FLEXURAL VIBRATIONS OF PRESTRESSED BERNOULLI-EULER BEAM RESTING ON ELASTIC FOUNDATION AND TRAVERSED BY MASSES TRAVELING AT VARIABLE SPEEDS

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CERTIFICATION

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DEDICATION

This work is affectionately dedicated to my Late father Prince Joseph Omolofe.

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ABSTRACT

The problem of flexural vibrations of prestressed Bernoulli-Euler beam resting on elastic foundation and traversed by concentrated masses traveling at variable speeds is studied in this thesis. Both cases of uniform and non-uniform Bernoulli-Euler beams involving fourth order partial differential equations having variable and singular coefficients are considered.

Foremost, closed form solutions are obtained for both the problems of uniform and non-uniform Bernoulli-Euler beams. The solution technique is based on the generalized integral transforms, the generalized Galerkin's method, the expansion of the Dirac Delta function in series form, a modification of the Struble's asymptotic method and the use of the generating functions of the Bessel functions. An important features of this robust technique is that it is applicable for all variants of classical boundary conditions for this class of problems. The closed form solutions are analyzed and numerical analysis in plotted curves are presented.

The results show that for the same natural frequency, the critical speed for the uniform Bernoulli-Euler beams traversed by moving force is greater than that under the influence of a moving mass for both uniform and non-uniform Bernoulli-Euler beams. Hence resonance is reached earlier in the moving mass problem. The same results are obtained for the non-uniform Bernoulli-Euler beams. Furthermore, for fixed values of axial force N and foundation modulus

K in all the illustrative examples considered, the moving force solution is not an upper bound for the accurate solution of the moving mass solution. It is also found that as the axial force N and the foundation modulus K increases, the amplitudes of both uniform and non-uniform Bernoulli-Euler beams under the action of moving loads traveling with variable velocities decrease. However, higher values of N and K are required for more noticeable effects in the case of other boundary conditions than those of simply supported end conditions.

Finally, it is observed that relying on the moving force problem as a good approximation to a moving mass problem is not only misleading, but it is tragic.

CHAPTER ONE

1.0 INTRODUCTION

The response of structural and flexible members to moving loads has been the subject of numerous investigations owing to its relevance in many diverse areas [1-3]. In most analytical studies in Engineering and Mathematical Physics, the structure has commonly been modeled either as a beam or as a plate. In such structural members under the passage of moving loads, the interaction between the passing load and the structure makes the dynamic response analysis very complex. The application of such moving load problems include [4,1] the response of railroad rails to moving trains, the response of bridges and elevated roadways to moving vehicles, machine chain and belt drives, computer tape drives, floppy disks and video cassette recorders (VCR). Moreover, in our time, modern means of transport are ever faster and heavier, while the structure over which they move are ever more slender and lighter. The dynamic stresses they produce are larger by far than the static ones. This has continued to motivate a lot of research activities in this area. Nevertheless, it appears that most of the recent studies focus on numerical simulations, possibly including the effects of train mass inertia; coupling with the train cars suspension systems, tracks stiffness, damping and roughness, especially for ballasted tracks, or rail wheel contact [3]. Comparatively, few studies concentrate on analytical developments, in fact when these are available, the inertia effects of the heavy mass are neglected. However, there are clearly many problems of great physical significance in which load inertia is not negligible and alters the dynamic behaviour of the system significantly [5]. Examples include the slab type bridges on which vehicles or trains travel and the decks of ships on which aircrafts land. These may be modeled as moving masses on plates which certainly have under laying beams as supports. The fundamental mathematical complexity encountered in this problem lies in the fact that one of the coefficients of linear operator describing the motion is a function of space and time. This is caused by the presence of a Dirac-delta function as a coefficient necessary for a proper description of the motion. Physically, this term represents the interplay of inertia forces due to moving mass inertia. Furthermore, the problem of assessing the dynamic behaviour of structures carrying moving loads has been almost exclusively reserved in literature for moving loads moving at constants speeds [6]. The more practical cases when velocities at which these loads move are no longer constants, but vary with time have received little attention in literature. This may be as result of the complex space-time dependencies inherent in such problems. Specifically, even when the inertia effects of the moving load is neglected, analytical solutions involving integral transform are both intractable and cumbersome [7]. However, such practical problems as acceleration and breaking of automobile on roadways and highway bridges, taking off and landing of air-crafts on runway and breaking and acceleration forces in the calculation of rails and railway bridges in which the motion is not uniform, but a function of time have intensified the need for the study of the behaviour of structures under the action of loads moving with variable velocities.

It should be remarked, at this juncture, that the behaviour of structural members differs from one end support to another. The end support conditions most commonly encountered in the analysis of structural motions are

(a) Simply supported end.

(b) Clamped end.

(c) Free end.

(d) Rayleigh end.

These are classified classical conditions. Other boundary conditions may arise, which are called non-classical end conditions. These are not covered in the theory proffered in this thesis. Among the classical end conditions listed above, employed most frequently is the simply supported end conditions in the analysis of structures under moving load. In almost all cases solution techniques employed using these boundary conditions are not suitable for other boundary conditions.

This thesis therefore, is concerned with the flexural vibrations of prestressed (uniform and non-uniform) Bernoulli-Euler beams resting on elastic foundation and traversed by masses moving at variable speeds. This work incorporates the inertia effect of the moving load, the effect of prestress and the effect of elastic foundation in the governing fourth order partial differential equations of the dynamical systems and sets at solving them. The objective is to analyze in each case of the uniform and non-uniform Bernoulli-Euler beams the effects of these parameters when it is being traversed by a heavy moving load, moving at variable speed.

1.1 REVIEW OF RELATED LITERATURE.

The problem of the response of an elastic system (beam or plate) to a moving load (moving force or moving mass) has been the objective of numerous investigation in Engineering, Mathematical Physics and Applied Mathematics for many years [7,8]. In particular, the dynamic response of a simply supported beam, traversed by a constant force moving at a uniform speed was first studied by Krylov [9]. His results were obtained by using the method of expansion of eigenfunctions. He assumed that the mass of the load is smaller than that of the beam. Later, Timoshenko [10] used energy methods to obtain solutions in series form for simply supported finite beam on an elastic foundation subjected to time dependent points loads moving with uniform velocities across the beam. Kenny [11] similarly investigated the dynamic response of infinite elastic beams on elastic foundation under the influence of load moving at constant speeds. He included the effects of viscous damping in the governing differential equation. Steel [12] also investigated the response of a finite simply supported Bernoulli-Euler beam to a unit force moving at a uniform velocity. He analysed the effects of this moving force on beams with and without an elastic foundation. Using a considerably simpler vector formulation with a Laplace rather than Fourier transformation, Steel [13] presented a review of the transient response of the Euler-Bernoulli-Euler beam and the Timoshenko beam on elastic foundation due to moving loads. The problem of a cylindrical shell with an engulfing axisymmetric pressure wave is shown to be generally quite analogous to Timoshenko beam. In a much latter development Oni [14] considered the problem of a harmonic time-variable concentrated force moving at a uniform velocity over a finite deep beam. The methods of integral transformations are used. Series solution which converges is obtained for the deflection of simply supported beams and analysed for various speeds of the load. Just as for elastic beams, the problem of dynamic response of elastic plates to moving loads when the mass effect of the moving load is neglected has been tackled by many authors. However, in comparison, plates subjected to moving loads have only attracted the attentions of few researchers. Among the earliest researches into this subject was Holl [15] who solved the problem of a rectangular plate carrying uniformly moving loads. He concluded that a critical velocity existed for each mode of vibration. Livesly [16] on the other hand, considered the problem of a uniformly traveling load on an infinite plate and showed that there exists a certain critical velocity, beyond which stresses and deflections become infinite. However, in these studies, the plates considered were idealized by one where mass is approximately neglected. Much later Stanisic et al [17] studied the problem of a simply supported non-Mindlin plate under a Multi-masses moving system they made use of an approximation of the Dirac Delta function and obtained in series form a closed form solution of the dynamical problem. For a plate structure, without an elastic foundation, Willis et al [18] used the finite element method to study the dynamic response under moving loads. He examined the effects of eccentricity, span length, acceleration and initial velocity of the moving load. The response to moving concentrated masses of elastic plates on a non Winkler elastic foundation was later taken up by Gbadeyan and Oni [19]. They found that, for the same natural frequency, the critical speed for a rectangular plate resting on a Pasternak foundation and subjected to a moving mass is smaller than that of the same plate traversed by a moving force. A one-dimensional analogue of the work in [17] was taken up by Milomir et al [20] who developed a theory describing the response of a Bernoulli-Euler beam under an arbitrary number of concentrated moving masses. The theory is based on the Fourier Technique and shows that, for a simply supported beam, the resonance frequency is lower with no corresponding decrease in maximum amplitude when the inertia is considered. The analytical and numerical solutions were shown to converge very rapidly. This work was later extended by Stanisic et al [21] to include all the components of the inertia term. Their method of solution as with other references earlier stated is suitable only for simply supported end conditions.

This deficiency was tackled by Sadiku and Leipholz [22] on the dynamic analysis of an elastic beam traversed by a concentrated mass. He developed a robust technique capable of solving Bernoulli-Euler moving load problems for all variant of classical boundary conditions. The technique involves transforming the differential equation by using the Green's function of the associated moving force problem. Although, this work is impressive, its application is limited only to the case of beams executing flexural vibrations according to the simple Bernoulli-Euler theory of flexure. Also to the best of the author's knowledge, this technique has not been extended to a two dimensional moving load problems. To this end a more robust technique was developed by Oni [2] and Gbadeyan and Oni [23] to solve the problem of a finite uniform Rayleigh beam (a thick beam) under an arbitrary number of moving concentrated masses. The theory advanced involves the development of an analytical versatile technique which is based on the modified generalized finite integral transform and the modified Struble's method. An important features of this technique is that it is applicable to all classical end conditions, as well as both thin and thick beam moving load problems. A two-dimensional analogous of this technique was employed by Oni [24] to solve the problem of the dynamic response of an elastic plate under the actions of several moving concentrated masses. It was observed that, for the same natural frequency, the critical speed for the system consisting of a rectangular plate resting on a pasternak's subgrade and traversed by a moving mass is smaller than that traversed by a moving force for both simply supported and simple-clamped rectangular plate. The analysis further show that for both simply supported and simple-clamped rectangular plates, the response amplitudes decrease with an increase in the value of shear modulus for the fixed value of foundation stiffness. However, for simple-clamped rectangular isotropic plate, greater values of the sub-grade's shear modulus for a fixed value of foundation stiffness are required for a noticeable effect on the response curves due to moving force or a moving mass.

It should be remarked at this juncture that in all the aforementioned investigations, the problem of assessing the dynamic behaviour of structures carrying moving loads has been restricted to the case when the loads are moving at constant speeds. The more realistic cases when velocities at which these loads move are not constants but vary with the time are almost virtually absent in literature Oni [6]. This class of problems was first tackled by Lowan [25] who solved the problem of the transverse oscillations of beams under the action of moving variable loads. Much later Kokhmanyuk and Filippov [26] treated the dynamic effects on the transverse motion of a uniform beam of a load moving at variable speed. In a more recent development, Gbadeyan and Ayesimi [27] undertook the analysis of the dynamic response of finite beam continuously supported by a visco elastic foundation to a moving load at variable speed. Only the force effect of the moving load was considered and the method of solution is only suitable for simply supported boundary conditions. It was found that the period of the resonating vibration decreases with increasing value of lateral frequency of load. Oni [6] made a bold attempt more recently on the motions of a uniform beam under the actions of a mass traveling with variable velocity. However, his method fails to cover the various practical cases of all classical boundary conditions. In particular, his method of solution is only suitable for simply supported end conditions and beams with uniform cross-sections. Thus, this study presents the problem of both uniform and non-uniform beams under the actions of concentrated masses whose speeds vary with time. The beams are assumed to be under tensile stress and on elastic foundation.

1.2 OBJECTIVES OF THE RESEARCH

The specific objectives of this work are to

(a) obtain closed form solutions of the fourth order differential equations, with variable and singular coefficients of uniform and non-uniform Bernoulli-Euler beams for all variants of classical boundary conditions.

(b) determine and classify the axial force influence on the response to moving masses of both uniform and non-uniform Bernoulli-Euler beams resting on elastic foundation.

(c) classify the effects of the elastic Winkler foundation on the transverse-displacement response of both Uniform and Non-Uniform Bernoulli-Euler beams for all variants of classical boundary conditions.

(d) indicate the reliability of the moving force solution as a safe approximation to the moving mass problem.

(e) establish the resonance conditions for both moving force and moving mass problems and the effect of axial force and foundation moduli on the resonance conditions.

1.3 DERIVATION OF GOVERNING DIFFERENTIAL EQUATION

Let us consider the motion of a straight, non-uniform beam as shown below in the diagram (fig 1a).

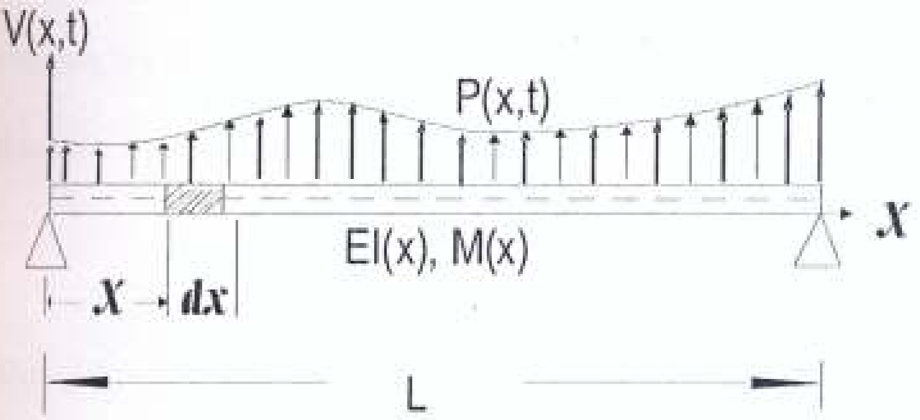

Fig 1a: Beam properties and coordinates

The significant physical properties of the beam: flexural stiffness $EI(x)$ and the mass per unit length $M(x)$, both vary arbitrarily with position x along the span L. The transverse load $P(x,t)$ is assumed to vary arbitrarily with position and time and the transverse displacement response $V(x,t)$ also is function of these variables. The end-support conditions for the beam are arbitrary, although they are pictured as simple supports for illustrative purposes.

Consider a free body diagram shown below (fig 1b).

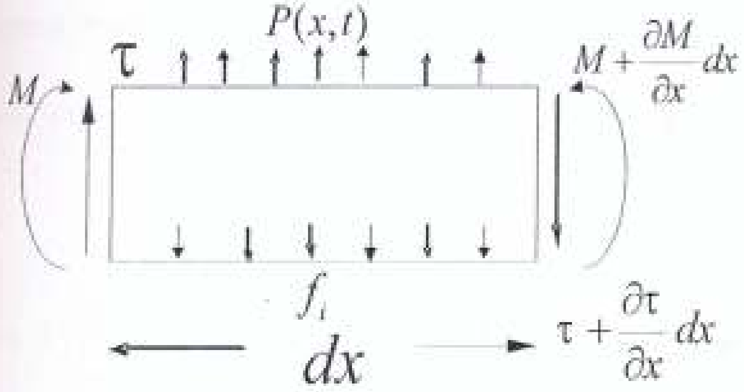

Fig 1b Forces acting on differential element.

The equation of motion of this simple system can readily be derived by considering the equilibrium of forces acting on the differential element of the beam shown above in fig 1b. Summing all forces acting vertically leads to the first dynamic-equilibrium relationship.

$$\tau + Pdx - \left(\tau + \frac{\partial \tau}{\partial x}dx\right) - f_i dx = 0 \tag{1.1}$$

in which $f_i dx$ represents the distributed transverse inertia force and is given by the product of the differential mass and the local acceleration:

$$f_i dx = m dx \frac{\partial^2 V}{\partial t^2} \tag{1.2}$$

Substituting (1.2) into equation (1.1) and simplifying yields

$$\frac{\partial \tau}{\partial x} = P - m\frac{\partial^2 V}{\partial t^2} \tag{1.3}$$

which may be recognized as the standard relationship between shear force and transverse load but with the transverse load now including the inertia force of the accelerating beam. The second equilibrium relationship is obtained by summing moments about the elastic axis at the right hand face of the segment as follows:

$$M + \tau dx - \left(M + \frac{\partial M}{\partial x} dx \right) = 0 \qquad (1.4)$$

where it has been noted that the distributed lateral force makes only a second order contribution to the moment. Simplifying (1.4) one arrives at

$$\frac{\partial M}{\partial x} = \tau \qquad (1.5)$$

No inertia force contributes in this case to the moment equilibrium. Differentiating (1.5) w.r.t x and substituting (1.3) yields, after rearrangement,

$$\frac{\partial^2 M}{\partial x^2} + m \frac{\partial^2 V}{\partial t^2} = P \qquad (1.6)$$

Finally, introducing the basic moment-curvature relationship of elementary beam theory that

$$M = EI(x) \frac{\partial^2 V}{\partial t^2} \qquad (1.7)$$

leads to the partial differential equation of motion for elementary case of beam flexure.

$$\frac{\partial^2}{\partial x^2} \left(EI(x) \frac{\partial^2 V}{\partial x^2} \right) + M \frac{\partial^2 V}{\partial t^2} = P(x,t) \qquad (1.8)$$

Now, if this beam is subjected to a force parallel to its axis in addition to the lateral loading, the beam in fig. 1a becomes

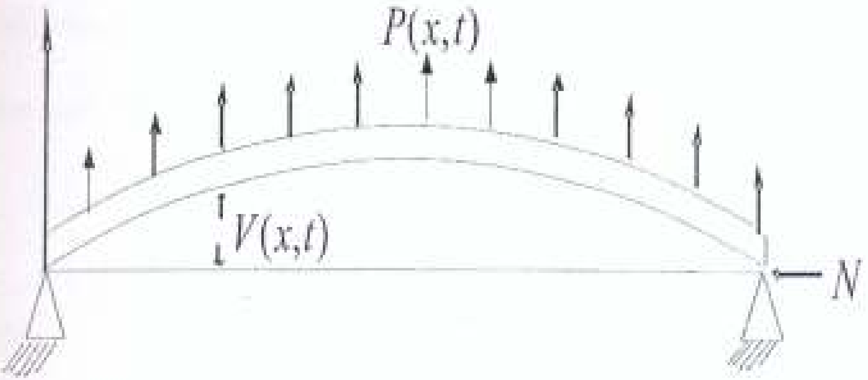

Fig. 1c

In this case, the local equilibrium of forces is altered because the axial force interacts with the lateral displacements to produce an additional term in the moment-equilibrium expression. A free body diagram of figure 1c is presented below.

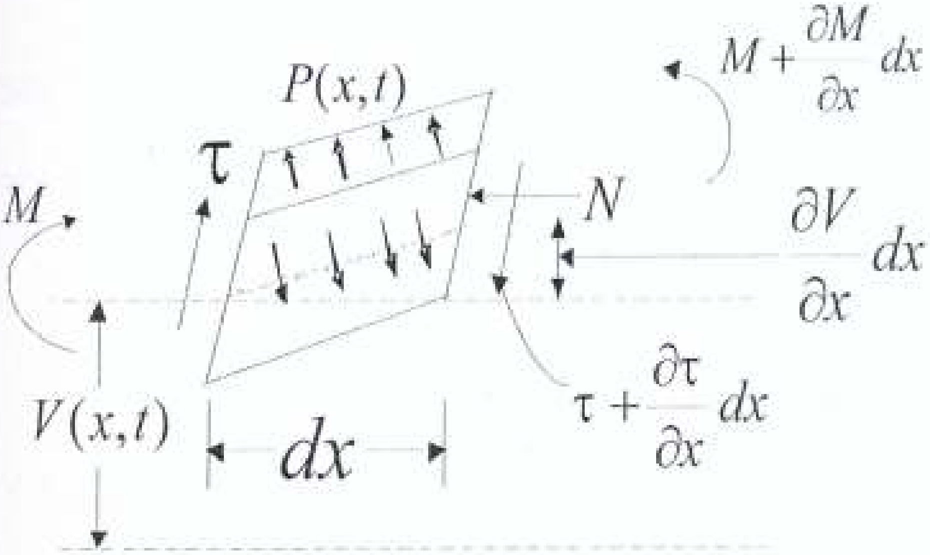

Fig 1d

It is apparent in fig 1d that the transverse equilibrium is not affected by the axial force because its direction does not change with the beam deflection; then equation (1.3) is still valid. However, the point of application of the axial force changes with the beam deflection so that the moment-equilibrium equation now becomes

$$M + \tau dx - N(x)\frac{\partial V}{\partial x} - \left(M + \frac{\partial M}{\partial x}dx\right) = 0 \tag{1.9}$$

this implies

$$\tau = N(x)\frac{\partial V}{\partial x} + \frac{\partial M}{\partial x} \tag{1.10}$$

Substituting the modified expression for τ into equation (1.3) and proceeding as before gives the final equation of motion including the effects of axial force as

$$\frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 V}{\partial x^2}\right) + \frac{\partial}{\partial x}\left(N(x)\frac{\partial V}{\partial x}\right) + M(x)\frac{\partial^2 V}{\partial t^2} = P(x,t) \tag{1.11}$$

When $EI(x)$, $M(x)$ and axial force $N(x)$ are assumed to be constant with respect to time and position, equation (1.11) becomes

$$EI\frac{\partial^4 V}{\partial x^4} + N\frac{\partial^2 V}{\partial x^2} + M\frac{\partial^2 V}{\partial t^2} = P(x,t) \tag{1.12}$$

1.4 FEATURES OF THE THESIS

The procedure adopted in the remaining part of this dissertation is as follows:

In chapter two, the initial-boundary value problem of prestressed uniform Bernoulli-Euler beams resting on elastic foundations and traversed by masses moving at variable speeds is solved in general form. Illustrative examples involving particular boundary conditions, numerical calculations and discussions of results are presented in chapter three. Chapter four considers the initial-boundary value problem of prestressed non-uniform Bernoulli-Euler beams resting on elastic foundations and traversed by masses moving at variable speeds. The closed form solution is obtained in general form. This is followed immediately by illustrative examples involving the various classical boundary conditions, numerical calculations and discussions of results in chapter five.

Finally, chapter six of the thesis contains the conclusions and suggestions for further work.

CHAPTER TWO

UNIFORM BERNOULLI-EULER BEAM RESTING ON ELASTIC FOUNDATION AND UNDER THE ACTION OF MOVING CONCENTRATED MASSES.

2.1 THE GOVERNING EQUATION.

In this chapter, the problem of the dynamic response to concentrated masses moving at non-uniform speed of uniform elastic beams resting on elastic foundation is considered. The fourth order partial differential equation governing this problem is given by

$$\frac{\partial^2}{\partial x^2}\left[EI\frac{\partial^2 V(x,t)}{\partial x^2}\right]-N\frac{\partial^2 V(x,t)}{\partial x^2}+\mu\frac{\partial^2 V(x,t)}{\partial t^2}+K(x)V(x,t)=P(x,t) \qquad (2.1)$$

where x is the spacial coordinate, t is the Time, V(x,t) is the Transverse Displacement, E is the Young's Modulus, I is the Moment of inertia, μ is the mass per unit length of the beam, N is the axial force and K(x) is the elastic foundation.

The moving load on the beam under consideration has mass commensurable with the mass of the beam. Thus, the load P(x,t) takes the form [2]

$$P(x,t)=P_f(x,t)\left[1-\frac{1}{g}\frac{d^2V(x,t)}{dt^2}\right] \qquad (2.2)$$

where the continuous moving force $P_f(x,t)$ acting on the beam model is given by

$$P_f(x,t)=Mg\delta\left[x-f(t)\right], \qquad (2.3)$$

g is an acceleration due to gravity and $\frac{d}{dx}$ is a convective acceleration operator defined as [7]

$$\frac{d}{dx} = \frac{\partial^2}{\partial t^2} + 2\frac{df(t)}{dt}\frac{\partial^2}{\partial x \partial t} + \left(\frac{df(t)}{dt}\right)^2 \frac{\partial}{\partial x^2} + \frac{d^2 f(t)}{dt^2}\frac{\partial}{\partial x} \qquad (2.4)$$

where f(t) is given by

$$f(t) = x_0 + \gamma \sin \beta t \qquad (2.5)$$

where x_0 is the equilibrium position of the logitudinally oscillating load, γ is the longitudinal amplitude of oscillation of the load and β is longitudinal frequency of the load. The load on the beam is assumed to be of mass M moving with non-uniform velocity. Time t is assumed to be limited to that interval of time within which the masses μ are on the beam, that is

$$0 \le f(t) \le L \qquad (2.6)$$

and $\delta[x - f(t)]$is the Dirac Delta function defined as

$$\delta[x - f(t)] = \begin{cases} 0 & x \neq f(t) \\ \infty & x = f(t) \end{cases} \qquad (2.7)$$

with the properties

(i) $$\delta(-x) = \delta(x) \qquad (2.8)$$

(ii) $$\int_a^b \delta(x-k) f(x) dx = \begin{cases} 0, & k < a < b \\ f(k), & a < k < b \\ 0, & a < b < k \end{cases} \qquad (2.9)$$

In Mechanics, the Dirac Delta function $\delta(x)$may be thought of as a unit concentrated force acting at point x = 0 Frybal [7]

In this chapter, the Bernoulli-Euler beam under consideration is assumed to be uniform, which implies, the beam's properties Young's Modulus E, the Moment of inertia I and the mass per unit length μ of the beam do not vary throughout the span L of the beam.

For simplicity, in this problem, a constant elastic foundation is considered. That is,

$$K(x)=K \tag{2.10}$$

where K is the foundation constant.

Substituting (2.2), (2.3), (2.4), (2.5), and (2.9) into (2.1), one obtains

$$EI\frac{\partial^4 V(x,t)}{\partial x^4} - N\frac{\partial^2 V(x,t)}{\partial x^2} + \mu\frac{\partial^2 V(x,t)}{\partial t^2} + KV(x,t) + M\delta[x-(x_o+\gamma Sin\beta t)]\left[\frac{\partial^2 V(x,t)}{\partial t^2}\right.$$

$$\left. + 2\gamma Cos\beta t\frac{\partial^2 V(x,t)}{\partial x\partial t} + (\gamma\beta Cos\beta t)^2\frac{\partial^2 V(x,t)}{\partial x^2} - \gamma\beta^2 Sin\beta t\frac{\partial V(x,t)}{\partial x}\right] = Mg\delta[x-(x_o+\gamma Sin\beta t)] \tag{2.11}$$

The boundary conditions of the above problem are assumed to be arbitrary, that is, it can take any form of the classical boundary conditions. The initial conditions without any loss of generality is given by

$$V(x,0) = 0 = \frac{\partial V(x,0)}{\partial t} \tag{2.12}$$

2.2 METHOD OF SOLUTION.

In this section, a general approach is developed in order to solve the initial-boundary value problem in equation (2.11). An interesting feature of this technique is that it is capable of solving moving mass beam problems involving

(i) Uniform beams other than Bernoulli-Euler beam.

(ii) Any choice of classical boundary condition often encountered in practice.

(iii) Moving loads moving with constant or variable velocities.

The approach involves expressing the Dirac delta function as a fourier cosine series and then reducing the modified form of the fourth order partial differential equation above using the generalized finite integral transforms. The resulting coupled transformed differential equation having some variable coefficients is then solved using the modified Struble's asymptotic technique.

2.3 THE GENERALIZED FINITE INTEGRAL TRANSFORM METHOD.

The Generalized finite integral transform is one of the best methods used in handling problems involving mechanical vibrations. This integral transform method is given by

$$\overline{V}(m,t) = \int_0^L V(x,t)U_m(x)dx \tag{2.13}$$

with the inverse

$$V(x,t) = \sum_{m=1}^{\infty} \frac{\mu}{V_m}\overline{V}(m,t)U_m(x) \tag{2.14}$$

where

$$V_m = \int_0^L \mu U_m^2(x)dx \tag{2.15}$$

Equation (2.13) is the transformation of the function V(x,t) while equation (2.14) is the inverse of this transformation. $U_m(x)$ is any function chosen such that the pertinent boundary conditions are satisfied. An appropriate selection of functions for beam problems are beam mode shapes. Thus, the mth normal mode of vibration of a uniform beam

$$U_m(x)=\sin\frac{\lambda_m x}{L}+A_m\cos\frac{\lambda_m x}{L}+B_m\sinh\frac{\lambda_m x}{L}+C_m\cosh\frac{\lambda_m x}{L} \quad (2.16)$$

is chosen as a suitable kernel of the integral transform (2.13) where, λ_m is the mode frequency, A_m, B_m, C_m are constants. The parameters λ_m, A_m, B_m, and C_m are obtained by substituting (2.16) into the appropriate boundary conditions.

2.4 OPERATIONAL SIMPLIFICATION.

By applying the generalized finite integral transform (2.13), equation (2.1) can be written as

$$T_1\theta(o,L,t)+T_1\theta_A(t)-T_2\theta_B(t)+\overline{V}_{tt}(m,t)+T_3\overline{V}(m,t)+\theta_C(t)+\theta_D(t)+\theta_E(t)-\theta_F(t)=Mg\delta[x-(x_0+\gamma Sin\beta t$$

(2.17)

where

$$T_1=\frac{EI}{\mu},\ T_2=\frac{N}{\mu},\ T_3=\frac{K}{\mu} \quad (2.18)$$

$$\theta(o,L,t)=\left[\frac{\partial^3V(x,t)}{\partial x^3}U_m(x)-\frac{\partial^2V(x,t)}{\partial x^2}\frac{dU_m(x)}{dx}+\frac{\partial V(x,t)}{\partial x}\frac{d^2U_m(x)}{dx^2}-V(x,t)\frac{d^3U_m(x)}{dx^3}\right]_0^L$$

(2.19)

$$\theta_A(t)=\int_0^L V(x,t)\frac{d^4U_m(x)}{dx^4}dx \quad (2.20)$$

$$\theta_B(t)=\int_0^L \frac{\partial^2V(x,t)}{\partial x^2}U_m(x)dx \quad (2.21)$$

$$\theta_C(t) = \int_0^L \frac{M}{\mu}\delta\left[x-(x_0+\gamma Sin\beta t)\right]\frac{\partial^2 V(x,t)}{\partial t^2}U_m(x)dx \qquad (2.22)$$

$$\theta_D(t) = \int_0^L \frac{2M\gamma\beta Cos\beta t}{\mu}\delta\left[x-(x_0+\gamma Sin\beta t)\right]\frac{\partial^2 V(x,t)}{\partial x\partial t}U_m(x)dx \qquad (2.23)$$

$$\theta_E(t) = \int_0^L \frac{M(\gamma\beta Cos\beta 2)^2}{\mu}\delta\left[x-(x_0+\gamma Sin\beta t)\right]\frac{\partial^2 V(x,t)}{\partial x^2}U_m(x)dx \qquad (2.24)$$

$$\theta_F(t) = \int_0^L \frac{M\gamma\beta^2 Sin\beta t}{\mu}\delta\left[x-(x_0+\gamma Sin\beta t)\right]\frac{\partial V(x,t)}{\partial x}U_m(x)dx \qquad (2.25)$$

It is well known that the natural modes

$$U_m(x) = \sin\frac{\lambda_m x}{L} + A_m\cos\frac{\lambda_m x}{L} + B_m\sinh\frac{\lambda_m x}{L} + C_m\cosh\frac{\lambda_m x}{L}$$

satisfy the homogeneous differential equation

$$EI\frac{d^4U_m(x)}{dx^4} - \mu\omega_m^2 U_m(x) = 0 \qquad (2.26)$$

for the Euler's beam. The parameter ω_m is the natural circular frequency defined by

$$\omega_m^2 = \frac{\lambda_m^4}{L^4}\frac{EI}{\mu} \qquad (2.27)$$

Equation (2.26) implies

$$\int_0^L V(x,t)\frac{d^4U_m(x)}{dx^4}dx = \frac{\mu}{EI}\omega_m^2\int_0^L V(x,t)U_m(x)dx \qquad (2.28)$$

Thus, by (2.13),

$$\theta_A(t) = \frac{\mu}{EI}\omega_m^2\overline{V}(m,t) \qquad (2.29)$$

Since $\overline{V}(k,t)$[2] is just the coefficient of the generalized finite integral transforms,

$$V(x,t)=\sum_{k=1}^{\infty}\frac{\mu}{V_k}\overline{V}(k,t)U_k(x) \qquad (2.30)$$

Thus,

$$\frac{\partial^2}{\partial x^2}V(x,t)=\sum_{k=1}^{\infty}\frac{\mu}{V_k}\overline{V}(k,t)\frac{d^2U_k(x)}{dx^2} \qquad (2.31)$$

Therefore, the integral (2.21) can be written as

$$\theta_B(t)=\sum_{k=1}^{\infty}\overline{V}(k,t)\int_0^L\frac{d^2U_k(x)}{dx^2}U_m(x)dx \qquad (2.32)$$

Similarly, the property of the Dirac Delta function as an even function can be used to express it in Fourier cosine series namely

$$\delta[x-(x_0+\gamma\sin\beta t)]=\frac{1}{L}+\frac{2}{L}\sum_{n=1}^{\infty}\cos\frac{n\pi}{L}(x_0+\gamma\sin\beta t)Cos\frac{n\pi x}{L} \qquad (2.33)$$

when use is made of equation (2.33) and equation (2.30), in equation (2.32), one obtains

$$\theta_C(t)=\frac{1}{L}\sum_{k=1}^{\infty}\frac{M}{\mu V_k}\overline{V}_u(k,t)\left[\int_0^L U_k(x)U_m(x)dx+2\sum_{n=1}^{\infty}Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)\int_0^L Cos\frac{n\pi x}{L}U_k(x)U_m(x)dx\right]$$
$$(2.34)$$

Using similar arguments in (2.30) and (2.33) it is straight forward to show that

$$\begin{aligned}\theta_D(t)=\frac{2}{L}\sum_{k=1}^{\infty}\frac{M\gamma\beta Cos\beta t}{\mu V_k}\overline{V}_r(k,t)\Bigg[\int_0^L\frac{dU_k(x)}{dx}U_m(x)dx\\ +2\sum_{n=1}^{\infty}Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)\int_0^L Cos\frac{n\pi x}{L}\frac{dU_k(x)}{dx}U_m(x)dx\Bigg]\end{aligned} \qquad (2.35)$$

$$\theta_c(t)=\frac{1}{L}\sum_{k=1}^{\infty}\frac{M(\gamma\beta Cos\beta t)^2}{\mu V_k'}\bar{V}(k,t)\left[\int_0^L\frac{d^2U_k(x)}{dx^2}U_m(x)dx\right.$$
$$\left.+2\sum_{n=1}^{\infty}Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)\int_0^L Cos\frac{n\pi x}{L}\frac{d^2U_k(x)}{dx^2}U_m(x)dx\right] \qquad (2.36)$$

and

$$\theta_d(t)=\frac{1}{L}\sum_{k=1}^{\infty}\frac{M\gamma\beta^2 Sin\beta t}{\mu V_k'}\bar{V}(k,t)\left[\int_0^L\frac{dU_k(x)}{dx}U_m(x)dx\right.$$
$$\left.+2\sum_{n=1}^{\infty}Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)\int_0^L Cos\frac{n\pi x}{L}\frac{dU_k(x)}{dx}U_m(x)dx\right] \qquad (2.37)$$

Substituting (2.29), (2.32), (2.34), (2.35), (2.36) and (2.37) into (2.17), after some simplifications and rearrangement yields

$$\bar{V}_{tt}(m,t)+\frac{EI\mu\omega_N^2}{EI\mu}\bar{V}(m,t)+\frac{K}{\mu}\bar{V}(m,t)-\frac{N}{\mu}\sum_{k=1}^{\infty}\bar{V}(k,t)H_a(k,m)$$
$$+\frac{M}{\mu L}\bar{V}_{tt}(k,t)H_b(k,m)+\frac{2M}{\mu L}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\bar{V}_{tt}(k,t)H_c(k,m,n)Cos\frac{n\pi}{L}(x_0+\gamma\sin\beta t)$$
$$+\frac{2M}{\mu L}2\gamma\beta Cos\beta t\sum_{k=1}^{\infty}\bar{V}_t(k,t)H_d(k,m)+\frac{4M}{\mu L}\gamma\beta Cos\beta t\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\bar{V}_t(k,t)H_e(k,m,n)Cos\frac{n\pi}{L}(x_0+\gamma\sin\beta t)$$
$$+\frac{M}{\mu L}(\gamma\beta Cos\beta t)^2\sum_{k=1}^{\infty}\bar{V}(k,t)H_f(k,m)+\frac{2M}{\mu L}(\gamma\beta Cos\beta t)^2\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\bar{V}(k,t)H_g(k,m,n)Cos\frac{n\pi}{L}(x_0+\gamma\sin\beta t)$$
$$-\frac{M}{\mu L}\gamma\beta^2 Sin\beta t\sum_{k=1}^{\infty}\bar{V}(k,t)H_i(k,m)-\frac{2M}{\mu L}\gamma\beta^2 Sin\beta t\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\bar{V}(k,t)H_j(k,m,n)Cos\frac{n\pi}{L}(x_0+\gamma\sin\beta t)$$
$$=\frac{P}{\mu}U_m(x_0+\gamma Sin\beta t)$$

(2.38)

where

$$H_a(k,m)=\frac{1}{\tau_k(x)}\int_0^L U_k''(x)U_m(x)dx \qquad (2.39)$$

$$H_b(k,m)=\frac{1}{\tau_k(x)}\int_0^L U_k(x)U_m(x)dx \qquad (2.40)$$

$$H_c(k,m,n)=\frac{1}{\tau_k(x)}\int_0^L U_k(x)U_m(x)Cos\frac{n\pi x}{L}dx \qquad (2.41)$$

$$H_d(k,m)=\frac{1}{\tau_k(x)}\int_0^L U_k^{'}(x)U_m(x)dx \tag{2.42}$$

$$H_e(k,m,n)=\frac{1}{\tau_k(x)}\int_0^L U_k^{'}(x)U_m(x)Cos\frac{n\pi x}{L}dx \tag{2.43}$$

$$H_f(k,m)=\frac{1}{\tau_k(x)}\int_0^L U_k^{''}(x)U_m(x)dx \tag{2.44}$$

$$H_g(k,m,n)=\frac{1}{\tau_k(x)}\int_0^L U_k^{''}(x)U_m(x)Cos\frac{n\pi x}{L}dx \tag{2.45}$$

$$H_h(k,m)=\frac{1}{\tau_k(x)}\int_0^L U_k^{'}(x)U_m(x)dx \tag{2.46}$$

$$H_i(k,m,n)=\frac{1}{\tau_k(x)}\int_0^L U_k^{'}(x)U_m(x)Cos\frac{n\pi x}{L}dx \tag{2.47}$$

Using (2.16) and its derivatives in integrals (2.39) to (2.47) one obtains

$$\begin{aligned}H_a(k,m)=\frac{\lambda_k^2}{\tau_k(x)L^2}[&-I_1-A_mI_2-B_mI_3-C_mI_4-A_mI_5-A_mA_kI_6-A_kB_mI_7-A_kC_mI_8+B_kI_9\\&+B_kA_mI_{10}+B_kB_mI_{11}+B_kC_mI_{12}+C_kI_{13}+C_kA_mI_{14}+C_kB_mI_{15}+C_kC_mI_{16}]\end{aligned}$$

(2.48)

$$\begin{aligned}H_b(k,m)=\frac{1}{\tau_k(x)}[&I_1+A_mI_2+B_mI_3+C_mI_4+A_mI_5+A_mA_kI_6+A_kB_mI_7+A_kC_mI_8+B_kI_9\\&+B_kA_mI_{10}+B_kB_mI_{11}+B_kC_mI_{12}+C_kI_{13}+C_kA_mI_{14}+C_kB_mI_{15}+C_kC_mI_{16}]\end{aligned}$$

(2.49)

$$\begin{aligned}H_c(k,m,n)=\frac{1}{\tau_k(x)}[&I_{17}+A_mI_{18}+B_mI_{19}+C_mI_{20}+A_mI_{21}+A_mA_kI_{22}+A_kB_mI_{23}+B_kI_{24}+A_kC_mI_{25}\\&+B_kA_mI_{26}+B_kB_mI_{27}+B_kC_mI_{28}+C_kI_{29}+C_kA_mI_{30}+C_kB_mI_{31}+C_kC_mI_{32}]\end{aligned}$$

(2.50)

$$H_3(k,m)=\frac{\lambda_k}{\tau_k(x)L}\Big[-A_kI_1-A_kA_mI_2-A_kB_mI_3-A_kC_mI_4+I_5+A_mI_6+B_wI_7+C_wI_8+C_kI_9$$
$$C_kA_mI_{10}+C_kB_mI_{11}+C_kC_mI_{12}+B_kI_{13}+B_kA_mI_{14}+B_kB_mI_{15}+B_kC_mI_{16}\Big]$$

(2.51)

$$B_3(k,m,n)=\frac{\lambda_k}{\tau_k(x)L}\Big[-A_kI_{17}-A_kA_mI_{18}-A_kB_mI_{19}-A_kC_mI_{20}+I_{21}+A_mI_{22}+B_mI_{23}+C_mI_{24}$$
$$+C_kI_{25}+C_kA_mI_{26}+C_kB_mI_{27}+C_kC_mI_{28}+B_kI_{29}+B_kA_mI_{30}+B_kB_mI_{31}+B_kC_mI_{32}\Big]$$

(2.52)

$$H_4(k,m)=\frac{\lambda_k^2}{\tau_k(x)L^2}\Big[-I_1-A_mI_2-B_mI_3-C_mI_4-A_mI_5-A_mA_kI_6-A_kB_mI_7-A_kC_mI_8+B_kI_9$$
$$+B_kA_mI_{10}+B_kB_mI_{11}+B_kC_mI_{12}+C_kI_{13}+C_kA_mI_{14}+C_kB_mI_{15}+C_kC_mI_{16}\Big]$$

(2.53)

$$B_4(k,m,n)=\frac{\lambda_k^2}{\tau_k(x)L^2}\Big[-I_{17}-A_mI_{18}-B_mI_{19}-C_mI_{20}-A_mI_{21}-A_mA_kI_{22}-A_kB_mI_{23}-B_kI_{24}+A_kC_mI_{25}$$
$$+B_kA_mI_{26}+B_kB_mI_{27}+B_kC_mI_{28}+C_kI_{29}+C_kA_mI_{30}+C_kB_mI_{31}+C_kC_mI_{32}\Big]$$

(2.54)

$$H_5(k,m)=\frac{\lambda_k}{\tau_k(x)L}\Big[-A_kI_1-A_kA_mI_2-A_kB_mI_3-A_kC_mI_4+I_5+A_mI_6+B_mI_7+C_mI_8+C_kI_9$$
$$C_kA_mI_{10}+C_kB_mI_{11}+C_kC_mI_{12}+B_kI_{13}+B_kA_mI_{14}+B_kB_mI_{15}+B_kC_mI_{16}\Big]$$

(2.55)

$$B_5(k,m,n)=\frac{\lambda_k}{\tau_k(x)L}\Big[-A_kI_{17}-A_kA_mI_{18}-A_kB_mI_{19}-A_kC_mI_{20}+I_{21}+A_mI_{22}+B_mI_{23}+C_mI_{24}$$
$$+C_kI_{25}+C_kA_mI_{26}+C_kB_mI_{27}+C_kC_mI_{28}+B_kI_{29}+B_kA_mI_{30}+B_kB_mI_{31}+B_kC_mI_{32}\Big]$$

(2.56)

where,

$$I_1 = \int_0^L Sin\frac{\lambda_k x}{L} Sin\frac{\lambda_m x}{L} dx, \qquad I_2 = \int_0^L Sin\frac{\lambda_k x}{L} Cos\frac{\lambda_m x}{L} dx,$$

$$I_3 = \int_0^L Sin\frac{\lambda_k x}{L} Sinh\frac{\lambda_m x}{L} dx, \qquad I_4 = \int_0^L Sin\frac{\lambda_k x}{L} Cosh\frac{\lambda_m x}{L} dx,$$

$$I_5 = \int_0^L Cos\frac{\lambda_k x}{L} Sin\frac{\lambda_m x}{L} dx, \qquad I_6 = \int_0^L Cos\frac{\lambda_k x}{L} Cos\frac{\lambda_m x}{L} dx,$$

$$I_7 = \int_0^L Cos\frac{\lambda_k x}{L} Sinh\frac{\lambda_m x}{L} dx, \qquad I_8 = \int_0^L Cos\frac{\lambda_k x}{L} Cosh\frac{\lambda_m x}{L} dx,$$

$$I_9 = \int_0^L Sinh\frac{\lambda_k x}{L} Sin\frac{\lambda_m x}{L} dx, \qquad I_{10} = \int_0^L Sinh\frac{\lambda_k x}{L} Cos\frac{\lambda_m x}{L} dx,$$

$$I_{11} = \int_0^L Sinh\frac{\lambda_k x}{L} Sinh\frac{\lambda_m x}{L} dx, \qquad I_{12} = \int_0^L Sinh\frac{\lambda_k x}{L} Cosh\frac{\lambda_m x}{L} dx,$$

$$I_{13} = \int_0^L Cosh\frac{\lambda_k x}{L} Sin\frac{\lambda_m x}{L} dx, \qquad I_{14} = \int_0^L Cosh\frac{\lambda_k x}{L} Cos\frac{\lambda_m x}{L} dx,$$

$$I_{15} = \int_0^L Cosh\frac{\lambda_k x}{L} Sinh\frac{\lambda_m x}{L} dx, \qquad I_{16} = \int_0^L Cosh\frac{\lambda_k x}{L} Cosh\frac{\lambda_m x}{L} dx,$$

$$I_{17} = \int_0^L Cos\frac{n\pi x}{L} Sin\frac{\lambda_k x}{L} Sin\frac{\lambda_m x}{L} dx, \qquad I_{18} = \int_0^L Cos\frac{n\pi x}{L} Sin\frac{\lambda_k x}{L} Cos\frac{\lambda_m x}{L} dx,$$

$$I_{19} = \int_0^L Cos\frac{n\pi x}{L} Sin\frac{\lambda_k x}{L} Sinh\frac{\lambda_m x}{L} dx, \qquad I_{20} = \int_0^L Cos\frac{n\pi x}{L} Sin\frac{\lambda_k x}{L} Cosh\frac{\lambda_m x}{L} dx,$$

$$I_{21} = \int_0^L Cos\frac{n\pi x}{L} Cos\frac{\lambda_k x}{L} Sin\frac{\lambda_m x}{L} dx, \qquad I_{22} = \int_0^L Cos\frac{n\pi x}{L} Cos\frac{\lambda_k x}{L} Cos\frac{\lambda_m x}{L} dx,$$

$$I_{23} = \int_0^L Cos\frac{n\pi x}{L} Cos\frac{\lambda_k x}{L} Sinh\frac{\lambda_m x}{L} dx, \qquad I_{24} = \int_0^L Cos\frac{n\pi x}{L} Cos\frac{\lambda_k x}{L} Cosh\frac{\lambda_m x}{L} dx,$$

$$I_{25} = \int_0^L Cos\frac{n\pi x}{L} Sinh\frac{\lambda_k x}{L} Sin\frac{\lambda_m x}{L} dx, \qquad I_{26} = \int_0^L Cos\frac{n\pi x}{L} Sinh\frac{\lambda_k x}{L} Cos\frac{\lambda_m x}{L} dx,$$

$$I_{27} = \int_0^L Cos\frac{n\pi x}{L} Sinh\frac{\lambda_k x}{L} Sinh\frac{\lambda_m x}{L} dx, \qquad I_{28} = \int_0^L Cos\frac{n\pi x}{L} Sinh\frac{\lambda_k x}{L} Cosh\frac{\lambda_m x}{L} dx,$$

$$I_{29} = \int_0^L Cos\frac{n\pi x}{L} Cosh\frac{\lambda_k x}{L} Sin\frac{\lambda_m x}{L} dx, \qquad I_{30} = \int_0^L Cos\frac{n\pi x}{L} Cosh\frac{\lambda_k x}{L} Cos\frac{\lambda_m x}{L} dx,$$

$$I_{11}=\int_0^L Cos\frac{n\pi x}{L}Cosh\frac{\lambda_k x}{L}Sinh\frac{\lambda_m x}{L}dx\,, \qquad I_{25}=\int_0^L Cos\frac{n\pi x}{L}Cosh\frac{\lambda_k x}{L}Cosh\frac{\lambda_m x}{L}dx\,,$$

(2.57)

The solutions to these integrals are listed under the appendix.

In view of equation (2.16) we have

$$U_m(x_0+\gamma Sin\beta t)=Sin\frac{\lambda_m}{L}(x_0+\gamma\sin\beta t)+A_m Cos\frac{\lambda_m}{L}(x_0+\gamma\sin\beta t)+B_m Sinh\frac{\lambda_m}{L}(x_0+\gamma\sin\beta t)$$
$$+C_m Cosh\frac{\lambda_m}{L}(x_0+\gamma\sin\beta t)$$

(2.58)

Using (2.58) in (2.38) one obtains

$$\bar{V}_{tt}(m,t)+\left[\omega_n^2+\frac{K}{\mu}\right]\bar{V}(m,t)-\frac{N}{\mu}\sum_{k=1}^{\infty}\bar{V}(k,t)H_a(k,m)$$
$$+\varepsilon_0\left\{\sum_{k=1}^{\infty}\bar{V}_{tt}(k,t)H_b(k,m)+2\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\bar{V}_{tt}(k,t)H_c(k,m,n)Cos\frac{n\pi}{L}(x_0+\gamma\sin\beta t)\right.$$
$$+2\gamma\beta Cos\beta t\sum_{k=1}^{\infty}\bar{V}_t(k,t)H_d(k,m)+4\gamma\beta Cos\beta t\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\bar{V}_t(k,t)H_e(k,m,n)Cos\frac{n\pi}{L}(x_0+\gamma\sin\beta t)$$
$$+(\gamma\beta Cos\beta t)^2\sum_{k=1}^{\infty}\bar{V}(k,t)H_f(k,m)+2(\gamma\beta Cos\beta t)^2\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\bar{V}(k,t)H_g(k,m,n)Cos\frac{n\pi}{L}(x_0+\gamma\sin\beta t)$$
$$\left.-2\gamma\beta^2 Sin\beta t\sum_{k=1}^{\infty}\bar{V}(k,t)H_i(k,m)-2\gamma\beta^2 Sin\beta t\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\bar{V}(k,t)H_j(k,m,n)Cos\frac{n\pi}{L}(x_0+\gamma\sin\beta t)\right\}$$
$$=\frac{P}{\mu}\left[Sin\frac{\lambda_m}{L}(x_0+\gamma\sin\beta t)+A_m Cos\frac{\lambda_m}{L}(x_0+\gamma\sin\beta t)+B_m Sinh\frac{\lambda_m}{L}(x_0+\gamma\sin\beta t)+C_m Cosh\frac{\lambda_m}{L}(x_0+\gamma\,\mathrm{sir}\right.$$

(2.59)

where

$$\varepsilon_0 = \frac{M}{L\mu} \qquad (2.60)$$

Equation (2.59) is the transformed equation governing the problem of a uniform Bernoulli-Euler beam on a constant elastic foundation. This coupled non-homogeneous Second order ordinary differential equation holds for all variants of the classical boundary conditions.

2.5 SOLUTION OF THE TRANSFORMED EQUATION

In this section, two special cases of equation (2.59) are considered namely ***Moving force*** problem and ***Moving mass*** problem.

2.5.1 The Moving Force Problem

Setting $\varepsilon_0 = 0$ in the transformed equation (2.59) yields

$$\overline{V}_{tt}(m,t) + \left[\omega_m^2 + \frac{K}{\mu}\right]\overline{V}(m,t) - \frac{N}{\mu}\sum_{k=1}^{\infty} \overline{V}(k,t)H_n(k,m)$$
$$= \frac{P}{\mu}\left[Sin\frac{\lambda_m}{L}(x_0 + \gamma \sin \beta t) + A_m Cos\frac{\lambda_m}{L}(x_0 + \gamma \sin \beta t) + B_m Sinh\frac{\lambda_m}{L}(x_0 + \gamma \sin \beta t) + C_m Cosh\frac{\lambda_m}{L}(x_0 + \gamma \sin\right.$$

2.61)

This represents the classical case of a moving force problem associated with our system. It is an approximate model, which assumes the inertia effect of the moving mass as negligible. Evidently, an exact analytical solution to equation

(2.61) is not possible. Though the equation yields readily to numerical technique, an analytical approximate method is desirable as solutions so obtained often shed light on vital information about the vibrating system. To this end, we are going to use a modification of the asymptotic method due to Struble's often used in treating weakly homogeneous and non-homogeneous nonlinear oscillatory systems.

For this purpose, equation (2.61) is rearranged to take the form.

$$\bar{V}_{tt}(m,t)+\left[\omega_n^2+\frac{K}{\mu}-\varepsilon^* H_n(m,m)\right]\bar{V}(m,t)-\varepsilon^*\sum_{k=1}^{r}\bar{V}(k,t)H_n(k,m)$$

$$=\frac{P}{\mu}\left[Sin\frac{\lambda_m}{L}(x_n+\gamma\sin\beta t)+A_m Cos\frac{\lambda_m}{L}(x_n+\gamma\sin\beta t)+B_m Sinh\frac{\lambda_m}{L}(x_n+\gamma\sin\beta t)+C_m Cosh\frac{\lambda_m}{L}(x_n+\gamma\sin\right.$$

(2.62)

where

$$\varepsilon^*=\frac{N}{\mu} \qquad (2.63)$$

Simplifying (2.62) further we have

$$\bar{V}_{tt}(m,t)+\left[\omega_{nf}^2-\varepsilon^* H_n(m,m)\right]\bar{V}(m,t)-\varepsilon^*\sum_{k=1}^{r}\bar{V}(k,t)H_n(k,m)$$

$$=\frac{P}{\mu}\left[Sin\frac{\lambda_m}{L}(x_n+\gamma\sin\beta t)+A_m Cos\frac{\lambda_m}{L}(x_n+\gamma\sin\beta t)+B_m Sinh\frac{\lambda_m}{L}(x_n+\gamma\sin\beta t)+C_m Cosh\frac{\lambda_m}{L}(x_n+\gamma\sin\right.$$

(2.64)

where

$$\omega_{nf}^2=\omega_n^2+\frac{K}{\mu} \qquad (2.65)$$

By this technique, one seeks the modified frequency corresponding to the frequency of the free system due to the presence of the effect of axial force N. An equivalent free system operator defined by the modified frequency then replaces equation (2.64)). Thus, we set the right-hand-side of (2.64) to zero and consider a parameter $\eta<1$ for any arbitrary ratio ε^*, defined as

$$\eta=\frac{\varepsilon^*}{1+\varepsilon^*} \qquad (2.66)$$

so that

$$\varepsilon^*=\eta+O(\eta^2) \qquad (2.67)$$

Substituting equation (2.67) into the homogeneous part of equation (2.64) one obtains

$$\frac{d^2\overline{V}(m,t)}{dt^2}+\left[\omega_{af}^2-\eta H_a(m,m)\right]\overline{V}(m,t)-\eta\sum_{k=1}^{\infty}\overline{V}(k,t)H_a(k,m)=0 \qquad (2.68)$$

When η is set to zero in equation (2.68) a situation corresponding to the case in which the axial force effect is regarded as negligible is obtained, then the solution of (2.68) becomes

$$\overline{V}_{af}(m,t)=C_{af}Cos\left[\omega_{af}t-\psi_{af}\right] \qquad (2.69)$$

where C_{af}, ω_{af} *and* ψ_{af} are constants

Furthermore as $\eta<1$ Struble's technique requires that the asymptotic solutions of the homogeneous part of the equation (2.64) be of the form

$$\overline{V}(m,t)=\Lambda(m,t)Cos\left[\omega_{af}t-\phi(m,t)\right]+\eta\Phi_1+O(\eta^2) \qquad (2.70)$$

where $\Lambda(m,t)$ and $\phi(m,t)$ are slowly varying functions of time or equivalently,

$$\frac{d\Lambda(m,t)}{dt} \to O(\lambda); \quad \frac{d^2\Lambda(m,t)}{dt^2} \to O(\lambda^2)$$

$$\frac{d\phi(m,t)}{dt} \to O(\lambda); \quad \frac{d^2\phi(m,t)}{dt^2} \to O(\lambda^2) \qquad (2.71)$$

where → implies " is of "

In view of (2.70), it can be shown that

$$\begin{aligned}\frac{d\overline{V}(m,t)}{dt} &= \dot{\Lambda}(m,t)Cos\left[\omega_{nf}t-\phi(m,t)\right]+\Lambda(m,t)\dot{\phi}(m,t)Sin\left[\omega_{nf}t-\phi(m,t)\right] \\ &-\Lambda(m,t)\omega_{nf}Sin\left[\omega_{nf}t-\phi(m,t)\right]+\eta\overline{\overline{V}}(1,t)\end{aligned} \qquad (2.72)$$

and

$$\begin{aligned}\frac{d^2\overline{V}(m,t)}{dt^2} &= \ddot{\Lambda}(m,t)Cos\left[\omega_{nf}t-\phi(m,t)\right]+\dot{\Lambda}(m,t)\dot{\phi}(m,t)Sin\left[\omega_{nf}t-\phi(m,t)\right] \\ &-\omega_{nf}\dot{\Lambda}(m,t)Sin\left[\omega_{nf}t-\phi(m,t)\right]+\dot{\Lambda}(m,t)\dot{\phi}(m,t)Sin\left[\omega_{nf}t-\phi(m,t)\right] \\ &+\Lambda(m,t)\ddot{\phi}(m,t)Sin\left[\omega_{nf}t-\phi(m,t)\right]-\Lambda(m,t)\left\{\dot{\phi}(m,t)\right\}^{2}Cos\left[\omega_{nf}t-\phi(m,t)\right] \\ &+\Lambda(m,t)\omega_{nf}\dot{\phi}(m,t)Cos\left[\omega_{nf}t-\phi(m,t)\right]-\dot{\Lambda}(m,t)\omega_{nf}Sin\left[\omega_{nf}t-\phi(m,t)\right] \\ &+\Lambda(m,t)\omega_{nf}\dot{\phi}(m,t)Cos\left[\omega_{nf}t-\phi(m,t)\right]-\Lambda(m,t)\omega_{nf}^{2}Cos\left[\omega_{nf}t-\phi(m,t)\right]+\eta\overline{\overline{V}}(1,m)\end{aligned}$$

$$(2.73)$$

To obtain the modified frequency, equation (2.70), (2.72) and (2.73) are substituted into the homogeneous part of equation (2.64). Subsequently, the variational part of the equation describing the axial force effect on the beam is extracted. Thus, substituting equations (2.70), (2.72) and (2.73) into the homogeneous part of the equation (2.64) one obtains

$$\begin{aligned}&2\Lambda(m,t)\omega_{nf}\dot{\phi}(m,t)Cos\left[\omega_{nf}t-\phi(m,t)\right]-2\dot{\Lambda}(m,t)\omega_{nf}Sin\left[\omega_{nf}t-\phi(m,t)\right] \\ &-\eta H_{a}(m,m)\Lambda(m,t)Cos\left[\omega_{nf}t-\phi(m,t)\right]=0\end{aligned} \qquad (2.74)$$

retaining terms to $O(\eta)$ only.

The variational equations are obtained by equating the coefficients of $Sin[\omega_{mf}t - \phi(m,t)]$ and $Cos[\omega_{mf}t - \phi(m,t)]$ on both sides of the equation (2.74). Thus,

$$-2\dot{\Lambda}(m,t)\omega_{mf} = 0 \tag{2.75}$$

and

$$2\Lambda(m,t)\omega_{mf}\dot{\phi}(m,t) - \eta H_a(m,m)\Lambda(m,t) = 0 \tag{2.76}$$

Rearranging (2.75) and (2.76) yields

$$\dot{\Lambda}(m,t) = 0 \tag{2.77}$$

and

$$\dot{\phi}(m,t) = \frac{\eta H_a(m,m)}{2\omega_{mf}} \tag{2.78}$$

Solving (2.77) and (2.78) respectively gives

$$\Lambda(m,t) = C_m''' \tag{2.79}$$

and

$$\phi(m,t) = \frac{\eta H_a(m,m)}{2\omega_{mf}}t + \psi_{mf} \tag{2.80}$$

where C_m''' and ψ_{mf} are constants.

Therefore, when the effect of the axial force is considered, the first approximation to the homogeneous system is

$$\overline{V}(m,t) = C_m'''Cos[\gamma_{mf}t - \psi_{mf}] \tag{2.81}$$

where

$$\gamma_{mf} = \omega_{mf}\left[1 - \frac{\eta H_a(m,m)}{2\omega_{mf}^2}\right] \tag{2.82}$$

The variational equations are obtained by equating the coefficients of $Sin[\omega_{mf}t-\phi(m,t)]$ and $Cos[\omega_{mf}t-\phi(m,t)]$ on both sides of the equation (2.74). Thus,

$$-2\dot{\Lambda}(m,t)\omega_{mf}=0 \tag{2.75}$$

and

$$2\Lambda(m,t)\omega_{mf}\dot{\phi}(m,t)-\eta H_a(m,m)\Lambda(m,t)=0 \tag{2.76}$$

Rearranging (2.75) and (2.76) yields

$$\dot{\Lambda}(m,t)=0 \tag{2.77}$$

and

$$\dot{\phi}(m,t)=\frac{\eta H_a(m,m)}{2\omega_{mf}} \tag{2.78}$$

Solving (2.77) and (2.78) respectively gives

$$\Lambda(m,t)=C_m^{m1} \tag{2.79}$$

and

$$\phi(m,t)=\frac{\eta H_a(m,m)}{2\omega_{mf}}t+\psi_{mf} \tag{2.80}$$

where C_m^{m1} and ψ_{mf} are constants.

Therefore, when the effect of the axial force is considered, the first approximation to the homogeneous system is

$$\overline{V}(m,t)=C_m^{m1}Cos\left[\gamma_{mf}t-\psi_{mf}\right] \tag{2.81}$$

where

$$\gamma_{mf}=\omega_{mf}\left[1-\frac{\eta H_a(m,m)}{2\omega_{mf}^2}\right] \tag{2.82}$$

represents the modified natural frequency due to the effect of axial force N. It is observed that when $\eta = 0$, we recover the frequency of the moving force problem when the axial force effect of the beam is neglected. Thus, to solve the non-homogeneous equation (2.64), the differential operator which acts on $\bar{V}(m,t)$ and $\bar{V}(k,t)$ is replaced by the equivalent free System operator defined by the modified frequency γ_{m}. Using equation (2.82) the homogeneous part of equation (2.64) can be written as

$$\frac{d^2\bar{V}(m,t)}{dt^2} + \gamma_{m}^{2}\bar{V}(m,t) = 0 \tag{2.83}$$

Hence, the entire equation (2.64) takes the form

$$\begin{aligned}\frac{d^2\bar{V}(m,t)}{dt^2} + \gamma_{m}^{2}\bar{V}(m,t) = \frac{P}{\mu}\Big[& Sin\frac{\lambda_m}{L}(x_0 + \gamma \sin \beta t) + A_m Cos\frac{\lambda_m}{L}(x_0 + \gamma \sin \beta t) \\ & + B_m Sinh\frac{\lambda_m}{L}(x_0 + \gamma \sin \beta t) + C_m Cosh\frac{\lambda_m}{L}(x_0 + \gamma \sin \beta t)\Big]\end{aligned} \tag{2.84}$$

Expanding and rearranging the RHS of equation (2.84) one obtains

$$\frac{d^2\bar{V}(m,t)}{dt^2} + \gamma_{m}^{2}\bar{V}(m,t) = Q\left[a_0 Sin(GSin\beta t) + a_1 Cos(GSin\beta t) + a_2 Cosh(GSin\beta t) + a_3 Sinh(G\sin \beta t)\right] \tag{2.85}$$

where

$$Q = \frac{P}{\mu} \tag{2.86}$$

$$a_1 = \left(Cos\frac{\lambda_m x_0}{L} - A_m Sin\frac{\lambda_m x_0}{L}\right), \qquad a_0 = \left(Sin\frac{\lambda_m x_0}{L} + A_m Cos\frac{\lambda_m x_0}{L}\right)$$

$$a_3 = \left(B_m Sinh\frac{\lambda_m x_0}{L} + C_m Cos\frac{\lambda_m x_0}{L}\right), \quad a_2 = \left(B_m Cosh\frac{\lambda_m x_0}{L} + C_m Sinh\frac{\lambda_m x_0}{L}\right) \tag{2.87}$$

Using variation of parameters, the general solution of equation (2.85) is obtained as

$$\overline{V}(m,t) = C_1 Cos\gamma_{mj} t + C_2 Sin\gamma_{mj} t + P_1(t) Cos\gamma'_{mj} t + P_2(t) Sin\gamma_{mj} t \tag{2.88}$$

where C_1 and C_2 are constants to be determined using initial conditions. The functions $P_1(t)$ and $P_2(t)$ are respectively given as

$$P_1(t) = -\frac{Q}{\gamma_{mj}} \int \left\{ a_0 Sin(GSin\beta t) + a_1 Cos(GSin\beta t) + S_1 e^{GSin\beta t} + S_2 e^{-GSin\beta t} \right\} Sin\gamma_{mj} t\, dt \tag{2.99}$$

and

$$P_2(t) = \frac{Q}{\gamma_{mj}} \int \left\{ a_0 Sin(GSin\beta t) + a_1 Cos(GSin\beta t) + S_1 e^{GSin\beta t} + S_2 e^{-GSin\beta t} \right\} Cos\gamma_{mj} t\, dt \tag{2.100}$$

where

$$G = \frac{\lambda_m \gamma}{L}, \; S_1 = \frac{a_2 + a_3}{2} \text{ and } S_2 = \frac{a_2 - a_3}{2} \tag{2.101}$$

In order to evaluate the integrals (2.99) and (2.100), it is pertinent at this juncture to state and proof the following theorems.

THEOREM I

$$Cos(ZSin\theta) = J_0(Z) + 2\sum_{k=1}^{\infty} J_{2k}(Z) Cos(2k\theta) \tag{2.102}$$

By Taylor's expansion,

$$e^Z = 1 + Z + \frac{Z^2}{2!} + \frac{Z^3}{3!} + \cdots\cdots = \sum_{r=0}^{\infty} \frac{Z^r}{r!} \tag{2.103}$$

Putting $Z = \frac{Zt}{2}$ into (2.103), one obtains

$$e^{\frac{Zt}{2}} = \sum_{r=0}^{\infty} \frac{Z^r t^r}{2^r r!} \tag{2.104}$$

Similarly, putting $Z = -\frac{Z}{2t}$ into (2.104) yields

$$e^{-\frac{Z}{2t}} = \sum_{m=0}^{\infty} \frac{(-)^m Z^m}{2^m t^m m!} \tag{2.105}$$

Multiplying equation (2.104) by (2.105) we have

$$e^{\frac{Z}{2}\left(t-\frac{1}{t}\right)} = \sum_{r=0}^{\infty} \frac{Z^r t^r}{2^r r!} \times \sum_{m=0}^{\infty} \frac{(-)^m Z^m}{2^m t^m m!} \tag{2.106}$$

In order to obtain the (t^k)th term of the series (2.106) above we replace r by (k+m). Thus,

$$\sum_{r=0}^{\infty} \frac{Z^{(k+m)} t^{(k+m)}}{2^{(k+m)}(k+m)!} \times \sum_{m=0}^{\infty} \frac{(-)^m Z^m}{2^m t^m m!} = \left[\sum_{m=0}^{\infty} (-)^m \left(\frac{Z}{2}\right)^{(k+2m)} \frac{1}{(k+m)!m!}\right] t^k \tag{2.107}$$

Evidently, the coefficient of (t^k)th term can be written as

$$\sum_{m=0}^{\infty} (-)^m \left(\frac{Z}{2}\right)^{(k+2m)} \frac{1}{(k+m)!m!} = J_k(Z) \tag{2.108}$$

Similarly, to obtain (t^{-k})th term, we replace m by (k+r) in (2.106) and we obtain

$$\sum_{r=0}^{\infty} \frac{Z^r t^r}{2^r r!} \times \sum_{S=0}^{\infty} \frac{(-)^{k+r} Z^{(k+r)}}{2^{(k+r)} t^{(k+r)} (k+r)!} = \left[\sum_{r=0}^{\infty} (-)^r \left(\frac{Z}{2}\right)^{k+2r} \frac{1}{(k+r)!r!}\right] (-1)^k t^{-k} \tag{2.109}$$

It is clearly seen from the foregoing that the coefficient of (t^{-k}) is given as

$$(-1)^k \sum_{r=0}^{\infty} (-1)^r \left(\frac{Z}{2}\right)^{k+2r} \frac{1}{r!(k+r)!} = (-1)^k \cdot J_k(Z) = J_{-k}(Z) \tag{2.110}$$

It is well known that

$$J_k(Z) = (-1)^k J_{-k}(Z) \tag{2.111}$$

where k is a positive integer. Then using equations (2.108) and (2.110), one obtains

$$e^{\frac{Z}{2}\left(t-\frac{1}{t}\right)}=\sum_{k=-\infty}^{\infty} t^{k} J_{k}(Z) \tag{2.112}$$

Setting $t=e^{i\theta}$ in equation (2.112), one obtains

$$e^{iZ\left(\frac{e^{i\theta}-e^{-i\theta}}{2}\right)}=\sum_{k=-\infty}^{\infty} e^{ik\theta} J_{k}(Z) \tag{2.113}$$

Equation (2.113) can be rewritten as

$$e^{iZ\sin\theta}=J_0(Z)+\left(J_1(Z)e^{i\theta}+J_{-1}(Z)e^{-i\theta}\right)+\left(J_2(Z)e^{i2\theta}+J_{-2}(Z)e^{-i2\theta}\right)+\left(J_3(Z)e^{i3\theta}+J_{-3}(Z)e^{-i3\theta}\right)+\cdots\cdots \tag{2.114}$$

Since $J_k(Z)=(-1)^k J_{-k}(Z)$, equation (2.114) can further be expressed as

$$e^{iZSin\theta}=Cos(Z\sin\theta)+iSin(ZSin\theta) \tag{2.115}$$

It therefore follows that

$$Cos(ZSin\theta)+iSin(ZSin\theta)=J_0(Z)+J_1(Z)\left[e^{i\theta}-e^{-i\theta}\right]+J_2(Z)\left[e^{i2\theta}-e^{-i2\theta}\right]+J_3(Z)\left[e^{i3\theta}-e^{-i3\theta}\right]+\cdots\cdots \tag{2.116}$$

which implies

$$\begin{aligned}Cos(ZSin\theta)+iSin(ZSin\theta)=&\,J_0(Z)+2iSin\theta J_1(Z)+2Cos2\theta J_2(Z)+2iSin3\theta J_3(Z)\\&+2Cos4\theta J_4(Z)+2iSin5\theta J_5(Z)+2Cos6\theta J_6(Z)+\cdots\cdots\cdots\end{aligned} \tag{2.117}$$

Equating the real parts of both sides of equation (2.117) one obtains

$$Cos(ZSin\theta)=J_0(Z)+2Cos2\theta J_2(Z)+2Cos4\theta J_4(Z)+2Cos6\theta J_6(Z)+\cdots\cdots\cdots \tag{2.118}$$

Equation (2.118) can be written as

$$Cos(ZSin\theta) = J_0(Z) + 2\sum_{k=1}^{\infty} J_{2k}(Z)Cos(2k\theta) \tag{2.119}$$

THEOREM II

$$Sin(ZSin\theta) = 2\sum_{k=0}^{\infty} J_{2k+1}(Z)Sin(2k+1)\theta \tag{2.120}$$

To establish theorem II we equate the imaginary parts of both sides of equation (2.117). Thus,

$$Sin(ZSin\theta) = 2Sin\theta J_1(Z) + 2J_3(Z)Sin3\theta + 2J_5(Z)Sin5\theta + \cdots\cdots\cdots \tag{2.121}$$

which in general gives

$$Sin(ZSin\theta) = 2\sum_{k=0}^{\infty} J_{2k+1}(Z)Sin(2k+1)\theta \tag{2.122}$$

THEOREM III

$$Cos(ZCos\theta) = J_0(Z) + 2\sum_{k=1}^{\infty} (-1)^k J_{2k}(Z)Cos(2k\theta) \tag{2.123}$$

we know that

$$Sin(90-\theta) = Cos\theta \tag{2.124}$$

Replacing θ by $(90-\theta)$ in (2.118) we have

$$Cos(ZCos\theta) = J_0(Z) - 2Cos2\theta J_2(Z) + 2Cos4\theta J_4(Z) - 2Cos6\theta J_6(Z) + \cdots\cdots \tag{2.125}$$

Equation (2.125) can be rewritten as

$$Cos(ZCos\theta) = J_0(Z) + 2\sum_{k=1}^{\infty} (-1)^k J_{2k}(Z)Cos(2k\theta) \tag{2.126}$$

THEOREM IV

$$Sin(ZCos\theta) = 2\sum_{k=0}^{\infty} (-1)^k J_{2k+1}(Z)Cos(2k+1)\theta \tag{2.127}$$

Replacing θ by $(90-\theta)$ in equation (2.121) one obtains

$$Sin(ZSin(90-\theta)) = 2Sin(90-\theta)J_1(Z) + 2Sin3(90-\theta)J_3(Z) + 2Sin5(90-\theta)J_5(Z) + \cdots\cdots \tag{2.128}$$

which can be written as

$$Sin(ZCos\theta) = 2J_1(Z)Cos\theta + 2J_3(Z)Cos3\theta + 2J_5(Z)Cos5\theta + \cdots\cdots\cdots\cdots \tag{2.129}$$

which in general gives

$$Sin(ZCos\theta) = 2\sum_{k=0}^{\infty} (-1)^k J_{2k+1}(Z)Cos(2k+1)\theta \tag{2.130}$$

Other useful relations that shall be employed in order to solve equation (2.85) are written below

(i) $$e^{\frac{z}{2}(t+\frac{1}{t})} = \sum_{k=-\infty}^{\infty} t^k I_{2k}(z) \qquad t \neq 0 \tag{2.131}$$

(ii) $$e^{ZCos\theta} = I_0(Z) + 2\sum_{k=1}^{\infty} I_{2k}(Z)Cos(2k\theta) \tag{2.132}$$

(iii) $$e^{ZSin\theta} = I_0(Z) + 2\sum_{k=0}^{\infty} (-)^k I_{2k+1}(Z)Sin(2k+1)\theta + 2\sum_{k=1}^{\infty} (-1)^k I_{2k}(Z)Cos(2k\theta) \tag{2.133}$$

Using theorems I and II and equations (2.131) and (2.133), equation (2.99) and (2.100) respectively become

$$\begin{aligned} P_1(t) = -\frac{Q}{\gamma_{oj}}\int\Bigg\{&\alpha_0\left[2\sum_{k=0}^{\infty} J_{2k+1}(G)Sin(2k+1)\beta t\right] + \alpha_1\left[J_0(G) + 2\sum_{k=1}^{\infty} J_{2k}(G)Cos2k\beta t\right] \\ &+ S_1\left[I_0(G) + 2\sum_{k=1}^{\infty} (-1)^k I_{2k+1}(G)Sin(2k+1)\beta t + 2\sum_{k=1}^{\infty} (-1)^k I_{2k}(G)Cos2k\beta t\right] \\ &+ S_2\left[I_0(-G) + 2\sum_{k=1}^{\infty} (-1)^k I_{2k+1}(-G)Sin(2k+1)\beta t + 2\sum_{k=1}^{\infty} (-1)^k I_{2k}(-G)Cos2k\beta t\right]\Bigg\}Sin\gamma_{oj}t\,dt \end{aligned} \tag{2.134}$$

and

$$P_2(t)=\frac{Q}{\gamma_{\alpha}}\int\left\{a_0\left[2\sum_{k=0}^{\infty}J_{2k+1}(G)Sin(2k+1)\beta t\right]+a_1\left[J_0(G)+2\sum_{k=1}^{\infty}J_{2k}(G)Cos2k\beta t\right]\right.$$
$$+S_1\left[I_0(G)+2\sum_{k=1}^{\infty}(-1)^k I_{2k+1}(G)Sin(2k+1)\beta t+2\sum_{k=1}^{\infty}(-1)^k I_{2k}(G)Cos2k\beta t\right]$$
$$\left.+S_2\left[I_0(-G)+2\sum_{k=1}^{\infty}(-1)^k I_{2k+1}(-G)Sin(2k+1)\beta t+2\sum_{k=1}^{\infty}(-1)^k I_{2k}(-G)Cos2k\beta t\right]\right\}Cos\gamma_{\alpha}t\,dt \tag{2.135}$$

Evaluating the integrals (2.134) and (2.135) one obtains

$$P_1(t)=-\frac{Q}{\gamma_{\alpha}}\left\{a_0\sum_{k=0}^{\infty}J_{2k+1}(G)\left[\frac{Sinb_4t}{b_4}-\frac{Sinb_3t}{b_3}\right]-a_1\left[J_0(G)\frac{Cosb_0t}{b_0}+\sum_{k=1}^{\infty}J_{2k}(G)\left(\frac{Cosb_1t}{b_1}+\frac{Cosb_2t}{b_2}\right)\right]\right.$$
$$+S_1\left[-I_0(G)\frac{Cosb_0t}{b_0}+\sum_{k=0}^{\infty}(-1)^k I_{2k+1}(G)\left[\frac{Sinb_4t}{b_4}-\frac{Sinb_3t}{b_3}\right]-\sum_{k=1}^{\infty}(-1)^k I_{2k}(G)\left(\frac{Cosb_1t}{b_1}+\frac{Cosb_2t}{b_2}\right)\right]$$
$$+S_2\left[-I_0(-G)\frac{Cosb_0t}{b_0}+\sum_{k=0}^{\infty}(-1)^k I_{2k+1}(-G)\left[\frac{Sinb_4t}{b_4}-\frac{Sinb_3t}{b_3}\right]-\sum_{k=1}^{\infty}(-1)^k I_{2k}(-G)\left(\frac{Cosb_1t}{b_1}+\frac{Cosb_2t}{b_2}\right)\right. \tag{2.136}$$

and

$$P_2(t)=\frac{Q}{\gamma_{\alpha}}\left\{a_0\sum_{k=0}^{\infty}J_{2k+1}(G)\left[\frac{Cosb_4t}{b_4}-\frac{Cosb_3t}{b_3}\right]-a_1\left[J_0(G)\frac{Sinb_0t}{b_0}+\sum_{k=1}^{\infty}J_{2k}(G)\left(\frac{Sinb_1t}{b_1}+\frac{Sinb_2t}{b_2}\right)\right]\right.$$
$$+S_1\left[I_0(G)\frac{Sinb_0t}{b_0}+\sum_{k=0}^{\infty}(-1)^k I_{2k+1}(G)\left[\frac{Cosb_4t}{b_4}-\frac{Cosb_3t}{b_3}\right]+\sum_{k=1}^{\infty}(-1)^k I_{2k}(G)\left(\frac{Sinb_1t}{b_1}+\frac{Sinb_2t}{b_2}\right)\right]$$
$$+S_2\left[I_0(-G)\frac{Sinb_0t}{b_0}+\sum_{k=0}^{\infty}(-1)^k I_{2k+1}(-G)\left[\frac{Cosb_4t}{b_4}-\frac{Cosb_3t}{b_3}\right]+\sum_{k=1}^{\infty}(-1)^k I_{2k}(-G)\left(\frac{Sinb_1t}{b_1}+\frac{\sin b_2t}{b_2}\right)\right. \tag{2.137}$$

where

$$b_0=\gamma_{\alpha} \tag{2.138}$$

$$b_1=\gamma_{\alpha}+2k\beta \tag{2.139}$$

$$b_2=\gamma_{\alpha}-2k\beta \tag{2.140}$$

$$b_3=\gamma_{\alpha}+(2k+1)\beta \tag{2.141}$$

$$b_4 = \gamma_{\alpha j} - (2k+1)\beta \tag{2.142}$$

Substituting equations (2.136) and (2.137) into (2.88) we have

$$\begin{aligned}
\overline{V}(m,t) = {} & C_1 Cos\gamma_{\alpha j} t + C_2 Sin\gamma_{\alpha j} t + \frac{Q}{\gamma_{\alpha j}}\left\{ a_0 \sum_{k=0}^{\infty} J_{2k+1}(G)\left[\frac{Sin(\gamma_{\alpha j} - b_4)t}{b_4} - \frac{Sin(\gamma_{\alpha j} - b_3)t}{b_3}\right]\right. \\
& + a_1 J_0(G)\left[\frac{Cos(\gamma_{\alpha j} - b_0)t}{b_0}\right] + a_1 \sum_{k=1}^{\infty} J_{2k}(G)\left[\frac{Cos(\gamma_{\alpha j} - b_1)t}{b_1} + \frac{Cos(\gamma_{\alpha j} - b_2)t}{b_2}\right] \\
& + S_1 I_0(G)\left[\frac{Cos(\gamma_{\alpha j} - b_0)t}{b_0}\right] + S_1 \sum_{k=1}^{\infty} (-1)^k I_{2k}(G)\left[\frac{Cos(\gamma_{\alpha j} - b_1)t}{b_1} + \frac{Cos(\gamma_{\alpha j} - b_2)t}{b_2}\right] \\
& + S_1 \sum_{k=0}^{\infty} (-1)^k I_{2k+1}(G)\left[\frac{Sin(\gamma_{\alpha j} - b_4)t}{b_4} - \frac{Sin(\gamma_{\alpha j} - b_3)t}{b_3}\right] \\
& + S_2 I_0(-G)\left[\frac{Cos(\gamma_{\alpha j} - b_0)t}{b_0}\right] + S_2 \sum_{k=1}^{\infty} (-1)^k I_{2k}(-G)\left[\frac{Cos(\gamma_{\alpha j} - b_1)t}{b_1} + \frac{Cos(\gamma_{\alpha j} - b_2)t}{b_2}\right] \\
& \left. + S_2 \sum_{k=0}^{\infty} (-1)^k I_{2k+1}(-G)\left[\frac{Sin(\gamma_{\alpha j} - b_4)t}{b_4} - \frac{Sin(\gamma_{\alpha j} - b_3)t}{b_3}\right]\right\}
\end{aligned} \tag{2.145}$$

Subjecting equation (2.145) to the initial conditions, one obtains

$$\begin{aligned}
C_1 = -\frac{Q}{\gamma_{\alpha j}}\left\{ a_1 \frac{J_0(G)}{b_0} + a_1 \sum_{k=1}^{\infty} J_{2k}(G)\left[\frac{1}{b_1} + \frac{1}{b_2}\right] + S_1 \frac{I_0(G)}{b_0} + S_1 \sum_{k=1}^{\infty} (-1)^k I_{2k}(G)\left[\frac{1}{b_1} + \frac{1}{b_2}\right]\right. \\
\left. + S_2 \frac{I_0(-G)}{b_0} + S_2 \sum_{k=1}^{\infty} (-1)^k I_{2k}(-G)\left[\frac{1}{b_1} + \frac{1}{b_2}\right]\right\}
\end{aligned} \tag{2.146}$$

and

$$\begin{aligned}
C_2 = -\frac{Q}{\gamma_{\alpha j}^2}\left\{ a_0 \sum_{k=0}^{\infty} J_{2k+1}(G)\left[\frac{(\gamma_{\alpha j} - b_4)}{b_4} + \frac{(b_3 - \gamma_{\alpha j})}{b_3}\right] + S_1 \sum_{k=0}^{\infty} (-1)^k I_{2k+1}(G)\left[\frac{(\gamma_{\alpha j} - b_4)}{b_4} + \frac{(b_3 - \gamma_{\alpha j})}{b_3}\right]\right. \\
\left. + S_2 \sum_{k=0}^{\infty} (-1)^k I_{2k+1}(-G)\left[\frac{(\gamma_{\alpha j} - b_4)}{b_4} + \frac{(b_3 - \gamma_{\alpha j})}{b_3}\right]\right\}
\end{aligned} \tag{2.147}$$

Substituting equations (2.146) and (2.147) into equation (2.145) and simplifying yield

$$
\begin{aligned}
\bar{V}(m,t) = \frac{P}{\mu\gamma_{ij}^{2}} \Bigg\{ & \gamma_{ij} a_1 J_0(G)\left[\frac{Cos(\gamma_{ij}-b_0)t - Cos\gamma_{ij}t}{b_0}\right] \\
+ \; & a_1\gamma_{ij}\sum_{k=1}^{\infty} J_{2k}(G)\left[\frac{Cos(\gamma_{ij}-b_1)t - Cos\gamma_{ij}t}{b_1} + \frac{Cos(\gamma_{ij}-b_2)t - Cos\gamma_{ij}t}{b_2}\right] \\
+ \; & a_0\sum_{k=0}^{\infty} J_{2k+1}(G)\left[\frac{\gamma_{ij}Sin(\gamma_{ij}-b_4)t - (\gamma_{ij}-b_4)Sin\gamma_{ij}t}{b_4} - \frac{\gamma_{ij}Sin(\gamma_{ij}-b_3)t - (\gamma_{ij}-b_3)Sin\gamma_{ij}t}{b_3}\right] \\
+ \; & S_1\gamma_{ij} I_0(G)\left[\frac{Cos(\gamma_{ij}-b_0)t - Cos\gamma_{ij}t}{b_0}\right] \\
+ \; & S_1\gamma_{ij}\sum_{k=1}^{\infty} (-1)^k I_{2k}(G)\left[\frac{Cos(\gamma_{ij}-b_1)t - Cos\gamma_{ij}t}{b_1} + \frac{Cos(\gamma_{ij}-b_2)t - Cos\gamma_{ij}t}{b_2}\right] \\
+ \; & S_1\sum_{k=0}^{\infty} (-1)^k I_{2k+1}(G)\left[\frac{\gamma_{ij}Sin(\gamma_{ij}-b_4)t - (\gamma_{ij}-b_4)Sin\gamma_{ij}t}{b_4} - \frac{\gamma_{ij}Sin(\gamma_{ij}-b_3)t - (\gamma_{ij}-b_3)Sin\gamma_{ij}t}{b_3}\right] \\
+ \; & S_2\gamma_{ij} I_0(-G)\left[\frac{Cos(\gamma_{ij}-b_0)t - Cos\gamma_{ij}t}{b_0}\right] \\
+ \; & S_2\gamma_{ij}\sum_{k=1}^{\infty} (-1)^k I_{2k}(-G)\left[\frac{Cos(\gamma_{ij}-b_1)t - Cos\gamma_{ij}t}{b_1} + \frac{Cos(\gamma_{ij}-b_2)t - Cos\gamma_{ij}t}{b_2}\right] \\
+ \; & S_2\sum_{k=0}^{\infty} (-1)^k I_{2k+1}(-G)\left[\frac{\gamma_{ij}Sin(\gamma_{ij}-b_4)t - (\gamma_{ij}-b_4)Sin\gamma_{ij}t}{b_4} - \frac{\gamma_{ij}Sin(\gamma_{ij}-b_3)t - (\gamma_{ij}-b_3)Sin\gamma}{b_3}\right.
\end{aligned}
\tag{2.148}
$$

Which on inversion yields

$$
\begin{aligned}
V(x,t) = \frac{1}{\tau_m(x)}\sum_{m=1}^{\infty} \frac{P}{\mu\gamma_{af}^2}\Bigg\{ & \gamma_{af}a_1 J_0(G)\left[\frac{Cos(\gamma_{af}-b_0)t - Cos\gamma_{af}t}{b_0}\right] \\
& + a_1\gamma_{af}\sum_{k=1}^{\infty} J_{2k}(G)\left[\frac{Cos(\gamma_{af}-b_1)t - Cos\gamma_{af}t}{b_1} + \frac{Cos(\gamma_{af}-b_2)t - Cos\gamma_{af}t}{b_2}\right] \\
& + a_0\sum_{k=0}^{\infty} J_{2k+1}(G)\left[\frac{\gamma_{af}Sin(\gamma_{af}-b_4)t - (\gamma_{af}-b_4)Sin\gamma_{af}t}{b_4} - \frac{\gamma_{af}Sin(\gamma_{af}-b_5)t - (\gamma_{af}-b_5)Sin\gamma_{af}t}{b_5}\right] \\
& + S_1\gamma_{af}I_0(G)\left[\frac{Cos(\gamma_{af}-b_0)t - Cos\gamma_{af}t}{b_0}\right] \\
& + S_1\gamma_{af}\sum_{k=1}^{\infty} (-1)^k I_{2k}(G)\left[\frac{Cos(\gamma_{af}-b_1)t - Cos\gamma_{af}t}{b_1} + \frac{Cos(\gamma_{af}-b_2)t - Cos\gamma_{af}t}{b_2}\right] \\
& + S_1\sum_{k=0}^{\infty} (-1)^k I_{2k+1}(G)\left[\frac{\gamma_{af}Sin(\gamma_{af}-b_4)t - (\gamma_{af}-b_4)Sin\gamma_{af}t}{b_4} - \frac{\gamma_{af}Sin(\gamma_{af}-b_5)t - (\gamma_{af}-b_5)Sin\gamma_{af}t}{b_5}\right. \\
& + S_2\gamma_{af}I_0(-G)\left[\frac{Cos(\gamma_{af}-b_0)t - Cos\gamma_{af}t}{b_0}\right] \\
& + S_2\gamma_{af}\sum_{k=1}^{\infty} (-1)^k I_{2k}(-G)\left[\frac{Cos(\gamma_{af}-b_1)t - Cos\gamma_{af}t}{b_1} + \frac{Cos(\gamma_{af}-b_2)t - Cos\gamma_{af}t}{b_2}\right] \\
& + S_2\sum_{k=0}^{\infty} (-1)^k I_{2k+1}(-G)\left[\frac{\gamma_{af}Sin(\gamma_{af}-b_4)t - (\gamma_{af}-b_4)Sin\gamma_{af}t}{b_4}\right. \\
& \left. - \frac{\gamma_{af}Sin(\gamma_{af}-b_5)t - (\gamma_{af}-b_5)Sin\gamma_{af}t}{b_5}\right]\Bigg\}\left(Sin\frac{\lambda_m x}{L} + A_m Cos\frac{\lambda_m x}{L} + B_m Sinh\frac{\lambda_m x}{L} + C_m Cosh\frac{\lambda_m x}{L}\right)
\end{aligned}
$$

(2.149)

where

$$\tau_m = \int_0^L U_m^2(x)dx \qquad (2.150)$$

Equation (2.150) represents the transverse displacement response to a moving force moving at variable velocity of a Uniform Bernoulli-Euler beam resting on elastic foundation and having arbitrary end support conditions.

2.5.2 The Moving Mass Problem

If the mass of the moving load is commensurable with that of the structure, the inertia effect of the moving mass is not negligible. Thus, $\varepsilon_0 \neq 0$ and the solution of the entire equation (2.59) is desired. This we term the moving mass problem. Like in the previous section, it is clearly seen that a closed form solution of equation (2.59) is not possible. Again, an approximate analytical method due to Struble is resorted to. We take note that, neglecting the terms representing the inertia effect of the moving mass, we obtain equation (2.61). The homogeneous part of this equation can be replaced by a free system operator defined by the modified frequency γ_{mf}, due to the presence of the effect of the axial force N.

Thus equation (2.59) can be written in the form

$$\bar{V}_{tt}(m,t)+\gamma_{mf}^{2}\bar{V}(m,t)+\varepsilon_0\left\{\sum_{k=1}^{\infty}\bar{V}_{tt}(k,t)H_b(k,m)+2\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\bar{V}_{tt}(k,t)H_c(k,m,n)Cos\frac{n\pi}{L}(x_0+\gamma\sin\beta t)\right.$$

$$+2\gamma\beta Cos\beta t\sum_{k=1}^{\infty}\bar{V}_t(k,t)H_d(k,m)+4\gamma\beta Cos\beta t\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\bar{V}_t(k,t)H_e(k,m,n)Cos\frac{n\pi}{L}(x_0+\gamma\sin\beta t)$$

$$+(\gamma\beta Cos\beta t)^2\sum_{k=1}^{\infty}\bar{V}(k,t)H_f(k,m)+2(\gamma\beta Cos\beta t)^2\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\bar{V}(k,t)H_g(k,m,n)Cos\frac{n\pi}{L}(x_0+\gamma\sin\beta t)$$

$$-2\gamma\beta^2 Sin\beta t\sum_{k=1}^{\infty}\bar{V}(k,t)H_i(k,m)-2\gamma\beta^2 Sin\beta t\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\bar{V}(k,t)H_j(k,m,n)Cos\frac{n\pi}{L}(x_0+\gamma\sin\beta t)$$

$$=\frac{P}{\mu}\left[Sin\frac{\lambda_m}{L}(x_0+\gamma\sin\beta t)+A_m Cos\frac{\lambda_m}{L}(x_0+\gamma\sin\beta t)+B_m Sinh\frac{\lambda_m}{L}(x_0+\gamma\sin\beta t)+C_m Cosh\frac{\lambda_m}{L}(x_0+\gamma\sin\right.$$

(2.151)

Rearranging equation (2.151), we have

$$\bar{V}_{tt}(m,t)+\frac{\varepsilon_0\left[2\gamma\beta Cos\beta t H_d(m,m)+4\gamma\beta Cos\beta t\sum_{n=1}^{\infty} Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)\right]H_e(m,m,n)}{\left[1+\varepsilon_0\left(H_b(m,m)+2\sum_{n=1}^{\infty} Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)H_c(m,m,n)\right)\right]}\bar{V}_t(m,t)$$

$$+\frac{\left[\gamma_m^2+\varepsilon_0\left((\gamma\beta Cos\beta t)^2 H_f(m,m)+2(\gamma\beta Cos)^2\sum_{n=1}^{\infty} Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)H_g(m,m,n)\right)\right]}{\left[1+\varepsilon_0\left(H_b(m,m)+2\sum_{n=1}^{\infty} Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)H_c(m,m,n)\right)\right]}\bar{V}(m,t)$$

$$-\frac{\varepsilon_0\left[\gamma\beta^2 Sin\beta t H_i(m,m)+2\gamma\beta^2 Sin\beta t\sum_{n=1}^{\infty} Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)H_j(m,m,n)\right]}{\left[1+\varepsilon_0\left(H_b(m,m)+2\sum_{n=1}^{\infty} Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)H_c(m,m,n)\right)\right]}\bar{V}(m,t)$$

$$+\sum_{\substack{k=1\\k\neq m}}^{\infty}\varepsilon_0\left\{\frac{H_b(k,m)+2\sum_{n=1}^{\infty} Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)H_c(k,m,n)}{\left[1+\varepsilon_0\left(H_b(m,m)+2\sum_{n=1}^{\infty} Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)H_c(m,m,n)\right)\right]}\bar{V}_{tt}(k,t)\right.$$

$$+\frac{\left[2\gamma\beta Cos\beta t H_a(k,m)+4\gamma\beta Cos\beta t\sum_{n=1}^{\infty} Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)\right]H_e(k,m,n)}{\left[1+\varepsilon_0\left(H_b(m,m)+2\sum_{n=1}^{\infty} Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)H_c(m,m,n)\right)\right]}\bar{V}_t(k,t)$$

$$+\frac{\left[(\gamma\beta Cos\beta t)^2 H_f(k,m)+2(\gamma\beta Cos)^2\sum_{n=1}^{\infty} Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)H_g(k,m,n)\right]}{\left[1+\varepsilon_0\left(H_b(m,m)+2\sum_{n=1}^{\infty} Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)H_c(m,m,n)\right)\right]}\bar{V}(k,t)$$

$$\left.-\frac{\left[\gamma\beta^2 Sin\beta t H_i(k,m)+2\gamma\beta^2 Sin\beta t\sum_{n=1}^{\infty} Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)H_j(k,m,n)\right]}{\left[1+\varepsilon_0\left(H_b(m,m)+2\sum_{n=1}^{\infty} Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)H_c(m,m,n)\right)\right]}\bar{V}(k,t)\right\}$$

$$=\frac{\varepsilon_0 Lg\left[Sin\frac{\lambda_m}{L}(x_0+\gamma Sin\beta t+A_m Cos\frac{\lambda_m}{L}(x_0+\gamma Sin\beta t+B_m Sinh\frac{\lambda_m}{L}(x_0+\gamma Sin\beta t+C_m Cosh\frac{\lambda_m}{L}(x_0+\gamma Sin\beta t\right.}{\left[1+\varepsilon_0\left(H_b(m,m)+2\sum_{n=1}^{\infty} Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)H_c(m,m,n)\right)\right]} \tag{2.152}$$

where

$$H_a(m,m)=H_a(k,m)\big|_{k=m} \qquad H_c(m,m,n)=H_c(k,m,n)\big|_{k=m}$$

$$H_d(m,m) = H_d(k,m)\big|_{k=m} \qquad H_e(m,m,n) = H_e(k,m,n)\big|_{k=m}$$

$$H_f(m,m) = H_f(k,m)\big|_{k=m} \qquad H_g(m,m,n) = H_g(k,m,n)\big|_{k=m}$$

$$H_i(m,m) = H_i(k,m)\big|_{k=m} \qquad H_j(m,m,n) = H_j(k,m,n)\big|_{k=m} \tag{2.153}$$

As in the previous sections, in the first instance, we consider the homogeneous part of equation (2.152) and obtain a modified frequency corresponding to the frequency of the free system due to the presence of the moving mass M. An equivalent free system operator defined by the modified frequency then replaces equation (2.152). Considering, as in the previous section, a parameter $\lambda < 1$ for any arbitrary mass ratio ε_0 defined as

$$\lambda = \frac{\varepsilon_0}{1+\varepsilon_0} \tag{2.154}$$

It can be shown that

$$\varepsilon_0 = \lambda\left[1 + O(\lambda) + O(\lambda^2) + \quad + \cdots\cdots\cdots\cdots \right]$$

(2.155)

and

$$\frac{1}{\left[1+\lambda\left(H_h(m,m) + 2\sum_{n=1}^{\infty} Cos\frac{n\pi}{L}(x_0 + \gamma Sin\beta t)H_e(m,m,n)\right)\right]}$$
$$= \left[1-\lambda\left(H_h(m,m) + 2\sum_{n=1}^{\infty} Cos\frac{n\pi}{L}(x_0 + \gamma Sin\beta t)H_e(m,m,n)\right) + O\left(\lambda^2\right)\right]$$

(2.156)

where

$$\left|\lambda\left(H_h(m,m) + 2\sum_{n=1}^{\infty} Cos\frac{n\pi}{L}(x_0 + \gamma Sin\beta t)H_e(m,m,n)\right)\right| < 1 \tag{2.157}$$

Now, using (2.155) and (2.156) in (2.152) we have

$$\overline{V}_{tt}(m,t) + \lambda\left[2\gamma\beta Cos\beta t H_a(m,m) + 4\gamma\beta Cos\beta t\sum_{n=1}^{\infty} Cos\frac{n\pi}{L}(x_0 + \gamma Sin\beta t)H_c(m,m,n)\right]\overline{V}_t(m,t)$$

$$+\left[\gamma_m^2 - \gamma_{mf}^2\lambda\left(H_b(m,m) + 2\sum_{n=1}^{\infty} Cos\frac{n\pi}{L}(x_0 + \gamma Sin\beta t)H_e(m,m,n)\right)\right]\overline{V}(m,t)$$

$$+\lambda\left[(\gamma\beta Cos\beta t)^2 H_f(m,m) + 2(\gamma\beta Cos\beta t)^2\sum_{n=1}^{\infty} Cos\frac{n\pi}{L}(x_0 + \gamma Sin\beta t)H_g(m,m,n)\right]\overline{V}(m,t)$$

$$-\lambda\left[\gamma\beta^2 Sin\beta t H_i(m,m) + 2\gamma\beta^2 Sin\beta t\sum_{n=1}^{\infty} Cos\frac{n\pi}{L}(x_0 + \gamma Sin\beta t)H_j(m,m,n)\right]\overline{V}(m,t)$$

$$+\sum_{\substack{k=1\\k\neq m}}^{\infty}\lambda\left\{\left[H_b(k,m) + 2\sum_{n=1}^{\infty} Cos\frac{n\pi}{L}(x_0 + \gamma Sin\beta t)H_e(k,m,n)\right]\overline{V}_{tt}(k,t)\right.$$

$$+\left[2\gamma\beta Cos\beta t H_a(k,m) + 4\gamma\beta Cos\beta t\sum_{n=1}^{\infty} Cos\frac{n\pi}{L}(x_0 + \gamma Sin\beta t)H_c(k,m,n)\right]\overline{V}_t(k,t)$$

$$+\left[(\gamma\beta Cos\beta t)^2 H_f(k,m) + 2(\gamma\beta Cos\beta t)^2\sum_{n=1}^{\infty} Cos\frac{n\pi}{L}(x_0 + \gamma Sin\beta t)H_g(k,m,n)\right]\overline{V}(k,t)$$

$$\left.-\left[\gamma\beta^2 Sin\beta t H_i(k,m) + 2\gamma\beta^2 Sin\beta t\sum_{n=1}^{\infty} Cos\frac{n\pi}{L}(x_0 + \gamma Sin\beta t)H_j(k,m,n)\right]\overline{V}(k,t)\right\}$$

$$= \lambda Lg\left[Sin\frac{\lambda_m}{L}(x_0 + \gamma Sin\beta t) + A_m Cos\frac{\lambda_m}{L}(x_0 + \gamma Sin\beta t) + B_m Sinh\frac{\lambda_m}{L}(x_0 + \gamma Sin\beta t) + C_m Cosh\frac{\lambda_m}{L}(x_0 + \gamma Sin\beta t)\right] \quad (2.158)$$

to $O(\lambda)$ only

When $\lambda = 0$, a case corresponding to the case when the inertia effect of the mass of the system is neglected, then the solution of (2.158) can be written as

$$\overline{V}(m,t) = C_m'' Cos\left[\gamma_{mf} t - \psi_m\right] \quad (2.159)$$

where γ_{mf} is as defined previously and C_m'' and ψ_m are constants. However, since for any arbitrary mass ratio ε_0 we always have $\lambda < 1$, the solution for the homogeneous part of equation (2.158) can be written as

$$\overline{V}(m,t) = A(m,t)Cos\left[\gamma_{mf} t - \phi(m,t)\right] + \lambda\overline{V}(1,t) + O(\lambda^2) \quad (2.160)$$

To obtain the modified frequency, equations (2.160) and its derivatives are substituted into the homogeneous part of equation (2.158). Subsequently, only the variational part of the equation describing the behaviours of $A(m,t)$ and $\phi(m,t)$ during the motion of the mass is extracted. Thus, substituting equations (2.160) and its derivatives into the homogeneous part of equation (2.158) and taking into account (2.71), one obtains

$$
\begin{aligned}
&-2\gamma_{mf}\dot{A}(m,t)Sin\left[\gamma_{mf}t-\phi(m,t)\right]+2A(m,t)\dot{\phi}(m,t)\gamma_{mf}Cos\left[\gamma_{mf}t-\phi(m,t)\right]-\gamma_{mf}^{2}A(m,t)Cos\left[\gamma_{mf}t-\phi(m,t)\right]\\
&+\gamma_{mf}^{2}A(m,t)Cos\left[\gamma_{mf}t-\phi(m,t)\right]-2\lambda\gamma_{mf}\gamma\beta A(m,t)H_{d}(m,m)Cos\beta tSin\left[\gamma_{mf}t-\phi(m,t)\right]\\
&-4\lambda A(m,t)\gamma_{mf}\gamma\beta\sum_{n=1}^{\infty}Cos\frac{n\pi}{L}(x_{0}+\gamma Sin\beta t)H_{e}(m,m,n)Cos\beta tSin\left[\gamma_{mf}t-\phi(m,t)\right]\\
&-\lambda\gamma_{mf}^{2}A(m,t)H_{b}(m,m)Cos\left[\gamma_{mf}t-\phi(m,t)\right]-2\lambda\gamma_{mf}^{2}A(m,t)\sum_{n=1}^{\infty}H_{c}(m,m,n)\Bigg[Cos\frac{n\pi x_{0}}{L}J_{0}(G_{1})+\\
&+2Cos\frac{n\pi x_{0}}{L}\sum_{k=1}^{\infty}J_{2k}(G_{1})Cos2k\beta t-2Sin\frac{n\pi x_{0}}{L}\sum_{k=0}^{\infty}J_{2k+1}(G_{1})Sin(2k+1)\Bigg]Cos\left[\gamma_{mf}t-\phi(m,t)\right]\\
&+\frac{1}{2}\lambda A(m,t)(\gamma\beta)^{2}H_{f}(m,m)Cos\left[\gamma_{mf}t-\phi(m,t)\right]+\frac{1}{2}\lambda A(m,t)(\gamma\beta)^{2}H_{f}(m,m)Cos2\beta tCos\left[\gamma_{mf}t-\phi(m,t)\right]\\
&+\lambda A(m,t)(\gamma\beta)^{2}\sum_{n=1}^{\infty}H_{g}(m,m,n)\Bigg[Cos\frac{n\pi x_{0}}{L}J_{0}(G_{1})\\
&+2Cos\frac{n\pi x_{0}}{L}\sum_{k=1}^{\infty}J_{2k}(G_{1})Cos2k\beta t-2Sin\frac{n\pi x_{0}}{L}\sum_{k=0}^{\infty}J_{2k+1}(G_{1})Sin(2k+1)\Bigg]Cos\left[\gamma_{mf}t-\phi(m,t)\right]\\
&+\lambda A(m,t)(\gamma\beta)^{2}\sum_{n=1}^{\infty}H_{g}(m,m,n)Cos\frac{n\pi}{L}(x_{0}+\gamma Sin\beta t)Cos2\beta tCos\left[\gamma_{mf}t-\phi(m,t)\right]\\
&+\lambda A(m,t)\gamma\beta^{2}H_{i}(m,m)Sin\beta tCos\left[\gamma_{mf}t-\phi(m,t)\right]\\
&-2\lambda A(m,t)\gamma\beta\sum_{n=1}^{\infty}H_{j}(m,m,n)Cos\frac{n\pi}{L}(x_{0}+\gamma Sin\beta t)Sin\beta tCos\left[\gamma_{mf}t-\phi(m,t)\right]\\
&+\sum_{\substack{k=1\\k\neq m}}^{\infty}\Bigg\{-\lambda\gamma_{mf}^{2}A(k,t)\left[H_{b}(k,m)+2\sum_{n=1}^{\infty}Cos\frac{n\pi}{L}(x_{0}+\gamma Sin\beta t)H_{c}(k,m,n)\right]Cos\left[\gamma_{mf}t-\phi(m,t)\right]\\
&-\lambda\gamma_{mf}A(k,t)\left[2\gamma\beta Cos\beta tH_{d}(k,m)+4\gamma\beta Cos\beta t\sum_{n=1}^{\infty}Cos\frac{n\pi}{L}(x_{0}+\gamma Sin\beta t)H_{e}(k,m,n)\right]Sin\left[\gamma_{mf}t-\phi(m,t)\right]\\
&+\lambda A(k,t)\left[(\gamma\beta Cos\beta t)^{2}H_{f}(k,m)+2(\gamma\beta Cos\beta t)^{2}\sum_{n=1}^{\infty}Cos\frac{n\pi}{L}(x_{0}+\gamma Sin\beta t)H_{g}(k,m,n)\right]Cos\left[\gamma_{mf}t-\phi(m,t)\right]\\
&-\lambda A(k,t)\left[\gamma\beta^{2}Sin\beta tH_{i}(k,m)+2\gamma\beta^{2}Sin\beta t\sum_{n=1}^{\infty}Cos\frac{n\pi}{L}(x_{0}+\gamma Sin\beta t)H_{j}(k,m,n)\right]Cos\left[\gamma_{mf}t-\phi(m,t)\right]\Bigg\}\\
&=0
\end{aligned}
\tag{2.161}
$$

retaining terms to order $O(\lambda)$ only.

The variational equations are obtained by equating the coefficients of $Sin[\gamma_{ij}t-\phi(m,t)]$ and $Cos[\gamma_{ij}t-\phi(m,t)]$ terms on both sides of the equation. Thus, noting the following identities

$$Cos\beta t Sin[\gamma_{ij}t-\phi(m,t)]=\frac{1}{2}[Sin[\beta t+\gamma_{ij}t-\phi(m,t)]-Sin[\beta t-\gamma_{ij}t+\phi(m,t)]]$$

$$Cos[\gamma_{ij}t-\phi(m,t)]Sin\beta t=\frac{1}{2}[Sin[\gamma_{ij}t-\phi(m,t)+\beta t]-Sin[\gamma_{ij}t-\phi(m,t)-\beta t]]$$

$$Sin[\gamma_{ij}t-\phi(m,t)]Cos2k\beta t=\frac{1}{2}[Sin[\gamma_{ij}t-\phi(m,t)+2k\beta t]-Sin[\gamma_{ij}t-\phi(m,t)-2k\beta t]]$$

$$Cos[\gamma_{ij}t-\phi(m,t)]Cos2k\beta t=\frac{1}{2}[Cos[\gamma_{ij}t-\phi(m,t)+2k\beta t]+Cos[\gamma_{ij}t-\phi(m,t)-2k\beta t]]$$

$$Sin[\gamma_{ij}t-\phi(m,t)]Sin(2k+1)\beta t=\frac{1}{2}[Cos[\gamma_{ij}t-\phi(m,t)-(2k+1)\beta t]-Cos[\gamma_{ij}t-\phi(m,t)+(2k+1)\beta t]]$$

$$Cos[\gamma_{ij}t-\phi(m,t)]Sin(2k+1)\beta t=\frac{1}{2}[Sin[\gamma_{ij}t-\phi(m,t)+(2k+1)\beta t]-Sin[\gamma_{ij}t-\phi(m,t)-(2k+1)\beta t]]$$

(2.162)

and neglecting those terms that do not contribute to the variational equations, equation (2.161) reduces to

$$-\gamma_{ij}\dot{A}(m,t)Sin[\gamma_{ij}t-\phi(m,t)]+[2A(m,t)\dot{\phi}(m,t)\gamma_{ij}-\lambda\gamma_{ij}^{2}A(m,t)H_{b}(m,m)$$
$$+\frac{1}{2}\lambda A(m,t)H_{f}(m,m)(\gamma\beta)^{2}-2\lambda\gamma_{ij}^{2}A(m,t)\sum_{n=1}^{\infty}H_{e}(m,m,n)Cos\frac{n\pi x_{0}}{L}J_{0}(G_{1})$$
$$+\lambda A(m,t)(\gamma\beta)^{2}\sum_{n=1}^{\infty}H_{g}(m,m,n)Cos\frac{n\pi x_{0}}{L}J_{0}(G_{1})]Cos[\gamma_{ij}t-\phi(m,t)]=0$$

(2.163)

Setting the coefficients of $Sin[\gamma_{ij}t-\phi(m,t)]$ and $Cos[\gamma_{ij}t-\phi(m,t)]$ to zero, we have,

$$-2\gamma_{ij}\dot{A}(m,t)=0 \qquad (2.164)$$

and

$$2A(m,t)\dot{\phi}(m,t)\gamma_m - \lambda\gamma_m^2 A(m,t)H_b(m,m) - 2\lambda\gamma_m^2 A(m,t)\sum_{n=1}^{\infty} H_c(m,m,n)Cos\frac{n\pi x_0}{L}J_0(G_1)$$
$$+\frac{1}{2}\lambda A(m,t)H_f(m,m)(\gamma\beta)^2 + \lambda A(m,t)(\gamma\beta)^2\sum_{n=1}^{\infty} H_g(m,m,n)Cos\frac{n\pi x_0}{L}J_0(G_1) = 0 \qquad (2.165)$$

Rearranging (2.164) and (2.165) one obtains

$$\dot{A}(m,t) = 0 \qquad (2.166)$$

and

$$\dot{\phi}(m,t) = \frac{\lambda}{2}\left[\gamma_m\left(H_b(m,m) + R_1(m,m,n)\right) - \frac{(\gamma\beta)^2\left(H_f(m,m) + 2R_2(m,m,n)\right)}{2\gamma_m}\right] \qquad (2.167)$$

where

$$R_1(m,m,n) = \sum_{n=1}^{\infty} H_c(m,m,n)Cos\frac{n\pi x_0}{L}J_0(G_1) \qquad (2.168)$$

$$R_2(m,m,n) = \sum_{n=1}^{\infty} H_g(m,m,n)Cos\frac{n\pi x_0}{L}J_0(G_1) \qquad (2.169)$$

and

$$G_1 = \frac{n\pi y}{L} \qquad (2.170)$$

Solving equations (2.166) and (2.167) respectively we have

$$A(m,t) = C_m \qquad (2.171)$$

and

$$\phi(m,t) = \frac{\lambda}{2}\left[\gamma_m\left(H_b(m,m) + R_1(m,m,n)\right) - \frac{(\gamma\beta)^2\left(H_f(m,m) + 2R_2(m,m,n)\right)}{2\gamma_m}\right]t + \psi_m$$

(2.172)

where

C_m and ψ_m are constants.

Therefore, when the mass effects of the particle is considered, the first approximation to the homogeneous system is given as

$$\overline{V}(m,t) = C_m Cos\left[\gamma_{bj}t - \psi_m\right] \qquad (2.173)$$

where

$$\gamma_{bj} = \gamma_{nj}\left\{1 - \frac{\lambda_0}{2}\left[(H_b(m,m) + R_1(m,m,n) - \frac{(\gamma\beta)^2\left[H_f(m,m) + 2R_2(m,m,n)\right]}{2\gamma_{nj}^2}\right]\right\} \qquad (2.174)$$

is called the modified frequency corresponding to the frequency of the free system due to the presence of the moving mass. Thus, the homogeneous part of (2.158) can be written as

$$\frac{d^2\overline{V}(m,t)}{dt^2} + \gamma_{bj}^2\overline{V}(m,t) = 0 \qquad (2.175)$$

Hence, the entire equation (2.151) taking into account (2.174) takes the form

$$\frac{d^2\overline{V}(m,t)}{dt^2} + \gamma_{bj}^2\overline{V}(m,t) = \varepsilon_0 Lg\left[a_0 Sin(GSin\beta t) + a_1 Cos(GSin\beta t) + a_2 Cosh(GSin\beta t) + a_3 Sinh(GSin\beta t)\right]$$

$$(2.176)$$

This is analogous to equation (2.85). Thus, using similar argument as in the previous section, the solution to equation (2.176) is given by

$$
\begin{aligned}
\bar{\bar{V}}(m,t) = \frac{P}{\mu\gamma_{bj}^{2}} \Bigg\{ & \gamma_{bj} a_1 J_0(G)\left[\frac{Cos(\gamma_{bj}-h_0)t - Cos\gamma_{bj}t}{h_0}\right] \\
& + a_1\gamma_{bj}\sum_{k=1}^{\infty} J_{2k}(G)\left[\frac{Cos(\gamma_{bj}-h_1)t - Cos\gamma_{bj}t}{h_1} + \frac{Cos(\gamma_{bj}-h_2)t - Cos\gamma_{bj}t}{h_2}\right] \\
& + a_0\sum_{k=0}^{\infty} J_{2k+1}(G)\left[\frac{\gamma_{bj}Sin(\gamma_{bj}-h_4)t - (\gamma_{bj}-h_4)Sin\gamma_{bj}t}{h_4} - \frac{\gamma_{bj}Sin(\gamma_{bj}-h_3)t - (\gamma_{bj}-h_3)Sin\gamma_{bj}t}{h_3}\right] \\
& + S_1\gamma_{bj} I_0(G)\left[\frac{Cos(\gamma_{bj}-h_0)t - Cos\gamma_{bj}t}{h_0}\right] \\
& + S_1\gamma_{bj}\sum_{k=1}^{\infty} (-1)^k I_{2k}(G)\left[\frac{Cos(\gamma_{bj}-h_1)t - Cos\gamma_{bj}t}{h_1} + \frac{Cos(\gamma_{bj}-h_2)t - Cos\gamma_{bj}t}{h_2}\right] \\
& + S_1\sum_{k=0}^{\infty} (-1)^k I_{2k+1}(G)\left[\frac{\gamma_{bj}Sin(\gamma_{bj}-h_4)t - (\gamma_{bj}-h_4)Sin\gamma_{bj}t}{h_4} - \frac{\gamma_{bj}Sin(\gamma_{bj}-h_3)t - (\gamma_{bj}-h_3)Sin\gamma_{bj}t}{h_3}\right] \\
& + S_2\gamma_{bj} I_0(-G)\left[\frac{Cos(\gamma_{bj}-h_0)t - Cos\gamma_{bj}t}{h_0}\right] \\
& + S_2\gamma_{bj}\sum_{k=1}^{\infty} (-1)^k I_{2k}(-G)\left[\frac{Cos(\gamma_{bj}-h_1)t - Cos\gamma_{bj}t}{h_1} + \frac{Cos(\gamma_{bj}-h_2)t - Cos\gamma_{bj}t}{h_2}\right] \\
& + S_2\sum_{k=0}^{\infty} (-1)^k I_{2k+1}(-G)\left[\frac{\gamma_{bj}Sin(\gamma_{bj}-h_4)t - (\gamma_{bj}-h_4)Sin\gamma_{bj}t}{h_4} - \frac{\gamma_{bj}Sin(\gamma_{bj}-h_3)t - (\gamma_{bj}-h_3)Sin\gamma_{bj}t}{h_3}\right]\Bigg\}
\end{aligned}
\tag{2.177}
$$

which on inversion gives

$$
\begin{aligned}
V(x,t) = {} & \frac{1}{\tau_m(x)} \sum_{m=1}^{\infty} \frac{\varepsilon_0 L g}{\gamma_{bj}^2} \Bigg\{ \gamma_{bj} a_0 J_0(G) \left[\frac{Cos(\gamma_{bj} - h_0)t - Cos\gamma_{bj} t}{h_0} \right] \\
& + a_1 \gamma_{bj} \sum_{k=1}^{\infty} J_{2k}(G) \left[\frac{Cos(\gamma_{bj} - h_1)t - Cos\gamma_{bj} t}{h_1} + \frac{Cos(\gamma_{bj} - h_2)t - Cos\gamma_{bj} t}{h_2} \right] \\
& + a_0 \sum_{k=0}^{\infty} J_{2k+1}(G) \left[\frac{\gamma_{bj} Sin(\gamma_{bj} - h_4)t - (\gamma_{bj} - h_4) Sin\gamma_{bj} t}{h_4} - \frac{\gamma_{bj} Sin(\gamma_{bj} - h_3)t - (\gamma_{bj} - h_3) Sin\gamma_{bj} t}{h_3} \right] \\
& + S_1 \gamma_{bj} I_0(G) \left[\frac{Cos(\gamma_{bj} - h_0)t - Cos\gamma_{bj} t}{h_0} \right] \\
& + S_1 \gamma_{bj} \sum_{k=1}^{\infty} (-1)^k I_{2k}(G) \left[\frac{Cos(\gamma_{bj} - h_1)t - Cos\gamma_{bj} t}{h_1} + \frac{Cos(\gamma_{bj} - h_2)t - Cos\gamma_{bj} t}{h_2} \right] \\
& + S_1 \sum_{k=0}^{\infty} (-1)^k I_{2k+1}(G) \left[\frac{\gamma_{bj} Sin(\gamma_{bj} - h_4)t - (\gamma_{bj} - h_4) Sin\gamma_{bj} t}{h_4} - \frac{\gamma_{bj} Sin(\gamma_{bj} - h_3)t - (\gamma_{bj} - h_3) Sin\gamma_{bj} t}{h_3} \right] \\
& + S_2 \gamma_{bj} I_0(-G) \left[\frac{Cos(\gamma_{bj} - h_0)t - Cos\gamma_{bj} t}{h_0} \right] \\
& + S_2 \gamma_{bj} \sum_{k=1}^{\infty} (-1)^k I_{2k}(-G) \left[\frac{Cos(\gamma_{bj} - h_1)t - Cos\gamma_{bj} t}{h_1} + \frac{Cos(\gamma_{bj} - h_2)t - Cos\gamma_{bj} t}{h_2} \right] \\
& + S_2 \sum_{k=0}^{\infty} (-1)^k I_{2k+1}(-G) \left[\frac{\gamma_{bj} Sin(\gamma_{bj} - h_4)t - (\gamma_{bj} - h_4) Sin\gamma_{bj} t}{h_4} \right. \\
& \left. - \frac{\gamma_{bj} Sin(\gamma_{bj} - h_3)t - (\gamma_{bj} - h_3) Sin\gamma_{bj} t}{h_3} \right] \Bigg\} \left(Sin\frac{\lambda_m x}{L} + A_m Cos\frac{\lambda_m x}{L} + B_m Sinh\frac{\lambda_m x}{L} + C_m Cosh\frac{\lambda_m x}{L} \right)
\end{aligned}
\tag{2.178}
$$

Equation (2.178) represents the transverse-displacement response to a moving mass moving at variable velocities of a uniform Bernoulli-Euler beam resting on elastic foundation when the boundary conditions are arbitrary.

CHAPTER THREE

ILLUSTRIATIVE EXAMPLES, NUMERICAL CALCULATIONS AND DISCUSSION OF RESULTS (UNIFORM BERNOULLI-EULER BEAM)

3.1.0 ILLUSTRIATIVE EXAMPLES

In this chapter, the foregoing analyses are illustrated by various practical examples. In particular, classical boundary conditions such as Simply Supported boundary conditions, Free ends condition, Clamped ends condition and Clamped-Free ends condition are considered.

3.1.1 Simply Supported Boundary Conditions

In this case, the uniform Bernoulli-Euler beam has simple supports at ends x=0 and x=L. The displacement and the bending moment vanish at a simply supported ends. Thus, the conditions are expressed as

$$V(0,t)=0=V(L,t), \qquad \frac{\partial^2 V(0,t)}{\partial x^2}=0=\frac{\partial^2 V(L,t)}{\partial x^2} \tag{3.1}$$

and hence for normal modes

$$U_m(0)=0=U_m(L), \qquad \frac{\partial^2 U_m(0)}{\partial x^2}=0=\frac{\partial^2 U_m(L)}{\partial x^2} \tag{3.2}$$

which implies that

$$U_k(0)=0=U_k(L), \qquad \frac{\partial^2 U_k(0)}{\partial x^2}=0=\frac{\partial^2 U_k(L)}{\partial x^2} \tag{3.3}$$

Thus, it can be shown that

$$A_m = A_k = 0; \quad B_m = B_k = 0; \quad C_m = C_k = 0 \tag{3.4}$$

and the frequency equation becomes

$$Sin\lambda_m = Sin\lambda_k = 0 \tag{3.5}$$

which implies

$$\lambda_m = m\pi \qquad and \qquad \lambda_k = k\pi \tag{3.6}$$

Thus, the moving force problem is reduced to a non-homogeneous second order ordinary differential equation

$$\frac{d^2\overline{V}(m,t)}{dt^2} + \gamma_{mf}^2 \overline{V}(m,t) = \frac{P}{\mu}\left[SinFCos(GSin\beta t) + CosFSin(GSin\beta t)\right] \tag{3.7}$$

where

$$\gamma_{mf}^2 = \left[\frac{EI\mu\omega_n^2}{EI\mu} + \frac{N}{\mu}\left(\frac{m\pi}{L}\right)^2 + \frac{K}{\mu}\right] \tag{3.8}$$

$$F = \frac{m\pi x_0}{L} \qquad and \qquad G = \frac{m\pi y}{L} \tag{3.9}$$

Equation (3.7) when solved in conjunction with the initial conditions, one obtains an expression for $\overline{V}(m,t)$ which on inversion yields

$$\begin{aligned} V(x,t) = \frac{2}{L}\sum_{m=1}^{\infty} \frac{P}{\mu\gamma_{mf}^2}\Bigg\{ & \gamma_{mf}\frac{SinFJ_0(G)}{b_0}\left[Cos(\gamma_{mf}-b_0)t - Cos\gamma_{mf}t\right] \\ & + \gamma_{mf}SinF\sum_{k=1}^{\infty} J_{2k}(G)\left[\frac{Cos(\gamma_{mf}-b_1)t - Cos\gamma_{mf}t}{b_1} + \frac{Cos(\gamma_{mf}-b_2)t - Cos\gamma_{mf}t}{b_2}\right] \\ & + CosF\sum_{k=0}^{\infty} J_{2k+1}(G)\Bigg[\frac{\gamma_{mf}Sin(\gamma_{mf}-b_4)t - (\gamma_{mf}-b_4)Sin\gamma_{mf}t}{b_4} \\ & - \frac{\gamma_{mf}Sin(\gamma_{mf}-b_3)t - (\gamma_{mf}-b_3)Sin\gamma_{mf}t}{b_3}\Bigg]\Bigg\} \times Sin\frac{m\pi x}{L} \end{aligned} \tag{3.10}$$

Equation (3.10) represents the transverse-displacement response to a moving force moving at a variable velocity of a simply supported uniform Bernoulli-Euler beam resting on elastic foundation. We consider next, the moving mass problem that is when $\varepsilon_0 \neq 0$ is considered. Following arguments used in the previous sections, the modified frequency corresponding to the frequency of the free system due to the presence of the moving mass of this model is obtained as

$$\gamma_{mm} = \gamma_{mf}\left\{1-\frac{\lambda}{2}\left[\left(2-J_0(G)Cos2F\right)-\frac{\left(2R_m-R_mJ_0(G)Cos2F\right)}{2\gamma_{mf}^2}\right]\right\} \qquad (3.11)$$

where

$$R_m = \left(\frac{\gamma\beta m\pi}{L}\right)^2 \qquad (3.12)$$

neglecting higher order terms of λ. Thus, the moving mass problem reduces to

$$\frac{d^2\overline{V}(m,t)}{dt^2}+\gamma_{mm}^2\overline{V}(m,t)=\varepsilon_0 Lg\left[SinFCos(GSin\beta t)+CosFSin(GSin\beta t)\right] \qquad (3.13)$$

which when solved in conjunction with the initial conditions yields expression for $\overline{V}(m,t)$ and on inversion becomes

$$\begin{aligned}
V(x,t)=\frac{2}{L}\sum_{m=1}^{\infty}\frac{\varepsilon_0 Lg}{\gamma_{mm}^2}\Bigg\{&\gamma_{mm}\frac{SinFJ_0(G)}{b_0}\left[Cos(\gamma_{mm}-b_0)t-Cos\gamma_{mm}t\right]\\
&+\gamma_{mm}SinF\sum_{k=1}^{\infty}J_{2k}(G)\left[\frac{Cos(\gamma_{mm}-b_1)t-Cos\gamma_{mm}t}{b_1}+\frac{Cos(\gamma_{mm}-b_2)t-Cos\gamma_{mm}t}{b_2}\right]\\
&+CosF\sum_{k=0}^{\infty}J_{2k+1}(G)\Bigg[\frac{\gamma_{mm}Sin(\gamma_{mm}-b_4)t-(\gamma_{mm}-b_4)Sin\gamma_{mm}t}{b_4}\\
&-\frac{\gamma_{mm}Sin(\gamma_{mm}-b_5)t-(\gamma_{mm}-b_5)Sin\gamma_{mm}t}{b_5}\Bigg]\Bigg\}\times Sin\frac{m\pi x}{L}
\end{aligned}$$

(3.14)

This represents the transverse-displacement response to a concentrated mass moving with variable velocity of simply supported uniform Bernoulli-Euler beam resting on elastic foundation.

3.1.2. Clamped/Fixed Ends Condition

At a clamped-clamped ends, both deflection and slope vanish. Thus, when the Long thin beam is clamped at x=0 and x=L, the conditions are expressed as

$$V(0,t) = 0 = V(L,t) \quad \text{and} \quad \frac{\partial V(0,t)}{\partial x} = 0 = \frac{\partial V(L,t)}{\partial x} \qquad (3.15)$$

And for normal modes

$$U_m(0) = 0 = U_m(L) \quad \text{and} \quad \frac{\partial U_m(0)}{\partial x} = 0 = \frac{\partial U_m(L)}{\partial x} \qquad (3.16)$$

which implies that

$$U_k(0) = 0 = U_k(L) \quad \text{and} \quad \frac{\partial U_k(0)}{\partial x} = 0 = \frac{\partial U_k(L)}{\partial x} \qquad (3.17)$$

Thus, it can be shown that

$$A_m = \frac{Sinh\lambda_m - Sin\lambda_m}{Cos\lambda_m - Cosh\lambda_m} = \frac{Cos\lambda_m - Cosh\lambda_m}{Sin\lambda_m + Sinh\lambda_m} = -C_m \quad \text{and} \quad B_m = -1 \qquad (3.18)$$

In view of (3.18), the frequency equation is given as

$$Cos\lambda_m Cosh\lambda_m = 1 \qquad (3.19)$$

It follows from equation (3.19), that

$$\lambda_1 = 4.73004\,, \ \lambda_2 = 7.85320\,, \ \lambda_3 = 10.99561 \qquad (3.20)$$

Substituting (3.18) and (3.20) into equations (2.149) and (2.178)one obtains the displacement response respectively to a moving force and a moving mass of a clamped/fixed long thing elastic beam resting on elastic foundation.

3.1.3. One End Clamped One End Free Condition

At end x=0, the beam is taking to be clamped and at the end x=L, the beam is free. Thus, the boundary conditions of the Bernoulli-Euler beam can be written as

$$V(0,t) = 0 = \frac{\partial V(0,t)}{\partial x} \quad \text{and} \quad \frac{\partial^2 V(L,t)}{\partial x^2} = 0 = \frac{\partial^3 V(L,t)}{\partial x^3} \tag{3.21}$$

And for normal modes

$$U_m(0) = 0 = \frac{dU_m(0)}{dx} \quad \text{and} \quad \frac{d^2 U_m(L)}{dx^2} = 0 = \frac{d^3 U_m(L)}{dx^3} \tag{3.22}$$

which implies that

$$U_k(0) = 0 = \frac{dU_k(0)}{dx} \quad \text{and} \quad \frac{d^2 U_k(L)}{dx^2} = 0 = \frac{d^3 U_k(L)}{dx^3} \tag{3.23}$$

Using (3.23), we can show that at x=0,

$$A_m = C_m \qquad \text{and} \qquad B_m = -1 \tag{3.24}$$

and at end x=L, using (3.24)

$$A_m = \frac{-Sin\lambda_m - Sinh\lambda_m}{Cos\lambda_m + Cosh\lambda_m} = \frac{-Cos\lambda_m - Cosh\lambda_m}{Sinh\lambda_m - Sin\lambda_m} = -C_m \qquad \text{and} \qquad B_m = -1 \tag{3.25}$$

and the frequency equation for both end conditions is

$$Cos\lambda_m Cosh\lambda_m = -1 \tag{3.26}$$

and we have that

$$\lambda_1 = 1.875,\ \lambda_2 = 4.694,\ \lambda_3 = 7.855 \tag{3.27}$$

Using (3.24), (3.25) and (3.27) in equations (2.149) and (2.178), one obtains the displacement response respectively to a moving force and a moving mass of a uniform clamped-free ends of Bernoulli-Euler beam resting on elastic foundation.

3.2.0 DISCUSSION OF THE ANALYTICAL SOLUTIONS

When an undamped system such as this is studied, one is interested in the resonance conditions of the system, because the transverse displacement of an elastic beam may increase without bound. Equation (3.10) clearly shows that the Simply Supported elastic beam resting on elastic foundation and traversed by moving force reaches a state of resonance whenever

$$\gamma_{mf} = 2k\beta \quad \text{and} \quad \gamma_{mf} = (2k+1)\beta \tag{3.28}$$

while equation (3.14) indicates that the same beam under the action of a moving mass experiences resonance effect when

$$\gamma_{mm} = 2k\beta \quad \text{and} \quad \gamma_{mm} = (2k+1)\beta \tag{3.29}$$

From equation (3.11),

$$\gamma_{mm} = \gamma_{mf}\left\{1 - \frac{\lambda}{2}\left[(2 - J_0(2G)Cos2F) + \frac{(2R_m - R_m J_0(G)Cos2F)}{2\gamma_{mf}^2}\right]\right\} \tag{3.30}$$

which implies

$$\gamma_{M} = \frac{2k\beta}{1 - \frac{\lambda}{2}\left[(2 - J_{0}(2G)Cos2F) + \frac{(2R_{m} - R_{m}J_{0}(G)Cos2F)}{2\gamma_{mf}^{2}}\right]} \qquad (3.31)$$

It is therefore evident, that for the same natural frequency, the critical speed for the system consisting of a Simply Supported elastic beam resting on an elastic foundation and traversed by a force moving with a non-uniform speed is greater than that of the moving mass problem. Thus, for the same natural frequency of an elastic beam, resonance is reached earlier in the moving mass system than in the moving force system.

For the resonance conditions for other classical boundary conditions, equation (2.149) clearly shows that the uniform elastic beam resting on an elastic foundation and traversed by a force moving with variable speed reaches a state of resonance whenever

$$\gamma_{sj} = 2k\beta \qquad \text{and} \qquad \gamma_{sj} = (2k+1)\beta \qquad (3.32)$$

while equation (2.178) shows that the same beam under the action of a moving mass experiences resonance effect whenever

$$\gamma_{hj} = 2k\beta \qquad \text{and} \qquad \gamma_{hj} = (2k+1)\beta \qquad (3.33)$$

From equation (2.174)

$$\gamma_{h} = \gamma_{sj}\left\{1 - \frac{\lambda}{2}\left[(H_{k}(m,m) + R_{1}(m,m,n)) + \frac{(\gamma\beta)^{2}(H_{j}(m,m) + 2R_{2}(m,m,n))}{2\gamma_{sj}^{2}}\right]\right\} \qquad (3.34)$$

which implies

$$\gamma_a = \frac{2k\beta}{1-\frac{\lambda}{2}\left[\left(H_A(m,m)+R_1(m,m,n)\right)+\frac{(\gamma\beta)^2\left(H_f(m,m)+2R_2(m,m,n)\right)}{2\gamma_a^2}\right]} \tag{3.35}$$

Evidently, from equation (3.33) and (3.35), the same results and analysis obtained in the case of a Simply Supported Bernoulli-Euler beam are obtained for all other examples of classical boundary conditions.

3.3.0 NUMERICAL CALCULATIONS AND DISCUSSION OF RESULTS

In order to illustrate the forgoing analysis, the uniform beam of length 12.192m is considered. Furthermore, $\frac{EI}{\mu} = 2200m^4/s^2$, $\gamma = 2\times10^{-4}m$, $\beta = \frac{3\pi}{4}$, $\varepsilon_0 = \frac{1}{20}$ and the ratio of the mass of the load to the mass of the beam is 0.25. The transverse deflections of the beam are calculated and plotted against time for various values of axial force N and subgrade K. Values of N between 0 and 20,000,000 were used while the values of K were varied between 0 N/m^3 and 400,000 N/m^3. The results are as shown on the various graphs below for the arious classical of boundary conditions considered.

.3.1 Simply Supported Ends

igure 3.1 displays transverse displacement response of a simply supported niform beam under the action of forces moving at variable velocities for various values of axial force N for fixed value of foundation moduli K = 40,000. The figure shows that as N increases the deflection of the of the uniform beam decreases. The same results is obtained when the simply supported beam is traversed by a concentrated masses moving at variable speed as shown in figure 3.3. Also, for various time t, the deflection profile of the beam for various values of foundation moduli K and for fixed axial force N are shown in figure 3.2. It is shown that higher values of foundation moduli reduce the deflection profile of the beam. The same behaviour characterizes the deflection profile of the simply supported beam under the action of concentrated masses moving at variable velocities for various values of foundation

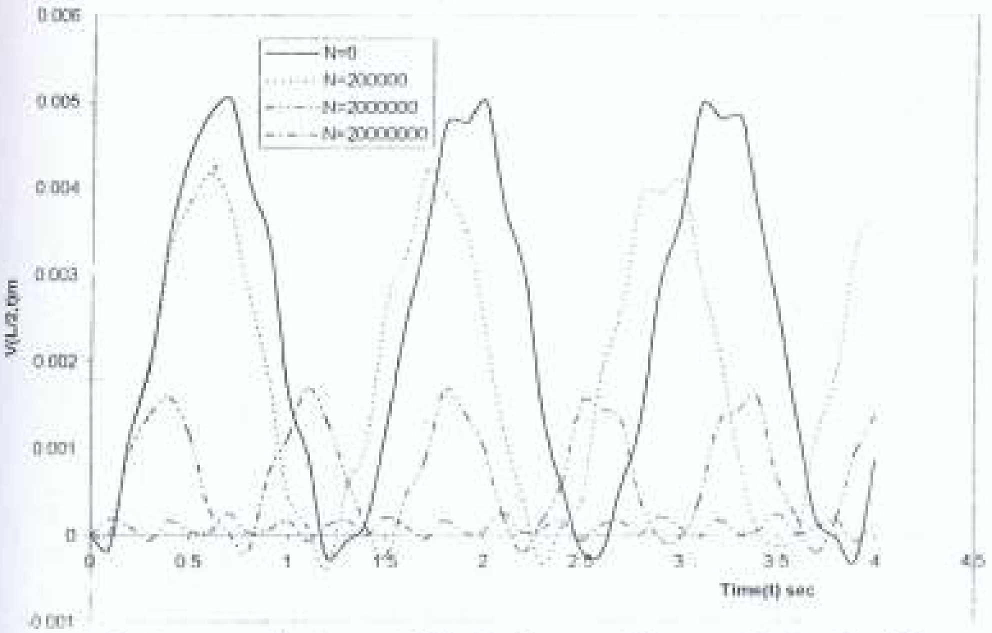

Fig 3.1: Transverse displacement of the simply supported beam under the action of forces moving at variable velocities for various values of axial force N for fixed value of foundation moduli K (40000)

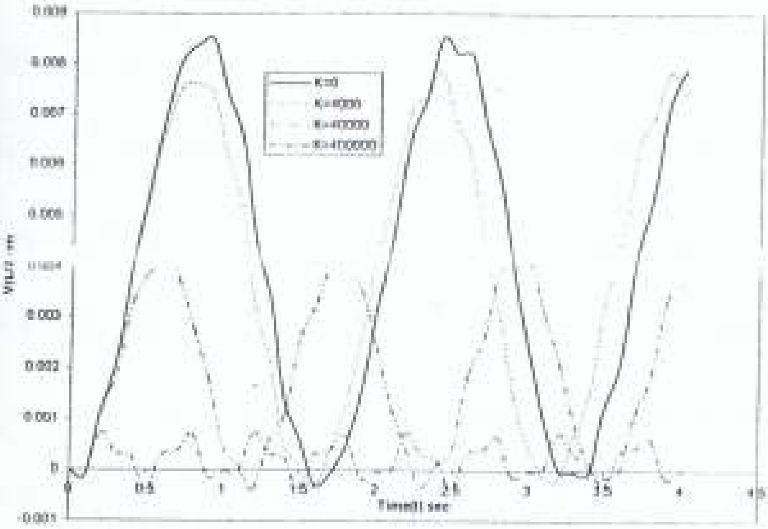

Fig 3.2 : Deflection profile of the simply supported beam under the action of forces moving at variable velocities for various values of foundation moduli K for fixed N (200000)

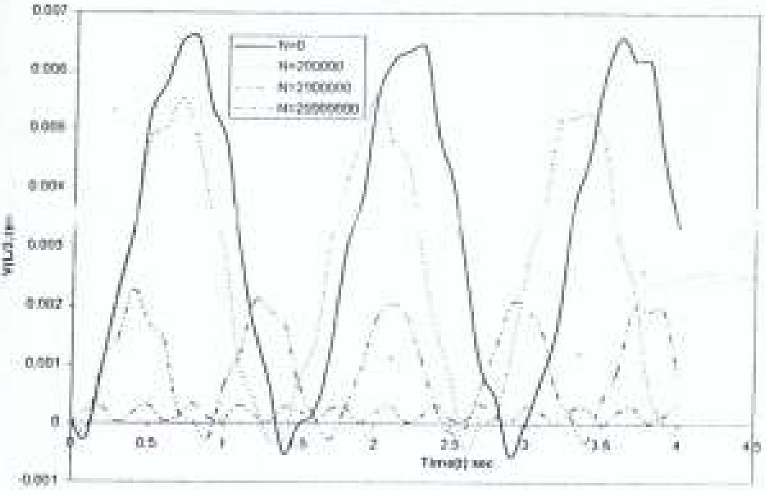

Fig 3.3 Transverse displacement of the simply supported beam under the action concentrated masses moving at variable velocities for various values of axial force N for fixed value of K (40000)

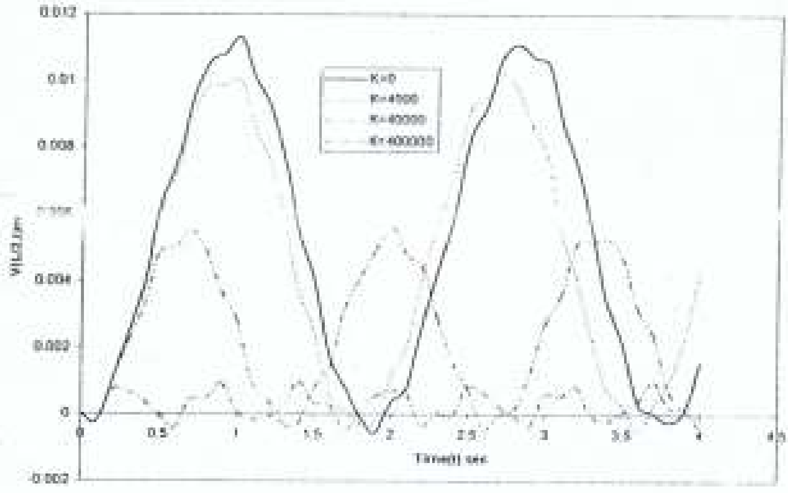

Fig 3.4 Deflection profile of the simply supported beam under the action of concentrated masses moving at variable velocities for various values of foundation moduli K for fixed N (200000)

moduli as shown in figure 3.4.

Finally, figure 3.5 depicts the comparison of the transverse displacement of moving force and moving mass cases for simply supported uniform beam traversed by a moving load moving at variable velocities for N = 200,000 and K = 40,000. Clearly, the response amplitudes of moving mass is higher than that of the moving force. This important result agrees with the existing result for cases when the traveling load is moving at constant velocities.

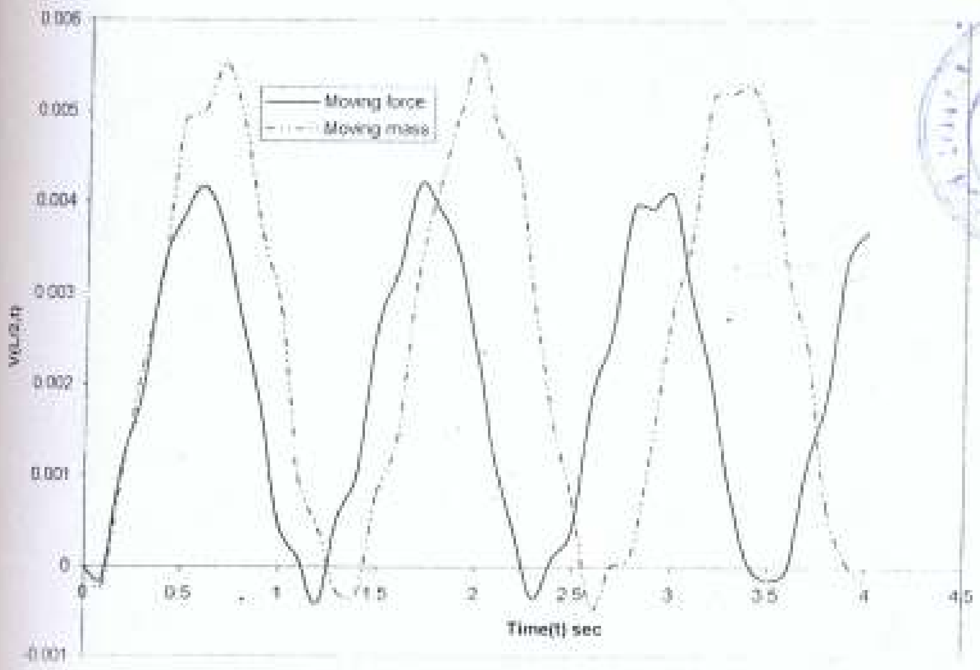

Fig 3.5 : Comparison of the displacement response of moving force and moving mass cases for simply supported beam for N=200000 and k=40000

.3.2 Clamped Ends

Figure 3.6 displays transverse displacement response of a clamped-lamped uniform beam under the action of forces moving at variable velocities r various values of axial force N for fixed value of foundation moduli

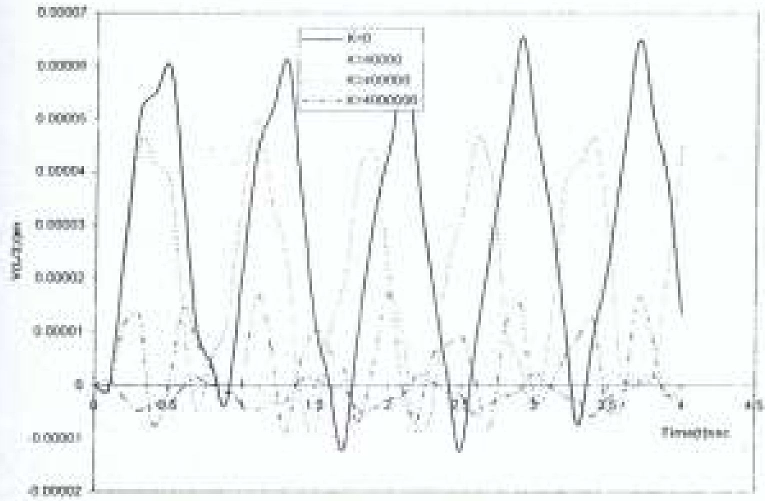

Fig 3.7: Deflection profile of the clamped-clamped uniform beam under the action of forces moving at variable velocities for various values of foundation moduli K for fixed N (200000)

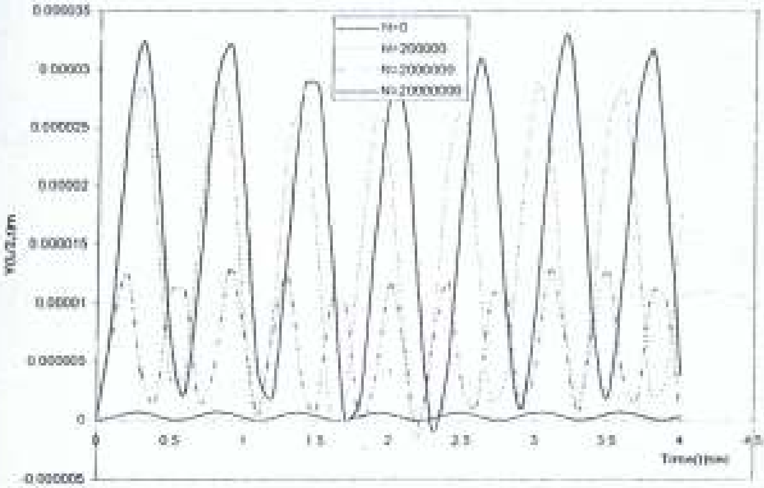

Fig 3.8: Transverse displacement of the clamped-clamped uniform beam under the action of concentrated masses moving at variable velocities for various values of axial force N for fixed foundation moduli k (40000)

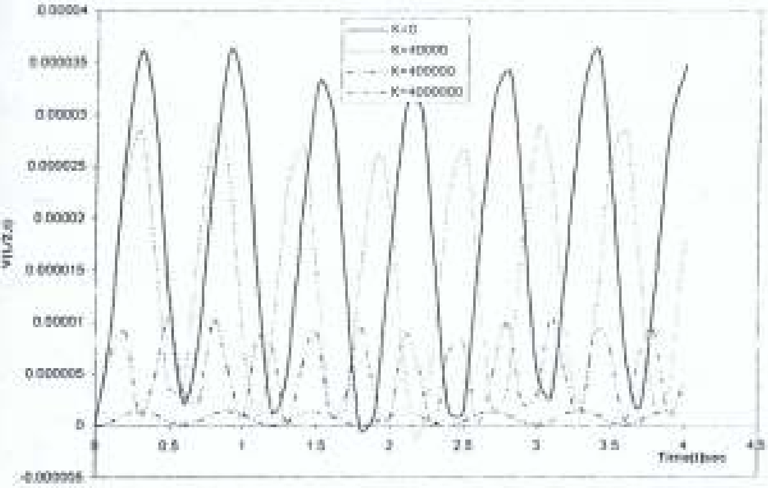

Fig 3.9: Deflection profile of the clamped-clamped uniform beam under the action of concentrated masses moving at variable velocities for various values of foundation moduli for fixed N (200000)

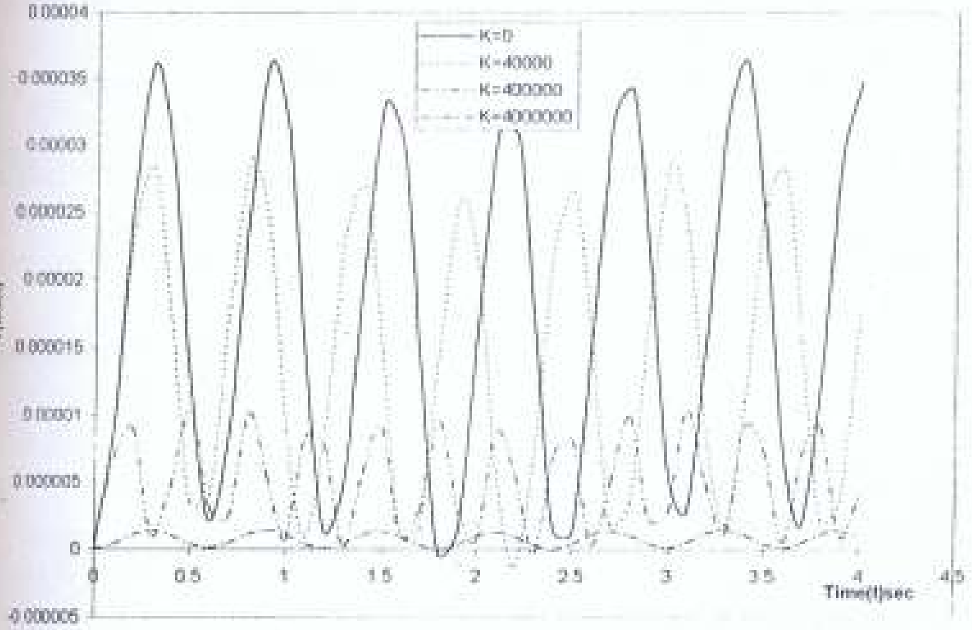

Fig 3.9: Deflection profile of the clamped-clamped uniform beam under the action of concentrated masses moving at variable velocities for various values of foundation moduli for fixed N (200000)

K = 40,000. The figure shows that as N increases the deflection of the uniform beam decreases. The same results is obtained when the simply supported beam is traversed by a concentrated mass moving at variable speed as shown in figure 3.8. Also, for various time t, the deflection profile of the beam for various values of foundation moduli K and for fixed axial force N are shown in figure 3.7. It is shown that higher values of foundation moduli reduce the deflection profile of the beam. The same behaviour characterizes the deflection profile of the clamped-clamped beam under the action of concentrated masses moving at variable velocities for various values of foundation moduli as shown in figure 3.9.

Finally, figure 3.10 depicts the comparison of the transverse displacement of moving force and moving mass cases for clamped-clamped uniform beam traversed by a moving load moving at variable velocity for fixed N = 200,000 and K = 40,000. Clearly, the response amplitudes of moving mass is higher than that of the moving force. This confirms the result in [17,23,24] that relying on the moving force problem as a good approximation to a moving mass problem is not only misleading but is tragic.

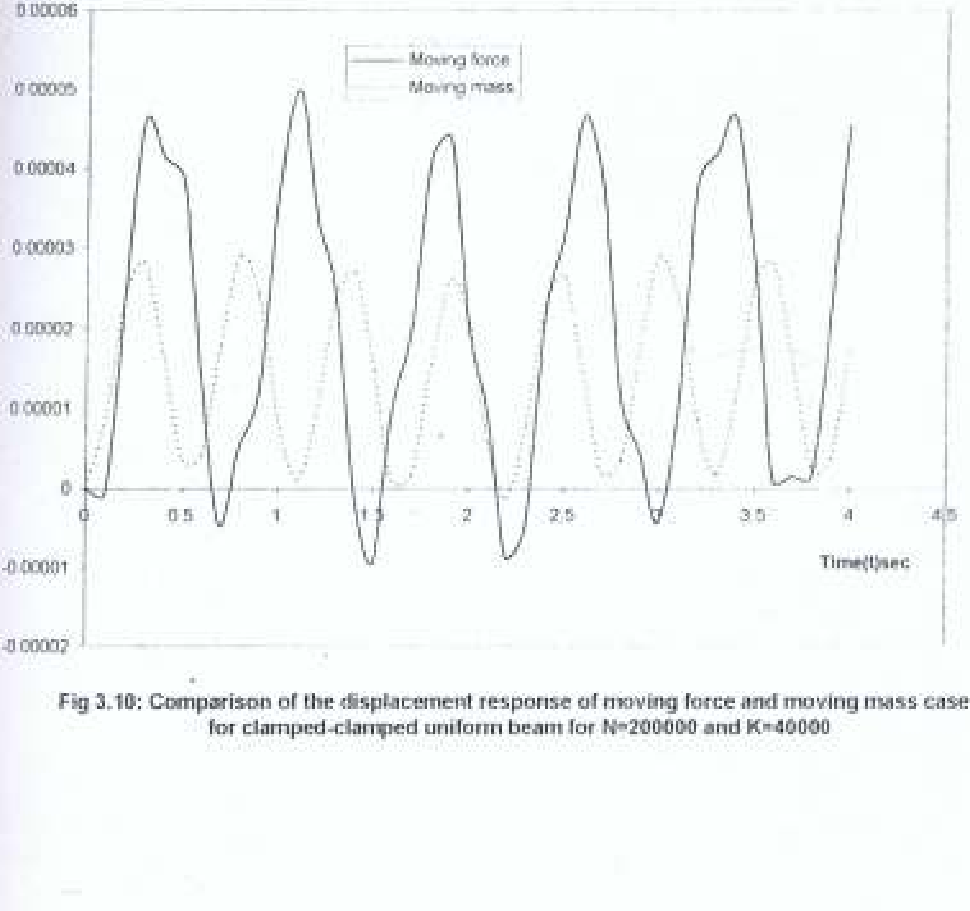

Fig 3.10: Comparison of the displacement response of moving force and moving mass cases for clamped-clamped uniform beam for N=200000 and K=40000

3.3.3 One End Clamped and One End Free.

Figure 3.11 displays transverse displacement response of a Cantilever uniform beam under the action of forces moving at variable velocities for various values of axial force N for fixed value of foundation modulus K = 40,000. The figure shows that as N increases the deflection of the of the uniform beam decreases.

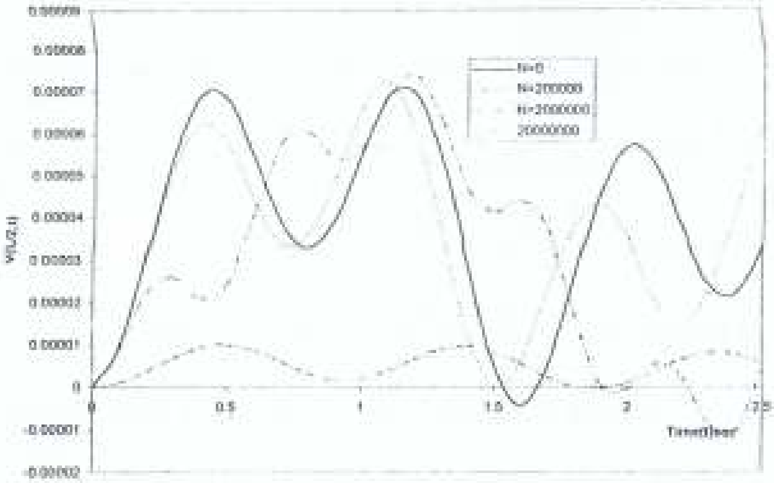

Fig 3.11: Transverse displacement of the clamped-free uniform beam under the action of forces moving at variable velocities for various values of axial force N for fixed foundation moduli K (40000)

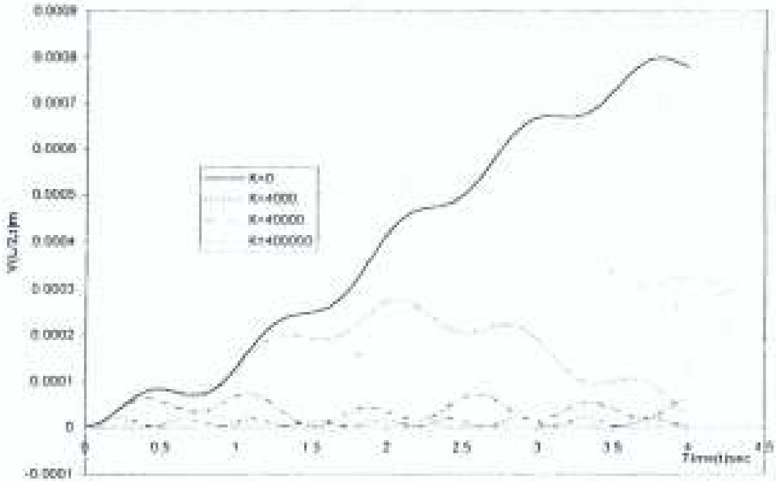

Fig 3.12: Deflection profile of the clamped-free uniform beam under the action of forces moving at variable velocities for various values of foundation moduli for fixed value of axial force N (200000)

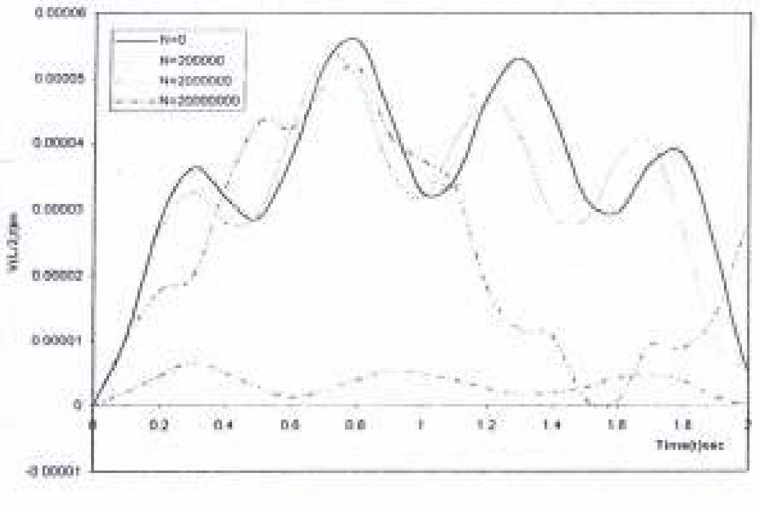

Fig 3.13: Transverse displacement of the clamped-free uniform beam under the action of concentrated masses moving at variable velocities for various values of axial force N for fixed foundation moduli K (40000)

The same results is obtained when the cantilever beam is traversed by a concentrated masses moving at variable speed as shown in figure 3.13. Also, for various time t, the deflection profiles of the beam for various values of foundation moduli K and for fixed axial force N are shown in figure 3.12. It is shown that as foundation modulus increases the deflection profile of the beam decreases. The same behaviour characterizes the deflection profile of the cantilever beam under the action of concentrated masses moving at variable velocities for various values of foundation moduli as shown in figure 3.14. Finally, figure 3.15 depicts the comparison of the transverse displacement of moving force and moving mass cases for cantilever uniform beam traversed by a moving load

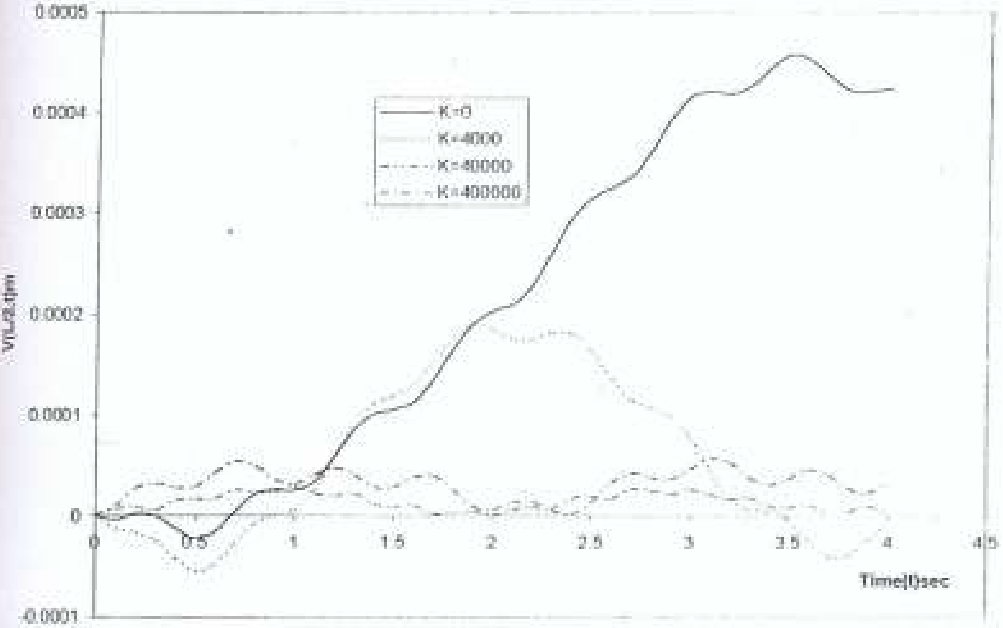

Fig 3.14: Deflection profile of the clamped-free uniform beam under the action of concentrated masses moving at variable velocities for various values of foundation moduli K for fixed value of axial force N (200000)

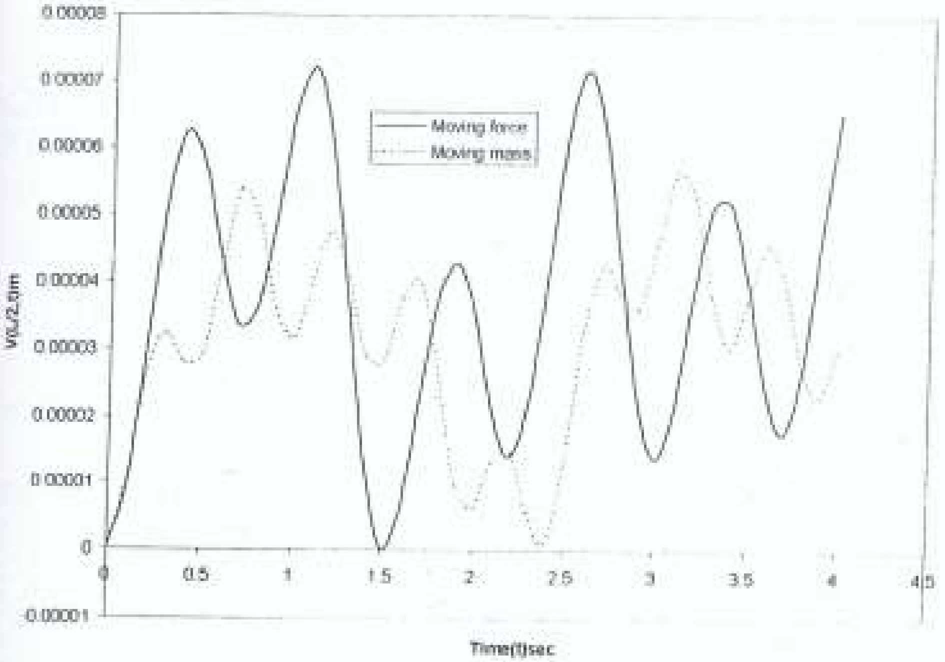

Fig 3.15: Comparison of the displacement response of moving force and moving mass cases for clamped-free uniform beam for axial force N=200000 and foundation moduli K=40000

moving at variable velocities for N = 200,000 and K = 40,000. Clearly, the response amplitude of moving mass is higher than that of the moving force.

CHAPTER FOUR

NON-PRISMATIC BERNOULLI-EULER BEAM RESTING ON ELASTIC FOUNDATION AND UNDER THE ACTIONS OF MOVING CONCENTRATED MASSES.

4.0 INTRODUCTION

In the previous chapter, the problem of the flexural motion of a uniform Bernoulli-Euler beam resting on an elastic foundation and traversed by masses moving with non-uniform speed was investigated. In the Mathematical model, the beam properties do not vary along the span L of the beam. However, in many practical problems involving dynamics of structures (beams or plates) under moving loads, the structures have variable cross-sections [28]. Thus, the problem of a non-uniform Bernoulli-Euler beam under the action of loads moving with variable speeds is considered in this chapter.

Unlike in the previous section, to tackle this problem the method of the Generalized finite integral transform is inapplicable and we resort to a modification of an approximate method best suited for solving diverse problems in dynamics of structures generally referred to as Galerkin's method. This we term Generalized Galerkin's Method GGM. This method is employed to simplify the governing fourth order partial differential equation with singular and variable coefficients. The resulting Galerkin's equations are solved via the modified Struble's asymptotic technique already alluded to in the previous chapter.

4.1 THE GOVERNING EQUATION

The problem of the flexural vibration of a non-uniform Bernoulli-Euler beam resting on elastic foundation and traversed by loads moving with variable speeds is investigated in this chapter. The beam's properties such as moment of inertia I and the mass per unit length of the beam μ vary along the span L of the beam. The transverse displacement of the beam when it is under the action of a moving load of mass M which is moving with a non-uniform velocity $f(t)$ is governed by the fourth order partial differential equation given by

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 V(x,t)}{\partial x^2}\right]+\mu(x)\frac{\partial^2 V(x,t)}{\partial t^2}-N\frac{\partial^2 V(x,t)}{\partial x^2}+K(x)V(x,t)+M\delta[x-(x_0+\gamma Sin\beta t)]\left[\frac{\partial^2 V(x,t)}{\partial t^2}\right.$$
$$\left.+2\frac{df(t)}{dt}\frac{\partial^2 V(x,t)}{\partial x\partial t}+\left(\frac{df(t)}{dt}\right)\frac{\partial^2 V(x,t)}{\partial x^2}+\frac{d^2 f(t)}{dt^2}\frac{\partial V(x,t)}{\partial x}\right]=P\delta[x-(x_0-\gamma Sin\beta t)] \tag{4.1}$$

where all parameters are as defined in the previous chapter.

As in the previous section, we shall take the elastic foundation to be of the form (2.10) and the velocity $f(t)$ of the moving loads to be of the form (2.5). We adopt the example in [28] and take $I(x)$ and $\mu(x)$ to be of the form

$$I(x)=I_0\left(1+Sin\frac{\pi x}{L}\right)^3 \tag{4.2}$$

and

$$\mu(x)=\mu_0\left(1+Sin\frac{\pi x}{L}\right) \tag{4.3}$$

Substituting equations (4.2) and (4.3) into equation (4.1) one obtains

$$\frac{1}{4}EI_0\frac{\partial^2}{\partial x^2}\left[\left(10+15Sin\frac{\pi x}{L}-6Cos\frac{2\pi x}{L}-Sin\frac{3\pi x}{L}\right)\frac{\partial^2 V(x,t)}{\partial x^2}\right]+\mu_0\left(1+Sin\frac{\pi x}{L}\right)\frac{\partial^2 V(x,t)}{\partial x^2}-N\frac{\partial^2 V(x,t)}{\partial x^2}$$
$$+KV(x,t)+M\delta[x-(x_0+\gamma Sin\beta t)]\left[\frac{\partial^2 V(x,t)}{\partial t^2}+2\gamma\beta Cos\beta t\frac{\partial^2 V(x,t)}{\partial x\partial t}+(\gamma\beta Cos\beta t)^2\frac{\partial^2 V(x,t)}{\partial x^2}\right.$$
$$\left.-\gamma\beta^2 Sin\beta t\frac{\partial V(x,t)}{\partial x}\right]=P\delta[x-(x_0+\gamma Sin\beta t)]$$

(4.4)

4.2.0 OPERATIONAL SIMPLIFICATION

It is evident that an exact closed form solution of the partial differential equation (4.4) is impossible. Thus, we generalize the Galerkin's method described in Oni and Awodola [29] and make use of this to reduce the partial differential equation to a sequence of ordinary differential equations. Thus, a solution of the form

$$V_n(x,t)=\sum_{m=1}^{n}Y_m(t)U_m(t) \qquad (4.5)$$

is sought where $U_m(x)$ is chosen such that pertinent boundary conditions are satisfied. Equation (4.5)) when substituted into the equation (4.4) yields

$$\sum_{m=1}^{n}\left\{\frac{1}{4}EI_0\frac{\partial^2}{\partial x^2}\left[10U''_m(x)+15Sin\frac{\pi x}{L}U''_m(x)-6Cos\frac{2\pi x}{L}U''_m(x)-Sin\frac{3\pi x}{L}U''_m(x)\right]Y_m(t)-NU''_m(x)Y_m(t)\right.$$
$$+KU_m(x)Y_m(t)+\mu_0\left[U_m(x)+U_m(x)Sin\frac{\pi x}{L}\right]\ddot{Y}_m(t)+M\delta[x-(x_0+\gamma Sin\beta t)][U_m(x)\ddot{Y}_m(t)+$$
$$\left.+2\gamma\beta Cos\beta tU'_m(x)\dot{Y}_m(t)+(\gamma\beta Cos\beta t)^2U''_m(x)Y_m(t)-\gamma\beta^2 Sin\beta tU'_m(x)Y_m(t)\ ]\right\}-P\delta[x-(x_0+\gamma Sin\beta t)]=0$$

(4.6)

A further simplification and rearrangement lead to

$$\sum_{m=1}^{\infty}\left\{\frac{1}{4}EI_0\frac{\partial^2}{\partial x^2}\left[\left(10+15Sin\frac{\pi x}{L}-6Cos\frac{2\pi x}{L}-Sin\frac{3\pi x}{L}\right)U_m'''(x)+\left(\frac{30\pi}{L}Cos\frac{\pi x}{L}+\frac{24\pi}{L}Sin\frac{2\pi x}{L}\right.\right.\right.$$

$$\left.-\frac{6\pi}{L}Cos\frac{3\pi x}{L}\right)U_m'''(x)+\left(\frac{9\pi^2}{L^2}Sin\frac{3\pi x}{L}+\frac{24\pi^2}{L^2}Cos\frac{2\pi x}{L}-\frac{15\pi^2}{L^2}Sin\frac{\pi x}{L}\right)U_m''(x)\Bigg]Y_m(t)-NU_m''(x)Y_m(t)$$

$$+K'U_m(x)Y_m(t)+\mu_0\left[U_m(x)+U_m(x)Sin\frac{\pi x}{L}\right]\ddot{Y}_m(t)+M\delta[x-(x_0+\gamma Sin\beta t)]\Big[U_m(x)\ddot{Y}_m(t)$$

$$\left.+2\gamma\beta Cos\beta t U_m'(x)\dot{Y}_m(t)+(\gamma\beta Cos\beta t)^2U_m''(x)Y_m(t)-\gamma\beta^2Sin\beta t U_m'(x)Y_m(t)\Big]-\right\}-P\delta[x-(x_0+\gamma Sin\beta t)]=0$$

(4.7)

n other to determine an expression for $Y_m(t)$, it is required that the expression on he left hand side of equation (4.7) be orthogonal to the function $U_k(x)$. Thus,

$$\int\left(\sum_{m=1}^{\infty}\left\{\frac{1}{4}EI_0\frac{\partial^2}{\partial x^2}\left[\left(10+15Sin\frac{\pi x}{L}-6Cos\frac{2\pi x}{L}-Sin\frac{3\pi x}{L}\right)U_m'''(x)+\left(\frac{30\pi}{L}Cos\frac{\pi x}{L}+\frac{24\pi}{L}Sin\frac{2\pi x}{L}\right.\right.\right.\right.$$

$$\left.-\frac{6\pi}{L}Cos\frac{3\pi x}{L}\right)U_m'''(x)+\left(\frac{9\pi^2}{L^2}Sin\frac{3\pi x}{L}+\frac{24\pi^2}{L^2}Cos\frac{2\pi x}{L}-\frac{15\pi^2}{L^2}Sin\frac{\pi x}{L}\right)U_m''(x)\Bigg]Y_m(t)-NU_m''(x)Y_m(t)$$

$$+K'U_m(x)Y_m(t)+\mu_0\left[U_m(x)+U_m(x)Sin\frac{\pi x}{L}\right]\ddot{Y}_m(t)+M\delta[x-(x_0+\gamma Sin\beta t)]\Big[U_m(x)\ddot{Y}_m(t)$$

$$\left.2\gamma\beta Cos\beta t U_m'(x)\dot{Y}_m(t)+(\gamma\beta Cos\beta t)^2U_m''(x)Y_m(t)-\gamma\beta^2Sin\beta t U_m'(x)Y_m(t)\Big]-\right\}$$

$$\left.P\delta[x-(x_0+\gamma Sin\beta t)]-\right)\times U_k(x)dx=0$$

(4.8)

e note that the Dirac Delta function as an even function can be expressed as

$$x-(x_0+\gamma Sin\beta t)]=\frac{1}{L}+\frac{2}{L}\sum_{n=1}^{\infty}Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)Cos\frac{n\pi x}{L} \qquad (4.9)$$

sing (4.9) in (4.8) one obtains

$$\int_0^L\left(\sum_{m=1}^{\infty}\left\{\frac{1}{4}EI_0\frac{\partial^2}{\partial x^2}\left[\left(10+15Sin\frac{\pi x}{L}-6Cos\frac{2\pi x}{L}-Sin\frac{3\pi x}{L}\right)U_m^{iv}(x)U_k(x)+\left(\frac{30\pi}{L}Cos\frac{\pi x}{L}+\frac{24\pi}{L}Sin\frac{2\pi x}{L}\right.\right.\right.\right.$$

$$\left.-\frac{6\pi}{L}Cos\frac{3\pi x}{L}\right)U_m'''(x)U_k(x)+\left(\frac{9\pi^2}{L^2}Sin\frac{3\pi x}{L}+\frac{24\pi^2}{L^2}Cos\frac{2\pi x}{L}-\frac{15\pi^2}{L^2}Sin\frac{\pi x}{L}\right)U_m''(x)U_k(x)\Bigg]Y_m(t)$$

$$-NU_m''(x)U_k(x)Y_m(t)+K^0U_m(x)U_k(x)Y_m(t)+\mu_0\left[U_m(x)U_k(x)+U_m(x)U_k(x)Sin\frac{\pi x}{L}\right]\ddot{Y}_m(t)$$

$$+M\delta[x-(x_0+\gamma Sin\beta t)]\left[U_m(x)U_k(x)\ddot{Y}_m(t)+2\gamma\beta Cos\beta t U_m'(x)U_k(x)\dot{Y}_m(t)+(\gamma\beta Cos\beta t)^2U_m''(x)U_k(x)Y_m(t)\right.$$

$$\left.\left.-\gamma\beta^2Sin\beta t U_m'(x)U_k(x)Y_m(t)\right]-\right\}-P\delta[x-(x_0+\gamma Sin\beta t)]U_k(x)-\Bigg)dx=0$$

$$(4.10)$$

Simplifying and rearranging (4.10) further, we have

$$\sum_{m=1}^{\infty}\left\{\frac{1}{\mu_0}\left[\frac{1}{4}EI_0(Q_0+Q_1+Q_2)-Q_3+Q_4\right]Y_m(t)+[Q_5+Q_6]\ddot{Y}_m(t)\right.$$

$$+\frac{M}{\mu_0L}\left(H_b^0(k,m)\ddot{Y}_m(t)+2\sum_{n=1}^{\infty}Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)H_c^n(k,m,n)\ddot{Y}_m(t)\right.$$

$$+2\gamma\beta Cos\beta t H_d^0(k,m)\dot{Y}_m(t)+4\gamma\beta Cos\beta t\sum_{n=1}^{\infty}Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)H_e^n(k,m,n)\dot{Y}_m(t)$$

$$+(\gamma\beta Cos\beta t)^2H_f^0(k,m)Y_m(t)+2(\gamma\beta Cos\beta t)^2\sum_{n=1}^{\infty}Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)H_g^n(k,m,n)Y_m(t)$$

$$\left.\left.-\gamma\beta^2Sin\beta t H_i^0(k,m)Y_m(t)-2\gamma\beta^2Sin\beta t\sum_{n=1}^{\infty}Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)H_j^n(k,m,n)Y_m(t)\right)\right\}$$

$$=\frac{P}{\mu_0}U_k(x-(x_0+\gamma Sin\beta t))$$

$$(4.11)$$

where

$Q_0=Z_a+Z_b-Z_c-Z_d$, $\quad Q_1=I_a+I_b-I_c$, $\quad Q_2=N_a+N_b-N_c$

$Z_a=10\int_0^L U_m^{iv}(x)U_k(x)dx$, $\quad Z_b=15\int_0^L Sin\frac{\pi x}{L}U_m^{iv}(x)U_k(x)dx$

$Z_c=6\int_0^L Cos\frac{2\pi x}{L}U_m^{iv}(x)U_k(x)dx$, $\quad Z_d=\int_0^L Sin\frac{3\pi x}{L}U_m^{iv}(x)U_k(x)dx$

$$L_b=\frac{30\pi}{L}\int_0^L Cos\frac{\pi x}{L}U_m'''(x)U_k(x)dx\,,\qquad L_a=\frac{24\pi}{L}\int_0^L \sin\frac{2\pi x}{L}U_m'''(x)U_k(x)dx$$

$$L_c=\frac{6\pi}{L}\int_0^L Cos\frac{3\pi x}{L}U_m'''(x)U_k(x)dx\,,\qquad N_a=\frac{9\pi^2}{L^2}\int_0^L Sin\frac{3\pi x}{L}U_m''(x)U_k(x)dx$$

$$N_b=\frac{24\pi^2}{L^2}\int_0^L Cos\frac{2\pi x}{L}U_m''(x)U_k(x)dx\,,\qquad N_a=\frac{15\pi^2}{L^2}\int_0^L Sin\frac{\pi x}{L}U_m''(x)U_k(x)dx$$

$$Q_3=N\int_0^L U_m''(x)U_k(x)dx\,,\qquad Q_4=K''\int_0^L U_m(x)U_k(x)dx$$

$$Q_5=\int_0^L U_m(x)U_k(x)dx\,,\qquad Q_6=\int_0^L Sin\frac{\pi x}{L}U_m(x)U_k(x)dx$$

$$H_b^0(k,m)=\int_0^L U_m(x)U_k(x)dx\,,\qquad H_c^0(k,m,n)=\int_0^L Cos\frac{n\pi x}{L}U_m(x)U_k(x)dx$$

$$H_d^0(k,m)=\int_0^L U_m'(x)U_k(x)dx\,,\qquad H_e^0(k,m,n)=\int_0^L Cos\frac{n\pi x}{L}U_m'(x)U_k(x)dx$$

$$H_f^0(k,m)=\int_0^L U_m''(x)U_k(x)dx\,,\qquad H_g^0(k,m,n)=\int_0^L Cos\frac{n\pi x}{L}U_m''(x)U_k(x)dx$$

$$H_i^0(k,m)=\int_0^L U_m'(x)U_k(x)dx\,,\qquad H_j^0(k,m,n)=\int_0^L Cos\frac{n\pi x}{L}U_m'(x)U_k(x)dx$$

(4.12)

Using (2.16) and its derivatives in integrals (4.12) it is straight forward to show that

$$\zeta_a=\frac{10\lambda_m^4}{L^4}\Big[I_1+A_mI_2+B_mI_3+C_mI_4+A_kI_5+A_kA_mI_6+A_kB_mI_7+A_kC_mI_8+B_kI_9$$
$$+B_kB_mI_{10}+B_kB_mI_{11}+B_kC_mI_{12}+C_kI_{13}+C_kA_mI_{14}+C_kB_mI_{15}+C_kC_mI_{16}\Big]$$

$$\zeta_b=\frac{15\lambda_m^4}{L^4}\Big[I_{33}+A_mI_{34}+B_mI_{35}+C_mI_{36}+A_kI_{37}+A_kA_mI_{38}+A_kB_mI_{39}+A_kC_mI_{40}+B_kI_{41}$$
$$+B_kB_mI_{42}+B_kB_mI_{43}+B_kC_mI_{44}+C_kI_{45}+C_kA_mI_{46}+C_kB_mI_{47}+C_kC_mI_{48}\Big]\Big|_{at\ n=1}$$

$$Z_c = \frac{6\lambda_m^4}{L^4}\Big[I_{17} + A_m I_{18} + B_m I_{19} + C_m I_{20} + A_k I_{21} + A_k A_m I_{22} + A_k B_m I_{23} + A_k C_m I_{24} + B_k I_{25}$$

$$+ B_k A_m I_{26} + B_k B_m I_{27} + B_k C_m I_{28} + C_k I_{29} + C_k A_m I_{30} + C_k B_m I_{31} + C_k C_m I_{32}\Big]\Big|_{at\ n=3}$$

$$Z_d = \frac{\lambda_m^4}{L^4}\Big[I_{33} + A_m I_{34} + B_m I_{35} + C_m I_{36} + A_k I_{37} + A_k A_m I_{38} + A_k B_m I_{39} + A_k C_m I_{40} + B_k I_{41}$$

$$+ B_k B_m I_{42} + B_k B_m I_{43} + B_k C_m I_{44} + C_k I_{45} + C_k A_m I_{46} + C_k B_m I_{47} + C_k C_m I_{48}\Big]\Big|_{at\ n=3}$$

$$L_a = \frac{30\pi\lambda_m^3}{L^4}\Big[A_m I_{17} - I_{18} + C_m I_{19} + B_m I_{20} + A_m A_k I_{21} - A_k I_{22} + A_k C_m I_{23} + A_k B_m I_{24} + A_m B_k I_{25}$$

$$- B_k I_{26} + B_k C_m I_{27} + B_m B_k I_{28} + A_m C_k I_{29} - C_k I_{30} + C_m C_k I_{31} + B_m C_k I_{32}\Big]\Big|_{at\ n=1}$$

$$L_b = \frac{24\pi\lambda_m^3}{L^4}\Big[A_m I_{33} - I_{34} + C_m I_{35} + B_m I_{36} + A_m A_k I_{37} - A_k I_{38} + A_k C_m I_{39} + A_k B_m I_{40} + A_m B_k I_{41}$$

$$- B_k I_{42} + B_k C_m I_{43} + B_m B_k I_{44} + A_m C_k I_{45} - C_k I_{46} + C_m C_k I_{47} + B_m C_k I_{48}\Big]\Big|_{at\ n=2}$$

$$L_c = \frac{6\pi\lambda_m^3}{L^4}\Big[A_m I_{17} - I_{18} + C_m I_{19} + B_m I_{20} + A_m A_k I_{21} - A_k I_{22} + A_k C_m I_{23} + A_k B_m I_{24} + A_m B_k I_{25}$$

$$- B_k I_{26} + B_k C_m I_{27} + B_m B_k I_{28} + A_m C_k I_{29} - C_k I_{30} + C_m C_k I_{31} + B_m C_k I_{32}\Big]\Big|_{at\ n=3}$$

$$N_a = \frac{9\pi^2\lambda_m^2}{L^4}\Big[- I_{33} - A_m I_{34} + B_m I_{35} + C_m I_{36} - A_k I_{37} - A_m A_k I_{38} + B_m A_k I_{39} + C_m A_k I_{40} - B_k I_{41}$$

$$- A_m B_k I_{42} + B_k B_m I_{43} + C_m B_k I_{44} - C_k I_{45} - A_m C_k I_{46} + B_m C_k I_{47} + C_m C_k I_{48}\Big]\Big|_{at\ n=3}$$

$$N_b = \frac{24\pi^2\lambda_m^2}{L^4}\Big[- I_{17} - A_m I_{18} + B_m I_{19} + C_m I_{20} - A_k I_{21} - A_k A_m I_{22} + A_k B_m I_{23} + A_k C_m I_{24} - B_k I_{25}$$

$$- B_k A_m I_{26} + B_k B_m I_{27} + B_k C_m I_{28} - C_k I_{29} + C_k A_m I_{30} + C_k B_m I_{31} + C_k C_m I_{32}\Big]\Big|_{at\ n=2}$$

$$H_s=\frac{15\pi^2\lambda_m^2}{L^4}\Big[-I_{33}-A_mI_{34}+B_mI_{35}+C_wI_{36}-A_kI_{37}-A_mA_kI_{38}+B_mA_kI_{39}+C_mA_kI_{40}-B_kI_{41}$$

$$-A_mB_kI_{42}+B_kB_wI_{43}+C_mB_kI_{44}-C_kI_{45}-A_mC_kI_{46}+B_mC_kI_{47}+C_wC_kI_{48}\Big]\Big|_{at\,n=1}$$

$$Q_3=\frac{N\lambda_m^2}{L^2}\Big[-I_1-A_mI_2+B_wI_3+C_mI_4-A_kI_5-A_kA_mI_6+A_kB_mI_7+A_kC_wI_8-B_kI_9$$

$$-B_kB_wI_{10}+B_kB_mI_{11}+B_kC_mI_{12}-C_kI_{13}-C_kA_mI_{14}+C_kB_wI_{15}+C_kC_mI_{16}\Big]$$

$$Q_4=K^o\Big[I_1+A_wI_2+B_mI_3+C_mI_4+A_kI_5+A_kA_wI_6+A_kB_wI_7+A_kC_mI_8+B_kI_9$$

$$+B_kB_wI_{10}+B_kB_mI_{11}+B_kC_mI_{12}+C_kI_{13}+C_kA_wI_{14}+C_kB_mI_{15}+C_kC_mI_{16}\Big]$$

$$Q_5=\Big[I_1+A_wI_2+B_wI_3+C_mI_4+A_kI_5+A_kA_mI_6+A_kB_wI_7+A_kC_wI_8+B_kI_9$$

$$+B_kB_wI_{10}+B_kB_wI_{11}+B_kC_wI_{12}+C_kI_{13}+C_kA_wI_{14}+C_kB_mI_{15}+C_kC_wI_{16}\Big]$$

$$Q_6=\Big[I_{33}+A_mI_{34}+B_wI_{35}+C_wI_{36}+A_kI_{37}+A_kA_mI_{38}+A_kB_wI_{39}+A_kC_mI_{40}+B_kI_{41}$$

$$+B_kB_wI_{42}+B_kB_mI_{43}+B_kC_mI_{44}+C_kI_{45}+C_kA_wI_{46}+C_kB_wI_{47}+C_kC_mI_{48}\Big]\Big|_{at\,n=1}$$

$$H_1^o(k,m)=\Big[I_1+A_mI_2+B_wI_3+C_wI_4+A_kI_5+A_kA_wI_6+A_kB_mI_7+A_kC_wI_8+B_kI_9$$

$$+B_kA_mI_{10}+B_kB_mI_{11}+B_kC_mI_{12}+C_kI_{13}+C_kA_mI_{14}+C_kB_wI_{15}+C_kC_mI_{16}\Big]$$

$$H_2^o(k,m,n)=\Big[I_{17}+A_wI_{18}+B_mI_{19}+C_wI_{20}+A_kI_{21}+A_kA_wI_{22}+A_kB_wI_{23}+A_kC_mI_{24}+B_kI_{25}$$

$$+B_kA_wI_{26}+B_kB_wI_{27}+B_kC_mI_{28}+C_kI_{29}+C_kA_wI_{30}+C_kB_wI_{31}+C_kC_wI_{32}\Big]$$

$$H_3^o(k,m)=\frac{\lambda_m}{L}\Big[-A_mI_1+I_2+C_mI_3+B_wI_4-A_mA_kI_5+A_kI_6+C_wA_kI_7+B_mA_kI_8-A_wB_kI_9$$

$$+B_kI_{10}+C_mB_kI_{11}+B_wB_kI_{12}-A_wC_kI_{13}+C_kI_{14}+C_mC_kI_{15}+B_wC_kI_{16}\Big]$$

$$\,^o_4(k,m,n)=\frac{\lambda_m}{L}\Big[-A_wI_{17}+I_{18}+C_wI_{19}+B_wI_{20}-A_mA_kI_{21}+A_kI_{22}+A_kC_wI_{23}+A_kB_mI_{24}$$

$$-A_mB_kI_{25}+B_kI_{26}+B_kC_mI_{27}+B_mB_kI_{28}-A_wC_kI_{29}+C_kI_{30}+C_mC_kI_{31}+B_wC_kI_{32}\Big]$$

$$\,^o_5(k,m)=\frac{\lambda_w^2}{L^2}\Big[-I_1-A_mI_2+B_wI_3+C_wI_4-A_kI_5-A_kA_mI_6+A_kB_mI_7+A_kC_wI_8-B_kI_9$$

$$-B_kA_mI_{10}+B_kB_mI_{11}+B_kC_mI_{12}-C_kI_{13}-C_kA_mI_{14}+C_kB_mI_{15}+C_kC_mI_{16}\Big]$$

$$W_1^s(k,m,n)=\frac{\lambda_m^2}{L^2}\Big[-I_{17}-A_mI_{18}+B_mI_{19}+C_mI_{20}-A_kI_{21}-A_kA_mI_{22}+A_kB_mI_{23}+A_kC_mI_{24}-B_kI_{25}$$
$$-B_kA_mI_{26}+B_kB_mI_{27}+B_kC_mI_{28}-C_kI_{29}-C_kA_mI_{30}+C_kB_mI_{31}+C_kC_mI_{32}\Big]$$

$$W(k,m)=\frac{\lambda_m}{L}\Big[-A_mI_1+I_2+C_mI_3+B_mI_4-A_mA_kI_5+A_kI_6+C_mA_kI_7+B_mA_kI_8-A_mB_kI_9$$
$$+B_kI_{10}+C_mB_kI_{11}+B_mB_kI_{12}-A_mC_kI_{13}+C_kI_{14}+C_mC_kI_{15}+B_mC_kI_{16}\Big]$$

$$W_1^s(k,m,n)=\frac{\lambda_m^2}{L^2}\Big[-I_{17}-A_mI_{18}+B_mI_{19}+C_mI_{20}-A_kI_{21}-A_kA_mI_{22}+A_kB_mI_{23}+A_kC_mI_{24}-B_kI_{25}$$
$$-B_kA_mI_{26}+B_kB_mI_{27}+B_kC_mI_{28}-C_kI_{29}-C_kA_mI_{30}+C_kB_mI_{31}+C_kC_mI_{32}\Big]$$

(4.13)

where I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_8, I_9, I_{10}, I_{11}, I_{12}, I_{13}, I_{14}, I_{15}, I_{16}, I_{17}, I_{18}, I_{19}, I_{20}, I_{21}, I_{22}, I_{23}, I_{24}, I_{25}, I_{26}, I_{27}, I_{28}, I_{29}, I_{30}, I_{31}, I_{32} are as defined in the previous chapter and

$$I_{33}=\int_0^L Sin\frac{n\pi x}{L}Sin\frac{\lambda_k x}{L}Sin\frac{\lambda_m x}{L}dx,\qquad I_{34}=\int_0^L Sin\frac{n\pi x}{L}Sin\frac{\lambda_k x}{L}Cos\frac{\lambda_n x}{L}dx,$$

$$I_{35}=\int_0^L Sin\frac{n\pi x}{L}Sin\frac{\lambda_k x}{L}Sinh\frac{\lambda_m x}{L}dx,\qquad I_{36}=\int_0^L Sin\frac{n\pi x}{L}Sin\frac{\lambda_k x}{L}Cosh\frac{\lambda_m x}{L}dx,$$

$$I_{37}=\int_0^L Sin\frac{n\pi x}{L}Cos\frac{\lambda_k x}{L}Sin\frac{\lambda_m x}{L}dx,\ I_{38}=\int_0^L Sin\frac{n\pi x}{L}Cos\frac{\lambda_k x}{L}Cos\frac{\lambda_m x}{L}dx,$$

$$I_{39}=\int_0^L Sin\frac{n\pi x}{L}Cos\frac{\lambda_k x}{L}Sinh\frac{\lambda_m x}{L}dx,\qquad I_{40}=\int_0^L Sin\frac{n\pi x}{L}Cos\frac{\lambda_k x}{L}Cosh\frac{\lambda_m x}{L}dx,$$

$$I_{41}=\int_0^L Sin\frac{n\pi x}{L}Sinh\frac{\lambda_k x}{L}Sin\frac{\lambda_m x}{L}dx,\qquad I_{42}=\int_0^L Sin\frac{n\pi x}{L}Sinh\frac{\lambda_k x}{L}Cos\frac{\lambda_m x}{L}dx,$$

$$I_{43}=\int_0^L Sin\frac{n\pi x}{L}Sinh\frac{\lambda_k x}{L}Sinh\frac{\lambda_m x}{L}dx,\qquad I_{44}=\int_0^L Sin\frac{n\pi x}{L}Sinh\frac{\lambda_k x}{L}Cosh\frac{\lambda_m x}{L}dx,$$

$$I_{45}=\int_0^L Sin\frac{n\pi x}{L}Cosh\frac{\lambda_k x}{L}Sin\frac{\lambda_m x}{L}dx,\qquad I_{46}=\int_0^L Sin\frac{n\pi x}{L}Cosh\frac{\lambda_k x}{L}Cos\frac{\lambda_m x}{L}dx,$$

$$I_{47}=\int_0^L Sin\frac{n\pi x}{L}Cosh\frac{\lambda_k x}{L}Sinh\frac{\lambda_m x}{L}dx,\qquad I_{48}=\int_0^L Sin\frac{n\pi x}{L}Cosh\frac{\lambda_k x}{L}Cosh\frac{\lambda_m x}{L}dx,$$

(4.14)

A further simplification and rearrangements of equation (4.11) yield

$$\sum_{m=1}^{\infty}\Bigg\{\Omega_0(k,m)\ddot{Y}_m(t)+\Omega_1(k,m)Y_m(t)+\varepsilon_1\Bigg[H_b^0(k,m)\ddot{Y}_m(t)+2\sum_{n=1}^{\infty}Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)H_c^0(k,m,n)\ddot{Y}_m(t)$$

$$-2\gamma\beta H_d^0(k,m)Cos\beta t\dot{Y}_m(t)+4\gamma\beta Cos\beta t\sum_{n=1}^{\infty}Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)H_e^0(k,m,n)\dot{Y}_m(t)$$

$$+(\gamma\beta Cos\beta t)^2H_f^0(k,m)Y_m(t)+2(\gamma\beta Cos\beta t)^2\sum_{n=1}^{\infty}Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)H_g^0(k,m,n)Y_m(t)$$

$$-\gamma\beta^2H_i^0(k,m)Sin\beta tY_m(t)-2\gamma\beta^2Sin\beta t\sum_{n=1}^{\infty}Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)H_j^0(k,m,n)Y_m(t)\Bigg]\Bigg\}$$

$$=\frac{P}{\mu_0}U_1\left(x-\lfloor x_0+\gamma Sin\beta t\rfloor\right)$$

(4.15)

where

$$\Omega_0(k,m)=Q_5+Q_6,\quad \Omega_1(k,m)=\frac{EI_0(Q_0+Q_1+Q_2)}{4\mu_0}-\frac{Q_3}{\mu_0}+\frac{Q_4}{\mu_0}\qquad(4.16)$$

and

$$\varepsilon_1=\frac{M}{\mu_0L}\qquad(4.17)$$

Equation (4.15) can be rewritten as

$$\sum_{m=1}^{\infty}\Bigg\{\ddot{Y}_m(t)+\frac{\Omega_1(k,m)}{\Omega_0(k,m)}Y_m(t)+\frac{\varepsilon_1}{\Omega_0(k,m)}\Bigg[H_b^0(k,m)\ddot{Y}_m(t)+2\sum_{n=1}^{\infty}Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)H_c^0(k,m,n)\ddot{Y}_m(t)$$

$$-2\gamma\beta H_d^0(k,m)Cos\beta t\dot{Y}_m(t)+4\gamma\beta Cos\beta t\sum_{n=1}^{\infty}Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)H_e^0(k,m,n)\dot{Y}_m(t)$$

$$+(\gamma\beta Cos\beta t)^2H_f^0(k,m)Y_m(t)+2(\gamma\beta Cos\beta t)^2\sum_{n=1}^{\infty}Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)H_g^0(k,m,n)Y_m(t)$$

$$-\gamma\beta^2H_i^0(k,m)Sin\beta tY_m(t)-2\gamma\beta^2Sin\beta t\sum_{n=1}^{\infty}Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)H_j^0(k,m,n)Y_m(t)\Bigg]\Bigg\}$$

$$=\frac{P}{\Omega_0(k,m)\mu_0}\left[Sin\frac{\lambda_k}{L}(x_0+\gamma Sin\beta t)+A_kCos\frac{\lambda_k}{L}(x_0+\gamma Sin\beta t)+B_kSinh\frac{\lambda_k}{L}(x_0+\gamma Sin\beta t)+C_kCosh\frac{\lambda_k}{L}(x_0+\gamma Sin\beta t)\right]$$

(4.18)

The second order ordinary differential equations (4.18) is the transformed equation governing the problem of a non-prismatic Bernoulli-Euler beam resting on an elastic foundation and traversed by a load moving with variable velocity. These second order differential equations are valid for all variants of the classical boundary conditions. Two cases of this equation are discussed in the following section.

4.3.0 SOLUTION OF THE TRANSFORMED EQUATION

Like in the previous chapter, we shall consider two special cases of the transformed equation namely, the ***moving force*** problem and the ***moving mass*** problem.

4.3.1 The moving force problem

Setting $\varepsilon_1 = 0$ in the transformed equation (4.18), one obtains

$$(t)+\frac{\Omega_1(k,m)}{\Omega_0(k,m)}Y_m(t)=\frac{P}{\Omega_0(k,m)\mu_0}\Bigg[Sin\frac{\lambda_k}{L}(x_0+\gamma Sin\beta t)+A_k Cos\frac{\lambda_k}{L}(x_0+\gamma Sin\beta t)$$
$$+B_k Sinh\frac{\lambda_k}{L}(x_0+\gamma Sin\beta t)+C_k Cosh\frac{\lambda_k}{L}(x_0+\gamma Sin\beta t)\Bigg] \quad (4.19)$$

This is the classical case of a moving force problem associated with the system. It is an approximate model which assumes the inertia effect of the moving mass is negligible. A rearrangement of equation (4.19) yields

$${}_{n}(t)+\omega_{nf}^{2}Y_{n}(t)=\frac{P}{\Omega_{0}(k,m)\mu_{0}}\left[Sin\frac{\lambda_{k}}{L}(x_{0}+\gamma Sin\beta t)+A_{k}Cos\frac{\lambda_{k}}{L}(x_{0}+\gamma Sin\beta t)\right.$$
$$\left.+B_{k}Sinh\frac{\lambda_{k}}{L}(x_{0}+\gamma Sin\beta t)+C_{k}Cosh\frac{\lambda_{k}}{L}(x_{0}+\gamma Sin\beta t)\right]$$

(4.20)

here

$$\omega_{nf}^{2}=\frac{\Omega_{1}(k,m)}{\Omega_{0}(k,m)} \tag{4.21}$$

$$a_{0}=\left(Cos\frac{\lambda_{k}x_{0}}{L}-A_{k}Sin\frac{\lambda_{k}x_{0}}{L}\right),\qquad a_{1}=\left(Sin\frac{\lambda_{k}x_{0}}{L}+A_{k}Cos\frac{\lambda_{k}x_{0}}{L}\right)$$

$$a_{2}=\left(B_{k}Sinh\frac{\lambda_{k}x_{0}}{L}+C_{k}Cos\frac{\lambda_{k}x_{0}}{L}\right),\quad a_{3}=\left(B_{k}Cosh\frac{\lambda_{k}x_{0}}{L}+C_{k}Sinh\frac{\lambda_{k}x_{0}}{L}\right) \tag{4.22}$$

arly, the second order non-homogeneous ordinary differential equation 20) is analogous to equation (2.85). Consequently, it can be shown that

$$
\begin{aligned}
=\frac{P_m}{\omega_{af}^2}\Bigg\{ & \omega_{af} a_1 J_0(G)\left[\frac{Cos(\omega_{af}-b_0)t-Cos\omega_{af}t}{b_0}\right] \\
& + a_1\omega_{af}\sum_{k=1}^{\infty} J_{2k}(G)\left[\frac{Cos(\omega_{af}-b_1)t-Cos\omega_{af}t}{b_1}+\frac{Cos(\omega_{af}-b_2)t-Cos\omega_{af}t}{b_2}\right] \\
& + a_0\sum_{k=0}^{\infty} J_{2k+1}(G)\left[\frac{\omega_{af}Sin(\omega_{af}-b_4)t-(\omega_{af}-b_4)Sin\omega_{af}t}{b_4}-\frac{\omega_{af}Sin(\omega_{af}-b_3)t-(\omega_{af}-b_3)Sin\omega_{af}t}{b_3}\right] \\
& + S_1\omega_{af} I_0(G)\left[\frac{Cos(\omega_{af}-b_0)t-Cos\omega_{af}t}{b_0}\right] \\
& + S_1\omega_{af}\sum_{k=1}^{\infty} (-1)^k I_{2k}(G)\left[\frac{Cos(\omega_{af}-b_1)t-Cos\omega_{af}t}{b_1}+\frac{Cos(\omega_{af}-b_2)t-Cos\omega_{af}t}{b_2}\right] \\
& + S_1\sum_{k=0}^{\infty} (-1)^k I_{2k+1}(G)\left[\frac{\omega_{af}Sin(\omega_{af}-b_4)t-(\omega_{af}-b_4)Sin\omega_{af}t}{b_4}-\frac{\omega_{af}Sin(\omega_{af}-b_3)t-(\omega_{af}-b_3)Sin\omega_{af}t}{b_3}\right] \\
& + S_2\omega_{af} I_0(-G)\left[\frac{Cos(\omega_{af}-b_0)t-Cos\omega_{af}t}{b_0}\right] \\
& + S_2\omega_{af}\sum_{k=1}^{\infty} (-1)^k I_{2k}(-G)\left[\frac{Cos(\omega_{af}-b_1)t-Cos\omega_{af}t}{b_1}+\frac{Cos(\omega_{af}-b_2)t-Cos\omega_{af}t}{b_2}\right] \\
& + S_2\sum_{k=0}^{\infty} (-1)^k I_{2k+1}(-G)\left[\frac{\omega_{af}Sin(\omega_{af}-b_4)t-(\omega_{af}-b_4)Sin\omega_{af}t}{b_4}-\frac{\omega_{af}Sin(\omega_{af}-b_3)t-(\omega_{af}-b_3)Sin\omega_{af}t}{b_3}\right.
\end{aligned}
\tag{4.23}
$$

where

$$
P_m=\frac{P}{\Omega_0(k,m)\mu_0} \quad \text{and} \quad G=\frac{\lambda_k\gamma}{L} \tag{4.24}
$$

iew of equations (4.5) and (4.24), one obtains

$$
\begin{aligned}
V_n(x,t) = \sum_{m=1}^{n}\frac{P_m}{\omega_m^2}\Bigg(& \omega_m a_1 J_0(G)\left[\frac{Cos(\omega_m-b_0)t-Cos\omega_m t}{b_0}\right] \\
+ & \ a_1\omega_m\sum_{k=1}^{\infty} J_{2k}(G)\left[\frac{Cos(\omega_m-b_1)t-Cos\omega_m t}{b_1}+\frac{Cos(\omega_m-b_2)t-Cos\omega_m t}{b_2}\right] \\
+ & \ a_0\sum_{k=0}^{\infty} J_{2k+1}(G)\left[\frac{\omega_m Sin(\omega_m-b_4)t-(\omega_m-b_4)Sin\omega_m t}{b_4}-\frac{\omega_m Sin(\omega_m-b_3)t-(\omega_m-b_3)Sin\omega_m t}{b_3}\right] \\
+ & \ S_1\omega_m I_0(G)\left[\frac{Cos(\omega_m-b_0)t-Cos\omega_m t}{b_0}\right] \\
+ & \ S_1\omega_m\sum_{k=1}^{\infty} (-1)^k I_{2k}(G)\left[\frac{Cos(\omega_m-b_1)t-Cos\omega_m t}{b_1}+\frac{Cos(\omega_m-b_2)t-Cos\omega_m t}{b_2}\right] \\
+ & \ S_1\sum_{k=0}^{\infty} (-1)^k I_{2k+1}(G)\left[\frac{\omega_m Sin(\omega_m-b_4)t-(\omega_m-b_4)Sin\omega_m t}{b_4}-\frac{\omega_m Sin(\omega_m-b_3)t-(\omega_m-b_3)Sin\omega_m t}{b_3}\right] \\
+ & \ S_2\omega_m I_0(-G)\left[\frac{Cos(\omega_m-b_0)t-Cos\omega_m t}{b_0}\right] \\
+ & \ S_2\omega_m\sum_{k=1}^{\infty} (-1)^k I_{2k}(-G)\left[\frac{Cos(\omega_m-b_1)t-Cos\omega_m t}{b_1}+\frac{Cos(\omega_m-b_2)t-Cos\omega_m t}{b_2}\right] \\
+ & \ S_2\sum_{k=0}^{\infty} (-1)^k I_{2k+1}(-G)\left[\frac{\omega_m Sin(\omega_m-b_4)t-(\omega_m-b_4)Sin\omega_m t}{b_4}\right. \\
& \left.-\frac{\omega_m Sin(\omega_m-b_3)t-(\omega_m-b_3)Sin\omega_m t}{b_3}\right]\Bigg)\times\left(Sin\frac{\lambda_m x}{L}+A_m Cos\frac{\lambda_m x}{L}+B_m Sinh\frac{\lambda_m x}{L}+C_m Cosh\frac{\lambda_m x}{L}\right)
\end{aligned}
\tag{4.25}
$$

as the transverse-displacement response to a moving force traveling with variable velocity of a non-uniform Bernoulli-Euler beam resting on elastic foundation.

4.3.2 The moving mass problem

As discussed in the previous chapter, if the mass of the moving load is commensurable with that of the structure, the inertia effect of the moving mass is not negligible. Thus, $\varepsilon_1 \neq 0$ and one is required to solve the entire equation

4.18). This we termed the ***moving mass problem***. Unlike in the case of the moving force problem, an exact analytical solution to this equation is not possible. Thus, we seek an approximate analytical technique due to Struble discussed in chapter two.

To this end equation (4.18) is rearranged to take the form

$$\ddot{Y}_m(t)+\omega_{\alpha}^2 Y_m(t)+\varepsilon_1\Bigg\{H_2(k,m)\ddot{Y}_m(t)+2\sum_{n=1}^{\infty} Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)H_3(k,m,n)\ddot{Y}_m(t)$$

$$+2\gamma\beta H_4(k,m)Cos\beta t\,\dot{Y}_m(t)+4\gamma\beta Cos\beta t\sum_{n=1}^{\infty} Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)H_5(k,m,n)\dot{Y}_m(t)$$

$$-(\gamma\beta Cos\beta t)^2 H_6(k,m)Y_m(t)+2(\gamma\beta Cos\beta t)^2\sum_{n=1}^{\infty} Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)H_7(k,m,n)Y_m(t)$$

$$-\gamma\beta^2 H_8(k,m)Sin\beta t\,Y_m(t)-2\gamma\beta^2 Sin\beta t\sum_{n=1}^{\infty} Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)H_9(k,m,n)Y_m(t)\Bigg\}$$

$$=\frac{\varepsilon_1 Lg}{\Omega_0(k,m)\mu_0}\left[a_0 Sin(GSin\beta t)+a_1 Cos(GSin\beta t)+a_2 Cosh(GSin\beta t)+a_3 Sinh(GSin\beta t)\right]$$

(4.26)

here

$$H_2(k,m)=\frac{H_b^{0}(k,m)}{\Omega_0(k,m)} \qquad H_3(k,m,n)=\frac{H_c^{0}(k,m,n)}{\Omega_0(k,m)}$$

$$H_4(k,m)=\frac{H_d^{0}(k,m)}{\Omega_0(k,m)} \qquad H_5(k,m,n)=\frac{H_e^{''}(k,m,n)}{\Omega_0(k,m)}$$

$$H_6(k,m)=\frac{H_f^{''}(k,m)}{\Omega_0(k,m)} \qquad H_7(k,m,n)=\frac{H_g^{''}(k,m,n)}{\Omega_0(k,m)}$$

$$H_8(k,m)=\frac{H_i^{''}(k,m)}{\Omega_0(k,m)} \qquad H_9(k,m,n)=\frac{H_j^{''}(k,m,n)}{\Omega_0(k,m)} \qquad (4.27)$$

uation (4.26) implies

$$Y_m(t)+\frac{\varepsilon_1\left[2\gamma\beta H_4(k,m)Cos\beta t+4\gamma\beta Cos\beta t\sum_{n=1}^{\infty}Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)H_5(k,m,n)\right]}{\left[1+\varepsilon_1\left(H_2(k,m)+2\sum_{n=1}^{\infty}Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)H_3(k,m,n)\right)\right]}\dot{Y}_m(t)$$

$$+\frac{\left[\omega_n^2+\varepsilon_1\left\{(\gamma\beta Cos\beta t)^2H_6(k,m)+2(\gamma\beta Cos\beta t)^2\sum_{n=1}^{\infty}Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)H_7(k,m,n)\right\}\right]}{\left[1+\varepsilon_1\left(H_2(k,m)+2\sum_{n=1}^{\infty}Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)H_3(k,m,n)\right)\right]}Y_m(t)$$

$$\frac{\varepsilon_1\left[\gamma\beta^2H_8(k,m)Sin\beta t+2\gamma\beta^2Sin\beta t\sum_{n=1}^{\infty}Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)H_9(k,m,n)\right]}{\left[1+\varepsilon_1\left(H_2(k,m)+2\sum_{n=1}^{\infty}Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)H_3(k,m,n)\right)\right]}Y_m(t)$$

$$=\frac{\varepsilon_1 Lg}{\Omega_n(k,m)}\frac{\left[a_0Sin(GSin\beta t)+a_1Cos(GSin\beta t)+a_2Cosh(GSin\beta t)+a_3Sinh(GSin\beta t)\right]}{\left[1+\varepsilon_1\left(H_2(k,m)+2\sum_{n=1}^{\infty}Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)H_3(k,m,n)\right)\right]}$$

(4.28)

As in the previous section, the homogeneous part of (4.28) is first considered and a modified frequency corresponding to the frequency of the free system due to the presence of moving mass is sought. An equivalent free system operator defined by the modified frequency then replaces equation (4.28).

To do this, consider a parameter $\eta < 1$ for any arbitrary mass ratio ε_1 defined as

$$\eta = \frac{\varepsilon_1}{1+\varepsilon_1} \tag{4.29}$$

It follows that

$$\varepsilon_1 = \eta\left[1+O(\eta)+O(\eta^2)+ \quad +\dots\dots\dots\dots\right] \tag{4.30}$$

and

$$\frac{1}{\left[1+\eta\left(H_2(k,m)+2\sum_{n=1}^{\infty}Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)H_3(k,m,n)\right)\right]}$$

$$=\left[1-\eta\left(H_2(k,m)+2\sum_{n=1}^{\infty}Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)H_3(k,m,n)\right)+O(\eta^2)\right]+\cdots\cdots \tag{4.31}$$

here

$$\left|\eta\left(H_2(k,m)+2\sum_{n=1}^{\infty}Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)H_3(k,m,n)\right)\right|<1 \tag{4.32}$$

Substituting (4.30) and (4.31) into equation (4.28) we have

$$(t)+\eta\left[2\gamma\beta H_4(k,m)Cos\beta t+4\gamma\beta Cos\beta t\sum_{n=1}^{\infty}Cos\frac{n\pi}{L}(x_0+\gamma\beta Sin\beta t)H_5(k,m,n)\right]\dot{Y}_m(t)$$
$$+\left[\omega_m^2-\eta\omega_m^2\left(H_2(k,m)+2\sum_{n=1}^{\infty}Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)H_3(k,m,n)\right)\right.$$
$$\left.+\eta\left\{(\gamma\beta Cos\beta t)^2H_6(k,m)+2(\gamma\beta Cos\beta t)^2\sum_{n=1}^{\infty}Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)H_7(k,m,n)\right\}\right]Y_m(t)$$
$$-\eta\left[\gamma\beta^2H_8(k,m)Sin\beta t+2\gamma\beta^2Sin\beta t\sum_{n=1}^{\infty}Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)H_9(k,m,n)\right]Y_m(t)$$
$$=\frac{\eta Lg}{\Omega_1(k,m)}\left[\alpha_0Sin(GSin\beta t)+\alpha_1Cos(GSin\beta t)+\alpha_2Cosh(GSin\beta t)+\alpha_3Sinh(GSin\beta t)\right] \tag{4.33}$$

$O(\eta)$ only.

hen η is set to zero in equation (4.33), a situation corresponding to the case in hich the inertia effect of the mass of the system is regarded as negligible is tained. In such a case, the solution is of the form.

$$Y_m(t)=C_m^{*}Cos(\omega_m t-\phi_m) \tag{4.34}$$

where; C_n^* and ϕ_m are constants and $\omega_{nf}^2 = \dfrac{\Omega_1(k,m)}{\Omega_0(k,m)}$.

However, since for any arbitrary mass ratio ε_1 we always have $\eta < 1$, then Struble's technique requires that the solution of the homogeneous part of equation (4.33) be given in an asymptotic form, namely

$$Y_m(t) = A(m,t)Cos\left[\omega_{nf}t - \phi(m,t)\right] + \eta Y_1(t) + O\left(\eta^2\right) \qquad (4.35)$$

To obtain the modified frequency, equation (4.35) and its first and second derivatives are substituted into the homogeneous part of equation (4.33). The modified frequency is determined by the resulting variational equations describing the behaviour of $A(m,t)$ and $\phi(m,t)$ during the motion of the mass. Thus, substituting (4.35) and its first derivatives into the homogeneous part of equation (4.33) and taking into account (4.30), (4.31) and (4.32) one obtains

$$-2\omega_{\omega}\dot{A}_m(t)Sin\left[\omega_{\omega}t-\phi_m(t)\right]+2A_m(t)\dot{\phi}_m(t)\omega_{\omega}Cos\left[\omega_{\omega}t-\phi_m(t)\right]-\omega_{\omega}^2A_m(t)Cos\left[\omega_{\omega}t-\phi_m(t)\right]$$
$$+\omega_{\omega}^2A_m(t)Cos\left[\omega_{\omega}t-\phi_m(t)\right]-2\eta A_m(t)\omega_{\omega}\gamma\beta H_3(k,m)Cos\beta t\,Sin\left[\omega_{\omega}t-\phi_m(t)\right]$$
$$-4\eta A_m(t)\gamma\beta\omega_{\omega}\sum_{n=1}^{\infty}H_3(k,m,n)Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)Cos\beta t\,Sin\left[\omega_{\omega}t-\phi_m(t)\right]$$
$$-\eta\omega_{\omega}^2A_m(t)H_4(k,m)Cos\left[\omega_{\omega}t-\phi_m(t)\right]-2\eta\omega_{\omega}^2A_m(t)\sum_{n=1}^{\infty}H_5(k,m,n)\left\{Cos\frac{n\pi x_0}{L}J_0(G_1)\right.$$
$$\left.+2Cos\frac{n\pi x_0}{L}\sum_{k=1}^{\infty}J_{2k}(G_1)Cos2k\beta t-2Sin\frac{n\pi x_0}{L}\sum_{k=0}^{\infty}J_{2k+1}(G_1)Sin(2k+1)\beta t\right\}Cos\left[\omega_{\omega}t-\phi_m(t)\right]$$
$$+\frac{1}{2}\eta A_m(t)(\gamma\beta)^2H_6(k,m)Cos\left[\omega_{\omega}t-\phi_m(t)\right]+\frac{1}{2}\eta A_m(t)(\gamma\beta)^2H_6(k,m)Cos2\beta t\,Cos\left[\omega_{\omega}t-\phi_m(t)\right]$$
$$+\frac{1}{2}\eta(\gamma\beta)^2A_m(t)\sum_{n=1}^{\infty}H_7(k,m,n)\left\{Cos\frac{n\pi x_0}{L}J_0(G_1)+2Cos\frac{n\pi x_0}{L}\sum_{k=1}^{\infty}J_{2k}(G_1)Cos2k\beta t\right.$$
$$\left.-2Sin\frac{n\pi x_0}{L}\sum_{k=0}^{\infty}J_{2k+1}(G_1)Sin(2k+1)\beta t\right\}Cos\left[\omega_{\omega}t-\phi_m(t)\right]$$
$$+\eta A_m(t)(\gamma\beta)^2\sum_{n=1}^{\infty}H_7(k,m,n)Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)Cos2\beta t\,Cos\left[\omega_{\omega}t-\phi_m(t)\right]$$
$$-\eta A_m(t)\gamma\beta^2H_8(k,m)Sin\beta t\,Cos\left[\omega_{\omega}t-\phi_m(t)\right]$$
$$-2\eta A_m(t)\gamma\beta^2\sum_{n=1}^{\infty}H_9(k,m,n)Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)Sin\beta t\,Cos\left[\omega_{\omega}t-\phi_m(t)\right]=0$$

(4.36)

where terms higher than $O(\eta^2)$ have been neglected and $G_1=\frac{n\pi\gamma}{L}$

The variational equations are obtained by equating the coefficients of $Sin\left[\omega_{\omega}t-\phi(m,t)\right]$ and $Cos\left[\omega_{\omega}t-\phi(m,t)\right]$ terms on both sides of the equation. Thus, we note the following trigonometric identities

$$Sin\left[\omega_{\omega}t-\phi(m,t)\right]Cos\beta t=\frac{1}{2}\left\{Sin\left[\omega_{\omega}t-\phi(m,t)+\beta t\right]+Sin\left[\omega_{\omega}t-\phi(m,t)-\beta t\right]\right\}$$

$$Cos\left[\omega_{\omega}t-\phi(m,t)\right]Sin\beta t=\frac{1}{2}\left\{Sin\left[\omega_{\omega}t-\phi(m,t)+\beta t\right]-Sin\left[\omega_{\omega}t-\phi(m,t)-\beta t\right]\right\}$$

$$Sin\left[\omega_{\omega}t-\phi(m,t)\right]Cos2k\beta t=\frac{1}{2}\left\{Sin\left[\omega_{\omega}t-\phi(m,t)+2k\beta t\right]+Sin\left[\omega_{\omega}t-\phi(m,t)-2k\beta t\right]\right\}$$

$$Sin\left[\omega_n t - \phi(m,t)\right]Sin(2k+1) = \frac{1}{2}\left\{Cos\left[\omega_n t - \phi(m,t) - (2k+1)\beta t\right] - Cos\left[\omega_n t - \phi(m,t) + (2k+1)\beta t\right]\right\}$$

$$Cos\left[\omega_n t - \phi(m,t)\right]Sin(2k+1)\beta t = \frac{1}{2}\left\{Sin\left[\omega_n t - \phi(m,t) + (2k+1)\beta t\right] - Sin\left[\omega_n t - \phi(m,t) - (2k+1)\beta t\right]\right\}$$

$$Cos\left[\omega_n t - \phi(m,t)\right]Cos 2k\beta t = \frac{1}{2}\left\{Cos\left[\omega_n t - \phi(m,t) + 2k\beta t\right] + Cos\left[\omega_n t - \phi(m,t) - 2k\beta t\right]\right\}$$

(4.37)

Using these, the terms that do not contribute to the variational equations ecome.

$$-2\omega_n \dot{A}(m,t)Sin\left[\omega_n t - \phi(m,t)\right] + 2\omega_n A(m,t)\dot{\phi}(m,t)Cos\left[\omega_n t - \phi(m,t)\right] - \eta\omega_n^2 A(m,t)H_2(k,m)Cos\left[\omega_n t - \phi(m,\right.$$
$$-2\eta\omega_n^2 A(m,t)\sum_{n=1}^{\infty} H_3(k,m,n)Cos\frac{n\pi x_0}{L}J_0(G_1)Cos\left[\omega_n t - \phi(m,t)\right] + \frac{1}{2}\eta A(m,t)(\gamma\beta)^2 H_6(k,m)Cos\left[\omega_n t - \phi(m,t\right.$$
$$+\eta A(m,t)(\gamma\beta)^2\sum_{n=1}^{\infty} H_7(k,m,n)Cos\frac{n\pi x_0}{L}J_0(G_1)Cos\left[\omega_n t - \phi(m,t)\right] = 0$$

(4.38)

The variational equations of the problem are obtained by setting the coefficients f $Sin\left[\omega_n t - \phi(m,t)\right]$ and $Cos\left[\omega_n t - \phi(m,t)\right]$ in equation (4.38) to zero. Thus, one btains

$$-2\omega_n \dot{A}(m,t) = 0 \qquad (4.39)$$

nd

$$\omega_n A(m,t)\dot{\phi}(m,t) - \eta\omega_n^2 A(m,t)H_2(k,m) - 2\eta\omega_n^2 A(m,t)\sum_{n=1}^{\infty} H_3(k,m,n)Cos\frac{n\pi x_0}{L}J_0(G_1)$$
$$\frac{1}{2}\eta A(m,t)(\gamma\beta)^2 H_6(k,m) + \eta A(m,t)(\gamma\beta)^2\sum_{n=1}^{\infty} H_7(k,m,n)Cos\frac{n\pi x_0}{L}J_0(G_1) = 0$$

(4.40)

earranging (4.39) and (4.40), we have

$$\dot{A}(m,t) = 0 \qquad (4.41)$$

nd

$$(m,t) = \frac{\eta}{2}\left[\omega_{nf}\left(H_2(k,m) + R_a(k,m,n)\right) - \left((\gamma\beta)^2 \frac{\{H_6(k,m) + 2R_b(k,m,n)\}}{2\omega_{nf}}\right)\right] \qquad (4.42)$$

where

$$R_a(k,m,n) = \sum_{n=1}^{\infty} H_3(k,m,n) J_0(G_1) Cos\frac{n\pi x_0}{L} \qquad (4.43)$$

$$R_b(k,m,n) = \sum_{n=1}^{\infty} H_7(k,m,n) J_0(G_1) Cos\frac{n\pi x_0}{L} \qquad (4.44)$$

Solving equations (4.41) and (4.42) respectively yields

$$A(m,t) = C_n^* \qquad (4.45)$$

where C_n^* is a constant and

$$(m,t) = \frac{\eta}{2}\left[\omega_{nf}\left(H_2(k,m) + R_a(k,m,n)\right) - \left((\gamma\beta)^2 \frac{\{H_6(k,m) + 2R_b(k,m,n)\}}{2\omega_{nf}}\right)\right] t + \psi_m \qquad (4.46)$$

o order one only. Where ψ_m is a constant.

Therefore, when the effect of the mass of the particle is considered, the first approximation to the homogeneous system is

$$Y_m(t) = C_n^* Cos\left[\omega_{nf} t - \varphi_m\right] \qquad (4.47)$$

where

$$= \omega_{nf}\left\{1 - \frac{\eta}{2}\left[\left(H_2(k,m) + R_a(k,m,n)\right) - (\gamma\beta)^2 \frac{\left[H_6(k,m) + 2R_b(k,m,n)\right]}{2\omega_{nf}^2}\right]\right\} \qquad (4.48)$$

s called the modified natural frequency representing the frequency of the free system due to the presence of the moving mass. Thus, the homogeneous part of he equation (4.33) can be written as

$$\frac{d^2Y_m(t)}{dt^2} + \omega_{bj}^2 Y_m(t) = 0 \qquad (4.49)$$

Thus, the entire equation (4.28), becomes

$$\frac{d^2Y_m(t)}{dt^2} + \omega_{bj}^2 Y_m(t) = \frac{\eta Lg}{\Omega_0(k,m)}\left[a_0 Sin(GSin\beta t) + a_1 Cos(GSin\beta t) + a_2 Cosh(GSin\beta t) + a_3 Sinh(GSin\beta t)\right]$$

(4.50)

Clearly, equations (4.5) is analogous to equation (2.85). Therefore, following the same arguments as in the previous sections, the solution to equation (4.50) is given by

$$\begin{aligned}
{}_m(t) = \frac{Q_m}{\omega_{bj}^2}\Bigg\{ & \omega_{bj} a_1 J_0(G)\left[\frac{Cos(\omega_{bj} - b_0)t - Cos\omega_{bj} t}{b_0}\right] \\
+ \; & a_1\omega_{bj}\sum_{k=1}^{\infty} J_{2k}(G)\left[\frac{Cos(\omega_{bj} - b_1)t - Cos\omega_{bj} t}{b_1} + \frac{Cos(\omega_{bj} - b_2)t - Cos\omega_{bj} t}{b_2}\right] \\
+ \; & a_0\sum_{k=0}^{\infty} J_{2k+1}(G)\left[\frac{\omega_{bj} Sin(\omega_{bj} - b_4)t - (\omega_{bj} - b_4)Sin\omega_{bj} t}{b_4} - \frac{\omega_{bj} Sin(\omega_{bj} - b_3)t - (\omega_{bj} - b_3)Sin\omega_{bj} t}{b_3}\right] \\
+ \; & S_1\omega_{bj} I_0(G)\left[\frac{Cos(\omega_{bj} - b_0)t - Cos\omega_{bj} t}{b_0}\right] \\
+ \; & S_1\omega_{bj}\sum_{k=1}^{\infty} (-1)^k I_{2k}(G)\left[\frac{Cos(\omega_{bj} - b_1)t - Cos\omega_{bj} t}{b_1} + \frac{Cos(\omega_{bj} - b_2)t - Cos\omega_{bj} t}{b_2}\right] \\
+ \; & S_1\sum_{k=0}^{\infty} (-1)^k I_{2k+1}(G)\left[\frac{\omega_{bj} Sin(\omega_{bj} - b_4)t - (\omega_{bj} - b_4)Sin\omega_{bj} t}{b_4} - \frac{\omega_{bj} Sin(\omega_{bj} - b_3)t - (\omega_{bj} - b_3)Sin\omega_{bj} t}{b_3}\right. \\
+ \; & S_2\omega_{bj} I_0(-G)\left[\frac{Cos(\omega_{bj} - b_0)t - Cos\omega_{bj} t}{b_0}\right] \\
+ \; & S_2\omega_{bj}\sum_{k=1}^{\infty} (-1)^k I_{2k}(-G)\left[\frac{Cos(\omega_{bj} - b_1)t - Cos\omega_{bj} t}{b_1} + \frac{Cos(\omega_{bj} - b_2)t - Cos\omega_{bj} t}{b_2}\right] \\
+ \; & S_2\sum_{k=0}^{\infty} (-1)^k I_{2k+1}(-G)\left[\frac{\omega_{bj} Sin(\omega_{bj} - b_4)t - (\omega_{bj} - b_4)Sin\omega_{bj} t}{b_4} - \frac{\omega_{bj} Sin(\omega_{bj} - b_3)t - (\omega_{bj} - b_3)Sin\omega_{bj}}{b_3}\right.
\end{aligned}$$

(4.51)

here

$$Q_m = \frac{\eta Lg}{\Omega_0(k,m)} \tag{4.52}$$

Hence, in view of equation (4.5) and (4.52) we have

$$\begin{aligned}
V_n(x,t) = \sum_{m=1}^{\infty} \frac{Q_m}{\omega_{bj}^2} \Bigg(& \omega_{bj} a_1 J_0(G) \left[\frac{Cos(\omega_{bj} - b_0)t - Cos\,\omega_{bj} t}{b_0} \right] \\
+ \; & a_1 \omega_{bj} \sum_{k=1}^{\infty} J_{2k}(G) \left[\frac{Cos(\omega_{bj} - b_1)t - Cos\,\omega_{bj} t}{b_1} + \frac{Cos(\omega_{bj} - b_2)t - Cos\,\omega_{bj} t}{b_2} \right] \\
+ \; & a_0 \sum_{k=0}^{\infty} J_{2k+1}(G) \left[\frac{\omega_{bj} Sin(\omega_{bj} - b_4)t - (\omega_{bj} - b_4) Sin\,\omega_{bj} t}{b_4} - \frac{\omega_{bj} Sin(\omega_{bj} - b_3)t - (\omega_{bj} - b_3) Sin\,\omega_{bj} t}{b_3} \right] \\
+ \; & S_1 \omega_{bj} I_0(G) \left[\frac{Cos(\omega_{bj} - b_0)t - Cos\,\omega_{bj} t}{b_0} \right] \\
+ \; & S_1 \omega_{bj} \sum_{k=1}^{\infty} (-1)^k I_{2k}(G) \left[\frac{Cos(\omega_{bj} - b_1)t - Cos\,\omega_{bj} t}{b_1} + \frac{Cos(\omega_{bj} - b_2)t - Cos\,\omega_{bj} t}{b_2} \right] \\
+ \; & S_1 \sum_{k=0}^{\infty} (-1)^k I_{2k+1}(G) \left[\frac{\omega_{bj} Sin(\omega_{bj} - b_4)t - (\omega_{bj} - b_4) Sin\,\omega_{bj} t}{b_4} - \frac{\omega_{bj} Sin(\omega_{bj} - b_3)t - (\omega_{bj} - b_3) Sin\,\omega_{j}}{b_3} \right. \\
+ \; & S_2 \omega_{bj} I_0(-G) \left[\frac{Cos(\omega_{bj} - b_0)t - Cos\,\omega_{bj} t}{b_0} \right] \\
+ \; & S_2 \omega_{bj} \sum_{k=1}^{\infty} (-1)^k I_{2k}(-G) \left[\frac{Cos(\omega_{bj} - b_1)t - Cos\,\omega_{bj} t}{b_1} + \frac{Cos(\omega_{bj} - b_2)t - Cos\,\omega_{bj} t}{b_2} \right] \\
+ \; & S_2 \sum_{k=0}^{\infty} (-1)^k I_{2k+1}(-G) \left[\frac{\omega_{bj} Sin(\omega_{bj} - b_4)t - (\omega_{bj} - b_4) Sin\,\omega_{bj} t}{b_4} \right. \\
& \left. - \frac{\omega_{bj} Sin(\omega_{bj} - b_3)t - (\omega_{bj} - b_3) Sin\,\omega_{bj} t}{b_3} \right] \Bigg) \times \left(Sin \frac{\lambda_m x}{L} + A_m Cos \frac{\lambda_m x}{L} + B_m Sinh \frac{\lambda_m x}{L} + C_m Cosh \frac{\lambda_m x}{L} \right)
\end{aligned} \tag{4.53}$$

This represents the transverse-displacement response to a mass moving at ariable velocity of a non-uniform elastic thin beam resting on an elastic oundation.

CHAPTER FIVE

ILLUSTRATIVE EXAMPLES, NUMERICAL CALCULATIONS AND DISCUSSIONS OF RESULTS (NON-PRISMATIC BERNOULLI-EULER BEAM)

5.1.0 ILLUSTRATIVE EXAMPLES

As in the previous chapter, we shall now illustrate the foregoing analysis y various practical examples. Classical boundary conditions such as simply upported boundary conditions, free ends condition, clamped-clamped ends ndition and clamped-free end conditions shall be considered.

5.1.1 Simply Supported Boundary Conditions

The deflection and the bending moment at both ends vanish for a non-iform Bernoulli-Euler beam having simple supports at ends x = 0 and x = L; hus,

$$V(0,t) = 0 = V(L,t), \qquad \frac{\partial^2 V(0,t)}{\partial x^2} = 0 = \frac{\partial^2 V(L,t)}{\partial x^2} \tag{5.1}$$

d hence for the normal modes

$$U_m(0) = 0 = U_m(L), \qquad \frac{\partial^2 U_m(0)}{\partial x^2} = 0 = \frac{\partial^2 U_m(L)}{\partial x^2} \tag{5.2}$$

ich implies that

$$U_k(0) = 0 = U_k(L), \qquad \frac{\partial^2 U_k(0)}{\partial x^2} = 0 = \frac{\partial^2 U_k(L)}{\partial x^2} \tag{5.3}$$

Thus, it can be shown that

$$A_m = A_k = 0; \quad B_m = B_k = 0; \quad C_m = C_k = 0 \tag{5.4}$$

and the frequency equation becomes

$$Sin\lambda_m = Sin\lambda_k = 0 \tag{5.5}$$

which implies

$\lambda_m = m\pi \ and \ \lambda_k = k\pi$ respectively.

It follows therefore that the moving force problem for the non-uniform Bernoulli-Euler beam reduces to a non-homogeneous second order ordinary differential equation given as

$$\ddot{Y}_m(t) + \frac{\Delta_1(k,m)}{\Delta_0(k,m)} Y_m(t) = \frac{P}{\Delta_0(k,m)\mu_0} Sin \frac{k\pi}{L}(x_0 + \gamma Sin\beta t) \tag{5.6}$$

where

$$\begin{aligned}\Delta_1(k,m) = \frac{EI_0}{4\mu_0}\Bigg[& \frac{5m^4\pi^4}{L^3} - \frac{60m^3\pi^3k}{L^3\left[(1-k)^2 - m^2\right]\left[(1+k)^2 - m^2\right]} - \frac{6m^4\pi^4}{L^4}\left(\frac{\alpha_1}{2} - \frac{\alpha_3}{2}\right) \\ & + \frac{12m^3\pi^3k}{L^3\left[(3-k)^2 - m^2\right]\left[(3+k)^2 - m^2\right]} + \frac{60\pi^3m^3k(m^2+1-k^2)}{L^3\left[(1+m)^2 - k^2\right]\left[(1-m)^2 - k^2\right]} - \frac{24\pi^4m^3}{L^4}\left(\frac{\alpha_1}{2} - \frac{\alpha_4}{2}\right) \\ & - \frac{12\pi^3m^3k(m^2+9-k^2)}{L^3\left[(3+m)^2 - k^2\right]\left[(3-m)^2 - k^2\right]} + \frac{108\pi^3m^3k}{L^3\left[(3-k)^2 - m^2\right]\left[(3+k)^2 - m^2\right]} - \frac{24m^2\pi^2}{L^4}\left(\frac{\alpha_1}{2} - \frac{\alpha_2}{2}\right) \\ & - \frac{60m^3\pi^3k}{L^3\left[(1-k)^2 - m^2\right]\left[(1+k)^2 - m^2\right]}\Bigg] + \frac{Nm^2\pi^2}{2\mu_0 L} + \frac{K^0 L}{2\mu_0}\end{aligned} \tag{5.7}$$

and

$$\Delta_0(k,m) = \frac{L}{2} - \frac{4mkL}{\pi\left[(1-k)^2 - m^2\right]\left[(1+k)^2 - m^2\right]} \tag{5.8}$$

Equation (5.6) can be rewritten as

$$\ddot{Y}_m(t)+\omega_{mf}^2 Y_m(t)=P_{mf}\left[SinF^0 Cos\left(G^0 Sin\beta t\right)+CosF^0 Sin\left(G^0 Sin\beta t\right)\right] \tag{5.9}$$

where

$$\omega_{mf}^2=\frac{\Delta_1(k,m)}{\Delta_0(k,m)},\ P_{mf}=\frac{P}{\Delta_0(k,m)\mu_0},\ F^0=\frac{k\pi x_0}{L}\text{ and }G^0=\frac{k\pi\gamma}{L} \tag{5.10}$$

When equation (5.9) is solved in conjunction with the initial conditions, one obtains expression for $Y_m(t)$. Thus, in view of (4.5) and (2.16) one obtains

$$\begin{aligned}
V_n(x,t)=\sum_{m=1}^{n}\frac{P_{mf}}{\omega_{mf}^2}\Bigg\{&\omega_{mf}\frac{SinF^0 J_0(G^0)}{b_0}\left[Cos(\omega_{mf}-b_0)t-Cos\omega_{mf}t\right]\\
&+\omega_{mf}SinF^0\sum_{k=1}^{\infty}J_{2k}(G^0)\left[\frac{Cos(\omega_{mf}-b_1)t-Cos\omega_{mf}t}{b_1}+\frac{Cos(\omega_{mf}-b_2)t-Cos\omega_{mf}t}{b_2}\right]\\
&+CosF^0\sum_{k=0}^{\infty}J_{2k+1}(G^0)\left[\frac{\omega_{mf}Sin(\omega_{mf}-b_4)t-(\omega_{mf}-b_4)Sin\omega_{mf}t}{b_4}\right.\\
&\left.-\frac{\omega_{mf}Sin(\omega_{mf}-b_3)t-(\omega_{mf}-b_3)Sin\omega_{mf}t}{b_3}\right]\Bigg\}\times Sin\frac{m\pi x}{L}
\end{aligned} \tag{5.11}$$

which is the transverse-displacement response to a moving force traveling with a non-uniform speed of a simply supported non-uniform Bernoulli-Euler beam resting on an elastic foundation.

In what follows, we consider the moving mass problem that is, when $\varepsilon_1\neq 0$. Following the same arguments as in the previous sections, the modified frequency corresponding to the frequency of the free system due to the presence of the moving mass of the model is obtained as

$$\omega_{mw} = \omega_{mf}\left\{1-\frac{\eta_0}{2}\left[\left(R_1(k,m)+R_2(k,m)B_n^*\right)+\frac{\left(R_3(k,m)+R_4(k,m)B_n^*\right)}{2\omega_{mf}^2}\right]\right\} \tag{5.12}$$

to order $O(\eta_0)$ only.

where

$$R_1(k,m)=\frac{L}{2\Delta_n(k,m)},\quad R_2(k,m)=2R_1(k,m),\ R_3=\frac{\gamma\beta m\pi}{2\Delta_n(k,m)},\ R_4=2R_3(k,m) \tag{5.13}$$

and

$$\begin{aligned} B_n^* = {} & SinFJ_0(G)SinF''J_0(G'')+2SinFSinF''\sum_{k=1}^{\infty}J_{2k}(G)\sum_{k=1}^{\infty}J_{2k}(G'') \\ & +2CosFCosF''\sum_{k=0}^{\infty}J_{2k+1}(G)\sum_{k=0}^{\infty}J_{2k+1}(G'') \end{aligned} \tag{5.14}$$

Thus, the moving mass problem becomes

$$\frac{d^2Y_m(t)}{dt^2}+\omega_{mw}^2Y_m(t)=\frac{\varepsilon_1 Lg}{\Delta_n(k,m)}Sin\frac{k\pi}{L}(x_0+\gamma Sin\beta t) \tag{5.15}$$

Equation (5.15) when solved in conjunction with the initial conditions yields an expression for $Y_m(t)$ which on inversion gives

$$\begin{aligned} Y_n(x,t)= {} & \sum_{m=1}^{n}\frac{\varepsilon_1 Lg}{\Delta_n(k,m)\omega_{mw}^2}\left\{\omega_{mw}\frac{SinF''J_0(G'')}{b_0}\left[Cos(\omega_{mw}-b_0)t-Cos\omega_{mw}t\right]\right. \\ & +\omega_{mw}SinF''\sum_{k=1}^{\infty}J_{2k}(G'')\left[\frac{Cos(\omega_{mw}-b_1)t-Cos\omega_{mw}t}{b_1}+\frac{Cos(\omega_{mw}-b_2)t-Cos\omega_{mw}t}{b_2}\right] \\ & +CosF''\sum_{k=0}^{\infty}J_{2k+1}(G'')\left[\frac{\omega_{mw}Sin(\omega_{mw}-b_4)t-(\omega_{mw}-b_4)Sin\omega_{mw}t}{b_4}\right. \\ & \left.\left.-\frac{\omega_{mw}Sin(\omega_{mw}-b_3)t-(\omega_{mw}-b_3)Sin\omega_{mw}t}{b_3}\right]\right\}\times Sin\frac{m\pi x}{L} \end{aligned} \tag{5.16}$$

This is the transverse-displacement response to a moving mass of simply upported non-uniform long thin elastic beam resting on an elastic foundation.

5.1.2. Clamped-Clamped Ends Condition

At a clamped ends, both the transverse deflection and slope vanish. Thus, for a non-uniform Bernoulli-Euler beam clamped at end x=0 and end x=L, the conditions are expressed as

$$V(0,t) = 0 = V(L,t) \quad \text{and} \quad \frac{\partial V(0,t)}{\partial x} = 0 = \frac{\partial V(L,t)}{\partial x} \qquad (5.17)$$

and for normal modes

$$U_m(0) = 0 = U_m(L) \quad \text{and} \quad \frac{\partial U_m(0)}{\partial x} = 0 = \frac{\partial U_m(L)}{\partial x} \qquad (5.18)$$

which implies that

$$U_k(0) = 0 = U_k(L) \quad \text{and} \quad \frac{\partial U_k(0)}{\partial x} = 0 = \frac{\partial U_k(L)}{\partial x} \qquad (5.19)$$

Thus, it can be shown that

$$A_m = \frac{Sinh\lambda_m - Sin\lambda_m}{Cos\lambda_m - Cosh\lambda_m} = \frac{Cos\lambda_m - Cosh\lambda_m}{Sin\lambda_m + Sinh\lambda_m} = -C_m \quad \text{and} \quad B_m = -1 \qquad (5.20)$$

and we obtain as the corresponding frequency equation,

$$Cos\lambda_m Cosh\lambda_m = 1 \qquad (5.21)$$

Using equation (5.21) we can show that

$$\lambda_1 = 4.73004,\ \lambda_2 = 7.85320,\ \lambda_3 = 10.99561 \qquad (5.22)$$

When (5.20) and (5.22) are substituted into equations (4.25) and (4.53) one btains the transverse-displacement response respectively to a moving force and a moving mass of a non-unifrom Bernoulli-Euler beam clamped/fixed on elastic foundation.

5.1.3. One End Clamped One End Free Condition

At end x=0, the elastic beam is taken to be clamped and at the end x=L, the beam model is free. Thus, the boundary conditions of the beam can be written as

$$V(0,t)=0=\frac{\partial V(0,t)}{\partial x} \quad \text{and} \quad \frac{\partial^2 V(L,t)}{\partial x^2}=0=\frac{\partial^3 V(L,t)}{\partial x^3} \tag{5.23}$$

and for normal modes

$$U_m(0)=0=\frac{dU_m(0)}{dx} \quad \text{and} \quad \frac{d^2 U_m(L)}{dx^2}=0=\frac{d^3 U_m(L)}{dx^3} \tag{5.24}$$

which implies that

$$U_k(0)=0=\frac{dU_k(0)}{dx} \quad \text{and} \quad \frac{d^2 U_k(L)}{dx^2}=0=\frac{d^3 U_k(L)}{dx^3} \tag{5.25}$$

As in the previous section, we can show that

$$A_m=\frac{-Sin\lambda_m-Sinh\lambda_m}{Cos\lambda_m+Cosh\lambda_m}=\frac{-Cos\lambda_m-Cosh\lambda_m}{Sinh\lambda_m-Sin\lambda_m}=-C_m \qquad \text{and} \qquad B_m=-1 \tag{5.26}$$

It then follows that, the frequency equation for the system is given by

$$Cos\lambda_m Cosh\lambda_m=-1 \tag{5.27}$$

and we note that

$$\lambda_1=1.875,\ \lambda_2=4.694,\ \lambda_3=7.855 \tag{5.28}$$

Substituting (5.26) and (5.28) into equations (4.25) and (4.53), one obtains the transverse-displacement response to a moving force and a moving mass of a non-uniform clamped-free- ends of Bernoulli-Euler beam resting on elastic foundation are respectively obtained.

5.2.0 DISCUSSION OF THE ANALYTICAL SOLUTIONS

As discussed in the previous chapter, one is interested in the resonance conditions, because the transverse-displacement of an elastic beam may increase without limit. Equation (5.11) clearly shows that the simply supported elastic thin beam resting on an elastic foundation and transverse by a moving force reaches a state of resonance whenever

$$\omega_{mf} = 2k\beta \text{ and } \omega_{mf} = (2k+1)\beta \qquad (5.29)$$

whereas, equation (5.16) depicts that the same beam under the action of a moving mass will grow without bound whenever

$$\omega_{mm} = 2k\beta \text{ and } \omega_{mm} = (2k+1)\beta \qquad (5.30)$$

From equation (5.12)

$$\omega_{mm} = \omega_{mf}\left\{1-\frac{\eta_0}{2}\left[\left(R_1(k,m)+R_2(k,m)B_0^*\right)+\frac{\left(R_3(k,m)+R_4(k,m)B_0^*\right)}{2\omega_{mf}^2}\right]\right\} \qquad (5.31)$$

which implies

$$\omega_{mf} = \frac{2k\beta}{1-\frac{\varepsilon_1}{2}\left\{\left(R_1(k,m)+R_2(k,m)B_0^*\right)+\frac{\left(R_3(k,m)+R_4(k,m)B_0^*\right)}{2\omega_{mf}^2}\right\}} \qquad (5.32)$$

Therefore, it is evident that for the same natural frequency, the critical speed for the moving mass problem is smaller than that of the moving force problem. Consequently, the resonance is attained earlier in the moving mass system than in the moving force system.

Next, we investigate the phenomenon of resonance for other classical boundary conditions. It is evident from equation (4.25) that a non-uniform elastic beam resting on an elastic foundation and transverse by a force moving with variable velocities will grow without bound whenever

$$\omega_{aj} = 2k\beta \text{ and } \omega_{bj} = (2k+1)\beta \tag{5.33}$$

while equation (4.53) shows that the same beam under the action of a moving mass experiences a resonance effect whenever

$$\omega_{bj} = 2k\beta \text{ and } \omega_{bj} = (2k+1)\beta \tag{5.34}$$

From equation (4.48)

$$\omega_{bj} = \omega_{aj}\left\{1-\frac{\eta}{2}\left[\left(H_2(k,m)+R_a(k,m,n)\right)-(\gamma\beta)^2\frac{\left[H_6(k,m)+2R_b(k,m,n)\right]}{2\omega_{aj}^2}\right]\right\} \tag{5.35}$$

It follows that,

$$\omega_{aj} = \frac{2k\beta}{1-\frac{\varepsilon_1}{2}\left\{\left(H_2(k,m)+R_a(k,m,n)\right)-\frac{(\gamma\beta)^2\left[H_6(k,m)+2R_b(k,m,n)\right]}{2\omega_{aj}^2}\right\}} \tag{5.36}$$

Thus, from equations (5.33) and (5.36) it is evident that the same results and analysis are obtained for simply supported end conditions are also obtained for other examples of classical end support conditions.

5.3.0 NUMERICAL CALCULATION AND DISCUSSIONS OF RESULTS

Again, to illustrate the forgoing analysis, the uniform beam of length 12.192m is considered. Furthermore, $\frac{EI}{\mu} = 2200m^4/s^2$, $\gamma = 2\times10^{-4}m$, $\beta = \frac{3\pi}{4}$, $x_0 = \frac{1}{20}$ and the ratio of the mass of the load to the mass of the beam is 0.25. The transverse deflections of the beam are calculated and plotted against time for various values of axial force N and subgrade K. Values of N between 0 and 20,000,000 were used while the values of K were varied between 0 N/m^3 and 400,000 N/m^3. The results are as shown on the various graphs below for the various classes of boundary conditions so far considered.

5.3.1 Simply Supported Ends

Figure 5.1 displays transverse displacement response of a simply supported uniform beam under the action of forces moving at variable velocities for various values of axial force N for fixed value of foundation modulus K = 40,000. The figure shows that as N increases the deflection of the of the uniform beam decreases. The same results is obtained when the simply supported beam is traversed by concentrated masses moving at variable speed as shown in figure 5.3. Also, for various time t, the deflection profile of the beam for various values of foundation moduli K and for fixed axial force N are shown in figure 5.2. It is

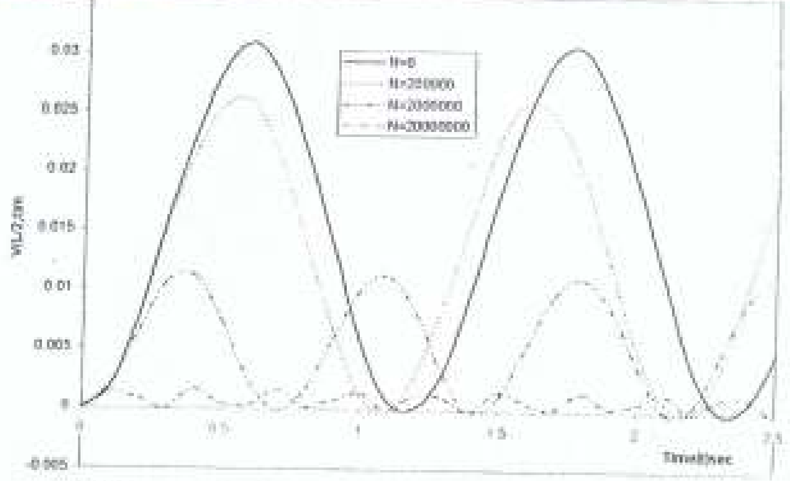

Fig 5.1: Transverse displacement of the simply supported non-uniform beam under the action of forces moving at variable velocities for various values of axial force N for fixed value of foundation moduli K (40000)

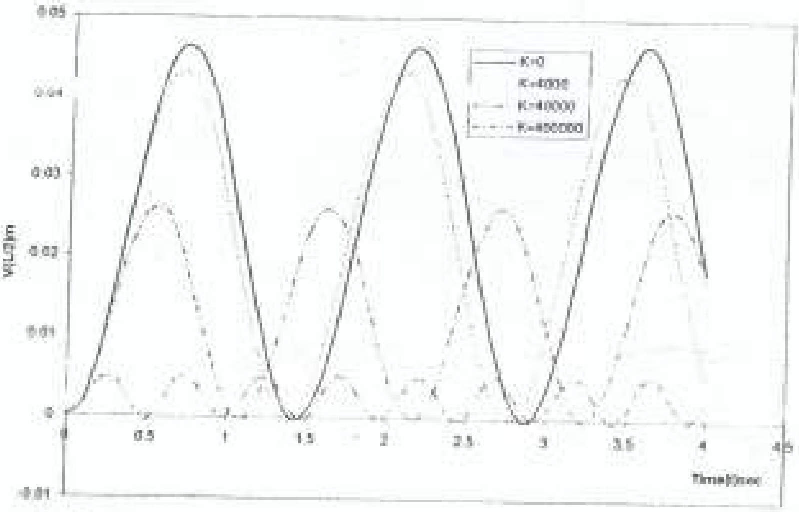

Fig 5.2: Deflection profile of the simply supported non-uniform beam under the action of forces moving at variable velocities for various values of foundation moduli K for fixed value of axial force N (200000)

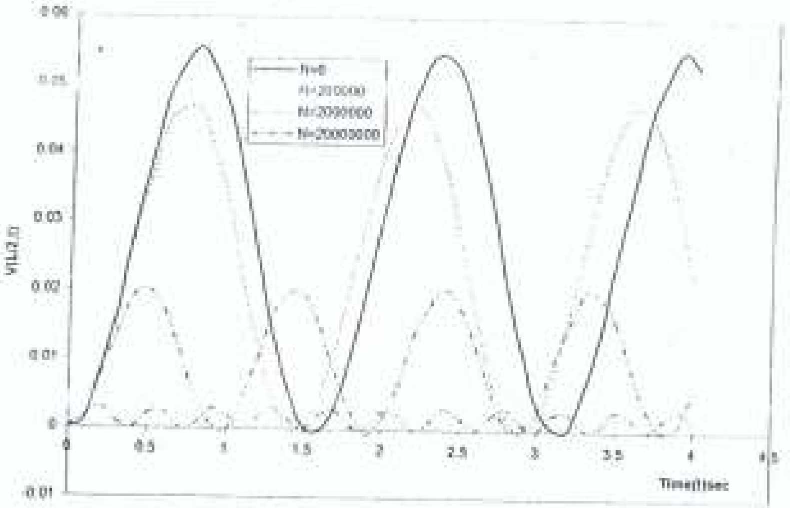

Fig 5.3: Transverse displacement of the simply supported non-uniform beam under the action of concentrated masses moving at variable velocities for various values of axial force N for fixed K (40000)

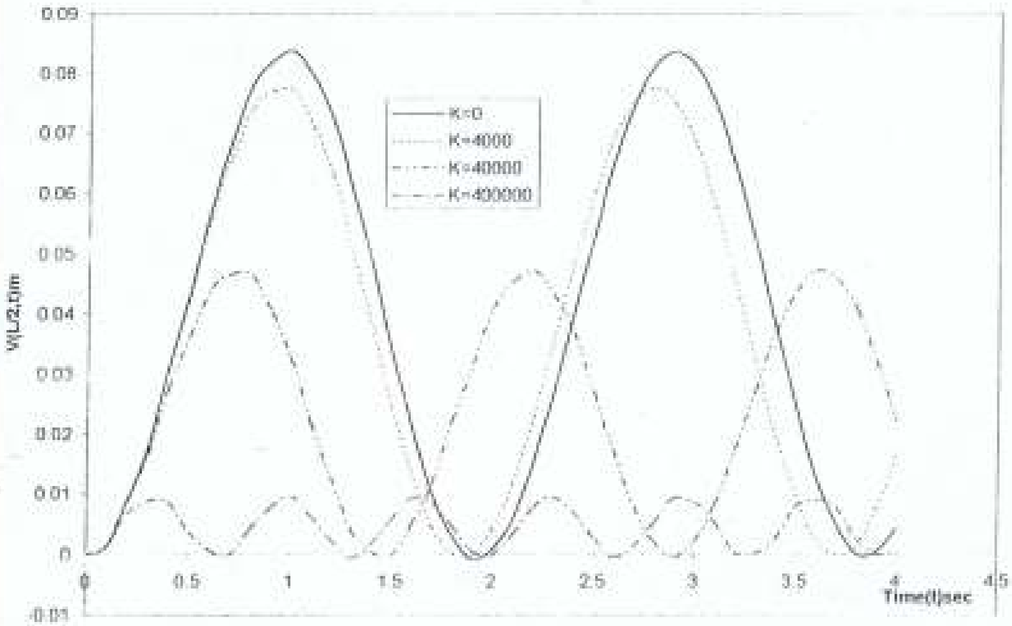

Fig 5.4: Deflection profile of the simply supported non-uniform beam under the action of concentrated masses moving at variable velocities for various values of foundation moduli K and for fixed N (200000)

shown that higher values of foundation moduli reduce the deflection profile of the beam. The same behaviour characterizes the deflection profile of the simply supported beam under the action of concentrated masses moving at variable velocities for various values of foundation moduli as shown in figure 5.4.

Finally, figure 5.5 depicts the comparison of the transverse displacement of moving force and moving mass cases for simply supported uniform beam traversed by a moving load moving at variable velocities for N = 200,000 and K = 40,000. Clearly, the response amplitudes of moving mass are higher than that of the moving force. Thus, the moving force result is not always an upper bound to a moving mass problem.

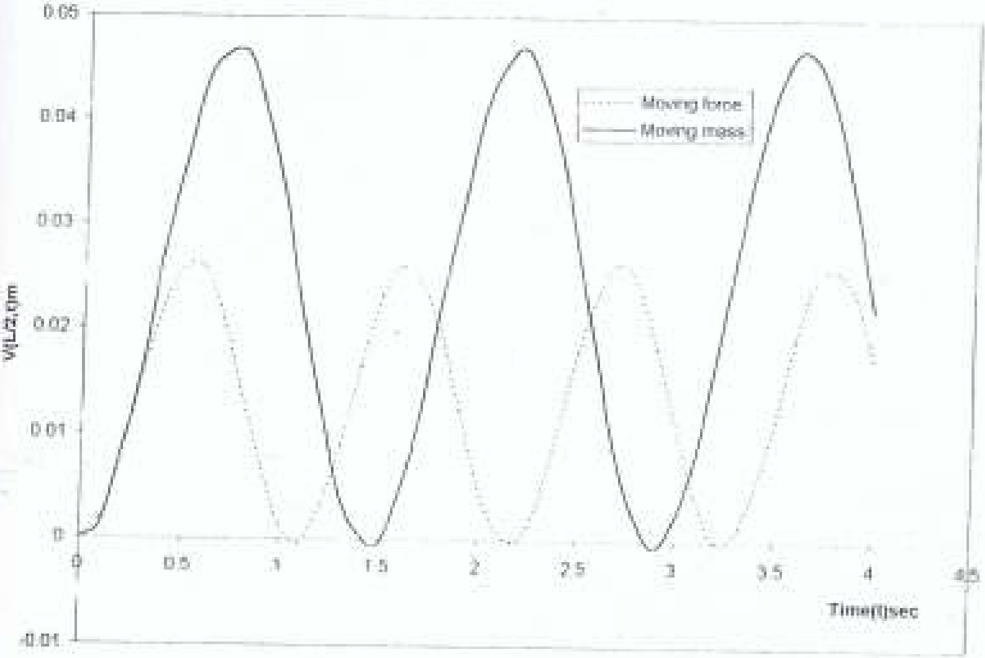

Fig 5.5: Comparison of the displacement of moving force and moving mass cases for simply supported non-uniform beam for N=200000 and K=40000

5.3.2 Clamped Ends

Figure 5.6 displays transverse displacement response of a clamped-clamped uniform beam under the action of forces moving at variable velocities for various values of axial force N for fixed value of foundation modulus

K = 40,000. The figure shows that as N increases the deflection of the of the uniform beam decreases. The same results is obtained when the clamped-clamped beam is traversed by a concentrated masses moving at variable speed as shown in figure 5.8. Also, for various time t, the deflection profile of the beam for various

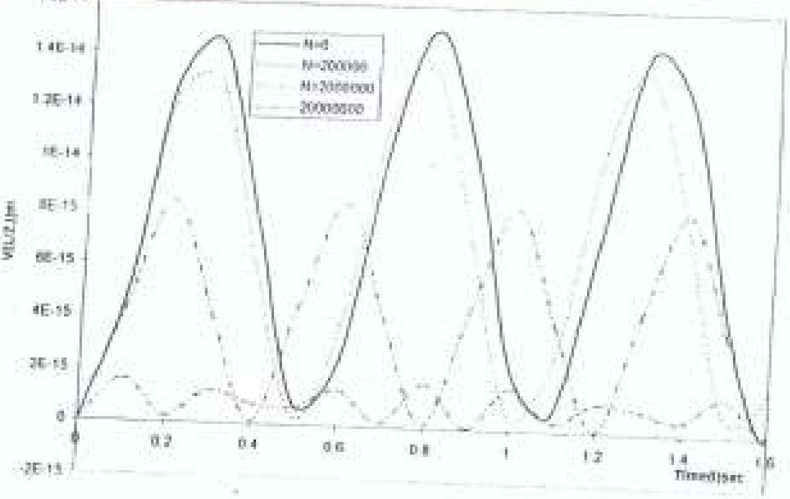

Fig. 5.6: Transverse displacement of the clamped-clamped non-unifrom beam under the action of forces moving at variable velocities for various values of axial force N for fixed K (40000)

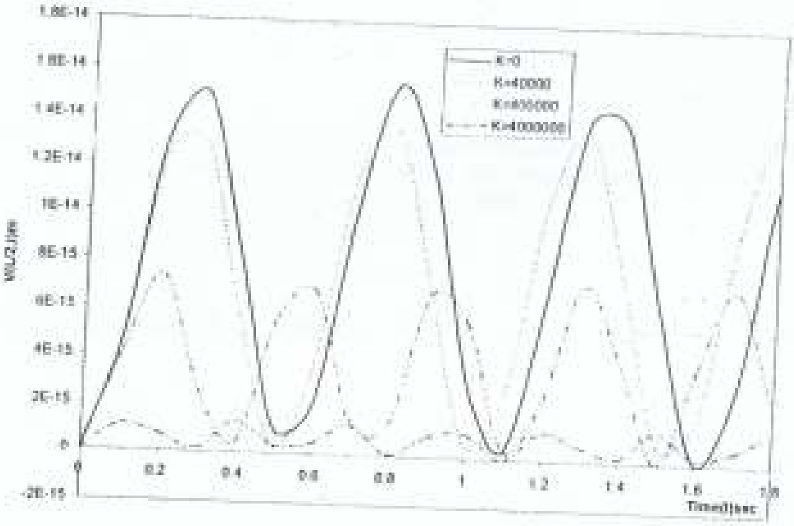

Fig. 5.7: Deflection profile of the clamped-clamped non-unifrom beam under the action of forces moving at variable velocities for various values of foundation moduli K for fixed value of axial force N (200000)

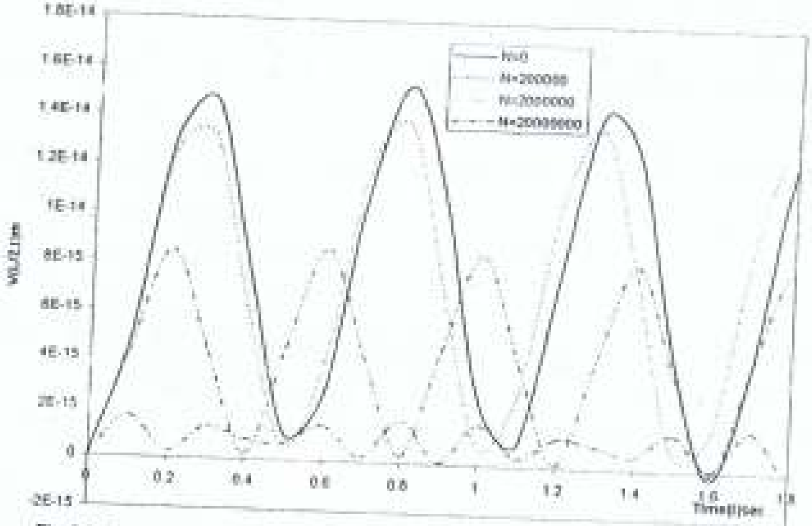

Fig. 5.8: Transverse displacement of the clamped-clamped non-uniform beam under the action of concentrated masses moving at variable velocities for various values of axial force N for fixed K (40000)

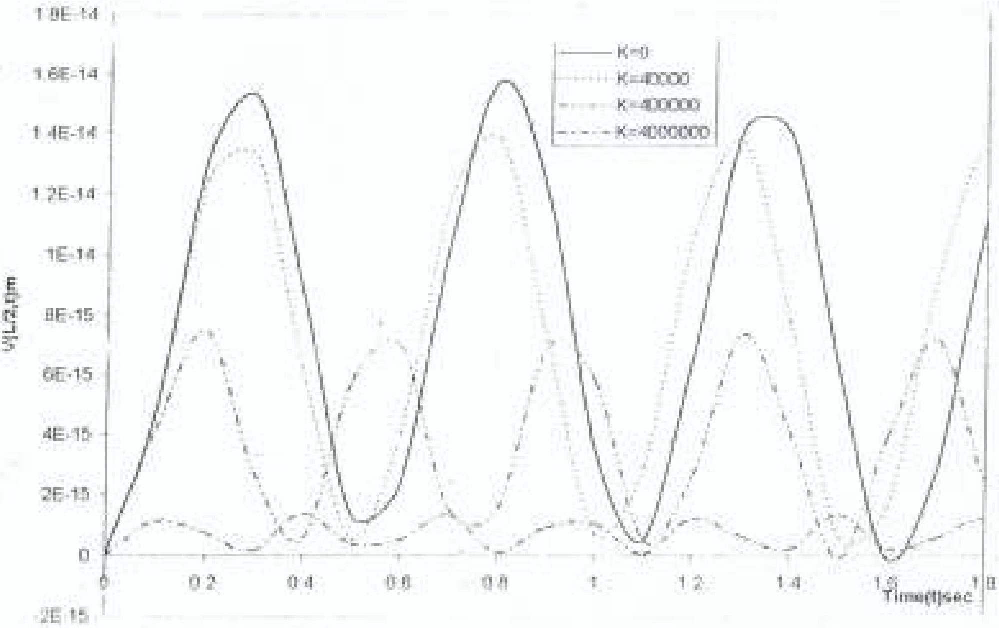

Fig. 5.9: Deflection profile of the clamped-clamped non-uniform beam under the action of concentrated masses moving at variable velocities for various values of foundation moduli K for fixed N (200000)

values of foundation moduli K and for fixed axial force N are shown in figure 5.7. It is shown that higher values of foundation moduli reduce the deflection profile of the beam. The same behaviour characterizes the deflection profile of the clamped-clamped beam under the action of concentrated masses moving at variable velocities for various values of foundation moduli as shown in figure 5.9. Finally, figure 5.10 depicts the comparison of the transverse displacement of moving force and moving mass cases for clamped-clamped uniform beam traversed by a moving load moving at variable velocities for N = 200,000 and K = 40,000. Clearly, the response amplitudes of moving mass is higher than that of the moving force. This important result shows that relying on moving force solution is seriously misleading.

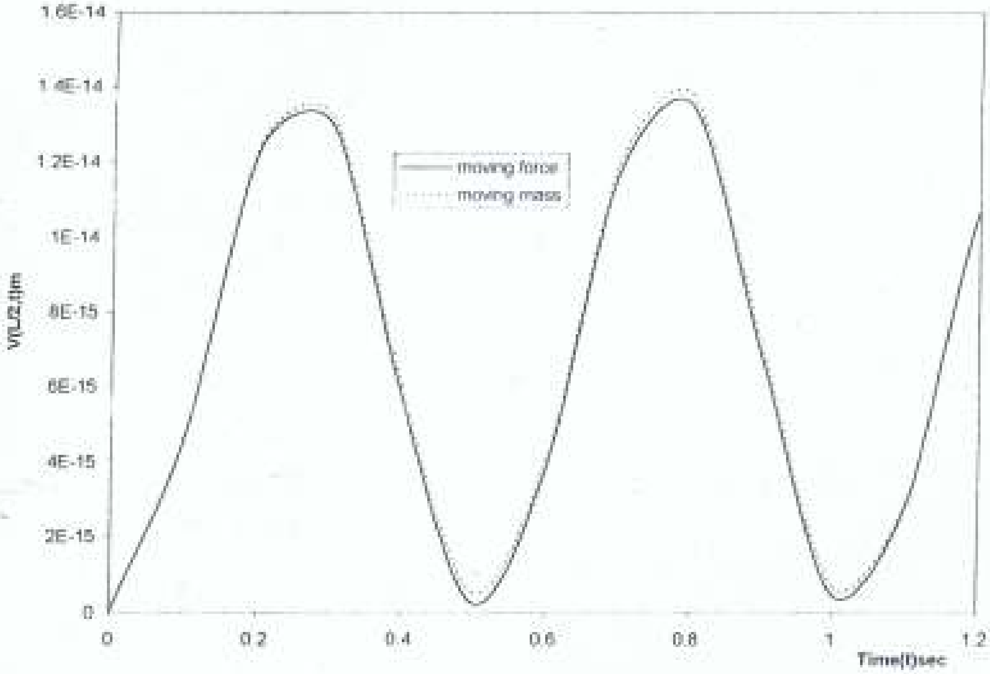

Fig. 5.10: Comparison of the displacement response of moving force and moving mass cases for clamped-clamped non-uniform beam for N=200000 and K=40000

5.3.3 One End Clamped and One End Free.

Figure 5.11 displays transverse displacement response of a cantilever uniform beam under the action of forces moving at variable velocities for various values of axial force N for fixed value of foundation modulus K = 40,000. The figure shows that as N increases the deflection of the of the uniform beam decreases. The same results is obtained when the cantilever beam is traversed by concentrated masses moving at variable speed as shown in figure 5.13. Also, for various time t, the deflection profile of the beam for various values of foundation moduli K and for fixed axial force N are shown in figure 5.12. It is shown that higher values of foundation moduli reduce the deflection profile of the beam. The same behaviour characterizes the deflection profile of the

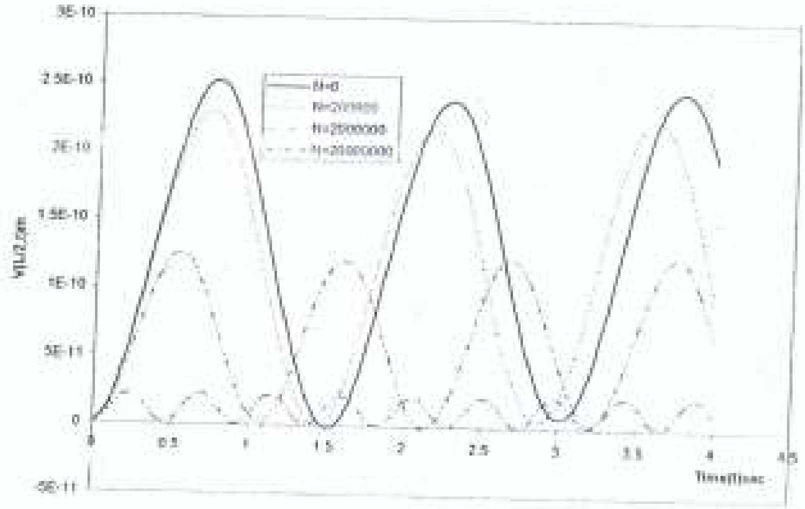

Fig. 5.11: Tranverse displacement of the clamped-free non-uniform beam under the action of forces moving at variable velocities for various values of axial force N for fixed value of foundation moduli K (40000)

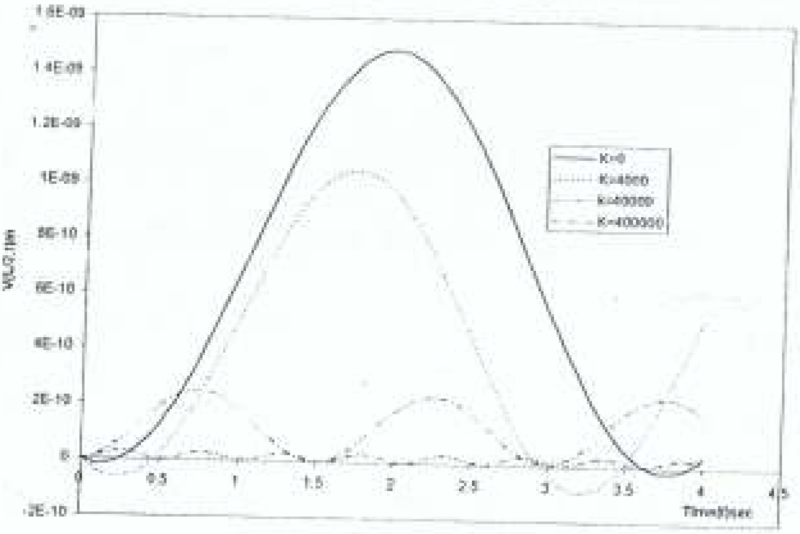

Fig. 5.12: Deflection profile of the clamped-free non-uniform beam under the action of forces moving at variable velocities for various values of foundation moduli for fixed N (20000)

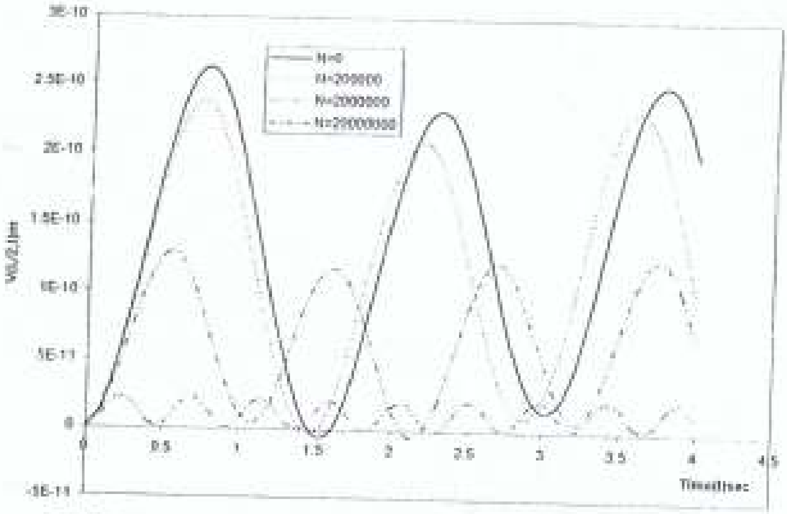

Fig. 5.13: Transverse displacement of the clamped-free non-uniform beam under the action of concentrated masses moving at variable velocities for various values of axial force N for fixed K (40000)

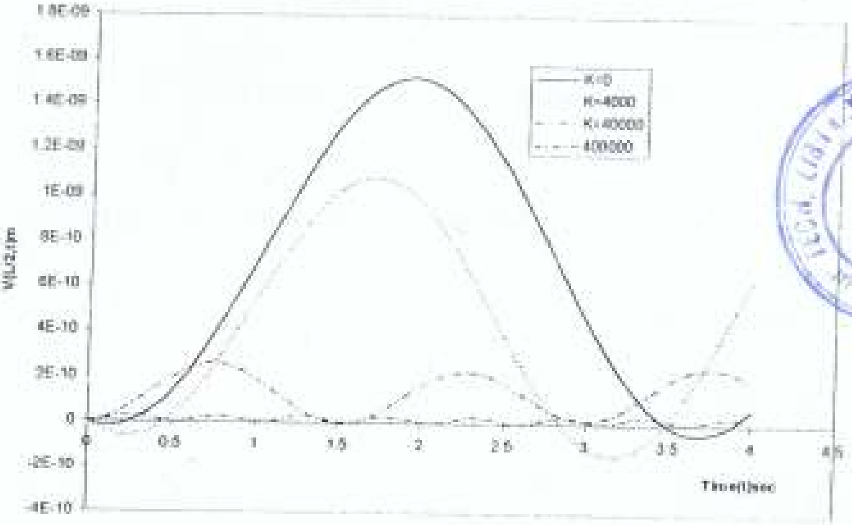

Fig. 5.14: Deflection profile of the clamped-free non uniform beam under the action of concentrated masses moving at variable velocities for various values of foundation moduli K for fixed N (20000)

moving at variable velocity for N = 200,000 and K = 40,000. Clearly, the response amplitudes of moving mass is higher than that of the moving force. This result shows that it is not safe to neglect the inertia effects of the moving loads in the dynamic analysis of moving load problems except the mass of the moving load is much smaller than the mass per unit length of the beam.

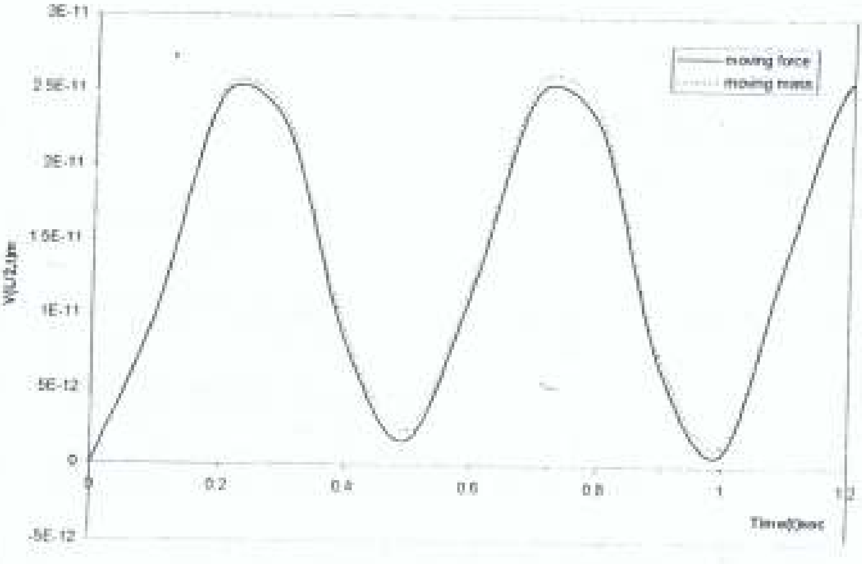

Fig. 5.15: Comparison of the displcaement response of the moving force and moving mass cases for clamped-free non-uniform beam for N=2000000 and K=400000.

CHAPTER SIX

GENERAL CONCLUSION

6.1 SUMMARY OF RESAERCH WORK

The problem of the flexural motion of a prestressed Bernoulli-Euler beam resting on elastic foundation and traversed by concentrated masses traveling at variable speeds has been investigated. Closed form solutions of the governing fourth order partial differential equations with variable and singular coefficients of

(i) Uniform Bernoulli-Euler beam and

(ii) Non-uniform Bernoulli-Euler beam

moving mass problems are presented.

Firstly, for the first problem involving uniform Bernoulli-Euler beam, the solution technique is based on generalized integral transformation, the expansion of the Dirac Delta function in series form, Galerkin's method, a modification of Struble's asymptotic method and the use of the generating functions of the Bessel functions.

Important features of this technique is that, is that is capable of solving moving mass beam problems involving

(i) Uniform beams other than Bernoulli-Euler beam.

(ii) Any choice of classical boundary condition often encountered in practice.

(iii) Moving loads moving with constant or variable velocities.

(iv) it can also handle moving mass problems of Rayleigh beams having uniform and non-uniform cross-section.

Unlike in the first problem, the second problem involving non-uniform Bernoulli-Euler beam is resistant to the generalized integral transformation technique. Thus, we resort to a modification of Galerkin's method to reduce the fourth order partial differential equation with singular and variable coefficients. The resulting Galerkin's equations are solved using the modified Struble's asymptotic method and the use of the generating functions of the Bessel function. This technique has the same features as the one used in tackling problem involving uniform Bernoulli-Euler beam.

In this work, illustrative examples involving

(i) Simply supported end conditions

(ii) Clamped end conditions

(iii) One end clamped and one end free conditions

are presented. The solutions hitherto obtained are analyzed and resonance conditions for the various problems obtained. Numerical analysis are carried out and the work exhibits the following interesting features:

(i) For all the four illustrative examples considered, the moving force solution is not an upper bound for the accurate solution of the moving mass solution in both uniform and non-uniform Euler-Bernoulli beams problems.

(ii) As the axial force N increases, the amplitudes of both uniform and non-uniform Bernoulli-Euler beams under the action of moving loads moving with variable velocities decrease.

(iii) When the axial force N is fixed, the displacements of a uniform Bernoulli-Euler beam resting on elastic foundation and traversed by masses traveling with variable speeds decrease as the foundation moduli increase for all variants of the boundary conditions. The same results obtain for non-uniform Bernoulli-Euler beams.

(iv) Higher values of axial force N and foundation modulus K are required for a more noticeable effect in the case of other boundary conditions than those of simply supported end conditions for both moving force and moving mass problems of both uniform and non-uniform beams.

(v) for fixed axial force and foundation modulus, the response amplitude for the moving mass problem is greater than that of the moving force problem for all illustrative end conditions considered whether the beam is uniform or non-uniform.

(vi) in all the illustrative examples considered, for the same natural frequency, the critical speed for moving mass problem is smaller than that of the moving force problem. Hence, resonance is reached earlier in moving mass problem.

(vii) In general, higher values of axial force N and foundation modulus K are required for a more noticeable effect on the response amplitudes of non-uniform Bernoulli-Euler beams than would be required for similar uniform Bernoulli-Euler beams moving mass problems for clamped-clamped and clamped-free end conditions.

Finally, this work has suggested valuable methods of analytical solution for this category of problems for all variants of classical boundary conditions.

6.2 CONTRIBUTIONS TO KNOWLEDGE

Apart from the fact that this study has provided a closed form solutions for both problems of uniform and non-uniform Bernoulli-Euler beams resting on elastic foundation and traversed by concentrated masses moving at variable velocities, this study has

(i) provided vital information on the effect of axial force on uniform and non-uniform Bernoulli-Euler beams under the action of concentrated masses moving at variable speeds.

(ii) classified the influence of the elastic foundation modulus on the displacement response of both uniform and non-uniform Bernoulli-Euler beams.

iii) provided useful information on the resonance conditions for both moving force and moving mass problems of uniform and non-uniform Bernoulli-Euler beam.

(iv) shed more light on the reliability of the moving force solution as a safe approximation to the moving mass problem for all variants of classical boundary conditions

6.3 LIMITATIONS TO STUDY AND RECOMMENDATIONS FOR FURTHER RESEARCH

The focus of this study is on the flexural motion of prestressed Bernoulli-Euler beam resting on elastic foundation and traversed by concentrated masses moving at variable velocities.

Illustrative examples have been limited to classical boundary conditions only. Non-classical boundary conditions such as (i) elastically supported end conditions (ii) time dependent boundary conditions are not considered and as such are suggested for future research. The two-dimensional analogue of the theory developed in this thesis could be extended to the corresponding moving load rectangular plate problems. Structures (beams or plates) on other foundation models are left for further research. Other beam models under the action of moving loads such as shear beams, Rayleigh beams and Timoshenko beams described by single equations resting on Winkler or non-Winkler elastic foundation and visco-elastic foundation are not considered in this study.

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APPENDIX

SOLUTIONS OF SOME INTEGRALS

This appendix presents the solutions of the definite integrals listed in chapters two and four of this thesis.

$$I_1 = \begin{cases} \dfrac{L}{2}\left[\dfrac{Sin(\lambda_k - \lambda_w)}{\lambda_k - \lambda_w} - \dfrac{Sin(\lambda_k + \lambda_w)}{\lambda_k + \lambda_w}\right] & , \quad \lambda_k \neq \lambda_w \\ \dfrac{L}{2}\left[1 - \dfrac{Sin2\lambda_m}{2\lambda_m}\right] & , \quad \lambda_k \neq \lambda_w \end{cases}$$

$$I_2 = \begin{cases} \dfrac{-L}{2}\left[\dfrac{(\lambda_k - \lambda_w)(Cos(\lambda_k + \lambda_{m)}) - 1) + (\lambda_k + \lambda_w)(Cos(\lambda_k - \lambda_{m)}) - 1)}{\lambda_k^2 - \lambda_m^2}\right] & , \quad \lambda_k \neq \lambda_m \\ \dfrac{-L}{2}\left[\dfrac{Cos2\lambda_w - 1}{2\lambda_m}\right] & , \quad \lambda_k = \lambda_m \end{cases}$$

$$I_3 = \frac{\lambda_m L}{\lambda_m^2 + \lambda_k^2}\left[Sin\lambda_k Cosh\lambda_m - \frac{\lambda_k}{\lambda_m} Cos\lambda_k Sinh\lambda_m\right]$$

$$I_4 = \frac{\lambda_m L}{\lambda_w^2 + \lambda_k^2}\left[Sin\lambda_k Sinh\lambda_w - \frac{\lambda_k}{\lambda_w}(Cos\lambda_k Cosh\lambda_m - 1)\right]$$

$$I_5 = \begin{cases} \dfrac{L}{2}\left[\dfrac{(\lambda_k + \lambda_m)(Cos(\lambda_k - \lambda_{w)}) - 1) + (\lambda_k - \lambda_w)(1 - Cos(\lambda_k + \lambda_{w)}))}{\lambda_k^2 - \lambda_w^2}\right] & , \quad \lambda_k \neq \lambda_m \\ \dfrac{-L}{2}\left[\dfrac{Cos2\lambda_w - 1}{2\lambda_m}\right] & , \quad \lambda_k = \lambda_m \end{cases}$$

$$I_6 = \begin{cases} \dfrac{L}{2}\left[\dfrac{Sin(\lambda_k + \lambda_m)}{\lambda_k + \lambda_m} + \dfrac{Sin(\lambda_k - \lambda_m)}{\lambda_k - \lambda_m}\right] & , \quad \lambda_k \neq \lambda_m \\ \dfrac{L}{2}\left[1 + \dfrac{Sin2\lambda_m}{2\lambda_m}\right] & , \quad \lambda_k \neq \lambda_w \end{cases}$$

$$I_7 = \frac{\lambda_m L}{\lambda_w^2 + \lambda_k^2}\left[Cos\lambda_k Cosh\lambda_w + \frac{\lambda_k}{\lambda_w} Sin\lambda_k Sinh\lambda_m - 1\right]$$

$$I_8 = \frac{\lambda_w L}{\lambda_w^2 + \lambda_k^2}\left[Cos\lambda_k Sinh\lambda_w + \frac{\lambda_k}{\lambda_w} Sin\lambda_k Cosh\lambda_m\right]$$

$$I_9 = \frac{\lambda_k L}{\lambda_k^2 + \lambda_m^2}\left[Sin\lambda_m Cosh\lambda_k - \frac{\lambda_m}{\lambda_k} Cos\lambda_m Sinh\lambda_k \right]$$

$$I_{10} = \frac{\lambda_k L}{\lambda_k^2 + \lambda_m^2}\left[Cos\lambda_m Cosh\lambda_k + \frac{\lambda_m}{\lambda_k} Sin\lambda_m Sinh\lambda_k - 1 \right]$$

$$I_{11} = \begin{cases} \frac{L}{2}\left[\frac{Sinh(\lambda_k + \lambda_m)}{\lambda_k + \lambda_m} + \frac{Sinh(\lambda_k - \lambda_m)}{\lambda_k - \lambda_m} \right] & , \quad \lambda_k \neq \lambda_m \\ \frac{L}{2}\left[\frac{Sinh 2\lambda_m}{2\lambda_m} - 1 \right] & , \quad \lambda_k \neq \lambda_m \end{cases}$$

$$I_{12} = \begin{cases} \frac{L}{2}\left[\frac{(\lambda_k - \lambda_m)(Cosh(\lambda_k + \lambda_m) - 1) + (\lambda_k + \lambda_m)(Cosh(\lambda_k - \lambda_m) - 1)}{\lambda_k^2 + \lambda_m^2} \right] & , \quad \lambda_k \neq \lambda_m \\ \frac{L}{2}\left[\frac{Cos 2\lambda_m - 1}{2\lambda_m} \right] & , \quad \lambda_k = \lambda_m \end{cases}$$

$$I_{13} = \frac{\lambda_k L}{\lambda_k^2 + \lambda_m^2}\left[Sin\lambda_m Sinh\lambda_k - \frac{\lambda_m}{\lambda_k}(Cos\lambda_m Cosh\lambda_k - 1) \right]$$

$$I_{14} = \frac{\lambda_k L}{\lambda_k^2 + \lambda_m^2}\left[Cos\lambda_m Sinh\lambda_k + \frac{\lambda_m}{\lambda_k} Sin\lambda_m Cosh\lambda_k \right]$$

$$I_{15} = \begin{cases} \frac{L}{2}\left[\frac{(\lambda_m - \lambda_k)(Cosh(\lambda_m + \lambda_k) - 1) + (\lambda_m + \lambda_k)(Cosh(\lambda_m - \lambda_k) - 1)}{\lambda_m^2 + \lambda_k^2} \right] & , \quad \lambda_k \neq \lambda_m \\ \frac{L}{2}\left[\frac{Cos 2\lambda_m - 1}{2\lambda_m} \right] & , \quad \lambda_k = \lambda_m \end{cases}$$

$$I_{16} = \begin{cases} \frac{L}{2}\left[\frac{Sinh(\lambda_m + \lambda_k)}{\lambda_k + \lambda_m} + \frac{Sinh(\lambda_k - \lambda_m)}{\lambda_k - \lambda_m} \right] & , \quad \lambda_k \neq \lambda_m \\ \frac{L}{2}\left[\frac{Sinh 2\lambda_m}{2\lambda_m} + 1 \right] & , \quad \lambda_k \neq \lambda_m \end{cases}$$

$$I_{17} = \begin{cases} \frac{L}{4}\left[\frac{Sin(n\pi + \lambda_k - \lambda_m)}{(n\pi + \lambda_k - \lambda_m)} + \frac{Sin(n\pi - \lambda_k + \lambda_m)}{(n\pi - \lambda_k + \lambda_m)} - \frac{Sin(n\pi + \lambda_k + \lambda_m)}{(n\pi + \lambda_k + \lambda_m)} - \frac{Sin(n\pi - \lambda_k - \lambda_m)}{(n\pi - \lambda_k - \lambda_m)} \right] \\ \frac{L}{4}\left[1 - \frac{Sin 2(n\pi + \lambda_k)}{2(n\pi + \lambda_k)} \right] \left| \begin{array}{l} \lambda_m = n\pi + \lambda_k \\ n\pi = \lambda_m - \lambda_k \end{array} \right. - \frac{L}{4}\left[1 - \frac{Sin 2(n\pi - \lambda_k)}{2(n\pi - \lambda_k)} \right] \left| \begin{array}{l} \lambda_m = n\pi + \lambda_k \\ n\pi = \lambda_m - \lambda_k \end{array} \right. \end{cases}$$

$$I_{18} = \frac{L}{4}\left[\frac{(Cos(n\pi - \lambda_k - \lambda_m) - 1)}{(n\pi - \lambda_k - \lambda_m)} + \frac{(1 - Cos(n\pi + \lambda_k + \lambda_m))}{(n\pi + \lambda_k + \lambda_m)} + \frac{(Cos(n\pi - \lambda_k + \lambda_m) - 1)}{(n\pi - \lambda_k + \lambda_m)} + \frac{(1 - Cos(n\pi + \lambda_k - \lambda_m))}{(n\pi + \lambda_k - \lambda_m)} \right]$$

$$I_{19} = \frac{L}{2}\left[\frac{\lambda_m}{\lambda_m^2 + (n\pi + \lambda_k)^2}\left(Sin(n\pi + \lambda_k)Cosh\lambda_m - \frac{(n\pi + \lambda_k)}{\lambda_m}Cos(n\pi + \lambda_k)Sinh\lambda_m\right)\right.$$

$$\left. + \frac{\lambda_m}{\lambda_m^2 + (n\pi - \lambda_k)^2}\left(Sin(n\pi - \lambda_k)Cosh\lambda_m - \frac{(n\pi - \lambda_k)}{\lambda_m}Cos(n\pi - \lambda_k)Sinh\lambda_m\right)\right]$$

$$I_{20} = \frac{L}{2}\left[\frac{\lambda_m}{\lambda_m^2 + (n\pi + \lambda_k)^2}\left(Sin(n\pi + \lambda_k)Sinh\lambda_m - \frac{(n\pi + \lambda_k)}{\lambda_m}(Cos(n\pi + \lambda_k)Cosh\lambda_m - 1)\right)\right.$$

$$\left. - \frac{\lambda_m}{\lambda_m^2 + (n\pi - \lambda_k)^2}\left(Sin(n\pi - \lambda_k)Sinh\lambda_m - \frac{(n\pi - \lambda_k)}{\lambda_m}(Cos(n\pi - \lambda_k)Cosh\lambda_m - 1)\right)\right]$$

$$I_{21} = \frac{L}{4}\left[\frac{(Cos(n\pi + \lambda_k - \lambda_m) - 1)}{(n\pi - \lambda_k + \lambda_m)} + \frac{(1 - Cos(n\pi + \lambda_k + \lambda_m))}{(n\pi + \lambda_k + \lambda_m)} + \frac{(Cos(n\pi - \lambda_k - \lambda_m) - 1)}{(n\pi - \lambda_k - \lambda_m)} + \frac{(1 - Cos(n\pi - \lambda_k + \lambda_m))}{(n\pi - \lambda_k + \lambda_m)}\right.$$

$$I_{22} = \frac{L}{4}\left[\frac{Sin(n\pi + \lambda_k + \lambda_m)}{(n\pi + \lambda_k + \lambda_m)} + \frac{Sin(n\pi - \lambda_k - \lambda_m)}{(n\pi - \lambda_k - \lambda_m)} + \frac{Sin(n\pi + \lambda_k - \lambda_m)}{(n\pi + \lambda_k - \lambda_m)} - \frac{Sin(n\pi - \lambda_k + \lambda_m)}{(n\pi - \lambda_k + \lambda_m)}\right]$$

$$I_{23} = \frac{L}{2}\left[\frac{\lambda_m}{\lambda_m^2 + (n\pi + \lambda_k)^2}\left(Cos(n\pi + \lambda_k)Cosh\lambda_m + \frac{(n\pi + \lambda_k)}{\lambda_m}Sin(n\pi + \lambda_k)Sinh\lambda_m - 1\right)\right.$$

$$\left. + \frac{\lambda_m}{\lambda_m^2 + (n\pi - \lambda_k)^2}\left(Cos(n\pi - \lambda_k)Cosh\lambda_m + \frac{(n\pi - \lambda_k)}{\lambda_m}Sin(n\pi - \lambda_k)Sinh\lambda_m - 1\right)\right]$$

$$I_{24} = \frac{L}{2}\left[\frac{\lambda_m}{\lambda_m^2 + (n\pi + \lambda_k)^2}\left(Cos(n\pi + \lambda_k)Sinh\lambda_m + \frac{(n\pi + \lambda_k)}{\lambda_m}Sin(n\pi + \lambda_k)Cosh\lambda_m\right)\right.$$

$$\left. + \frac{\lambda_m}{\lambda_m^2 + (n\pi - \lambda_k)^2}\left(Cos(n\pi - \lambda_k)Sinh\lambda_m + \frac{(n\pi - \lambda_k)}{\lambda_m}Sin(n\pi - \lambda_k)Cosh\lambda_m\right)\right]$$

$$I_{25} = \frac{L}{2}\left[\frac{\lambda_k}{\lambda_k^2 + (n\pi + \lambda_m)^2}\left(Sin(n\pi + \lambda_m)Cosh\lambda_k - \frac{(n\pi + \lambda_m)}{\lambda_k}Cos(n\pi + \lambda_m)Sinh\lambda_k\right)\right.$$

$$\left. + \frac{\lambda_k}{\lambda_k^2 + (n\pi - \lambda_m)^2}\left(Sin(n\pi - \lambda_m)Cosh\lambda_k - \frac{(n\pi - \lambda_m)}{\lambda_k}Cos(n\pi - \lambda_m)Sinh\lambda_k\right)\right]$$

$$I_{26} = \frac{L}{2}\left[\frac{\lambda_k}{\lambda_k^2 + (n\pi + \lambda_m)^2}\left(Cos(n\pi + \lambda_m)Cosh\lambda_k + \frac{(n\pi + \lambda_m)}{\lambda_k}Sin(n\pi + \lambda_m)Sinh\lambda_k - 1\right)\right.$$

$$\left. + \frac{\lambda_k}{\lambda_k^2 + (n\pi - \lambda_m)^2}\left(Cos(n\pi - \lambda_m)Cosh\lambda_k + \frac{(n\pi - \lambda_m)}{\lambda_k}Sin(n\pi - \lambda_m)Sinh\lambda_k - 1\right)\right]$$

$$I_{27} = \frac{L}{2}\left[\frac{(\lambda_k + \lambda_m)(-1)^n Sinh(\lambda_k + \lambda_m)}{(\lambda_k + \lambda_m)^2 + (n\pi)^2} - \frac{(\lambda_k - \lambda_m)(-1)^n Sinh(\lambda_k - \lambda_m)}{(\lambda_k - \lambda_m)^2 + (n\pi)^2}\right]$$

$$I_{28} = \frac{L}{2}\left[\frac{(\lambda_k + \lambda_m)((-1)^n Cosh(\lambda_k + \lambda_m) - 1)}{(\lambda_k + \lambda_m)^2 + (n\pi)^2} - \frac{(\lambda_k - \lambda_m)((-1)^n Cosh(\lambda_k - \lambda_m) - 1)}{(\lambda_k - \lambda_m)^2 + (n\pi)^2}\right]$$

$$I_{29} = \frac{L}{2}\left[\frac{\lambda_k}{\lambda_k^2 + (n\pi + \lambda_m)^2}\left(Sin(n\pi + \lambda_m)Sinh\lambda_k - \frac{(n\pi + \lambda_m)}{\lambda_k}(Cos(n\pi + \lambda_m)Cosh\lambda_k - 1)\right)\right.$$
$$\left. - \frac{\lambda_k}{\lambda_k^2 + (n\pi - \lambda_m)^2}\left(Sin(n\pi - \lambda_m)Sinh\lambda_k - \frac{(n\pi - \lambda_m)}{\lambda_k}(Cos(n\pi - \lambda_m)Cosh\lambda_k - 1)\right)\right]$$

$$I_{30} = \frac{L}{2}\left[\frac{\lambda_k}{\lambda_k^2 + (n\pi + \lambda_m)^2}\left(Cos(n\pi + \lambda_m)Sinh\lambda_k + \frac{(n\pi + \lambda_m)}{\lambda_k}Sin(n\pi + \lambda_m)Cosh\lambda_k\right)\right.$$
$$\left. + \frac{\lambda_k}{\lambda_k^2 + (n\pi - \lambda_m)^2}\left(Cos(n\pi - \lambda_m)Sinh\lambda_k + \frac{(n\pi - \lambda_m)}{\lambda_k}Sin(n\pi - \lambda_m)Cosh\lambda_k\right)\right]$$

$$I_{31} = \frac{L}{2}\left[\frac{(\lambda_m + \lambda_k)((-1)^n Cosh(\lambda_m + \lambda_k) - 1)}{(\lambda_m + \lambda_k)^2 + (n\pi)^2} - \frac{(\lambda_m - \lambda_k)((-1)^n Cosh(\lambda_m - \lambda_k) - 1)}{(\lambda_m - \lambda_k)^2 + (n\pi)^2}\right]$$

$$I_{32} = \frac{L}{2}\left[\frac{(\lambda_k + \lambda_m)(-1)^n Sinh(\lambda_k + \lambda_m)}{(\lambda_k + \lambda_m)^2 + (n\pi)^2} - \frac{(\lambda_m - \lambda_k)(-1)^n Sinh(\lambda_m - \lambda_k)}{(\lambda_m - \lambda_k)^2 + (n\pi)^2}\right]$$

$$I_{33} = \frac{L}{4}\left[\frac{(Cos(n\pi + \lambda_k + \lambda_m) - 1)}{(n\pi + \lambda_k + \lambda_m)} - \frac{(Cos(n\pi + \lambda_k - \lambda_m) - 1)}{(n\pi + \lambda_k - \lambda_m)} + \frac{(Cos(n\pi - \lambda_k - \lambda_m) - 1)}{(n\pi - \lambda_k - \lambda_m)}\right.$$
$$\left. - \frac{(Cos(n\pi - \lambda_k + \lambda_m) - 1)}{(n\pi - \lambda_k + \lambda_m)}\right]$$

$$I_{34} = \frac{L}{4}\left[\frac{Sin(n\pi - \lambda_k - \lambda_m)}{(n\pi - \lambda_k - \lambda_m)} - \frac{Sin(n\pi + \lambda_k + \lambda_m)}{(n\pi + \lambda_k + \lambda_m)} + \frac{Sin(n\pi - \lambda_k + \lambda_m)}{(n\pi - \lambda_k + \lambda_m)} - \frac{Sin(n\pi + \lambda_k - \lambda_m)}{(n\pi + \lambda_k - \lambda_m)}\right]$$

$$I_{35} = \frac{L}{2}\left[\frac{\lambda_m}{\lambda_m^2 + (n\pi - \lambda_k)^2}\left(Cos(n\pi - \lambda_k)Cosh\lambda_m + \frac{(n\pi - \lambda_k)}{\lambda_m}Sin(n\pi - \lambda_k)Sinh\lambda_m - 1\right)\right.$$
$$\left. - \frac{\lambda_m}{\lambda_m^2 + (n\pi + \lambda_k)^2}\left(Cos(n\pi + \lambda_k)Cosh\lambda_m + \frac{(n\pi + \lambda_k)}{\lambda_m}Sin(n\pi + \lambda_k)Sinh\lambda_m - 1\right)\right]$$

$$I_{36} = \frac{L}{2}\left[\frac{\lambda_m \cdot}{\lambda_m^2 + (n\pi - \lambda_k)^2}\left(Cos(n\pi - \lambda_k)Sinh\lambda_m + \frac{(n\pi - \lambda_k)}{\lambda_m}Sin(n\pi - \lambda_k)Cosh\lambda_m\right)\right.$$
$$\left. - \frac{\lambda_m}{\lambda_m^2 + (n\pi + \lambda_k)^2}\left(Cos(n\pi + \lambda_k)Sinh\lambda_m + \frac{(n\pi + \lambda_k)}{\lambda_m}Sin(n\pi + \lambda_k)Cosh\lambda_m\right)\right]$$

$$I_{37} = \frac{L}{4}\left[\frac{Sin(n\pi - \lambda_m - \lambda_k)}{(n\pi - \lambda_m - \lambda_k)} - \frac{Sin(n\pi + \lambda_m + \lambda_k)}{(n\pi + \lambda_m + \lambda_k)} + \frac{Sin(n\pi - \lambda_m + \lambda_k)}{(n\pi - \lambda_m + \lambda_k)} - \frac{Sin(n\pi + \lambda_m - \lambda_k)}{(n\pi + \lambda_m - \lambda_k)}\right]$$

$$I_{38} = -\frac{L}{4}\left[\frac{(Cos(n\pi + \lambda_k + \lambda_m) - 1)}{(n\pi + \lambda_k + \lambda_m)} + \frac{(Cos(n\pi - \lambda_k - \lambda_m) - 1)}{(n\pi - \lambda_k - \lambda_m)} + \frac{(Cos(n\pi + \lambda_k - \lambda_m) - 1)}{(n\pi + \lambda_k - \lambda_m)}\right.$$

$$\left. + \frac{(Cos(n\pi - \lambda_k + \lambda_m) - 1)}{(n\pi - \lambda_k + \lambda_m)}\right]$$

$$I_{39} = \frac{L}{2}\left[\frac{\lambda_m}{\lambda_m^2 + (n\pi + \lambda_k)^2}\left(Sin(n\pi + \lambda_k)Cosh\lambda_m - \frac{(n\pi + \lambda_k)}{\lambda_m}Cos(n\pi + \lambda_k)Sinh\lambda_m\right)\right.$$

$$\left. + \frac{\lambda_m}{\lambda_m^2 + (\lambda_k - n\pi)^2}\left(Sin(\lambda_k - n\pi)Cosh\lambda_m - \frac{(n\pi - \lambda_k)}{\lambda_m}Cos(\lambda_k - n\pi)Sinh\lambda_m\right)\right]$$

$$I_{40} = \frac{L}{2}\left[\frac{\lambda_m}{\lambda_m^2 + (n\pi + \lambda_k)^2}\left(Sin(n\pi + \lambda_k)Sinh\lambda_m - \frac{(n\pi + \lambda_k)}{\lambda_m}(Cos(n\pi + \lambda_k)Cosh\lambda_m - 1)\right)\right.$$

$$\left. - \frac{\lambda_m}{\lambda_m^2 + (\lambda_k - n\pi)^2}\left(Sin(\lambda_k - n\pi)Sinh\lambda_m - \frac{(\lambda_k - n\pi)}{\lambda_m}(Cos(\lambda_k - n\pi)Cosh\lambda_m - 1)\right)\right]$$

$$I_{41} = \frac{L}{2}\left[\frac{\lambda_k}{\lambda_k^2 + (n\pi - \lambda_m)^2}\left(Cos(n\pi - \lambda_m)Cosh\lambda_k + \frac{(n\pi - \lambda_m)}{\lambda_k}Sin(n\pi - \lambda_m)Sinh\lambda_k - 1\right)\right.$$

$$\left. - \frac{\lambda_k}{\lambda_k^2 + (n\pi + \lambda_m)^2}\left(Cos(n\pi + \lambda_m)Cosh\lambda_k + \frac{(n\pi + \lambda_m)}{\lambda_k}Sin(n\pi + \lambda_m)Sinh\lambda_k - 1\right)\right]$$

$$I_{42} = \frac{L}{2}\left[\frac{\lambda_k}{\lambda_k^2 + (n\pi + \lambda_m)^2}\left(Sin(n\pi + \lambda_m)Cosh\lambda_k - \frac{(n\pi + \lambda_m)}{\lambda_k}Cos(n\pi + \lambda_m)Sinh\lambda_k\right)\right.$$

$$\left. + \frac{\lambda_k}{\lambda_k^2 + (n\pi - \lambda_m)^2}\left(Sin(n\pi - \lambda_m)Cosh\lambda_k - \frac{(n\pi - \lambda_m)}{\lambda_k}Cos(n\pi - \lambda_m)Sinh\lambda_k\right)\right]$$

$$I_{43} = \frac{L}{2}\left[\frac{(\lambda_k + \lambda_m)}{(\lambda_k + \lambda_m)^2 + (n\pi)^2}\left(Sin(n\pi)Sinh(\lambda_k + \lambda_m) - \frac{n\pi}{(\lambda_k + \lambda_m)}(Cos(n\pi)Cosh(\lambda_k + \lambda_m) - 1)\right)\right.$$

$$\left. - \frac{(\lambda_k - \lambda_m)}{(\lambda_k - \lambda_m)^2 + (n\pi)^2}\left(Sin(n\pi)Sinh(\lambda_k - \lambda_m) - \frac{n\pi}{(\lambda_k - \lambda_m)}(Cos(n\pi)Cosh(\lambda_k - \lambda_m) - 1)\right)\right]$$

$$I_{44} = \frac{L}{2}\left[\frac{(\lambda_k + \lambda_m)}{(\lambda_k + \lambda_m)^2 + (n\pi)^2}\left(Sin(n\pi)Cosh(\lambda_k + \lambda_m) - \frac{n\pi}{(\lambda_k + \lambda_m)}Cos(n\pi)Sinh(\lambda_k + \lambda_m)\right)\right.$$

$$\left. + \frac{(\lambda_k - \lambda_m)}{(\lambda_k - \lambda_m)^2 + (n\pi)^2}\left(Sin(n\pi)Cosh(\lambda_k - \lambda_m) - \frac{n\pi}{(\lambda_k - \lambda_m)}Cos(n\pi)Sinh(\lambda_k - \lambda_m)\right)\right]$$

$$I_{45} = \frac{L}{2}\left[\frac{\lambda_k}{\lambda_k^2 + (n\pi - \lambda_m)^2}\left(Cos(n\pi - \lambda_m)Sinh\lambda_k + \frac{(n\pi - \lambda_m)}{\lambda_k}Sin(n\pi - \lambda_m)Cosh\lambda_k\right)\right.$$

$$\left. - \frac{\lambda_k}{\lambda_k^2 + (n\pi + \lambda_m)^2}\left(Cos(n\pi + \lambda_m)Sinh\lambda_k + \frac{(n\pi + \lambda_m)}{\lambda_k}Sin(n\pi + \lambda_m)Cosh\lambda_k\right)\right]$$

$$I_{46} = \frac{L}{2}\left[\frac{\lambda_k}{\lambda_k^2 + (n\pi + \lambda_m)^2}\left(Sin(n\pi + \lambda_m)Sinh\lambda_k - \frac{(n\pi + \lambda_m)}{\lambda_k}(Cos(n\pi + \lambda_m)Cosh\lambda_k - 1)\right)\right.$$

$$\left. + \frac{\lambda_k}{\lambda_k^2 + (n\pi - \lambda_m)^2}\left(Sin(n\pi - \lambda_m)Sinh\lambda_k - \frac{(n\pi - \lambda_m)}{\lambda_k}(Cos(n\pi - \lambda_m)Cosh\lambda_k - 1)\right)\right]$$

$$I_{47} = \frac{L}{2}\left[\frac{(\lambda_k + \lambda_m)}{(\lambda_k + \lambda_m)^2 + (n\pi)^2}\left(Sin(n\pi)Cosh(\lambda_m + \lambda_k) - \frac{n\pi}{(\lambda_m + \lambda_k)}Cos(n\pi)Sinh(\lambda_m + \lambda_k)\right)\right.$$

$$\left. + \frac{(\lambda_m - \lambda_k)}{(\lambda_m - \lambda_k)^2 + (n\pi)^2}\left(Sin(n\pi)Cosh(\lambda_m - \lambda_k) - \frac{n\pi}{(\lambda_k - \lambda_m)}Cos(n\pi)Sinh(\lambda_m - \lambda_k)\right)\right]$$

$$I_{48} = \frac{L}{2}\left[\frac{(\lambda_k + \lambda_m)}{(\lambda_k + \lambda_m)^2 + (n\pi)^2}\left(Sin(n\pi)Sinh(\lambda_k + \lambda_m) - \frac{n\pi}{(\lambda_k + \lambda_m)}(Cos(n\pi)Cosh(\lambda_k + \lambda_m) - 1)\right)\right.$$

$$\left. + \frac{(\lambda_k - \lambda_m)}{(\lambda_k - \lambda_m)^2 + (n\pi)^2}\left(Sin(n\pi)Sinh(\lambda_k - \lambda_m) - \frac{n\pi}{(\lambda_k - \lambda_m)}(Cos(n\pi)Cosh(\lambda_k - \lambda_m) - 1)\right)\right]$$

```
CLS
REM  THIS PROGRAM IS WRITTEN BY OMOLOFE BABATOPE
REM  IT IS WRITTEN TO EVALUATE THE TRANSVERESE DISPLACEMENT OF MOVING FORC
REM  UNIFORM BEAM FOR SIMPLY SUPPORTED ENDS CONDITION.
10   DIM J(35), V(10)
20   OPEN "33a.BAS" FOR OUTPUT AS #1
30 '  FOR mm = 1 TO 4
40 '  PRINT "Enter the value of FM K", mm
50 '  INPUT FM
60   L = 12.192
70   P = 8407.27 * 9.81
80   MIU = 2758.291
90   E = 2.10924E+09
100  IN = .00287698#
120  GA = .0002
130  X0 = 1 / 20
140  PI = 22 / 7
150  BE = 3 * PI / 4
160  AF = 200000
170  FM = 40000
180 'PRINT #1, "RESULT FOR K=", FM
190  PRINT #1,
200  PRINT #1, "TIME(t)", SPC(2); "DEFLECTION V(x,t)"
210  PRINT #1,
220  FOR t = 0 TO 4 STEP .1
230  FOR m = 1 TO 3
240  REM *****EVALUATING GAMMA MF SQUARE*****
250  GMF1 = (E * IN / MIU) * ((m * PI) / L) ^ 4
260  GMF2 = (AF / MIU) * ((m * PI) / L) ^ 2
270  GMF3 = FM / MIU
280  GMF = SQR(GMF1 + GMF2 + GMF3)
290  MS = P / (MIU * (GMF) ^ 2)
300  B0 = GMF
310  F = m * PI * X0 / L
320  G = m * PI * GA / L
330  REM ******EVALUATING THE BESSELS FUNCTIONS******
340  FOR n = 0 TO 7
350  FOR r = 0 TO 3
360  IF r = 0 THEN
370  rf = 1
380  ELSE
390  rf = 1
400  FOR I = 1 TO r
410  rf = rf * I
420  NEXT I
430  END IF
440  SJ = r + n
450  IF SJ = 0 THEN
460  nprf = 1
470  ELSE
480  nprf = 1
490  FOR I = 1 TO SJ
500  nprf = nprf * I
510  NEXT I
520  END IF
```

```
530  SUM(r) = (-1) ^ r * (1 / (rf * nprf)) * (G / 2) ^ (n + 2 * r)
540  NEXT r
550  J(n) = SUM(0) + SUM(1) + SUM(2) + SUM(3)
560  NEXT n
570  REM ******EVALUATING THE EXPRESSION V(x,t) TERM BY TERM******
580  SBO1 = (J(0) * GMF * SIN(F)) / B0
590  SBO2 = COS((GMF - B0) * t) - COS(GMF * t)
600  SB0 = SBO1 * SBO2
610  SUM2 = 0
620  FOR K = 1 TO 3
630  B1 = GMF + 2 * K * BE
640  B2 = GMF - 2 * K * BE
650  COB1 = (COS((GMF - B1) * t) - COS(GMF * t)) / B1
660  COB2 = (COS((GMF - B2) * t) - COS(GMF * t)) / B2
670  SES = J(2 * K) * (COB1 + COB2)
680  SUM2 = SUM2 + SES
690  NEXT K
700  SB12 = GMF * SUM2 * SIN(F)
710  SUM3 = 0
720  FOR K1 = 0 TO 3
730  B3 = GMF + (2 * K1 + 1) * BE
740  B4 = GMF - (2 * K1 + 1) * BE
750  SINB3 = (GMF * SIN((GMF - B3) * t) - (GMF - B3) * SIN(GMF * t)) / B3
760  SINB4 = (GMF * SIN((GMF - B4) * t) - (GMF - B4) * SIN(GMF * t)) / B4
770  THS = J(2 * K1 + 1) * (SINB4 - SINB3)
780  SUM3 = SUM3 + THS
790  NEXT K1
800  SB34 = SUM3 * COS(F)
810  V(m) = (2 / L) * (SB0 + SB12 + SB34) * MS * SIN(m * PI / 2)
820  NEXT m
830  V1 = V(1) + V(2) + V(3)
840  PRINT #1, t, V1
850  NEXT t
860  PRINT #1,
'870  NEXT mm
880  END
```

```
CLS
REM  THIS PROGRAM IS WRITTEN BY OMOLOFE BABATOPE
REM  IT IS WRITTEN TO EVALUATE THE TRANSVERESE DISPLACEMENT OF MOVING MASS
REM  UNIFORM BEAM FOR SIMPLY SUPPORTED ENDS CONDITION.
10   DIM J(30), V(3), JI(35)
20   OPEN "35.BAS" FOR OUTPUT AS #1
30   FOR TT = 1 TO 4
40   PRINT "ENTER the value of FM ", TT
50   INPUT FM
60   L = 12.192
70   P = 8407.27 * 9.81
80   MIU = 2758.291
90   E = 2.10924E+09
100  IN = .00287698#
110  GA = .0002
120  X0 = 1 / 20
130  PI = 22 / 7
```

```
140  BE = 3 * PI / 4
150  AF = 200000
160  'FM = 40000
170  GRA = 9.81
180  M1 = 8407.27
190  E0 = M1 / (MIU * L)
200  PRINT #1, "RESULT FOR K="; FM
210  PRINT #1,
220  PRINT #1, "TIME(t)", SPC(2); "DEFLECTION V(x,t)"
230  PRINT #1,
240  FOR t = 0 TO 4 STEP .1
250  FOR M = 1 TO 3
260  REM *****EVALUATING GAMMA MF SQUARE*****
270  GMF1 = (E * IN / MIU) * ((M * PI) / L) ^ 4
280  GMF2 = (AF / MIU) * (M * PI / L) ^ 2
290  GMF3 = FM / MIU
300  GMF = SQR(GMF1 + GMF2 + GMF3)
310  F = M * PI * X0 / L
320  G = M * PI * GA / L
330 REM ******EVALUATING THE BESSELS FUNCTIONS******
340  FOR n = 0 TO 7
350  FOR r = 0 TO 3
360  IF r = 0 THEN
365  RF = 1
380  ELSE
370  RF = 1
390  FOR I = 1 TO r
405  RF = RF * I
400  NEXT I
410  END IF
420  FF = n + r
430  IF FF = 0 THEN
440  NPRF = 1
450  ELSE
460  NPRF = 1
470  FOR I = 1 TO FF
480  NPRF = NPRF'* I
490  NEXT I
500  END IF
510  SUM(r) = (-1) ^ r * (1 / (RF * NPRF)) * (G / 2) ^ (n + 2 * r)
520  SUM0(r) = (-1) ^ r * (1 / (RF * NPRF)) * (G) ^ (n + 2 * r)
530  NEXT r
540  J(n) = SUM(0) + SUM(1) + SUM(2) + SUM(3)
550  JI(n) = SUM(0) + SUM(1) + SUM(2) + SUM(3)
560  NEXT n
570  REM *****EVALUATING GAMMA MM SQUARE*****
580  GMM1 = 2 - JI(0) * COS(2 * F)
590  GMM2 = ((BE * GA * M * PI) / L) ^ 2
600  SA = (GMM1 * GMM2) / (2 * GMF ^ 2)
610  SB = (GMM1 + SA) * E0 / 2
620  GMM = (1 - SB) * GMF
630    MS = (E0 * GRA * L) / (GMM ^ 2)
640     B0 = GMM
650  REM ******EVALUATING THE EXPRESSION V(x,t) TERM BY TERM******
```

```
660   SINBO1 = (J(0) * GMM * SIN(F)) / B0
670   SINBO2 = COS((GMM - B0) * t) - COS(GMM * t)
680   SB0 = SINBO1 * SINBO2
690   SUM2 = 0
700   FOR K = 1 TO 3
710   B1 = GMM + 2 * K * BE
720   B2 = GMM - 2 * K * BE
730   COSB1 = (COS((GMM - B1) * t) - COS(GMM * t)) / B1
740   COSB2 = (COS((GMM - B2) * t) - COS(GMM * t)) / B2
750   SES = J(2 * K) * (COSB1 + COSB2)
760   SUM2 = SUM2 + SES
770   NEXT K
780   SB12 = GMM * SUM2 * SIN(F)
790   SUM3 = 0
800   FOR K1 = 0 TO 3
810   B3 = GMM + (2 * K1 + 1) * BE
820   B4 = GMM - (2 * K1 + 1) * BE
830   SINB3 = (GMM * SIN((GMM - B3) * t) - (GMM - B3) * SIN(GMM * t)) / B3
840   SINB4 = (GMM * SIN((GMM - B4) * t) - (GMM - B4) * SIN(GMM * t)) / B4
850   THS = J(2 * K1 + 1) * (SINB4 - SINB3)
860   SUM3 = SUM3 + THS
870   NEXT K1
880   SB34 = SUM3 * COS(F)
890   V(M) = (2 / L) * (SB0 + SB12 + SB34) * MS * SIN(M * PI / 2)
900   NEXT M
910   V1 = V(1) + V(2) + V(3)
920   PRINT #1, t, V1
930   NEXT t
940   PRINT #1,
950   NEXT TT
960   END


CLS
REM   THIS PROGRAM IS WRITTEN BY OMOLOFE BABATOPE
REM   IT IS WRITTEN TO EVALUATE THE TRANSVERESE DISPLACEMENT OF MOVING FORC
REM   UNIFORM BEAM FOR OTHER BOUNDARY CONDITIONS
10    DIM J(35), I(35), IM(35), Lm(3)
20    OPEN "317.BAS" FOR OUTPUT AS #1
30    FOR MM = 1 TO 4
40    PRINT "ENTER THE VALUE OF FM", MM
50    INPUT FM
60    L = 12.192
70    E = 2.10924E+09
80    P = 8407.27 * 9.81
90    GRA = 9.81
100   MIU = 2758.291
110   INA = .00287698#
120   GA = .0002
130   M1 = 8407.27
140   X0 = 1 / 20
150   PI = 22 / 7
160   BE = 3 * PI / 4
170   AF = 200000
180   'FM = 40000
```

```
190  X = L / 2
200  PRINT #1, "RESULT FOR K=", FM
210  PRINT #1,
220  PRINT #1, "TIME(t)", SPC(2); "DEFLECTION V(x,t)"
230  PRINT #1,
240     'Lm(1) = 4.73004
250     'Lm(2) = 7.8532
260     'Lm(3) = 10.99561
      Lm(1) = 1.875
      Lm(2) = 4.694
      Lm(3) = 7.855
270  FOR t = 0 TO 4 STEP .1
280  FOR m = 1 TO 3
290  REM *****EVALUATING GAMMA AJ SQUARE*****
300  COSH = (EXP(Lm(m)) + EXP(-Lm(m))) / 2
310  SINH = (EXP(Lm(m)) - EXP(-Lm(m))) / 2
320  COSH1 = (EXP(2 * Lm(m)) + EXP(-2 * Lm(m))) / 2
330  SINH1 = (EXP(2 * Lm(m)) - EXP(-2 * Lm(m))) / 2
340  AM1 = -SINH - SIN(Lm(m))
360  AM2 = COS(Lm(m)) + COSH
370  AM = AM1 / AM2
380  BM = -1
390  CM = -AM
400  I1a = SIN(2 * Lm(m)) / (2 * Lm(m))
410  I1 = L / 2 * (1 - I1a)
420  I2a = (COS(2 * Lm(m)) - 1) / (2 * Lm(m))
430  I2 = -L / 2 * I2a
440  I31 = L / (2 * Lm(m))
450  I32 = SIN(Lm(m)) * COSH
460  I33 = COS(Lm(m)) * SINH
470  I3 = I31 * (I32 - I33)
480  I41 = L / (2 * Lm(m))
490  I42 = SIN(Lm(m)) * SINH
500  I43 = COS(Lm(m)) * (COSH - 1)
510  I4 = I41 * (I42 - I43)
520  I5 = I2
530  I6a = SIN(2 * Lm(m)) / (2 * Lm(m))
540  I6 = L / 2 * (1 + I6a)
550  I71 = L / (2 * Lm(m))
560  I72 = COS(Lm(m)) * COSH
570  I73 = SIN(Lm(m)) * SINH
580  I7 = I71 * (I72 + I73 - 1)
590  I81 = L / (2 * Lm(m))
600  I82 = COS(Lm(m)) * SINH
610  I83 = SIN(Lm(m)) * COSH
620  I8 = I81 * (I82 + I83)
630  I9 = I3
640  I101 = L / (2 * Lm(m))
650  I102 = COS(Lm(m)) * COSH
660  I103 = SIN(Lm(m)) * SINH
670  I10 = I101 * (I102 + I103 - 1)
680  I11a = SINH1 / (2 * Lm(m))
690  I11 = L / 2 * (I11a - 1)
700  I12a = (COSH1 - 1) / (2 * Lm(m))
710  I12 = L / 2 * I12a
```

```
720  I13 = I4
730  I14 = I8
740  I15 = I12
750  I16a = SINH1 / (2 * Lm(m))
760  I16 = L / 2 * (I16a + 1)
770  T01 = I1 + 2 * AM * I2 + 2 * BM * I3 + 2 * CM * I4 + AM ^ 2 * I6
780  T02 = BM ^ 2 * I11 + CM ^ 2 * I16 + 2 * AM * BM * I10
790  T03 = 2 * AM * CM * I14 + 2 * BM * CM * I12
800  T0 = T01 + T02 + T03
810  HA1 = -I1 - AM * I2 - BM * I3 - CM * I4 - AM * I5 - AM * AM * I6
820  HA2 = -AM * BM * I7 - AM * CM * I8 + BM * I9 + BM * AM * I10
830  HA3 = BM * BM * I11 + BM * CM * I12 + CM * I13 + CM * AM * I14
840  HA4 = CM * BM * I15 + CM * CM * I16
850  HAMM = (((Lm(m) / L) ^ 2)) * (HA1 + HA2 + HA3 + HA4) / T0
860  GJ11 = (E * INA / MIU) * (Lm(m) / L) ^ 4
870  GJ12 = FM / MIU
880  GJ21 = GJ11 + GJ12
890  GJ22 = SQR(GJ21)
900  GAJ0 = (AF * HAMM / (2 * MIU * GJ21))
910  GAJ = GJ22 * (1 - GAJ0)
920  MS = P / (MIU * (GAJ) ^ 2)
930  B0 = GAJ
940  F = m * PI * X0 / L
950  G = Lm(m) * GA / L
960  a0 = COS(Lm(m) * X0 / L) - AM * SIN(Lm(m) * X0 / L)
970  a1 = SIN(Lm(m) * X0 / L) + AM * COS(Lm(m) * X0 / L)
980  a21 = BM * (EXP(Lm(m) * X0 / L) - EXP(-Lm(m) * X0 / L)) / 2
990  a22 = CM * (EXP(Lm(m) * X0 / L) + EXP(-Lm(m) * X0 / L)) / 2
1000 a2 = a21 + a22
1010 a31 = CM * (EXP(Lm(m) * X0 / L) - EXP(-Lm(m) * X0 / L)) / 2
1020 a32 = BM * (EXP(Lm(m) * X0 / L) + EXP(-Lm(m) * X0 / L)) / 2
1030 a3 = a31 + a32
1040 S1 = (a2 + a3) / 2
1050 S2 = (a2 - a3) / 2
REM ******EVALUATING THE BESSELS FUNCTIONS******
1060  FOR n = 0 TO 7
1070  FOR r = 0 TO 3
1080  IF r = 0 THEN
1090  rf = 1
1100  ELSE
1110  rf = 1
1120  FOR I = 1 TO r
1130  rf = rf * I
1140  NEXT I
1150  END IF
1160  SJ = r + n
1170  IF SJ = 0 THEN
1180  nprf = 1
1190  ELSE
1200  nprf = 1
1210  FOR I = 1 TO SJ
1220  nprf = nprf * I
1230  NEXT I
1240  END IF
1250  SUM1(r) = (-1) ^ r * (1 / (rf * nprf)) * (G / 2) ^ (n + 2 * r)
```

```
1260  NEXT r
1270  J(n) = SUM1(0) + SUM1(1) + SUM1(2) + SUM1(3)
1280  NEXT n
REM ******EVALUATING THE EXPRESSION V(x,t) TERM BY TERM******
1290  SBO1 = (J(0) * GAJ * a1) / B0
1300  SBO2 = COS((GAJ - B0) * t) - COS(GAJ * t)
1310  SB0 = SBO1 * SBO2
1320  SUM2 = 0
1330  FOR K = 1 TO 3
1340  B1 = GAJ + 2 * K * BE
1350  B2 = GAJ - 2 * K * BE
1360  COB1 = (COS((GAJ - B1) * t) - COS(GAJ * t)) / B1
1370  COB2 = (COS((GAJ - B2) * t) - COS(GAJ * t)) / B2
1380  SES = J(2 * K) * (COB1 + COB2)
1390  SUM2 = SUM2 + SES
1400  NEXT K
1410  SB12 = GAJ * SUM2 * a1
1420  SUM3 = 0
1430  FOR K1 = 0 TO 3
1440  B3 = GAJ + (2 * K1 + 1) * BE
1450  B4 = GAJ - (2 * K1 + 1) * BE
1460  SINB3 = (GAJ * SIN((GAJ - B3) * t) - (GAJ - B3) * SIN(GAJ *
1470  SINB4 = (GAJ * SIN((GAJ - B4) * t) - (GAJ - B4) * SIN(GAJ *
1480  THS = J(2 * K1 + 1) * (SINB4 - SINB3)
1490  SUM3 = SUM3 + THS
1500  NEXT K1
1510  SB34 = SUM3 * a0
1520  FOR in = 0 TO 7
1530  FOR ir = 0 TO 3
1540  IF ir = 0 THEN
1550  irf = 1
1560  ELSE
1570  irf = 1
1580  FOR q = 1 TO ir
1590  irf = irf * q
1600  NEXT q
1610  END IF
1620  SJ = ir + in
1630 IF SJ = 0 THEN
1640  inpirf = 1
1650  ELSE
1660  inpirf = 1
1670  FOR q = 1 TO SJ
1680  inpirf = inpirf * q
1690  NEXT q
1700  END IF
1710  SU = (1 / (2 ^ (in + 2 * ir) * (irf * inpirf)))
1720  SUM2(ir) = SU * (G / 2) ^ (in + 2 * ir)
1730  NEXT ir
1740  I(in) = SUM2(0) + SUM2(1) + SUM2(2) + SUM2(3)
1750  NEXT in
1760  SB2O1 = (I(0) * GAJ * S1) / B0
1770  SB2O2 = COS((GAJ - B0) * t) - COS(GAJ * t)
1780  SB20 = SB2O1 * SB2O2
1790  SUM4 = 0
```

```
1800  FOR K = 1 TO 3
1810  B1 = GAJ + 2 * K * BE
1820  B2 = GAJ - 2 * K * BE
1830  COB21 = (COS((GAJ - B1) * t) - COS(GAJ * t)) / B1
1840  COB22 = (COS((GAJ - B2) * t) - COS(GAJ * t)) / B2
1850  SES2 = (-1) ^ K * I(2 * K) * (COB21 + COB22)
1860  SUM4 = SUM4 + SES2
1870  NEXT K
1880  SB212 = GAJ * SUM4 * S1
1890  SUM5 = 0
1900  FOR K1 = 0 TO 3
1910  B3 = GAJ + (2 * K1 + 1) * BE
1920  B4 = GAJ - (2 * K1 + 1) * BE
1930  SINB23 = (GAJ * SIN((GAJ - B3) * t) - (GAJ - B3) * SIN(GAJ * t
1940  SINB24 = (GAJ * SIN((GAJ - B4) * t) - (GAJ - B4) * SIN(GAJ * t
1950  THS2 = (-1) ^ K1 * I(2 * K1 + 1) * (SINB24 - SINB23)
1960  SUM5 = SUM5 + THS2
1970  NEXT K1
1980  SB234 = SUM5 * S1
1990  FOR in = 0 TO 7
2000  FOR ir = 0 TO 3
2010  IF ir = 0 THEN
2020  irf = 1
2030  ELSE
2040  irf = 1
2050  FOR q = 1 TO ir
2060  irf = irf * q
2070  NEXT q
2080  END IF
2090  SJ = ir + in
2100  IF SJ = 0 THEN
2110  inpirf = 1
2120  ELSE
2130  inpirf = 1
2140  FOR q = 1 TO SJ
2150  inpirf = inpirf * q
2160  NEXT q
2170  END IF
2180  SU = (1 / (2 ^ (in + 2 * ir) * (irf * inpirf)))
2190  SUM3(ir) = SU * (-G / 2) ^ (in + 2 * ir)
2200  NEXT ir
2210  IM(in) = SUM3(0) + SUM3(1) + SUM3(2) + SUM3(3)
2220  NEXT in
2230  SB3O1 = (IM(0) * GAJ * S2) / B0
2240  SB3O2 = COS((GAJ - B0) * t) - COS(GAJ * t)
2250  SB30 = SB3O1 * SB3O2
2260  SUM6 = 0
2270  FOR K = 1 TO 3
2280  B1 = GAJ + 2 * K * BE
2290  B2 = GAJ - 2 * K * BE
2300  COB31 = (COS((GAJ - B1) * t) - COS(GAJ * t)) / B1
2310  COB32 = (COS((GAJ - B2) * t) - COS(GAJ * t)) / B2
2320  SES3 = (-1) ^ K * IM(2 * K) * (COB31 + COB32)
2330  SUM6 = SUM6 + SES3
2340  NEXT K
```

```
2350  SB312 = GAJ * SUM6 * S2
2360  SUM7 = 0
2370  FOR K1 = 0 TO 3
2380  B3 = GAJ + (2 * K1 + 1) * BE
2390  B4 = GAJ - (2 * K1 + 1) * BE
2400  SINB33 = (GAJ * SIN((GAJ - B3) * t) - (GAJ - B3) * SIN(GAJ *
2410  SINB34 = (GAJ * SIN((GAJ - B4) * t) - (GAJ - B4) * SIN(GAJ *
2420  THS3 = (-1) ^ K1 * IM(2 * K1 + 1) * (SINB34 - SINB33)
2430  SUM7 = SUM7 + THS3
2440  NEXT K1
2450  SB334 = SUM7 * S2
2460  U11 = SB0 + SB12 + SB34 + SB20 + SB212
2470  U12 = SB234 + SB30 + SB312 + SB334
2480  U1 = U11 + U12
2490  U21 = SIN(Lm(m) * X / L) + AM * COS(Lm(m) * X / L)
2500  U22 = BM * (EXP(Lm(m) * X / L) - EXP(-Lm(m) * X / L)) / 2
2510  U23 = CM * (EXP(Lm(m) * X / L) + EXP(-Lm(m) * X / L)) / 2
2520  U2 = U21 + U22 + U23
2530  V(m) = U1 * U2 * MS / T0
2540  NEXT m
2550  V1 = (V(1) + V(2) + V(3))
2570  PRINT #1, t, V1
2580  NEXT t
2590  PRINT #1,
2600  NEXT MM
2610  END


CLS
REM  THIS PROGRAM IS WRITTEN BY OMOLOFE BABATOPE
REM  IT IS WRITTEN TO EVALUATE THE TRANSVERESE DISPLACEMENT OF M
REM  UNIFORM BEAM FOR OTHER BOUNDARY CONDITIONS
10   DIM J(35), I(35), IM(35), Lm(3), V(3)
20   OPEN "320.bas" FOR OUTPUT AS #1
30   FOR mm = 1 TO 4
40   PRINT "Enter the value of FM", mm
50   INPUT FM
60   L = 12.192
70   P = 8407.27 * 9.81
80   MIU = 2758.291
90   E = 2.10924E+09
100  INA = .00287698#
120  GA = .0002
130  X0 = 1 / 20
140  M1 = 8407.27
150  E0 = M1 / (MIU * L)
160  GRA = 9.81
170  PI = 22 / 7
180  BE = 3 * PI / 4
190  AF = 200000
200  'FM = 40000
210  X = L / 2
220  PRINT #1, "RESULT FOR K=", FM
230  PRINT #1,
240  PRINT #1, "TIME(t)", SPC(2); "DEFLECTION V(x,t)"
```

```
250  PRINT #1,
260      'Lm(1) = 4.73004
270      'Lm(2) = 7.8532
280      'Lm(3) = 10.99561
290    Lm(1) = 1.875
300    Lm(2) = 4.694
310    Lm(3) = 7.855
320  FOR t = 0 TO 4 STEP .1
330  FOR m = 1 TO 3
340  REM *****EVALUATING GAMMA AJ SQUARE*****
350  COSH = (EXP(Lm(m)) + EXP(-Lm(m))) / 2
360  SINH = (EXP(Lm(m)) - EXP(-Lm(m))) / 2
370  COSH1 = (EXP(2 * Lm(m)) + EXP(-2 * Lm(m))) / 2
380  SINH1 = (EXP(2 * Lm(m)) - EXP(-2 * Lm(m))) / 2
390  AM1 = -SINH - SIN(Lm(m))
400  AM2 = COS(Lm(m)) + COSH
410  AM = AM1 / AM2
420  BM = -1
430  CM = -AM
440  I1a = SIN(2 * Lm(m)) / (2 * Lm(m))
450  I1 = (L / 2) * (1 - I1a)
460  I2a = (COS(2 * Lm(m)) - 1) / (2 * Lm(m))
470  I2 = -(L / 2) * I2a
480  I31 = L / (2 * Lm(m))
490  I32 = SIN(Lm(m)) * COSH
500  I33 = COS(Lm(m)) * SINH
510  I3 = I31 * (I32 - I33)
520  I41 = L / (2 * Lm(m))
530  I42 = SIN(Lm(m)) * SINH
540  I43 = COS(Lm(m)) * (COSH - 1)
550  I4 = I41 * (I42 - I43)
560  I5 = I2
570  I6a = SIN(2 * Lm(m)) / (2 * Lm(m))
580  I6 = (L / 2) * (1 + I6a)
590  I71 = L / (2 * Lm(m))
600  I72 = COS(Lm(m)) * COSH
610  I73 = SIN(Lm(m)) * SINH
620  I7 = I71 * (I72 + I73 - 1)
630  I81 = L / (2 * Lm(m))
640  I82 = COS(Lm(m)) * SINH
650  I83 = SIN(Lm(m)) * COSH
660  I8 = I81 * (I82 + I83)
670  I9 = I3
680  I101 = L / (2 * Lm(m))
690  I102 = COS(Lm(m)) * COSH
700  I103 = SIN(Lm(m)) * SINH
710  I10 = I101 * (I102 + I103 - 1)
720  I11a = SINH1 / (2 * Lm(m))
730  I11 = (L / 2) * (I11a - 1)
740  I12a = (COSH1 - 1) / (2 * Lm(m))
750  I12 = (L / 2) * I12a
760  I13 = I4
770  I14 = I8
780  I15 = I12
790  I16a = SINH1 / (2 * Lm(m))
```

```
800  I16 = (L / 2) * (I16a + 1)
810  T01 = I1 + 2 * AM * I2 + 2 * BM * I3 + 2 * CM * I4 + AM ^ 2 * I6
820  T02 = BM ^ 2 * I11 + CM ^ 2 * I16 + 2 * AM * BM * I10
830  T03 = 2 * AM * CM * I14 + 2 * BM * CM * I12
840  TO = T01 + T02 + T03
850  HA1 = -I1 - AM * I2 - BM * I3 - CM * I4 - AM * I5 - AM * AM * I6
860  HA2 = -AM * BM * I7 - AM * CM * I8 + BM * I9 + BM * AM * I10
870  HA3 = BM * BM * I11 + BM * CM * I12 + CM * I13 + CM * AM * I14
880  HA4 = CM * BM * I15 + CM * CM * I16
890  HAMM = (((Lm(m) / L) ^ 2)) * (HA1 + HA2 + HA3 + HA4) / TO
900  GJ11 = (E * INA / MIU) * (Lm(m) / L) ^ 4
920  GJ12 = FM / MIU
930  GJ21 = GJ11 + GJ12
940  GJ22 = SQR(GJ21)
950  GAJO = (AF * HAMM / (2 * MIU * GJ21))
960  GAJ = GJ22 * (1 - GAJO)
970  HB1 = I1 + AM * I2 + BM * I3 + CM * I4 + AM * I5 + AM * AM * I6
980  HB2 = AM * BM * I7 + AM * CM * I8 + BM * I9 + BM * AM * I10
990  HB3 = BM * BM * I11 + BM * CM * I12 + CM * I13 + CM * AM * I14
1000 HB4 = CM * BM * I15 + CM * CM * I16
1010 HBMM = (HB1 + HB2 + HB3 + HB4) / TO
1020 HFMM = HAMM
1030 RMMN1 = 0
1040 RMMN2 = 0
1050 'FOR n = 1 TO 5
1060 n = 1
1070 I171 = (SIN(n * PI)) / (n * PI)
1080 I172 = SIN(n * PI) / (n * PI)
1090 I173 = SIN((n * PI) + (2 * Lm(m))) / ((n * PI) + (2 * Lm(m)))
1100 I174 = SIN((n * PI) - (2 * Lm(m))) / ((n * PI) - (2 * Lm(m)))
1110 I17 = (L / 4) * (I171 + I172 - I173 - I174)
1120 I181 = (COS((n * PI) - (2 * Lm(m))) - 1) / ((n * PI) - (2 * Lm(m)))
1130 I182 = (1 - COS((n * PI) + (2 * Lm(m)))) / ((n * PI) + (2 * Lm(m)))
1140 I183 = (COS(n * PI) - 1) / (n * PI)
1150 I184 = (1 - COS(n * PI)) / (n * PI)
1160 I18 = (L / 4) * (I181 + I182 + I183 + I184)
1170 I1911 = SIN((n * PI) + Lm(m)) * COSH
1180 I1912 = ((n * PI) + Lm(m)) * COS((n * PI) + Lm(m)) * SINH / Lm(m)
1190 I191 = (Lm(m) / (Lm(m) ^ 2 + ((n * PI) + Lm(m)) ^ 2)) * (I1911 - I191
1200 I1921 = SIN((n * PI) - Lm(m)) * COSH
1210 I1922 = ((n * PI) - Lm(m)) * COS((n * PI) - Lm(m)) * SINH / Lm(m)
1220 I192 = (Lm(m) / (Lm(m) ^ 2 + ((n * PI) - Lm(m)) ^ 2)) * (I1921 - I192
1230 I19 = (I191 + I192) * (L / 2)
1240 I2011 = (SIN((n * PI) + Lm(m)) * SINH)
1250 I2012 = (((n * PI) + Lm(m)) * (COS((n * PI) + Lm(m)) * COSH - 1)) /
Lm(m)
1260 I201 = (Lm(m) / (Lm(m) ^ 2 + ((n * PI) + Lm(m)) ^ 2)) * (I2011 - I201
1270 I2021 = SIN((n * PI) - Lm(m)) * SINH
1280 I2022 = (((n * PI) - Lm(m)) * (COS((n * PI) - Lm(m)) * COSH - 1)) /
Lm(m)
1290 I202 = (Lm(m) / (Lm(m) ^ 2 + ((n * PI) - Lm(m)) ^ 2)) * (I2021 - I202
1300 I20 = (I201 - I202) * (L / 2)
1310 I2111 = (COS(n * PI) - 1) / (n * PI)
1320 I2112 = (1 - COS((n * PI) + (2 * Lm(m)))) / ((n * PI) + (2 * Lm(m)))
1330 I2113 = (COS((n * PI) - (2 * Lm(m))) - 1) / ((n * PI) - (2 * Lm(m)))
```

```
1340 I2114 = (1 - COS(n * PI)) / (n * PI)
1350 I21 = (L / 4) * (I2111 + I2112 + I2113 + I2114)
1360 I2211 = SIN((n * PI) + (2 * Lm(m))) / ((n * PI) + (2
1370 I2212 = SIN((n * PI) - (2 * Lm(m))) / ((n * PI) - (2
1380 I2213 = SIN(n * PI) / (n * PI)
1390 I2214 = SIN(n * PI) / (n * PI)
1400 I22 = (L / 4) * (I2211 + I2212 + I2213 + I2214)
1410 I2311 = COS((n * PI) + Lm(m)) * COSH
1420 I2312 = ((n * PI + Lm(m)) * SIN((n * PI) + Lm(m)) * S
1430 I231 = (Lm(m) / (Lm(m) ^ 2 + ((n * PI) + Lm(m)) ^ 2))
1440 I2321 = COS((n * PI) - Lm(m)) * COSH
1450 I2322 = (((n * PI) - Lm(m)) * SIN((n * PI) - Lm(m)) *
1460 I232 = (Lm(m) / (Lm(m) ^ 2 + ((n * PI) - Lm(m)) ^ 2))
1470 I23 = (L / 2) * (I231 + I232)
1480 I2411 = COS((n * PI) + Lm(m)) * SINH
1490 I2412 = ((n * PI) + Lm(m)) * SIN((n * PI) + Lm(m)) * C
1500 I241 = (Lm(m) / (Lm(m) ^ 2 + ((n * PI) + Lm(m)) ^ 2))
1510 I2421 = COS((n * PI) - Lm(m)) * SINH
1520 I2422 = ((n * PI) - Lm(m)) * SIN((n * PI) - Lm(m)) * C
1530 I242 = (Lm(m) / (Lm(m) ^ 2 + ((n * PI) - Lm(m)) ^ 2))
1540 I24 = (L / 2) * (I241 + I242)
1550 I2511 = (SIN((n * PI) + Lm(m)) * COSH)
1560 I2512 = (((n * PI) + Lm(m)) * COS((n * PI) + Lm(m)) *
1570 I251 = (Lm(m) / (Lm(m) ^ 2 + ((n * PI) + Lm(m)) ^ 2))
1580 I2521 = SIN((n * PI) - Lm(m)) * COSH
1590 I2522 = (((n * PI) - Lm(m)) * COS((n * PI) - Lm(m)) *
1600 I252 = (Lm(m) / (Lm(m) ^ 2 + ((n * PI) - Lm(m)) ^ 2)) *
1610 I25 = (I251 + I252) * (L / 2)                              / B3
1620 I2611 = COS((n * PI) + Lm(m)) * COSH                       / B4
1630 I2612 = (((n * PI) + Lm(m)) * SIN((n * PI) + Lm(m)) * S
1640 I261 = (Lm(m) / (Lm(m) ^ 2 + ((n * PI) + Lm(m)) ^ 2)) *
1650 I2621 = COS((n * PI) - Lm(m)) * COSH
1660 I2622 = (((n * PI) - Lm(m)) * SIN((n * PI) - Lm(m)) * S
1670 I262 = (Lm(m) / (Lm(m) ^ 2 + ((n * PI) - Lm(m)) ^ 2)) *
1680 I26 = (L / 2) * (I261 + I262)
1690 I271 = (-1) ^ n * (2 * Lm(m)) * SINH1
1700 I272 = (2 * Lm(m)) ^ 2 + (n * PI) ^ 2
1710 I27 = (L / 2) * (I271 / I272)
1720 I281 = (2 * Lm(m)) * ((-1) ^ n * COSH1 - 1)
1730 I282 = (2 * Lm(m)) ^ 2 + (n * PI) ^ 2
1740 I28 = (L / 2) * (I281 / I282)
1750 I2911 = (SIN((n * PI) + Lm(m)) * SINH)
1760 I2912 = (((n * PI) + Lm(m)) * (COS((n * PI) + Lm(m)) * C
Lm(m)
1770 I291 = (Lm(m) / (Lm(m) ^ 2 + ((n * PI) + Lm(m)) ^ 2)) *
1780 I2921 = SIN((n * PI) - Lm(m)) * SINH
1790 I2922 = (((n * PI) - Lm(m)) * (COS((n * PI) - Lm(m)) * C
Lm(m)
1800 I292 = (Lm(m) / (Lm(m) ^ 2 + ((n * PI) - Lm(m)) ^ 2)) *
1810 I29 = (L / 2) * (I291 - I292)
1820 I3011 = COS((n * PI) + Lm(m)) * SINH
1830 I3012 = ((n * PI) + Lm(m)) * SIN((n * PI) + Lm(m)) * COS
1840 I301 = (Lm(m) / (Lm(m) ^ 2 + ((n * PI) + Lm(m)) ^ 2)) *
1850 I3021 = COS((n * PI) - Lm(m)) * SINH
1860 I3022 = ((n * PI) - Lm(m)) * SIN((n * PI) - Lm(m)) * COSH
```

```
1870 I302 = (Lm(m) / (Lm(m) ^ 2 + ((n * PI) - Lm(m)) ^ 2)) * (I3021 + I302
1880 I30 = (L / 2) * (I301 + I302)
1890 I31 = I28
1900 I32 = I27
1910 HCM1 = I17 + AM * I18 + BM * I19 + CM * I20 + AM * I21 + AM * AM * I2
1920 HCM2 = AM * BM * I23 + AM * CM * I24 + BM * I25 + BM * AM * I26
1930 HCM3 = BM * BM * I27 + BM * CM * I28 + CM * I29 + CM * AM * I30
1940 HCM4 = CM * BM * I31 + CM * CM * I32
1950 HCMN = (HCM1 + HCM2 + HCM3 + HCM4) / TO
1960 HGM1 = -I17 - AM * I18 - BM * I19 - CM * I20 - AM * I21 - AM * AM * I
1970 HGM2 = -AM * BM * I23 - AM * CM * I24 + BM * I25 + BM * AM * I26
1980 HGM3 = BM * BM * I27 + BM * CM * I28 + CM * I29 + CM * AM * I30
1990 HGM4 = CM * BM * I31 + CM * CM * I32
2000 HGMN = (Lm(m) / L) ^ 2 * (HGM1 + HGM2 + HGM3 + HGM4) / TO
2010 J0 = 1
2020 'RMN1 = HCMN * COS(PI * X0 / L) * J0
2030 'RMMN1 = RMMN1 + RMN1
2040 'RMN2 = HGMN * COS(PI * X0 / L) * J0
2050 'RMMN2 = RMMN2 + RMN2
2060 'NEXT n
2070 RMMN1 = J0 * HCMN * COS(PI * X0 / L)
2080 RMMN2 = J0 * HGMN * COS(PI * X0 / L)
2090 GB1 = HBMM + RMMN1
2100 GB2 = (GA * BE) ^ 2 * (HFMM + 2 * RMMN2) / (2 * GAJ ^ 2)
2110 GB3 = E0 * (GB1 - GB2) / 2
2120 GBJ = GAJ * (1 - GB3)
2130 MS = E0 * L * GRA / (GBJ) ^ 2
2140 B0 = GBJ
2150 F = m * PI * X0 / L
2160 G = Lm(m) * GA / L
2170 a0 = COS(Lm(m) * X0 / L) - (AM * SIN(Lm(m) * X0 / L))
2180 a1 = SIN(Lm(m) * X0 / L) + (AM * COS(Lm(m) * X0 / L))
2190 a21 = BM * (EXP(Lm(m) * X0 / L) - EXP(-Lm(m) * X0 / L)) / 2
2200 a22 = CM * (EXP(Lm(m) * X0 / L) + EXP(-Lm(m) * X0 / L)) / 2
2210 a2 = a21 + a22
2220 a31 = CM * (EXP(Lm(m) * X0 / L) - EXP(-Lm(m) * X0 / L)) / 2
2230 a32 = BM * (EXP(Lm(m) * X0 / L) + EXP(-Lm(m) * X0 / L)) / 2
2240 a3 = a31 + a32
2250 S1 = (a2 + a3) / 2
2260 S2 = (a2 - a3) / 2
  REM ******EVALUATING THE BESSELS FUNCTIONS******
2270 FOR n = 0 TO 7
2280 FOR r = 0 TO 3
2290 IF r = 0 THEN
2300 rf = 1
2310 ELSE
2320 rf = 1
2330 FOR I = 1 TO r
2340 rf = rf * I
2350 NEXT I
2360 END IF
2370 SJ = r + n
2380 IF SJ = 0 THEN
2390 nprf = 1
2400 ELSE
```

```
2410 nprf = 1
2420 FOR I = 1 TO SJ
2430 nprf = nprf * I
2440 NEXT I
2450 END IF
2460 SUM1(r) = (-1) ^ r * (1 / (rf * nprf)) * (G / 2) ^ (n + 2 * r)
2470 NEXT r
2480 J(n) = SUM1(0) + SUM1(1) + SUM1(2) + SUM1(3)
2490 NEXT n
  REM ******EVALUATING THE EXPRESSION V(x,t) TERM BY TERM******
2500 SBO1 = (J(0) * GBJ * a1) / B0
2510 SBO2 = COS((GBJ - B0) * t) - COS(GBJ * t)
2520 SB0 = SBO1 * SBO2
2530 SUM2 = 0
2540 FOR K = 1 TO 3
2550 B1 = GBJ + 2 * K * BE
2560 B2 = GBJ - 2 * K * BE
2570 COB1 = (COS((GBJ - B1) * t) - COS(GBJ * t)) / B1
2580 COB2 = (COS((GBJ - B2) * t) - COS(GBJ * t)) / B2
2590 SES = J(2 * K) * (COB1 + COB2)
2600 SUM2 = SUM2 + SES
2610 NEXT K
2620 SB12 = GBJ * SUM2 * a1
2630 SUM3 = 0
2640 FOR K1 = 0 TO 3
2650 B3 = GBJ + (2 * K1 + 1) * BE
2660 B4 = GBJ - (2 * K1 + 1) * BE
2670 SINB3 = (GBJ * SIN((GBJ - B3) * t) - (GBJ - B3) * SIN(GBJ * t)) / B
2680 SINB4 = (GBJ * SIN((GBJ - B4) * t) + (GBJ - B4) * SIN(GBJ * t)) / B
2690 THS = J(2 * K1 + 1) * (SINB4 - SINB3)
2700 SUM3 = SUM3 + THS
2710 NEXT K1
2720 SB34 = SUM3 * a0
2730 FOR in = 0 TO 7
2740 FOR ir = 0 TO 3
2750 IF ir = 0 THEN
2760 irf = 1
2770 ELSE
2780 irf = 1
2790 FOR q = 1 TO ir
2800 irf = irf * q
2810 NEXT q
2820 END IF
2830 SJ = ir + in
2840 IF SJ = 0 THEN
2850 inpirf = 1
2860 ELSE
2870 inpirf = 1
2880 FOR q = 1 TO SJ
2890 inpirf = inpirf * q
2900 NEXT q
2910 END IF
2920 SU = (1 / (2 ^ (in + 2 * ir) * (irf * inpirf)))
2930 SUM2(ir) = SU * (G / 2) ^ (in + 2 * ir)
2940 NEXT ir
```

```
2950 I(in) = SUM2(0) + SUM2(1) + SUM2(2) + SUM2(3)
2960 NEXT in
2970 SB201 = (I(0) * GBJ * S1) / B0
2980 SB202 = COS((GBJ - B0) * t) - COS(GBJ * t)
2990 SB20 = SB201 * SB202
3000 SUM4 = 0
3010 FOR K = 1 TO 3
3020 B1 = GBJ + 2 * K * BE
3030 B2 = GBJ - 2 * K * BE
3040 COB21 = (COS((GBJ - B1) * t) - COS(GBJ * t)) / B1
3050 COB22 = (COS((GBJ - B2) * t) - COS(GBJ * t)) / B2
3060 SES2 = (-1) ^ K * I(2 * K) * (COB21 + COB22)
3070 SUM4 = SUM4 + SES2
3080 NEXT K
3090 SB212 = GBJ * SUM4 * S1
3100 SUM5 = 0
3110 FOR K1 = 0 TO 3
3120 B3 = GBJ + (2 * K1 + 1) * BE
3130 B4 = GBJ - (2 * K1 + 1) * BE
3140 SINB23 = (GBJ * SIN((GBJ - B3) * t) - (GBJ - B3) * SIN(GBJ * t)) / B
3150 SINB24 = (GBJ * SIN((GBJ - B4) * t) - (GBJ - B4) * SIN(GBJ * t)) / B
3160 THS2 = (-1) ^ K1 * I(2 * K1 + 1) * (SINB24 - SINB23)
3170 SUM5 = SUM5 + THS2
3180 NEXT K1
3190 SB234 = SUM5 * S1
3200 G0 = -G
3210 FOR in = 0 TO 7
3220 FOR ir = 0 TO 3
3230 IF ir = 0 THEN
3240 irf = 1
3250 ELSE
3260 irf = 1
3270 FOR q = 1 TO ir
3280 irf = irf * q
3290 NEXT q
3300 END IF
3310 SJ = ir + in
3320 IF SJ = 0 THEN
3330 inpirf = 1
3340 ELSE
3350 inpirf = 1
3360 FOR q = 1 TO SJ
3370 inpirf = inpirf * q
3380 NEXT q
3390 END IF
3400 SU = (1 / (2 ^ (in + 2 * ir) * (irf * inpirf)))
3410 SUM3(ir) = SU * (-G / 2) ^ (in + 2 * ir)
3420 NEXT ir
3430 IM(in) = SUM3(0) + SUM3(1) + SUM3(2) + SUM3(3)
3440 NEXT in
3450 SB301 = (IM(0) * GBJ * S2) / B0
3460 SB302 = COS((GBJ - B0) * t) - COS(GBJ * t)
3470 SB30 = SB301 * SB302
3480 SUM6 = 0
3490 FOR K = 1 TO 3
```

```
3500 B1 = GBJ + 2 * K * BE
3510 B2 = GBJ - 2 * K * BE
3520 COB31 = (COS((GBJ - B1) * t) - COS(GBJ * t)) / B1
3530 COB32 = (COS((GBJ - B2) * t) - COS(GBJ * t)) / B2
3540 SES3 = (-1) ^ K * IM(2 * K) * (COB31 + COB32)
3550 SUM6 = SUM6 + SES3
3560 NEXT K
3570 SB312 = GBJ * SUM6 * S2
3580 SUM7 = 0
3590 FOR K1 = 0 TO 3
3600 B3 = GBJ + (2 * K1 + 1) * BE
3610 B4 = GBJ - (2 * K1 + 1) * BE
3620 SINB33 = (GBJ * SIN((GBJ - B3) * t) - (GBJ - B3) * SIN(GBJ * t)) / B
3630 SINB34 = (GBJ * SIN((GBJ - B4) * t) - (GBJ - B4) * SIN(GBJ * t)) / B
3640 THS3 = (-1) ^ K1 * IM(2 * K1 + 1) * (SINB34 - SINB33)
3650 SUM7 = SUM7 + THS3
3660 NEXT K1
3670 SB334 = SUM7 * S2
3680 U11 = SB0 + SB12 + SB34 + SB20 + SB212
3690 U12 = SB234 + SB30 + SB312 + SB334
3700 U1 = U11 + U12
3710 U21 = SIN(Lm(m) * X / L) + (AM * COS(Lm(m) * X / L))
3720 U22 = BM * (EXP(Lm(m) * X / L) - EXP(-Lm(m) * X / L)) / 2
3730 U23 = CM * (EXP(Lm(m) * X / L) + EXP(-Lm(m) * X / L)) / 2
3740 U2 = U21 + U22 + U23
3750 V(m) = (U1 * MS * U2) / T0
3760 NEXT m
3770 V1 = (V(1) + V(2) + V(3))
3780 PRINT #1, t, V1
3790 NEXT t
3800 PRINT #1,
3810 NEXT mm
3820 END
```