfrac{L}{2}\left[1 + \dfrac{Sin2\alpha_p}{2\alpha_p}\right] & ,\alpha_j = \alpha_p \end{cases}$$

$$I_7 = \frac{\beta_p L_x}{\beta_p^2 + \alpha_j^2}\left[Cos\alpha_j Cosh\beta_p + \frac{\alpha_j}{\beta_p} Sin\alpha_j Sinh\beta_p - 1\right]$$

$$I_8 = \frac{\beta_p L_x}{\beta_p^2 + \alpha_j^2}\left[Cos\alpha_j Sinh\beta_p + \frac{\alpha_j}{\beta_p} Sin\alpha_j Cosh\beta_p\right]$$

$$I_9 = \frac{\beta_j L_x}{\beta_j^2 + \alpha_p^2}\left[Sin\alpha_p Cosh\beta_j - \frac{\alpha_p}{\beta_j} Cos\alpha_p Sinh\beta_j\right]$$

$$I_{10} = \frac{\beta_j L_x}{\beta_j^2 + \alpha_p^2}\left[Cos\alpha_p Cosh\beta_j + \frac{\alpha_p}{\beta_j} Sin\alpha_p Sinh\beta_j - 1\right]$$

$$I_{11} = \begin{cases} \frac{L_x}{2}\left[\frac{Sinh(\beta_j + \beta_p)}{\beta_j + \beta_p} + \frac{Sinh(\beta_j - \beta_p)}{\beta_j - \beta_p}\right] & , \lambda_k \neq \lambda_m \\ \frac{L_x}{2}\left[\frac{Sinh 2\beta_p}{2\beta_p} - 1\right] & , \lambda_k = \lambda_m \end{cases}$$

$$I_{12} = \begin{cases} \frac{L_x}{2}\left[\frac{(\beta_j - \beta_p)(Cosh(\beta_j + \beta_p) - 1) + (\beta_j + \beta_p)(Cosh(\beta_j - \beta_p) - 1)}{\beta_j^2 + \beta_p^2}\right] & , \beta_j \neq \beta_p \\ \frac{L_x}{2}\left[\frac{Cosh 2\beta_p - 1}{2\beta_p}\right] & , \beta_j = \beta_p \end{cases}$$

$$I_{13} = \frac{\beta_j L}{\beta_j^2 + \alpha_p^2}\left[Sin\alpha_p Sinh\beta_j - \frac{\alpha_p}{\beta_j}(Cos\alpha_p Cosh\beta_j - 1)\right]$$

$$I_{14} = \frac{\beta_j L_x}{\beta_j^2 + \alpha_p^2}\left[Cos\alpha_p Sinh\beta_j + \frac{\alpha_p}{\beta_j} Sin\alpha_p Cosh\beta_j\right]$$

$$
I_{15} = \begin{cases} \dfrac{L_x}{2}\left[\dfrac{(\beta_p-\beta_j)(Cosh(\beta_p+\beta_j)-1)+(\beta_p+\beta_j)(Cosh(\beta_p-\beta_j)-1)}{\beta_p^2+\beta_j^2}\right] \\ \dfrac{L_x}{2}\left[\dfrac{Cosh2\beta_p-1}{2\beta_p}\right] \end{cases}
$$

$$
I_{16} = \begin{cases} \dfrac{L}{2}\left[\dfrac{Sinh(\lambda_k+\lambda_m)}{\lambda_k+\lambda_m}+\dfrac{Sinh(\lambda_k-\lambda_m)}{\lambda_k-\lambda_m}\right] & ,\beta_j \neq \beta_p \\ \dfrac{L}{2}\left[\dfrac{Sinh2\beta_p}{2\beta_p}+1\right] & ,\beta_j = \beta_p \end{cases}
$$

$$
I_{17} = \begin{cases} \dfrac{4}{L_x}\left[\dfrac{\sin(n\Pi+\alpha_j-\alpha_p)}{(n\Pi+\alpha_j-\alpha_p)}+\dfrac{\sin(n\Pi+\alpha_j-\alpha_p)}{(n\Pi-\alpha_j+\alpha_p)}-\dfrac{\sin(n\Pi+\alpha_j+\alpha_p)}{(n\Pi+\alpha_j+\alpha_p)}-\dfrac{\sin(n\Pi-\alpha_j-\alpha_p)}{(n\Pi-\alpha_j-\alpha_p)}\right]_{\alpha_j \neq \alpha_p} \\ \dfrac{4}{L_x}\left[1-\dfrac{\sin 2(n\Pi+\alpha_j)}{2(n\Pi+\alpha_j)}\right]_{\alpha_p = n\Pi+\alpha_j} \qquad -\dfrac{L_x}{4}\left[1-\dfrac{\sin 2(n\Pi-\alpha_j)}{2(n\Pi-\alpha_j)}\right]_{\alpha_p = n\Pi-\alpha_j} \end{cases}
$$

$$
\begin{aligned} &\alpha_j \neq \alpha_p \\ &\alpha_p = n\Pi + \alpha_j \\ &\alpha_p = n\Pi - \alpha_j \end{aligned}
$$

$$
I_{18} = \begin{cases} \dfrac{L_x}{4}\left[\dfrac{(n\Pi+\alpha_j+\alpha_p)(\cos(n\Pi-\alpha_j-\alpha_p)-1)+(n\Pi-\alpha_j-\alpha_p)(1-\cos(n\Pi+\alpha_j+\alpha_p))}{n^2\Pi^2-(\alpha_j+\alpha_p)^2}+\right. \\ \left.+\dfrac{(n\Pi+\alpha_j-\alpha_p)(\cos(n\Pi-\alpha_j+\alpha_p)-1)+(n\Pi-\alpha_j+\alpha_p)(1-\cos(n\Pi+\alpha_j-\alpha_p))}{n^2\Pi^2-(\alpha_j-\alpha_p)^2}\right] \end{cases}
$$

$$I_{19}=\left\{\begin{array}{l}\frac{L_x}{2}\left[\frac{\beta_p}{\beta_p^2+(n\Pi+\alpha_j)^2}\left(\sin(n\Pi+\alpha_j)\cosh\beta_p-\frac{(n\Pi+\alpha_j)}{\beta_p}\cos(n\Pi+\alpha_j)\sinh\beta_p\right)+\right.\\ \left.\left(\frac{\beta_p}{\beta_p^2+(n\Pi-\alpha_j)^2}\left(\sin(n\Pi-\alpha_j)\cosh\beta_p-\frac{(n\Pi-\alpha_j)}{\beta_p}\cos(n\Pi-\alpha_j)\sinh\beta_p\right)\right)\right]\end{array}\right.$$

$$I_{20}=\left\{\begin{array}{l}\frac{L_x}{2}\left[\frac{\beta_p}{\beta_p^2+(n\Pi+\alpha_j)^2}\left(\sin(n\Pi+\alpha_j)\sinh h\beta_p-\frac{(n\Pi+\alpha_j)}{\beta_p}\cos(n\Pi+\alpha_j)\cosh\beta_p-1\right)+\right.\\ \left.\left(\frac{\beta_p}{\beta_p^2+(n\Pi-\alpha_j)^2}\left(\sin(n\Pi-\alpha_j)\sinh\beta_p-\frac{(n\Pi-\alpha_j)}{\beta_p}\cos(n\Pi-\alpha_j)\cosh\beta_p-1\right)\right)\right]\end{array}\right.$$

$$I_{21}=\left\{\begin{array}{l}\frac{L_x}{4}\left[\left(\frac{(n\Pi+\alpha_j+\alpha_p)(\cos(n\Pi+\alpha_j+\alpha_p)-1)+(n\Pi+\alpha_j+\alpha_p)(1-\cos(n\Pi+\alpha_j+\alpha_p))}{(n\Pi+\alpha_j)^2-\alpha_p^2}\right)\right.\\ \left.+\left(\frac{(n\Pi-\alpha_j+\alpha_p)(\cos(n\Pi-\alpha_j-\alpha_p)-1)+(n\Pi-\alpha_j-\alpha_p)(1-\cos(n\Pi-\alpha_j+\alpha_p))}{(n\Pi-\alpha_j)^2-\alpha_p^2}\right)\right]\end{array}\right.$$

$$I_{22}=\left\{\begin{array}{l}\frac{4}{L_x}\left[\frac{\sin(n\Pi+\alpha_j+\alpha_p)}{(n\Pi+\alpha_j+\alpha_p)}+\frac{\sin(n\Pi-\alpha_j-\alpha_p)}{(n\Pi-\alpha_j-\alpha_p)}-\frac{\sin(n\Pi+\alpha_j-\alpha_p)}{(n\Pi+\alpha_j-\alpha_p)}-\frac{\sin(n\Pi-\alpha_j+\alpha_p)}{(n\Pi-\alpha_j+\alpha_p)}\right]_{\alpha_j\neq\alpha_p}\\ \frac{4}{L_x}\left[1-\frac{\sin 2(n\Pi+\alpha_j)}{2(n\Pi+\alpha_j)}\right]_{\alpha_p=n\Pi+\alpha_j}-\frac{L_x}{4}\left[1-\frac{\sin 2(n\Pi-\alpha_j)}{2(n\Pi-\alpha_j)}\right]_{\alpha_p=n\Pi-\alpha_j}\end{array}\right.$$

$$\alpha_j\neq\alpha_p$$
$$\alpha_p=n\Pi+\alpha_j$$
$$\alpha_p=n\Pi-\alpha$$

$$I_{23} = \begin{cases} \dfrac{L_x}{2}\left[\dfrac{\beta_p}{\beta_p^2 + (n\Pi + \alpha_j)^2}\left(\cos(n\Pi + \alpha_j)\cosh h\beta_p + \dfrac{(n\Pi + \alpha_j)}{\beta_p}\sin(n\Pi + \alpha_j)\sinh\beta_p - 1 \right) + \right. \\ \left(\dfrac{\beta_p}{\beta_p^2 + (n\Pi - \alpha_j)^2}\left(\cos(n\Pi - \alpha_j)\cosh\beta_p - \dfrac{(n\Pi - \alpha_j)}{\beta_p}\sin(n\Pi - \alpha_j)\cosh\beta_p - 1 \right) \right] \end{cases}$$

$$I_{24} = \begin{cases} \dfrac{L_x}{2}\left[\dfrac{\beta_p}{\beta_p^2 + (n\Pi + \alpha_j)^2}\left(\cos(n\Pi + \alpha_j)\sinh h\beta_p - \dfrac{(n\Pi + \alpha_j)}{\beta_p}\sin(n\Pi + \alpha_j)\cosh\beta_p \right) + \right. \\ \left(\dfrac{\beta_p}{\beta_p^2 + (n\Pi - \alpha_j)^2}\left(\cos(n\Pi - \alpha_j)\sinh\beta_p + \dfrac{(n\Pi - \alpha_j)}{\beta_p}\sin(n\Pi - \alpha_j)\cosh\beta_p \right) \right] \end{cases}$$

$$I_{25} = \begin{cases} \dfrac{L_x}{2}\left[\dfrac{\beta_j}{\beta_j^2 + (n\Pi + \alpha_p)^2}\left(\sin(n\Pi + \alpha_p)\cosh\beta_j - \dfrac{(n\Pi + \alpha_p)}{\beta_j}\cos(n\Pi + \alpha_p)\sinh\beta_j \right) \right. \\ + \dfrac{\beta_j}{\beta_j^2 + (n\Pi - \alpha_p)^2}\left(\sin(n\Pi - \alpha_p)\cosh\beta_j - \dfrac{(n\Pi - \alpha_p)}{\beta_j}\cos(n\Pi - \alpha_p)\sinh\beta_j \right) \Bigg] \end{cases}$$

$$I_{26} = \begin{cases} \dfrac{L_x}{2}\left[\dfrac{\beta_j}{\beta_j^2 + (n\Pi + \alpha_p)^2}\left(\cos(n\Pi + \alpha_p)\cosh\beta_j + \dfrac{(n\Pi + \alpha_p)}{\beta_j}\sin(n\Pi + \alpha_p)\sinh\beta_j - 1 \right) \right. \\ + \dfrac{\beta_j}{\beta_j^2 + (n\Pi - \alpha_p)^2}\left(\cos(n\Pi - \alpha_p)\cosh\beta_j - \dfrac{(n\Pi - \alpha_p)}{\beta_j}\sin(n\Pi - \alpha_p)\sinh\beta_j - 1 \right) \Bigg] \end{cases}$$

$$I_{27} = \frac{L_x}{2}\left[\frac{(\beta_j + \beta_p)(-1)^n \sinh(\beta_j + \beta_p)}{(\beta_j + \beta_p)^2 + n^2\Pi^2} + \frac{(\beta_j - \beta_p)(-1)^n \sinh(\beta_j - \beta_p)}{(\beta_j - \beta_p)^2 + n^2\Pi^2} \right]$$

$$I_{28} = \frac{L_x}{2}\left[\frac{(\beta_j + \beta_p)^n \left((-1)^n \cosh(\beta_j + \beta_p) - 1\right)}{(\beta_j + \beta_p)^2 + n^2\Pi^2} + \frac{(\beta_j - \beta_p)\left((-1)^n \cosh(\beta_j - \beta_p) - 1\right)}{(\beta_j - \beta_p)^2 + n^2\Pi^2} \right]$$

$$I_{29} = \begin{cases} \dfrac{L_x}{2}\left[\dfrac{\beta_j}{\beta_j^2 + (n\Pi + \alpha_p)^2}\left(\sin(n\Pi + \alpha_p)\sinh\beta_j - \dfrac{(n\Pi + \alpha_p)}{\beta_j}\cos(n\Pi + \alpha_p)\cosh\beta_j - 1 \right) \right. \\ - \dfrac{\beta_j}{\beta_j^2 + (n\Pi - \alpha_p)^2}\left(\sin(n\Pi - \alpha_p)\sinh\beta_j - \dfrac{(n\Pi - \alpha_p)}{\beta_j}\cos(n\Pi - \alpha_p)\cosh\beta_j - 1 \right) \Bigg] \end{cases}$$

$$I_{30} = \begin{cases} \dfrac{L_x}{2}\left[\dfrac{\beta_j}{\beta_j^2 + (n\Pi + \alpha_p)^2}\left(\cos(n\Pi + \alpha_p)\sinh\beta_j + \dfrac{(n\Pi + \alpha_p)}{\beta_j}\sin(n\Pi + \alpha_p)\cosh\beta_j\right)\right. \\ \left. + \dfrac{\beta_j}{\beta_j^2 + (n\Pi - \alpha_p)^2}\left(\cos(n\Pi - \alpha_p)\sinh\beta_j + \dfrac{(n\Pi - \alpha_p)}{\beta_j}\sin(n\Pi - \alpha_p)\cosh\beta_j\right)\right] \end{cases}$$

$$I_{31} = \frac{L_x}{2}\left[\frac{(\beta_j + \beta_p)^n((-1)^n\cosh(\beta_j + \beta_p) - 1)}{(\beta_j + \beta_p)^2 + n^2\Pi^2} + \frac{(\beta_p - \beta_j)((-1)^n\cosh(\beta_p - \beta_j) - 1)}{(\beta_p - \beta_j)^2 + n^2\Pi^2}\right]$$

$$I_{33} = \frac{L_x}{2}\left[\frac{(\beta_j + \beta_p)(-1)^n\sinh(\beta_j + \beta_p)}{(\beta_j + \beta_p)^2 + n^2\Pi^2} + \frac{(\beta_j - \beta_p)(-1)^n\sinh(\beta_j - \beta_p)}{(\beta_j - \beta_p)^2 + n^2\Pi^2}\right]$$

```
REM THIS PROGRAM IS WRITTEN BY ADEDOWOLE ALIMI
REM IT IS WRITTEN TO EVALUATE THE TRANVERSE DISPLACEMENT
OF MOVING
REM FORCE PROBLEM OF RECTANGULAR PLATE FOR SIMPLY
CLAMPED END CONDITION
CLS
10 DIM W1(5), W12(3)
20 OPEN "SCMFR2.BAS" FOR OUTPUT AS #1
30 PRINT #1,
40 FOR MM = 1 TO 3
50 PRINT "SUPPLY THE VALUE OF  RO"
60 INPUT RO
PRINT "THE VALUE OF RO = ", RO
' RO = 10
70 LX = .457
80 LY = .914
90 M1 = 8407.27
100 MIU = 2758.291
110 P = 8407.27 * 9.81 / MIU
120 FX = 2000000
130 AFX = FX / MIU
140 FY = 2500000
150 AFY = FY / MIU
160 GRA = 9.81
170 X = LX / 2
180 Y = LY / 2
190 C = 1.5
200 E = 2.109E+09
205 PI = 22 / 7
210 FS = 2000000
220 FU = FS / MIU
225 Y1 = .4
230 LT = .2
240 D = 10000
245 DM = D / MIU
247 PRINT #1,
250 k = 1
  PRINT #1, "THIS IS THE RESULT FOR RO=", RO
251 AF(1) = 1.5
252 AF(2) = 1.4
253 AF(3) = 1.2
254 BE(1) = 1.6
255 BE(2) = 1.7
```

```
256 BE(3) = 1.9
260 FOR t = 0 TO 1.5 STEP .1
265 FOR j = 1 TO 3
270 JP = j * PI / LX
280 JK = k * PI / LY
290 JP2 = JP ^ 2
300 JK2 = JK ^ 2
310 OME = DM * (JP2 ^ 2 + 2 * JP2 * JK2 + JK2 ^ 2)
380 SA = SIN(AF(j))
390 SHA = SINH(AF(j))
400 SB = SIN(BE(j))
410 SHB = SINH(BE(j))
420 CA = COS(AF(j))
430 CHA = COSH(AF(j))
440 CB = COS(BE(j))
450 CHB = COSH(BE(j))
460 S2A = (SIN(2 * AF(j))) / (2 * AF(j))
470 C2A = (COS(2 * AF(j))) / (2 * AF(j))
480 BLA = (BE(j) * LX) / (BE(j) ^ 2 + AF(j) ^ 2)
490 S3B = (SIN(2 * BE(j))) / (2 * BE(j))
500 C3B = (COS(2 * BE(j))) / (2 * BE(j))
510 ALX = AF(j) / LX
520 BLX = BE(j) / LX
530 JB = AF(j) / BE(j)
540 BJ = (-JB)
550 AJ = (JB * SHB - SA) / (CA - CHB)
560 I1 = X * (1 - S2A)
570 I2 = (-X) * (C2A - 1)
580 I3 = BLA * (SA * CHB - JB * CA * SHB)
590 I4 = BLA * (SA * SHA - JB * (CA * CHB - 1))
600 I5 = (-X) * (C2A - 1)
610 I6 = X * (1 + S2A)
620 I7 = BLA * (CA * CHB + JB * (SA * SHB - 1))
630 I8 = BLA * (CA * SHB + JB * SA * CHB)
640 I9 = BLA * (SA * CHB - JB * CA * SHB)
650 I10 = BLA * (CA * CHB + JB * (SA * SHB - 1))
660 I11 = X * (S3B - 1)
670 I12 = X * (C3B - 1)
680 I13 = BLA * (SA * SHB - JB * (CA * CHB - 1))
690 I14 = BLA * (CA * SHB + JB * SA * CHB)
700 I15 = X * (C3B - 1)
710 I16 = X * (S3B + 1)
720 UA = AJ * (I2 - I4 + I5 - I13)
```

```
730 UB = JB * (I3 + I9)
740 UC = AJ ^ 2 * (I14 + I16 - I6 - I8)
750 UD = AJ * JB * (I7 - I10 - I12 - I15)
760 UE = JB ^ 2 * I11
770 UF = AJ * (I2 + I4 + I5)
780 UG = JB * (I3 - I9)
790 GW = (I1 + UA - UB + UC - UD - UE)
800 GW2 = ALX ^ 2 * (-I1 - UF - UG + UC - UD + UE)
810 JKW = SQR(OME + FU)
820 EC1 = (JK2 * GW - GW2)
830 EC2 = (AFX * GW2 - AFY * JK2 * GW) / (JKW * RO)
840 BEJK = JKW * (1 - LT * (EC1 + EC2))
850 JFA = ALX * C
860 JFB = BLX * C
870 AP = 2 * P / (LX * LY)
880 AA = 1 / (BEJK ^ 2 - JFA ^ 2)
885 AX = JFA / BEJK * SIN(BEJK * t)
890 AB = AJ * COS(JFA * t) - AJ * COS(BEJK * t) + SIN(JFA * t) - AX
900 AO = BEJK / JFB
910 AC = 1 / (AO * (JFB ^ 2 + BEJK ^ 2))
920 XD = BJ * AO * SINH(JFB * t) - BJ * SIN(BEJK * t)
930 AD = XD - AJ * AO * COSH(JFB * t) + AJ * AO * COS(BEJK * t)
940 XC = AJ * COSH(BLX * X)
950 AE = SIN(ALX * X) + AJ * COS(ALX * X) + BJ * SINH(BLX * X) - XC
960 AG = SIN(JK * Y) * SIN(JK * Y1)
970 BG = (AA * AB + AC * AD)
980 W1(j) = AP * BG * AG * AE
990 NEXT j
1000 W = W1(1) + W1(2) + W1(3)
1010 PRINT t, W
1020 PRINT #1, t, W
1030 NEXT t
1040 PRINT #1,
1050 NEXT MM
1060 END
```