DERIVATION OF CONTINUOUS LINEAR MULTISTEP METHODS FOR INITIAL VALUE PROBLEMS OF FIRST ORDER ORDINARY DIFFERENTIAL EQUATION

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CERTIFICATION

This work has not been presented elsewhere for the award of a degree, or any other purpose.

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We hereby certify that this work has been carried out by Oluwatusin, Ebenezer Ayodele and submitted to the Department of Mathematical Sciences of The Federal University of Technology, Akure, in partial fulfillment of the award of M. Tech Degree in Mathematics. To the best of our knowledge it has not been submitted elsewhere for the award of a degree.

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DEDICATION

This thesis is dedicated to God the Father, God the Son and God the Holy Spirit

ABSTRACT

In this work, a class of numerical methods with step numbers k > 1 is studied to solve initial value problems of first order ordinary differential equations. Four new linear multistep collocation methods were developed. Collocations were taken at even grid points for even k, while for odd k, collocations were taken at odd grid points. For maximal order method k = 4 collocation were taken at all grid points. In both cases interpolations were taken at all the grid points except the last grid point. At $x=x_{n+k}$ the discrete methods obtained are symmetric for even k and of order p = 3, 4, 6 and 8 respectively.

The methods are consistent but not zero stable. This is so because the interpolation points were not restricted to a single point x_{n+k-1} or x_n as it is in Adams methods. The accuracy of the methods was tested, using linear and non-linear first order sample problems. The results show a significant improvement over the existing methods in terms of accuracy. It is also striking to note that orders of accuracy of the correctors and predictor are equal for corresponding step numbers.

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CHAPTER ONE

INTRODUCTION

1.1 Differential Equations

A differential equation is an equation involving the dependent variables and their derivatives. Examples are:

$$\frac{dy}{dx} = 2x + 7 \qquad 1.1$$

$$\frac{dy}{dx} = (1-x)y^2 - y \qquad 1.2$$

$$\frac{dy}{dx} = x\sin y + y\sin x \qquad 1.3$$

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + y = 0 \qquad 1.4$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 3x^2 + y \qquad 1.5$$

The order of a differential equation is the order of its highest derivative. From the above examples, equations (1.1) – (1.3) are first order while equations (1.4) and (1.5) are second order differential equations.

A differential equation with respect to a single independent variable is called an ordinary differential equation (ODE). On the other hand, if there exist more than one independent variable the equation is called a partial differential equation (PDE). Equations (1.1) – (1.4) are ordinary differential equations while equation (1.5) is a partial differential equation.

The derivative of a function does not give enough information to determine it completely there is a need therefore to seek either a solution to a differential equation or a general solution (which usually has a constant for each order of the equation in it) or a solution subject to some additional conditions.

The general solution to any separable first order differential equation is to integrate both sides of the equation but if a first order differential equation is not separable, its solution cannot be reduced to quadraures directly, then the numerical techniques should be applied.

1.2 Ordinary differential equation (ODE)

Ordinary differential Equation is an equation involving a function and its derivative. An ODE of order n is of the form

$$F(x, y, y^1 \ldots\ldots\ldots, y^{(n)}) = 0, \qquad 1.6$$

where $y^r = \frac{d^r y}{dx^r}$, r = 0(1)n.

An ODE of order n is said to be linear if it is of the form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_n(x)y^{(2)} + a_1(x)y' + a_0(x)y = f(x) \qquad 1.7$$

This means that if there is no product of the dependent variable with itself or its derivatives then the ODE is linear, otherwise it is non–linear. For example, equations (1.1), (1.3) and (1.4) are linear (ODE) while (1.2) is non-linear ODE.

It has been discovered that mathematical models resulting in single or system of first order ODE's are largely applied in nearly all disciplines most especially in Sciences, Engineering and Economics.

Any system whose behaviour can be modeled by a first order differential equations or even by a set of linear first order ODE can be solved numerically to any desired degree of accuracy.

According to Lurgi Bringnano and Cecelia Maghermi (2003), the numerical solutions of ODEs remain an active field of investigation, though, the areas of research vary significantly.

1.3 Initial value problems

Ordinary differential equations with a set of initial conditions are called initial value problems. The general form of nth order initial value problem is:

$y^{(n)} = f(x, y, y^{/}........y^{n-1}), y^{(i)}(x) = y^{i}_{0}, i = 0(1)n-1$ 1.8

if n = 1, (1.8) becomes a first order initial value problem in ODE's and it is of the form;

$y^{/} = f(x,y), y(x_0) = y_o$ 1.9

where x is the independent variable, y (x) is the unknown quantity (or dependent variable) that should be determined, f(x, y) is a known function that depend on both x and y, y_0 is called the initial value or initial condition. Since it provides a value for the solution at an initial stage x = 0 (the initial value is required so that the problem has a unique solution).

Equation (1.9) involves the first derivative of the solution and also initial values for x and y are provided, hence the name first order initial value problem. If n >1, equation (1.8) can lead to a system of n first order initial value simultaneous equations. Such systems arise naturally in a large number of applications in Sciences and Engineering.

The existence and uniqueness of solutions of equation (1.8) is guaranteed by the theorem popularly referred to as "Existence and Uniqueness Theorem." The theorem is stated below without a proof.

Theorem 1.1

Let f(x, y) be defined and continuous for all points (x, y) in the region D defined by $a \leq x \leq b, -\infty < y < \infty$, a and b are finite, and let there exists a constant L for any x in (a,b) and number y_1, y_2, in R^n such that

$|f(x_1, y_1) - f(x_2, y_2)| = L |y_1 - y_2|.$ (1.10)

The condition (1.10) is known as Lipschitz conditions of order one and the constant L as the Lipschitz constant [Lambert (1973)].

This implies that if the function f is continuous and satisfies the Lipschitz condition (Theorem 1.1) on some closed region U x V with $x \in U$ and $y \in V$, containing the points (x_0, y_0), then there exists a unique solution y(x) and sub interval U_0 containing x_0 and contained in U such that $y(x_0) = y_0$ and (x, y(x)) is contained in U x V for x in U_0. The condition $y(x_0) = y_0$ is called the initial condition. If the function f is differentiable in the closed interval [a, b], then the Lipschitz condition is automatically satisfied.

It should be noted that if the Lipschitz condition is not satisfied, then either equation (1.8) has no solution or there may exist more than one solutions.

1.4 Basic concepts and principles

Some of the basic concepts and principles used in this work are defined here under.

1.4.1 Step length or mesh size

The numerical solution of problem (1.9) are often based on the principle of discretisation in which an approximation to an unknown function y (x) are sought on a certain discrete points x_i, i = 0, 1, ..., n of set X. Consider the set of points $\{x_i\}$ in the closed interval I= [a, b], defined by $a = x_0 < x_1 < x_2 < \ldots < x_n = b$

$h = x_{i+1} - x_i$, i = 0 (1) n −1.

The parameter h is called the step length or mesh size. In this work the step length h is constant

Fig. 1.1: Grid or mesh points

The points x_{n+i}, i = 0 (1) k are called the grid or mesh points (Fig.1.1). Each grid point is given in terms of the previous point in the form

$$x_{n+i} = x_n + i\,h, \quad i = 0\ (1)\ k \qquad 1.11$$

The last grid point is the step number k, in fig.1.1 k = 4.

1.4.2 Linear multistep method

The general linear multistep method for first order ODEs is of the form

$$y_k(x) = \sum_{j=0}^{k-1} \alpha_j y_{n+j} + \sum_{j=0}^{k} \beta_j f_{n+j}, \quad j = 0(1)4, \qquad 1.12$$

where α_j, β_j are constants, $f_{n+j} = f(x_{n+j}, y_{n+j})$, $\alpha_k \neq 0$ and that both α_0 and β_0 are non zero.

The linear multistep method is said to be explicit if $\beta_k = 0$ and "implicit" if otherwise.

The problem with linear multistep method is that they need help getting started which is not encountered in single step methods.

The usual way of solving this problem is to use a one-step explicit method such as Runge – Kutta of the same order of accuracy until enough values have been generated for multistep method to take off. This helper required for getting started is called a predictor.

In this work, the initial approximation y_{n+i}, i = 1(1)k were generated as starting values. These starting values for y_{n+i}, i = 1(1)k are called Predictors for (1.12) while the equation (1.12) is called the corrector, hence the procedure is called predictor - corrector method. The predictors are explicit while the correctors are implicit methods.

1.4.3 Stability

Numerical stability is discussed in depth in Lambert (1973) and Burden Fave (1993) in Kokler (1994). It deals with growth or decay of error as numerical computation progresses.

If any error introduced into the computation is amplified as computation progresses, it means that the method used is unstable but if the error decays as the computation progresses, it shows that the method is stable.

1.5 The existing methods

There are many general techniques for analytical solution of ODEs but the only practical solution technique for complicated equations is the use of numerical methods Milne (1970).

A vast amount of research and huge numbers of publications have been devoted to the numerical solution of differential equations. Most of the existing methods could be grouped into two categories viz: (i) single step method and (ii) multistep method.

In general, a single –step method is of the form

$$y_{n+1} - y_n = h\,\varphi(x_n, y_n, h), \qquad 1.13$$

where φ is called an increament function.

Methods in this category use only the information from the previous step and they are self starting. Amongst the existing popular single step methods are Euler method, Heun method, Trapazoidal method, Taylor series method and Runge Kulta method. Lambert (1973), Jacques and Judd (1987), Ademiluyi (1987), Ademiluyi and Kayode (2001), Kockler (1994) Bruganam and Magherimi (2003) are among others who have developed single step methods to solve problems (1.9).

The general multistep method of the form (1.12) include Simpson method, Adam-Bashforth method and Adam – Molton methods. A number of researchers have developed methods of the form (1.12) for solving problem (1.9). These include Henrici (1962), Gear (1971), Lambert (1973), Lambert and Watson (1976), Awoyemi (1992).

All the Adams' methods are regarded as constant coefficient method but in this study, Linear multistep methods with variable coefficients are generated. Such methods are often called continuous collocation methods in the literature. They are of the form:

$$y_k(t) = \sum_{j=0}^{k-1} \alpha_j(t)\, y_{n+j} + \sum_{j=0}^{k} \beta_j(t)\, f_{n+j} \quad j = 0(1)4 \qquad 1.14$$

The parameters of these methods are determined by the collocation approach. Much work has been done on discrete method but less attention was given to continuous collocation methods.

In his work, Awoyemi (1992) stated the advantages of continuous schemes over the discrete ones, which include;

I. Provision of better global error estimate

II. Usefulness for further analytical work in a simpler form than the discrete ones.

III. Provision of approximation at all interior points.

Another added advantage of continuous scheme is that infinite number of schemes could emerge from one continuous scheme.

The main objective of this study is to develop some continuous multistep collocation methods with all collocation points taken at the selected grid points.

1.6 Collocation method

According to Encyclopedia of Mathematics (Vol. 1, pg 704), collocation is defined as a projection method for solving integral and differential equation in which the approximate solution is determined from the condition that the equation must be stratified at certain given point. It involves the determination of an approximate solution in a suitable set of functions called the basis functions. The basis function is taken to be power series in x of degree n in this work.

Awoyemi (1992) attributed the discovery of collocation method to Kantorovich (1934) who developed spline collocation procedure for the solution of partial differential equation Frazer et al (1937, 1938) developed collocation method for the solution of ODEs and partial differential equations (PDEs).

Awoyemi (1992, 1999, 2001), Collatz (1960) Lanczos (1973), Cresswell (1971), Kayode (2004), Oladele (1991) Onumanyi (1994) are amongst researchers who have adopted collocation method for solving ODEs.

1.7 Specific objectives of the research:

The specific objectives are to:

(a) Develop continuous linear multistep methods of the form

$$y_k(t) = \sum_{j=0}^{k-1} \alpha_j(t)\, y_{n+j} + \sum_{j=0}^{k} \beta_j(t)\, f_{n+j} \quad j = 0(1)4 \qquad 1.15$$

for solving initial value problems of first order ODE's

(b) Develop predictors for calculating $y_{n+j},\ j = 1(1)k$, in the main schemes

(c) Analyse the accuracy, order, consistency, convergence and stability of the schemes,

(d) Develop computer programmes using Fortran 77 compilers to test the accuracy of the schemes by solving linear and nonlinear first order ODEs.

(e) Compare the results with the results from some existing methods (Runge-Kutta of order 4).

1.8 Research methodology:

Assuming an approximate solution to problem (1.9) is given as

$$y(x) = \sum_{j=0}^{\frac{3}{2}k} a_j x^j \qquad 1.16$$

for even K and its first derivative is

$$y'(x) = \sum_{j=0}^{\frac{3}{2}k} j a_j x^{j-1} = f(x, y) \qquad 1.17$$

and

$$y(x) = \sum_{j=0}^{k+i} a_j x^j,\ i = \frac{k-1}{2} \qquad 1.18$$

for odd k and its first derivative is

$$y'(x) = \sum_{j=0}^{k+i} j a_j x^{j-1} = f(x, y) \qquad 1.19$$

where a_j's are the parameters to be determined. Collocation and interpolation at the selected grid points were used to obtain system of linear equations involving the unknown parameters. The system was solved by Gaussian elimination method. Four different methods were developed for the step number k =2,3, and 4. For even step number k, collocations was taken at even grid points while for odd step number k, collocation was at odd number grid points and maximal order for k = 4 consisted of collocation at all grid points.

In all cases, interpolation of the approximate solution was done at

$x = x_{n+j}$, j = 0(1)k-1 and evaluation was done at the last grid point to obtain the new schemes.

Analysis of the properties of the methods, which includes order, error term, consistency, convergence and stability, was carried out. Taylor series expansion was adopted to determine the order and error term while boundary locus method was used to determine the region of absolute stability of the methods.

The algorithm was coded in Fortran programming language and tested with sample problems to establish the applicability and accuracy of the new methods. The results obtained were compared with the results from existing R-K method.

CHAPTER TWO

DERIVATION OF THE METHODS

2.1 INTRODUCTION

In this section, continuous linear multistep method are derived for ODE problems of the form

$$y' = f(x,y),\ y(x) = y_0 \tag{2.1}$$

whose approximate solutions are given as

$$y(x) = \sum_{j=0}^{\frac{3}{2}k} a_j x^j \tag{2.2}$$

for even k and its first derivative is

$$y'(x) = \sum_{j=0}^{\frac{3}{2}k} j a_j x^{j-1} = f(x,y), \tag{2.3}$$

and

$$y(x) = \sum_{j=0}^{k+i} a_j x^j,\ i = \frac{k-1}{2} \tag{2.4}$$

for odd k and its first derivative is

$$y'(x) = \sum_{j=0}^{k+i} j a_j x^{j-1} = f(x,y). \tag{2.5}$$

The a_js are the parameters to be determined.

In this work, it is assumed that problem (2.1) satisfies the existence and uniqueness theorem as stated in theorem (1.1)

2.2 Derivation of 2–step method

The method is derived by collocating the differential system (2.3) at even grid points x = x_n, x_{n+2} and interpolating the approximate equation (2.2) is at all grid points other than the last grid point to obtain the following system of equations

$$\left.\begin{aligned}
&a_0 + a_1x_n + a_2x_n^2 + a_3x_n^3 = y_n \\
&a_0 + a_1x_{n+1} + a_2x_{n+1}^2 + a_3x_{n+1}^3 = y_{n+1} \\
&a_1 + 2a_2x_n + 3a_3x_n^2 = f_n \\
&a_1 + 2a_2x_{n+2} + 3a_3x_{n+2}^2 = f_{n+2}
\end{aligned}\right\} \qquad (2.6)$$

By using Gaussian elimination techniques, the values of a_j's are determined.

$$\left.\begin{aligned}
&a_3 = \frac{1}{8h^3}\{hf_{n+2} + 3hf_n - 4y_{n+1} + 4y_n\} \\
&a_2 = \frac{1}{h^2}\{-hf_n + y_{n+1} - y_n - a_3(3h^2x_n + h^3)\} \\
&a_1 = \frac{1}{h}\{y_{n+1} - y_n - a_2(2hx_n + h^2) - a_3(3hx_n^2 + 3h^2x_n + h^3)\} \\
&a_0 = y_n - a_1x_n - a_2x_n^2 + a_3x_n^3
\end{aligned}\right\} \qquad (2.7)$$

Substituting the values of a_j's into the equation (2.2), and after algebraic manipulation a continuous scheme is obtained in the form

$$\sum_{j=0}^{kk} \alpha_j(x)y_{n+j} = \sum_{j=0}^{k} \beta_j(x)f_{n+j}, \ k = 2, \qquad (2.8)$$

where $f_{n+j} = f(x_{n+j}, y_{n+j})$.

By choosing

$$t = \frac{(x - x_{n+k-1})}{h} \qquad (2.9)$$

and substituting this into (2.8), the continuous methods in power series of t is obtained in the form

$$y_k(t)\sum_{j=0}^{kk}\alpha_j(t)y_{n+j}+\sum_{j=0}^{k}\beta_j(t)f_{n+j} \qquad (2.10)$$

where $f_{n+j}=f(x_{n+j},y_{n+j})$ and

$$\alpha_1(t)=\frac{1}{2}(2+3t-t^3)$$

$$\alpha_0(t)=\frac{1}{2}(-3t+t^3)$$

$$\beta_2(t)=\frac{h}{8}(t+2t^2+t^3)$$

$$\beta_0(t)=\frac{h}{8}(-5t-2t^2+3t^3)$$

Evaluating (2.10), for t =1, yields a discrete symmetric scheme

$$y_{n+2}-2y_{n+1}+y_n=\frac{h}{2}(f_{n+2}-f_n) \qquad (2.11)$$

2.3 Derivation of 3-step method

The collocation of the differential system (2.5) at odd grid points $x = x_{n+1}, x_{n+3}$ and interpolation of the approximate equation (2.4) at all grid point other than the last grid point yields the following system of equations

$$\left.\begin{aligned}
&a_o+a_1x_n+a_2x_n^2+a_3x_n^3+a_4x_n^4=y_n\\
&a_o+a_1x_{n+1}+a_2x_{n+1}^2+a_3x_{n+1}^3+ax_{n+1}^4=y_{n+1}\\
&a_o+a_1x_{n+2}+a_2x_{n+2}^2+a_3x_{n+2}^3+ax_{n+2}^4=y_{n+2}\\
&a_1+2a_2x_{n+1}+3a_3x_{n+1}^2+4a_4x_{n+1}^3=f_{n+1}\\
&a_1+2a_2x_{n+3}+3a_3x_{n+3}^2+4a_4x_{n+3}^3=f_{n+3}
\end{aligned}\right\} \qquad (2.12)$$

By solving equation (2.12), the values of a_j's are obtained to be

$$\left.\begin{aligned}
a_4 &= \frac{1}{28h^4}\{hf_{n+3} + 11hf_{n+1} - 8y_{n+2} + 4y_{n+1} + 4y_n\} \\
a_3 &= \frac{1}{2h^3}\{-2hf_{n+1} + y_{n+2} - y_n - a_4(8h^3x_n + 8h^4)\} \\
a_2 &= \frac{1}{2h^2}\{y_{n+2} - y_{n+1} + y_n - a_3(6h^2x_n + 6h^3) - a_4(12h^2x_n^2 + 24h^3x_n + 14h^4)\} \\
a_1 &= \frac{1}{h}\{y_{n+1} - y_n - a_2(2hx_n + h^2) - a_3(3hx_n^2 + 3h^2x_n + h^3) - a(4hx_n^3 + 6h^2x_n^2 \\
&\quad + 4h^3x_n + h^4\} \\
a_0 &= y_n - a_1x_n - a_2x_n^2 - a_3x_n^3 - a_4x_n^4
\end{aligned}\right\} \qquad (2.13)$$

Substituting the values of a_j's into equation (2.4), and after algebraic manipulation a continuous scheme is obtained in the form

$$y_k(x) = \sum_{j=0}^{k-1}\alpha_j(x)y_{n+j} + \sum_{j=0}^{k}\beta_j(x)f_{n+j} \qquad (2.14)$$

where $f_{n+j} = f(x_{n+j}, y_{n+j})$.

By using the transformation (2.9) in (2.14), the continuous implicit methods in power series of t are obtained in the form

$$y_k(t) = \sum_{j=0}^{k-1}\alpha_j(t)y_{n+j} + \sum_{j=0}^{k}\beta_j(t)f_{n+j}\text{ , k = 3,} \qquad (2.15)$$

where $f_{n+j} = f(x_{n+j}, y_{n+j})$ and

$$\alpha_2(t) = \frac{1}{14}(14 + 27t + 8t^2 - 9t^3 - 4t^4)$$

$$\alpha_1(t) = \frac{1}{7}(-12t - 2t^2 + 4t^3 + t^4)$$

$$\alpha_0(t) = \frac{1}{14}(-3t - 4t^2 + t^3 + 2t^4)$$

$$\beta_3(t) = \frac{h}{28}(2t + 5t^2 + 4t^3 + t^4)$$

$$\beta_1(t) = \frac{h}{28}(-34t - 29t^2 + 16t^3 + 11t^4)$$

Evaluating (2.15), for t =1, yields a discrete scheme

$$7y_{n+3} - 18y_{n+2} + 9y_{n+1} + 2y_n = 3h(f_{n+3} - 3f_{n+1}) \qquad (2.16)$$

2.4 Derivation of 4-step method

In this section, equation (2.3) is collocated at even grid points $x = x_{n+j}$, $j = 0(2)\,4$, and the approximate equation (2.2) is interpolated at $x = x_{n+j}$, $j = 0(1)\,3$, to yield a system of algebraic equations

$$a_0 + a_1x_1 + a_2x_n^2 + a_3x_n^3 + a_4x_n^4 + a_5x_n^5 + a_6x_n^6 = y_n$$

$$a_0 + a_1x_{n+1} + a_2x_{n+1}^2 + a_3x_{n+1}^3 + a_4x_{n+1}^4 + a_5x_{n+1}^5 + a_6x_{n+1}^6 = y_{n+1}$$

$$a_0 + a_1x_{n+2} + a_2x_{n+2}^2 + a_3x_{n+2}^3 + a_4x_{n+2}^4 + a_5x_{n+2}^5 + a_6x_{n+2}^6 = y_{n+2}$$

$$a_0 + a_1x_{n+3} + a_2x_{n+3}^2 + a_3x_{n+3}^3 + a_4x_{n+3}^4 + a_5x_{n+3}^5 + a_6x_{n+3}^6 = y_{n+3}$$

$$a_1 + 2a_2x_n + 3a_3x_n^2 + 4a_4x_n^3 + 5a_5x_n^4 + 6a_6x_n^5 = f_n$$

$$a_1 + 2a_2x_{n+2} + 3a_3x_{n+2}^2 + 4a_4x_{n+2}^3 + 5a_5x_{n+2}^4 + 6a_6x_{n+2}^5 = f_{n+2}$$

$$a_1 + 2a_2x_{n+4} + 3a_3x_{n+4}^2 + 4a_4x_{n+4}^3 + 5a_5x_{n+4}^4 + 6a_6x_{n+4}^5 = f_{n+4} \qquad (2.17)$$

by using Gaussian elimination techniques the values of a_j's are determined.

$a_6 = \frac{1}{5385h^6}$ $\{1936\, y_{n+3} + 3366y_{n+2} - 7920y_{n+1} + 261y_n - 99hf_{n+4} - 5544hf_{n+2} + 1023hf_n\}$,

$a_5 = \frac{1}{36h^5}$ $\{-2y_{n+3} - 9y_{n+2} + 18y_{n+7} - 7y_n + 9hf_{n+2} - 3hf_n + a_6(11880h^5x_n + 15840h^6)\}$,

$a_4 = \frac{-1}{36h^4}$ $\{-2y_{n+3} + 9y_{n+2} - 18y_{n+1} + 11y_n + 6hf_{n+a}\, 5(180h^4x_2 + 216h^5) + a_6\,(540h^4x_n + 1296h^5x_n + 900h^6)\}$,

$$a^3 = \frac{1}{6h^3}\{y_{n+3}-3y_{n+2}+3y_{n+1}-y_n-a_4(24h^3x_n+36h^4)-a_5(60h^3x^2n+180h^4x_{n+1}50h^5)-$$

$$a_6(120h^3x^3_n+540h^4x_n^2+900h^5x_n+540h^6)\},$$

$$a_2 = \frac{1}{2h^2}\{y_{n+2}-2y_{n+1}+y_n-a_3(6h^2x_n+6h^3)-a_4(12h^2x2_n+24h^3x_n+$$

$$24h^3x_n+14h^4)-a_5(70h^2x^3_n+60h^3x^2_n+70h^4x_n+30h^5)-$$

$$a_6(30h^2x^4_n+120h^3x^3_n+210h^4x^2_n+180h^5x_n+62h^6)\},$$

$$a_1 = \frac{1}{h}\{y_{n+1}-y_n-a_2(2hx_n+h^2)-\alpha_3(3hx^2_n+3h^2x_n+h^3)-$$

$$\alpha_4(4hx_n^3+6h^2x_n^2+4h^3x_n+h^4)(5hx_n^4+10h^2x_n^3+10h^3x_n^2+5h^4x_n+h^5)$$

$$a_2(6hx^5_n+15h^2x^2_n+20h^3x^3_n+15h^4x^2_n+6h^5x_n+h^6)$$

$$a_0 = y_n - a_1x_n - a_2x^2_n - a_3x^3_n - a_4x^4_n - a_5x^5_n - a_6x^6_n \qquad (2.18)$$

Substituting the values of a_j's into the expression (2.2), and after algebraic manipulation a continuous scheme is obtained in the form

$$\sum_{j=0}^{k} \alpha_j(x)\, y_{n+j} = \sum_{j=0}^{k} \beta_j(x)\, f_{n+j} \qquad (2.19)$$

where $f_{n+j} = f(x_{n+j}, y_{n+j})$.

By choosing $t = (x - x_{n+3})/h$ (2.20)

and substituting this into (2.19), the continuous methods in power series of t is obtained in the form

$$y_k(t) = \sum_{j=0}^{k-1} \alpha j(t)\, y_{n+j} + \sum_{j=0}^{k} \beta j(t)\, f_{n+j}, \qquad j = 0(1)4 \qquad (2.21)$$

$$\alpha_0(t) = \frac{1}{144}(66t + 137t^2 + 48t^3 - 58t^4 - 42t^5 - 7t^6),$$

$$\alpha_1(t) = \frac{1}{34}(-63t - 123t^2 - 34t^3 + 54t^4 + 33t^5 + 5t^6),$$

$$\alpha_2(t) = \frac{1}{16}(-18t + 15t^2 + 16t^3 - 6t^4 - 6t^5 - t^6),$$

$\alpha_3(t) = \frac{1}{306}(306 + 771t + 529t^2 - 102t^3 - 248t^4 - 93t^5 + 7t^6)$,

$\beta_4(t) = \frac{h}{544}(18t + 57t^2 + 68t^3 + 38t^4 + 10t^5 + t^6)$,

$\beta_2(t) = \frac{h}{68}(-180t - 264t^2 - 17t^3 + 113t^4 + 53t^5 + 7t^6)$,

$\beta_0(t) = \frac{h}{1632}(258t + 545t^2 + 204t^3 - 226t^4 - 174t^5 - 31t^6)$

Evaluating at t=1 and substituting the results in (2.19) yields a discrete symmetric scheme.

$$17y_{n+4} - 64y_{n+3} + 64y_{n+1} - 17y_n = 6h(f_{n+4} - 12f_{n+2} + f_n) \qquad (2.22)$$

2.5 DERIVATION OF MAXIMAL ORDER METHOD

In addition to the methods earlier developed where collocation points are taken at selected points, derivation of maximal order method is also investigated for k = 4.

The approximate solution to problem (2.1) is given as

$$y(x) = \sum_{j=0}^{2k} a_j x^j \qquad (2.23)$$

and its first derivative is

$$y'(x) = \sum_{j=0}^{2k} j a_j x^{j-1} = f(x, y), \qquad (2.24)$$

By collocating the differential system (2.24) at all the grid points and interpolating (2.23) at $x = x_{n+i}$, i = 0(1) 3 a system of algebraic equations is obtained

$$a_0 + a_1 x_n + a_2 x_n^2 + a_3 x_n^3 + a_4 x_n^4 + a_5 x_n^5 + a_6 x_n^6 + a_7 x_n^7 + a_8 x_n^8 = y_n$$

$$a_0 + a_1 x_{n+1} + a_2 x_{n+1}^2 + a_3 x_{n+1}^3 + a_4 x_{n+1}^4 + a_5 x_{n+1}^5 + a_6 x_{n+1}^6 + a_7 x_{n+1}^7 + a_8 x_{n+1}^8 = y_{n+1}$$

$$a_0 + a_1 x_{n+2} + a_2 x_{n+2}^2 + a_3 x_{n+2}^3 + a_4 x_{n+2}^4 + a_5 x_{n+2}^5 + a_6 x_{n+2}^6 + a_7 x_{n+2}^7 + a_8 x_{n+2}^8 = y_{n+2}$$

$$a_0 + a_1 x_{n+3} + a_2 x_{n+3}^2 + a_3 x_{n+3}^3 + a_4 x_{n+3}^4 + a_5 x_{n+3}^5 + a_6 x_{n+3}^6 + a_7 x_{n+3}^7 + a_8 x_{n+3}^8 = y_{n+3}$$

$$a_1 + 2a_2 x_n + 3a_3 x_n^2 + 4a_4 x_n^3 + 5a_5 x_n^4 + 6a_6 x_n^5 + 7a_7 x_n^6 + 8a_8 x_n^7 = f_n$$

$$a_1 + 2a_2 x_{n+1} + 3a_3 x_{n+1}^2 + 4a_4 x_{n+1}^3 + 5a_5 x_{n+1}^4 + 6a_6 x_{n+1}^5 + 7a_7 x_{n+1}^6 + 8a_8 x_{n+1}^7 = f_{n+1}$$

$$a_1 + 2a_2x_{n+2} + 3a_3x_{n+2}^2 + 4a_4x_{n+2}^3 + 5a_5x_{n+2}^4 + 6a_6x_{n+2}^5 + 7a_7x_{n+2}^6 + 8a_8x_{n+2}^7 = f_{n+2}$$

$$a_1 + 2a_2x_{n+3} + 3a_3x_{n+3}^2 + 4a_4x_{n+3}^3 + 5a_5x_{n+3}^4 + 6a_6x_{n+3}^5 + 7a_7x_{n+3}^6 + 8a_8x_{n+3}^7 = f_{n+3}$$

$$a_1 + 2a_2x_{n+4} + 3a_3x_{n+4}^2 + 4a_4x_{n+4}^3 + 5a_5x_{n+4}^4 + 6a_6x_{n+4}^5 + 7a_7x_{n+4}^6 + 8a_8x_{n+4}^7 = f_{n+4}$$

... (2.25)

Using Gaussian elimination techniques, the values of a_j's are given as follows

$$a_8 = \frac{1}{7200}h^8[\frac{1360}{3}y_{n+3} + 450y_{n+2} - 720y_{n+1} - \frac{550}{3}y_n + 3hf_{n+4} - 152hf_{n+3} - 792hf_{n+2} - 552hf_{n+1} - 47hf_n]$$

$$a_7 = \{\frac{1}{1692}h^7(281y_{n+3} + 27y_{n+2} - 297y_{n+1} - 11y_n + 3hf_{n+4} - 105hf_{n+3} - 369hf_{n+2} - 129hf_n) - a_8(8x_n + \frac{764}{47}h)\}$$

$$a_6 = \{\frac{1}{9720}h^6(720y_{n+1} - 720y_n + 19hf_{n+4} - 106hf_{n+3} + 264hf_{n+2} - 646f_{n+1} - 251hf_n) - a_7(7x_n + \frac{986}{81}h) - a_8(28x_n^2 + \frac{7888}{81}hx_n + \frac{7514}{81}h^2)\}$$

$$a_5 = \{\frac{1}{120}h^5(f_{n+4} - 4f_{n+3} + 6f_{n+2} - 4f_{n+1} + f_n) - a_6(6x_n + 12h) - a_7(21x_n^2 + 84hx_n + 91h^2) - a_8(56x_n^3 + 336hx_n^2 + 728h^2x_n + 560h^3)\}$$

$$a_4 = \{\frac{1}{24}h^4(f_{n+3} - 3f_{n+2} - f_n) - a_5(5x_n + \frac{15}{2}h) - a_6(15x_n^2 + 45hx_n + \frac{75}{2}h^2) - a_7(35x_n^3 + \frac{315}{2}hx_n + \frac{525}{2}h^2x_n + \frac{315}{2}h^3) - a_8(70x_n^4$$
$$+420hx_n^3 + 1050h^2x_n^2 + 1260h^3x_n + 602h^4)\}$$

$$a_3 = \{\frac{1}{6}h^3(f_{n+2} - 2f_{n+1} + f_n) - a_4(4x_n + 4h) - a_5(10x_n^2 + 20hx_n + \frac{35}{3}h^2) - a_6(20x_n^3 + 60hx_n^2 + 30h^3) - a_7(35x_n^4$$
$$+140hx_n^3 + 25xh^2x_n^2 + 210h^3x_n + \frac{217}{3}h^4) - a^8(56x_n^5 + 280hx_n^4 + \frac{1960}{3}h^2x_n^3 + 840h^3x_n^2 + \frac{1736}{3}h^4x_n + 168h^5)\}$$

$$a_2 = \{\frac{1}{2}h^2(f_{n+1} - f_n) - a_3(3x_n + 3h) - a_4(6x_n^2 + 6hx_n + 2h^2) - a_5(10x_n^3 + 15hx_n^2 + 10h^2x_n + \frac{5}{2}h^3) - a_6(15x_n^4$$
$$+30hx_n^3 + 30h^2x_n^2 + 15h^3x_n + 3h^4) - a_7(21x_n^5 + \frac{105}{2}hx_n^4 + 70h^2x_n^3 + \frac{105}{2}hx_n^4 + 70h^2x_n^3 + \frac{105}{2}h^2x_n^2$$
$$+21h^4x_n + \frac{7}{2}h^5) - a^8(28x_n^6 + 84hx_n^5 + 140h^2x_n^4 + 140h^3x_n^3 + 84h^4x_n^2 + 28h^5x_n + 4h^6)\}$$

$$a_1 = f_n - 2a_nx_n - 3xa_{3n}^2 - 4a_4x_n^3 - 5a_5x_n^4 - 6a_6x_n^5 - 7a_7x_n^6 - 8a_8x_n^7$$

$$a_0 = y_n - a_1x_n - a_2x_n^2 - a_3x_n^3 - a_4x_n^4 - a_5x_n^5 - a_6x_n^6 - a_7x_n^7 - a_8x_n^8$$

... (2.26)

Following the same procedure for obtaining (2.19) the expressions for α_j;s and β_j;s in

$$\sum_{j=0}^{k}\alpha_j(x)y_{n+j} = \sum_{j=0}^{k}\beta_j(x)f_{n+j}. \qquad (2.27)$$

taking t = $(x-x_{n+3})/h$, are

$$\alpha_3 \frac{1}{540}(540 - 3141t^2 - 3967t^3 - 488t^4 + 1886t^5 + 1327t^6 + 3538t^7 + 34t^8)$$

$$\alpha_2 \frac{1}{16}(36t^2 - 12t^3 - 47t^4 + 4t^5 + 18t^6 + 8t^7 + t^8)$$

$$\alpha_1 \frac{1}{20}(63t^2 + 141t^3 + 64t^4 - 58t^5 - 61t^6 - 13t^7 - t^8)$$

$$\alpha_o = \frac{1}{270}(180t^2 + 452t^3 + 277t^4 - 148t^5 - 230t^6 - 88t^7 - 11t^8)$$

$$\beta_4 = \frac{h}{2400}(36t^2 + 132t^3 + 193t^4 + 144t^5 + 58t^6 + 12t^7 + t^8)$$

$$\beta_3 = \frac{h}{900}(900t + 2616t^2 + 2317t^3 - 67t^4 - 1286t^5 - 802t^6 - 703t^7 - 19t^8)$$

$$\beta_2 = \frac{h}{120}(504t^2 + 948t^3 + 302t^4 - 409t^5 - 363t^6 - 107t^7 - 11t^8)$$

$$\beta_1 = \frac{h}{300}(600 + 522t^2 + 1231t^3 + 661t^4 - 462t^5 - 584t^6 - 201t^7 - 23t^8)$$

$$\beta_0 = \frac{h}{7200}(7200t + 708t^2 + 1796t^3 + 1129t^4 - 568t^5 - 926t^6 - 364t^7 - 47t^8)$$

Put t = 1 implies that $x = x_{n+4}$ and substitute the result in (2.21) yields a symmetric discrete scheme

$$5y_{n+4} + 33y_{n+3} - 33y_{n+1} - 5y_n = \frac{6h}{5}(f_{n+4} + 16f_{n+3} + 36f_{n+2} + 16f_{n+1} + f_n) \qquad (2.28)$$

2.6 Derivations of predictors for the proposed methods

To implement the proposed methods, the predictors are required. Hence, the derivations of the predictors are considered in this section. Using the same procedure as for the main methods the following predictors are obtained for k = 2, 3 and 4

2.6.1 Development of predictors for 2-step method

Taking the approximate solution to problem (2.1) to be

$$y(x) = \sum_{j=0}^{2k-1} a_j x^j \qquad (2.29)$$

The first derivate of (2.29) is given as

$$y_{(x)} = \sum_{j=0}^{2k-1} j a_j x^{j-1} \qquad (2.30)$$

By collocating (2. 30) at $\chi = \chi_{n+i}$ I = 0 (1) k – 1 and interpolating (2.29) at $\chi = \chi_{n+1}$ i = 0 (1) k – 1 to obtain a system of non – linear equations

$a_0 + a_1\chi_n + a_2\chi^2_n + a_3\chi^3_n = y_n$

$$a_0 + a_n\chi_{n+1} + a_2\chi^2_{n+1} + a_3\chi^3_{n+1} = y_{n+1}$$

$$a_1 + 2a_2\chi_{n+1} + 3a_3\chi^2_n = f_n$$

$$a_1 + 2a_2\chi_{n+1} + 3a_3\chi^2_{n+1} = f_{n+1} \qquad (2.31)$$

by using Gaussian elimination techniques the values of a_j's are determined

$$a_3 = \frac{1}{h^3}(hf_{n+1} + hf_n - y_{n+1} + y_n)$$

$$a_2 = \frac{1}{h^3}(-hfn + y_{n+1} - yn - a_3(3h^2\chi n + h^3)$$

$$a_1 = \frac{1}{h}(y_{n+1} - y_n - a_2(2h\chi_n + h^2) - a_3(3h\chi n^2 + 3h^2\chi_n + h^3)$$

$$a_0 = y_n - a_n\chi_n - a_2\chi^2_n - a_3\chi^3 n \qquad (2.32)$$

Substituting the value of a_j,s into equation (2.29), and after algebraic manipulation a continuous scheme is obtained in the form

$$y(t) = \sum_{j=0}^{k-1} \alpha_2(\chi)\, y_{n+j} + \sum_{j=0}^{k-1} \beta j(\chi)\, f_{n+j}, \qquad (2.33)$$

where $f_{n+j} = f(\chi_{n+j}, y_{n+j})$

by choosing $t = (\chi - \chi_{n+k-1})/h$, $t = (0, 1]$ (2.34)

By using the transformation (2.34) in (2.33), the continuous explicit methods in power series of t are obtained in the form

$$y(t) = \sum_{j=0}^{k-1} \alpha_2(t)\, y_{n+j} + \sum_{j=0}^{k-1} \beta j(t)\, f_{n+j}, \qquad (2.35)$$

where

$\alpha_0(t) = 3t^2 + 2t^3$

$\alpha_1(t) = 1 - 3t^2 - 2t^3$

$\beta_1(t) = t + 2t^2 + t^3$

$\beta_3(t) = t^2 + t^3$

Evaluating (2.35), for t = 1, yields a discrete scheme

$y_{n+2} = -4y_{n+1} + 5y_n + 2h(2f_{n+1} + f_n)$ (2.36)

By adopting Taylor Series expansion for y_{n+1} about (x_n, y_n) yields

$$y_{n+1} = y_n + hf_n + \frac{h^2}{2}(\delta f/\delta_x + f_n(\delta f/\delta_y) \quad (2.37)$$

2.6.2 Development of predictors for 3 – step methods

Let the approximate solution to problem (1.9) be (2.29) and its derivate be (2.30) then by collocating the differential system (2.30) at χ_{n+i}, i = 0 (1) 2 and interpolate the approximate solution (2.29) at χ_{n+i}, i = 0 (1) 2 to obtain a system of non – linear equation.

$a_0 + a_1\chi_n + a_2\chi^2_n + a_3\chi^3_n + a_4\chi^3_n + a_5\chi^3_n = y_n$

$a_0 + a_{1n}\chi_{n+1} + a_2\chi^2_{n+1} + a_3\chi^3_{n+1} + a_4\chi^4_{n+1} + a_5\chi^5_{n+1} = y_{n+1}$

$a_0 + a_1\chi_{n+2} + a_2\chi^2_{n+2} + a_3\chi^3_{n+2} + a_4\chi^4_{n+2} + a_5\chi^5_{n+2} = y_{n+2}$

$a_1 + 2a_2\chi_n + 3a_3\chi^2_n + 4a_4\chi^3_n + 5a_5\chi^4_n = f_n$

$a_1 + 2a_2\chi_{n+1} + 3a_3\chi^2_{n+1} + 4a_4\chi^3_{n+1} + 5a_5\chi^4_{n+1} = f_{n+1}$

$a_1 + 2a_2\chi_{n+2} + 3a_3\chi^2_{n+2} + 4a_4\chi^3_{n+2} + 5a_5\chi^4_{n+2} = f_{n+2}$ (2.38)

By using Guassian elimination techniques the values of a_i's are determined

$a_5 = \frac{1}{18h^5}(-3y_{n+2} + 3y_n + hf_{n+2} + 4hf_{n+1} + hf_n)$

$a_4 = \frac{1}{4h^4}(y_{n+2} + 4y_{n+1} - 3y_n - 4hf\, y_{n+1} - 2hf_n) - a_5(20h^4\chi_n + 16h^5)$

$a_3 = \frac{1}{4h^3}(y_{n+2} + 4y_{n+1} + 3y_n + 2hf_n) - a_4(4h^3\chi_n + 3h^4)\, a_5(10h^3\chi_n + 15h^4\chi_n + 7h^5)$

$a_2 = \frac{1}{2h^2}(y_{n+2} - 2y_{n+1} + y_n) - a_3(3h^2\chi_n + 3h^3) - a_4(6h^2\chi^2_n + 12h^3\chi^2_n + 12h^3\chi_n + 7h^4)$

$\quad - a_5(10h^2\chi^3_n + 30h^3\chi^2_n + 35h^4\chi_n + 15h^5)$

$a_1 = \frac{1}{h}(y_{n+1} - y_n) - a_2(2h^2\chi_n + h^2) - a_3(3h\chi^2 + 3h^3\chi_n + h^3) - a_4(4h^2\chi^3_n + 6h^2\chi^2_n$

$\quad + 4h^3\chi_n + h^4) - a_5(5h\chi^4_n + 10h^2\chi^3_n + 10h^3\chi^2_n + 10h^3\chi^2_n + 5h^4\chi_n + h^5)$

$a_0 = y_n - a_1\chi_n - a_2\chi^2_n - a_3\chi^3_n - a_4\chi^4_n - a_5\chi^5_n$ (2.39)

Substituting the value of a_j,s into equation (2.30), and after algebraic manipulation a continuous scheme is obtained in the form

$$y(t) = \sum_{j=0}^{k-1} \alpha_2 (\chi)\, y_{n+j} + \sum_{j=0}^{k-1} \beta j(\chi)\, f_{n+j}, \qquad (2.40)$$

where $f_{n+j} = f\,(\chi_{n+j}, y_{n+j})$

by choosing $t = (\chi - \chi_{n+k-1})/h,\ t = (0, 1]$ (2.41)

By using the transformation (2.34) in (2.33), the continuous explicit methods in power series of t are obtained in the form

$$y(t) = \sum_{j=0}^{k-1} \alpha_2 (t)\, y_{n+j} + \sum_{j=0}^{k-1} \beta j(t)\, f_{n+j}, \qquad (2.42)$$

where

$$\alpha_2 (t) = \frac{1}{4} (4 - 23t^3 - 33t^3 - 6t^5 - t^6)$$

$$\alpha_1 (t) = 4t^2 + 4t^3 + t^4$$

$$\alpha_0 (t) = \frac{1}{4} (7t^2 + 17t^3 + 13t^4 + 3t^5)$$

$$\beta_2(t) = \frac{h}{4} (4t + 12t^2 + 13t^3 + 6t^4 + t^5)$$

$$\beta_1 (t) = h\,(4t^2 + 8t^3 + 5t^4 + t^5)$$

$$\beta_0(t) = \frac{h}{4} (2t^2 + 5t^3 + 4t^5 + t^5)$$

Evaluating (2.42), for t = 1, yields a discrete scheme

$$y_{n+3} = 3y_{n+2} + 9_{n+1} - 11y_n + 2h(f_{n+2} - 5f_{n+1} - 2f_n) \qquad (2.43)$$

2.6.3 Development of predictors for 4 – step methods

Let the approximate solution to problem (1.9) be (2.29) and its derivate be (2.30) then by collocating the differential system (2.30) at χ_{n+i}, i = 0 (1) 2 and interpolate the approximate solution (2.29) at χ_{n+i}, i = 0 (1) 2 to obtain a system of non – linear equation.

$$a_0 + a_1\chi_n + a_2\chi^2_n + a_3\chi^3_n + a_4\chi^4_n + a_5\chi^5_n + a_6\chi^6_n = y_n$$

$$a_0 + a_{1n}\chi_{n+1} + a_2\chi^2_{n+1} + a_3\chi^3_{n+1} + a_4\chi^4_{n+1} + a_5\chi^5_{n+1} + a_6\chi^6_{n+1} = y_{n+1}$$

$$a_0 + a_1\chi_{n+2} + a_2\chi^2_{n+2} + a_3\chi^3_{n+2} + a_4\chi^4_{n+2} + a_5\chi^5_{n+2} + a_6\chi^6_{n+2} = y_{n+2}$$

$$a_0 + a_1\chi_{n+3} + a_2\chi^2_{n+3} + a_3\chi^3_{n+3} + a_4\chi^4_{n+3} + a_5\chi^5_{n+3} + a_6\chi^6_{n+3} = y_{n+3}$$

$$a_1 + 2a_2\chi_{n+1} + 3a_3\chi^2_{n+1} + 4a_4\chi^3_{n+1} + 5a_5\chi^4_{n+1} + 6a_6\chi^6_{n+1} = f_{n+1}$$

$$a_1 + 2a_2\chi_{n+2} + 3a_3\chi^2_{n+2} + 4a_4\chi^3_{n+2} + 5a_5\chi^4_{n+2} + 6a_6\chi^6_{n+2} = f_{n+2}$$

$$a_1 + 2a_2\chi_{n+3} + 3a_3\chi^2_{n+3} + 4a_4\chi^3_{n+3} + 5a_5\chi^4_{n+3} + 6a_6\chi^6_{n+3} = f_{n+3} \qquad (2.44)$$

By using Guassian elimination techniques the values of a_j's are determined

$$a_6 = \frac{1}{36h^6}(-3y_{n+3} - 16_{n+2} + 18y_n + y_n + 3hf(f_{n+3} + 6f_n + 3f_{n+1})$$

$$a_5 = \frac{1}{24h^5}[-6y_{n+3} + 26y_{n+3} - 18y_n - 2y_n - 12h + (f_{n+2} + f_{n+1})\} - a_6(144h^5\chi_n + 216h^6)$$

$$a_4 = \frac{1}{12h^4}\{4y_{n+3} + 9y_{n+2} + 3y_{n+1} + 2y_n + 6hf_{n+1}\} - a_5(60h^4\chi_n + 84h^5)$$

$$- a_6(180h^4\chi^2_n + 504h^4\chi_n + 384h^6)$$

$$a_3 = \frac{1}{6h^3}\{y_{n+3} - 3y_{n+2} + 3y_{n+1} - y_n\} - a_4(624h^3\chi_n + 36h^4) - a_5(60h^3\chi^2_n + 180h^4\chi_n +$$

$$150h^5) - a_6(120h^3\chi^3_n + 540h^4\chi^2_n + 900h^5\chi_n + 540h^6)$$

$$a_2 = \frac{1}{2h^2}\{y_{n+2} - 2y_{n+1} + y_n) - a_3(6h^2\chi_n + 6h^3) - a_4(12h^2\chi^2_n + 24h^3\chi_n + 14h^4$$

$$- a_5(20h^2\chi^3_n + 60h^3\chi^2_n + 70h^4\chi_n + 30h^5) - a_6(30h^2\chi^4_n + 120h^3\chi^3_n + 210h^4\chi^3_n$$

$$+ 180h^5\chi_n + 62h^6)$$

$$a_1 = \frac{1}{h}\{y_{n+1} - y_n - a_2(2h\chi_n + h^2) - a_3(3h\chi^2 + 3h^2\chi_n + h^3) - a_4(4h\chi^3_n + 6h^2\chi^2_n$$

$$+ 4h^3\chi_n + h^4) - a_5(5h\chi^4_n + 10h^2\chi^3_n + 10h^3\chi^2_n + 5h^4\chi_n + h^5)$$

$$a_0 = y_n - a_1\chi_n - a_2\chi^2_n - a_3\chi^3_n - a_4\chi^4_n - a_5\chi^5_n - a_6\chi^6_n \qquad (2.45)$$

Substituting the value of a_j,s into equation (2.30), and after algebraic manipulation a continuous scheme is obtained in the form

$$y(t) = \sum_{j=0}^{k-1} \alpha_2 (\chi)\, y_{n+j} + \sum_{j=0}^{k-1} \beta j(\chi)\, f_{n+j}, \qquad (2.46)$$

where $f_{n+j} = f(\chi_{n+j}, y_{n+j})$

by choosing $t = (\chi - \chi_{n+k-1})/h,\ t = (0, 1]$ (2.47)

By using the transformation (2.47) in (2.46), the continuous explicit methods in power series of t are obtained in the form

$$y(t) = \sum_{j=0}^{k-1} \alpha_2 (t)\, y_{n+j} + \sum_{j=0}^{k-1} \beta j(t)\, f_{n+j}, \qquad (2.48)$$

where

$$\alpha_3 (t) = \frac{1}{36} (36 - 247t^2 - 417t^3 - 283t^4 - 87t^5 - 10t^6)$$

$$\alpha_2 (t) = \frac{1}{4} (12t^2 + 4t^3 - 9t^4 - 6t^5 - t^6)$$

$$\alpha_1 (t) = \frac{1}{4} (15t^2 + 41t^3 + 39t^4 + 15t^5 + 2t^6)$$

$$\alpha_0 (t) = \frac{1}{36} (4t^2 + 12t^3 + 13t^4 + 6t^5 + t^6)$$

$$\beta_3(t) = \frac{h}{12} (12t + 41t^2 + 51t^3 + 31t^4 + 9t^3 + t^6)$$

$$\beta_2 (t) = \frac{h}{2} (12t^2 + 28t^3 + 23t^4 + 8t^5 + t^6)$$

$$\beta_1(t) = \frac{h}{4} (6t^2 + 17t^3 + 17t^4 + 7t^5 + t^6) \qquad (2.49)$$

Evaluating (2.48) at $t = 1$, that is, $x = x_{n+4}$, yields a symmetric discrete explicit scheme

$$y_{n+4} = -28y_{n+3} + 28y_{n+1} + y_n + 12h\,(f_{n+3} + 3f_{n+2} + f_{n+1}) \qquad (2.50)$$

CHAPTER THREE

ANALYSIS OF BASIC PROPERTIES OF THE METHODS

3.1 Basic properties of 2 -step method (2.11)

The basic properties of numerical methods are order of accuracy, consistency, convergence and stability. These are examined for the derived methods.

3.1.1 Order and Error Term

A characteristic of discretisation methods is that errors are generated when they are adopted for the solution of ODEs. Consequently, these errors have to be examined to ensure the accuracy of the method. A solution is said to be accurate when the computed value does not deviate significantly from the exact solution as the iteration progresses. The magnitude of these errors is referred to as global error denoted by

$$e_n = y(x_n) - y_n, \tag{3.1}$$

where $y(x_n)$ and y_n are the exact and computed solutions at x_n respectively.

This is supposed to be small since the magnitude of the error determines the accuracy of the scheme. Therefore, the objective of this work is to ensure that the local truncation error is minimized

Defined the local truncation error as

$$L[y(x),h] = \sum_{j=0}^{k}[\alpha_j y(x_{n+j}) - h\beta_j y'(x_{n+j})], \tag{3.2}$$

where $a_k = 1$, α_o and β_o are not both zero and $y(x) \varepsilon C^{p+1}[a, b]$, $y(x_{n+j}) = y(x_n+jh)$, a, b and $p \varepsilon R$

If y (x) represents the true solution of (2.17), adopting Taylor Series expansion of $y(x_{n+j})$ and $y^1(x_{n+j})$, j = 0(1) k about $x = x_n$, and combining terms of equal powers of h we obtain:

$$L[y(x), h] = C_o y(x_n) + C_1 h y^1(x_n) + \ldots + C_{p+1} h^{p+1} y^{p+1} + O(h^{p+2}) \tag{3.3}$$

where

$C_0 = 1 - 2 + 1 = 0$

$C_1 = 2 - 2 - 1/2 + 1/2 = 0$

$C_2 = 2 - 1 - 1 = 0$

$C_3 = \frac{4}{3} - \frac{1}{3} - 1 = 0$

$C_4 = \frac{2}{3} - \frac{1}{12} + \frac{2}{3} = \frac{-1}{12}$

Definition: A Linear Multistep Method (2.11) for first order ODEs is of order p if the local truncation error (3.2) satisfies

$$T_{n+k} = C_{p+1}h^{p+1}y^{p+1} + O(h^{p+2}) \qquad (3.3)$$

where $C_0 = C_1 = C_2 = C_p = 0$, $C_{p+1} \neq 0$ is called the principal error constant (see Lambert (1973,1991), Fatunla (1988), Kayode (2004)).

Hence the result above shows that the scheme (2.11) is of order $p = 3$ and the error constant = $\frac{-1}{12}$.

Following the same procedure as above, the order and error constant of the predictor (2.36) is found to be $p = 3$ and $C_{p+1} = 1/6$ respectively.

3.1.2 Consistency

The numerical method (2.11) is said to be consistent if its stability polynomial equation

$$\pi(r, \bar{h}) = \rho(r) - h\,\delta(r) \qquad (3.4)$$

satisfies the following conditions

(i) The order $P \geq 1$ (3.5)

(ii) $\rho(r) = 0$ (3.6)

(iii) $\rho'(r) = \delta(r)$ (3.7)

where

$\rho(r)$ and $\delta(r)$ are first and second characteristics polynomials of the method respectively.

[Lambert (1973, 1991) and Awoyemi (2001)].

For

$$\rho(r) = \sum_{j=0}^{k} \alpha_j r^j \quad (3.8)$$

The first derivative of (3.8) is

$$\rho'(r) = \sum_{j=0}^{k} j\alpha_j r^{j-1} \quad (3.9)$$

also

$$\delta(r) = \sum_{j=0}^{k} \beta_j r^j \quad (3.10)$$

For the method (2.11) the following are obtained

(i.) The order $p = 3$

(ii.) $\rho(r) = r^2 - 2r + 1 = 0$ (3.11)

$\rho(1) = 1 - 2 + 1 = 0$

(iii.) $\rho'(r) = 2r - 2$

$\rho'(1) = 2 - 2 = 0$

$$\delta(1) = \frac{1}{2}(r^2 - 1)$$

$$\delta(r) = \frac{1}{2}(1 - 1) = 0$$

Hence

$\rho'(1) = \delta(1)$

This shows that all the conditions for consistency are fulfilled, thus method (2.11) is consistent.

3.1.3 Zero stability

Solving the first characteristics polynomial equation (3.11)

$\rho(r) = r^2 - 2r + 1 = 0$

the following roots are obtained,

$r = 1, 1,$

According to Lambert (1973), a linear multistep method is said to be zero stable if no root of the first characteristic polynomial $\rho(r)$ has modulus greater than one, and if every root of modulus one is simple.

But Kayode (2004), said that a LM numerical methods with $k>1$ for ODEs whose first characteristic polynomial equations cannot be expressed in the form $(r\pm1)^k$ will have the modulus of one of its spurious roots greater than one. From the foregoing, the method (2.11) with k=2 is not zero stable.

Theorem (Henric (1962)

The necessary and sufficient condition for a linear multistep method to be convergent is for it to be consistent and zero stable. Hence method (2.11) is not convergent.

3.1.4 Region of absolute stability of the method (2.11)

To determine the region of absolute stability of method (2.11) we adopt the boundary locus method as discussed in Lambert (1973)

That is,

$$\bar{h}(r) = \frac{\rho(r)}{\delta(r)}, \qquad (3.12)$$

where ρ (r) and δ(r) are first and second characteristic polynomials respectively.

$$\bar{h}(r) = \frac{r^2 - 2r + 1}{1/2\,(r^2 - 1)} \qquad (3.13)$$

The roots, r, of the stability polynomial are in generally complex numbers. Thus we have $r = e^{i\theta}$, $0 \le \theta \le \pi$

where

$e^{i\theta} = \cos\theta + i\sin\theta$ (by Euler's rule) (3.14)

and after transformation $r = e^{i\theta}$ (3.13) becomes

$$\bar{h}(\theta) = \frac{\rho\{\exp(i\theta)\}}{\delta\{\exp(i\theta)\}} \tag{3.15}$$

and (3.14) becomes

$$\bar{h}(\theta) = \frac{(e^{i\theta})^2 - 2(e^{i\theta}) + 1}{1/2\,[(e^{i\theta})^2 - 1]} \tag{3.16}$$

$$= \frac{2\{\text{Cos } 2\theta + 1 \text{ Sin } 2\theta - 2 \text{ Cos}\theta - 2i \text{ Sin}\theta + 1\}}{\text{Cos } 2\theta + i\text{Sin } 2\theta - 1} \tag{3.17}$$

Write (3.17) in standard form

$$\bar{h}\,\theta = X(\theta) + iy(\theta) \tag{3.18}$$

$$\bar{h}\theta = \frac{(2 \cos 2\theta - 4 \text{ Cos}\theta + 2) + i(2 \text{ Sin } 2\theta - 4 \sin\theta)}{(\text{Cos}2\theta - 1) + i \text{ Sin}2\theta}$$

$$\bar{X}(\theta) = \frac{(2 \cos 2\theta - 4 \text{ Cos}\theta + 2)(\text{Cos}2\theta - 1) + (2 \text{ Sin } 2\theta - 4 \sin\theta) \text{ Sin}2\theta}{(\text{Cos}2\theta - 1)^2 + (\text{Sin}2\theta)^2}$$

$$= 0$$

$$iy(\theta) = \frac{(2 \cos 2\theta - 4 \text{ Cos}\theta + 2)(i\text{Sin}2\theta) + (2\text{Sin}2\theta - 4 \text{ Sin}\theta)(\text{Cos}2\theta) - 1)}{(\text{Cos}2\theta - 1)^2 + (\text{Sin}2\theta)^2}$$

$$iy(\theta) = \frac{2 \text{ Sin}4\theta - 4\text{Sin}3\theta + 4 \text{ Sin}\theta}{2 - 2\text{Cos}\theta}$$

Table 1: Interval of absolute stability of method (2.11)

θ	0	30^0	60^0	90^0	120^0	150^0	180^0
X (θ)	0	0	0	0	0	0	0
Iy (θ)	0	-1.0	1.73	4.00	1.00	-1	0

Evaluation of x (θ) and iy (θ) within the range, $0 \le \theta \le 180^0$ at intervals of 30^0 shows that the interval of absolute stability of the method (2.11) is (0, 0)

3.2 Basic Properties of 3 - step Method (2.16)

In this section the basic properties of method (2.16) is examined by adopting a similar approach in section (3.1)

3.2.1 Order and Error Term.

A linear operator L (truncated error) associated with (2. 16) as previously defined by equation (3.2) for K = 3, and adopting Taylor series expansion of y (x_{n+j}) and $y'(x_{n+j})$, j =1(1) 3 about $x = x_n$ and combining of terms of equal powers of h to obtain,

$$L[y(x), h] = C_0 y(x_n) + C_1 h y^1(x_n) + \dots + C_{p+1} h^{p+1} y^{p+1} + O(h^{p+2}) \qquad (3.19)$$

Where

$C_0 = 7 - 18 + 9 + 2 = 0$

$C_1 = 21 - 36 + 18 - 3 + 9 = 0$

$C_2 = \frac{63}{2} - \frac{72}{2} + \frac{9}{2} - 9 + 9 = 0$

$C_3 = \frac{189}{6} + \frac{144}{6} - \frac{9}{6} - \frac{27}{2} + \frac{9}{2} = 0$

$C_4 = \frac{567}{24} - \frac{288}{24} + \frac{9}{24} - \frac{81}{6} + \frac{9}{6} = 0$

$C_5 = \frac{170}{120} - \frac{576}{120} + \frac{9}{120} - \frac{243}{24} + \frac{9}{24} = -\frac{3}{10}$

Thus:

$$C_0 = C_1 = C_2 = C_3 = C_4 = 0, \; C_5 = -3/10$$

Hence method (2.16) is of order 4 and the principal error constant C_{p+1} =-3/10

Following the same procedure as above, the order and error constant of the predictor (2.43) is found to be p = 4 and C_{p+1} = 7/30 respectively.

3.2.2 Consistency

The stability polynomial method (2.16) is given by

$\Pi(r, \bar{h}) = \{7r^3 - 18r^2 + 9r + 2\} - 3h(r^3 - 3r)$ (3.20)

The characteristic polynomial (3.20) satisfies the condition for consistency stated in section (3.1.2). Thus the method is consistent

3.2.3 Zero stability

Solving the first characteristic polynomial equation

$P(r) = 7r^3 - 18r^2 + 9r + 2 = 0$

The following roots are obtained

r = 1, - 0.164, 1.736. since the modules of one of its spurious roots is 1.736, by section (3.1.3), method (2.16) is not Zero Stable

3.2.4 Region of absolute stability (ras) of the method (2. 16)

To examine the region of f absolute stability of the method (2.16) the same procedure for method (2.11) in section 3.1. was followed.

where

$$\bar{h}(r) = \frac{7r^3 - 18r^2 + 9r + 2}{3(r^3 - 3r)} \quad (3.2.1)$$

The roots, r, of the stability polynomial are in generally complex numbers. Thus we have $r = e^{i\theta}$, $0 \le \theta \le \pi$

where

$$e^{i\theta} = \cos\theta + i\sin\theta \text{ (by Euler's rule)} \quad (3.2.2)$$

and (3.21) becomes

$$\bar{h}(\theta) = \frac{7(e^{i\theta})^3 - 18(e^{i\theta})^2 + 9(e^{i\theta}) + 2}{3(e^{i\theta})^3 - 3(e^{i\theta})}$$

$$= \frac{\{7\text{Cos } 3\theta - 18 \text{ Cos } 2\theta + 9 \text{ Cos}\theta + 2) + i(\text{Sin}\theta - 18 \sin 2\theta + 9 \sin\theta\}}{(3 \text{ Cos } 3\theta - 9 \text{ Cos } \theta) + I(3 \sin 3\theta - 9 \sin\theta)}$$

which when further simplified and reduced to standard form

$\bar{h}\,\theta = X(\theta) + i\,y(\theta)$ (3.23)

gives

$$x(\theta) = \frac{-10 + 15 \cos\theta - 6 \cos 2\theta + \cos 3}{15 - 9 \cos 2\theta}$$

and

$$i\,y(\theta) = \frac{(-18 \sin\theta - 81 \quad \sin 2\theta + 168 \sin 3\theta - 54 \sin 5\theta + 21 \sin 6\theta)}{15 - 9 \cos 2\theta}$$

Table 2. Interval of absolute stability of method 2.16

θ	0	30^0	60^0	90^0	120^0	150^0	180^0
X (θ)	0	-0.00091	-0.026	-0.167	-0.692	-2..48	-5.33
Iy (θ)	0	2.92	-0.40	-10.00	3.60	22.6	0

The interval of absolute stability for method 2.16 is x (θ) = (-5.33,0)

3.3 Basic properties of 4 - step method (2.22)

In this section, the basic properties of method (2.22) is examined by adopted of the same procedure for method (2.11) in section (3.1).

3.3.1 Order and error term

If y (x) represents the true solution of (2.22), adopting Taylor Series expansion of $y(x_{n+j})$ and $y^1(x_{n+j})$, j = 0(1) k about $x = x_n$, and combining terms of equal powers of h to have:

$$L[y(x), h] = C_0 y(x_n) + C_1 h y^1(x_n) + \ldots + C_{p+1} h^{p+1} y^{p+1} + O(h^{p+2}) \qquad (3.24)$$

where

$$C_0 = -1 + 64 - 64 + 1 = 0$$

$$C_1 = \frac{4}{17}\left\{(-1) + 64\,(3) - 64 + 6 - 72 + 6\right\} = 0$$

$$C_2 = \frac{8}{17}\left\{(-17) + 9\,(32) - 32 + 24 - 144\right\} = 0$$

$$C_3 = \frac{1}{17}\left\{\frac{64}{3!}(-17) + \frac{27}{3!}(64) + \frac{64}{3!} + 48 - 144\right\} = 0$$

$$C_4 = \frac{1}{17}\left\{\frac{256}{4!}(-17) + \frac{81}{4!}(64) + \frac{1}{4!}(-64) + 64 - 96\right\} = 0$$

$$C_5 = \frac{1}{17}\left\{\frac{-17408}{5!} + \frac{15552}{5!} - \frac{64}{5!} + \frac{1536}{4!} - \frac{1152}{4!}\right\} = 0$$

$$C_6 = \frac{1}{17}\left\{\frac{-69632}{6!} + \frac{46656}{6!} - \frac{64}{6!} + \frac{6144}{5!} - \frac{3304}{5!}\right\} = 0$$

$$C_7 = \frac{1}{17}\left\{\frac{-278528}{7!} + \frac{139968}{7!} - \frac{64}{7!} + \frac{24576}{6!} - \frac{4608}{6!}\right\} = \frac{-8}{596}$$

$$C_{p+1} = \frac{-8}{596} = -0.0134$$

Definition: A Linear Multistep Method (2.22) for first order ODEs is of order p if the local truncation error (3.2) satisfies

$$T_{n+k} = C_{p+1}h^{p+1}y^{p+1} + O(h^{p+2}) \qquad (3.25)$$

where $C_0 = C_1 = C_2 = C_3 = C_4 = C_5 = C_6 = 0$, $C_7 = -0.0134$

Hence the result above shows that the scheme (2.22) is of order $\rho = 6$ and the error constant $= \frac{-8}{596}$

Following the same procedure as above, the order and error constant of the predictor (2.50) is found to be p = 6 and $C_{p+1} = 1/35$ respectively.

3.3.2 Consistency

The numerical method (2.22) is said to be consistent if its stability polynomial equation

$$\pi(r, \bar{h}) = \rho(r) - h\,\delta(r) \qquad (3.26)$$

satisfies the condition for consistency stated in section (3.12).

For the scheme (2.22), the following are obtained

(i) The order P = 6 which is greater than one

(ii) $\rho(r) = 17r^4 - 64r^3 + 64r - 17$

$\rho(1) = 17 - 64 + 64 - 17 = 0$

(iii) $\rho^1(r) = 68r^3 - 192r^2 + 64$

$\rho^1(1) = 68 - 192 + 64 = -60$

$\delta(r) = 6(r^4 - 12r^2 + 1)$

$\delta(1) = 6 - 72 + 6 = -60$

This shows that,

$\rho^1(1) = \delta(1)$

This implies that the scheme (2.22) is consistent.

3.3.3 Zero stability

Solving the first characteristics polynomial equation

$\rho(r) = 17r^4 - 64r^3 + 64r - 17 = 0$

the following roots are obtained,

$r = 1, -1, 0.2876$ and 3.477

From the foregoing, the method (2.22) with $k = 4$ has one of its roots greater than one, hence it is not zero stable.

3.3.4 Region of absolute stability of the method (2.22)

To determine the region of absolute stability of method (2.22) we adopt the boundary locus method as discussed in Lambert (1973)

That is,

$$\bar{h}(r) = \frac{\rho(r)}{\delta(r)} \quad (3.27)$$

where ρ (r) and $\delta(r)$ are first and second characteristic polynomials respectively.

$$\bar{h}(r) = \frac{17r^4 - 64r^3 + 64r - 17}{6(r^4 - 12r^2 + 1)} \quad (3.28)$$

The roots, r, of the stability polynomial are in generally complex numbers. Thus we have $r = e^{i\theta}$, $0 \le \theta \le \pi$

where

$$e^{i\theta} = \cos\theta + i\sin\theta \quad (3.29)$$

and after transformation $r = e^{i\theta}$ (3.27) becomes

$$\bar{h}(\theta)=\frac{\rho\{\exp(i\theta)\}}{\delta\{\exp(i\theta)\}} \qquad (3.30)$$

and (3.28) becomes

$$\bar{h}(\theta) = \frac{17(e^{i\theta})^4 - 64(e^{i\theta})^3 + 64(e^{i\theta}) - 17}{6[(e^{i\theta})^4 - 12(e^{i\theta})^2 + 1]} \qquad (3.31)$$

$$\bar{h}(\theta) = \frac{17(\cos4\theta + i\sin4\theta) - 64(\cos3\theta+i\sin3\theta)+64(\cos\theta+i\sin\theta)-17}{6[(\cos4\theta+i\sin4\theta)-12(\cos2\theta+ i\sin2\theta) + 1]}$$

Simplifying in the form

$\bar{h}(\theta) = x(\theta) + iy(\theta)$, yields

$$\bar{h}(\theta)= \frac{1}{6}\left\{\frac{(17\cos 4\theta - 64\cos 3\theta+64\cos\theta -17)+i(17\sin4\theta -64\sin3\theta+64 \sin\theta)}{(\cos 4\theta - 12\cos 2\theta + 1) + i(\sin 4\theta - 12 \sin 2\theta)}\right\}$$

We now have $x(\theta) = \frac{N(\theta)}{D(\theta)}$ and $iy(\theta) = \frac{iN(\theta)}{D(\theta)}$

$N(\theta)=(17\cos4\theta-64\cos3\theta+64\cos\theta-17)(\cos4\theta-12\cos2\theta+1) +(17\sin4\theta-64\sin3\theta+64\sin\theta)(\sin4\theta - 12\sin 2\theta)$

$= 0$

$D(\theta) = 6\ (\cos 4\theta - 12\cos 2\theta + 1)^2 + (\sin 4\theta - 12 \sin 2\theta)^2$

$=146 + 2\cos4\theta - 48\cos2\theta$

$iN(\theta) = i(17\cos4\theta-64\cos3\theta+64\cos\theta-17)(-\sin4\theta+12\sin2\theta)+i(17\sin4\theta- 64\sin3\theta + 64\sin\theta)(\cos4\theta -12\cos2\theta + 1)$

$= 34\sin4\theta - 128\sin3\theta + 128\sin\theta$

$x(\theta)= 0$

$$iy(\theta)=\left\{\frac{34\sin4\theta - 128\sin3\theta + 128\sin\theta}{146 + 2\cos4\theta - 48\cos2\theta}\, i\right\}$$

Table 3: Interval of absolute stability method 2.22

θ	0	30	60	90	120	150	180
X (θ)	0	0	0	0	0	0	0
Y(θ)	0	-0.048	0.803	0.218	0.139	-0.129	0

Evaluation of x (θ) and iy(θ) within the range, $0 \le \theta \le 180^0$ at intervals of 30^0 shows that the interval of absolute stability of the method (2.22)

is x (θ) = (0, 0)

3.4 Analysis of basic properties of maximal order method (2.28), K = 4

Following the same procedure as discussed in section 3.1 the analysis of basic properties of scheme (2.28) are as follows:

3.4.1 Order and error term

The linear operator L associated with method (2.28) as defined in equation (3.24)

where

$$C_0 = 1 + \frac{32}{5} - \frac{32}{5} - 1 = 0$$

$$C_1 = 4 + \frac{96}{5} - \frac{32}{5} - \frac{6}{25} - \frac{96}{25} - \frac{216}{25} - \frac{96}{25} - \frac{6}{25} = 0$$

$$C_2 = \frac{16}{2} + \frac{288}{10} - \frac{32}{10} - \frac{24}{25} - \frac{288}{25} - \frac{432}{25} - \frac{96}{25} = 0$$

$$C_3 = \frac{64}{2} + \frac{864}{30} - \frac{32}{30} - \frac{96}{50} - \frac{864}{50} - \frac{864}{50} - \frac{96}{50} = 0$$

$$C_4 = \frac{256}{24} + \frac{2592}{120} - \frac{32}{120} - \frac{384}{150} - \frac{2592}{150} - \frac{1728}{150} - \frac{96}{150} = 0$$

$$C_5 = \frac{1025}{120} + \frac{7776}{600} - \frac{32}{600} - \frac{1536}{600} - \frac{7776}{600} - \frac{3456}{600} - \frac{96}{600} = 0$$

$$C_6 = \frac{4096}{720} + \frac{23328}{3600} - \frac{32}{3600} - \frac{6144}{3000} - \frac{23328}{3000} - \frac{6912}{3000} - \frac{96}{3000} = 0$$

$$C_7 = \frac{16384}{5040} + \frac{69984}{5200} - \frac{32}{25200} - \frac{24576}{18000} - \frac{69984}{18000} - \frac{13824}{18000} - \frac{96}{18000} = 0$$

$$C_8 = 65536 + 209952 - 32 - 98304 - 209962 - 27648 - 96 = 0$$

40320 201600 201600 126000 126000 26000 126000

$$C_9 = \frac{262144}{362880} + \frac{629856}{1814400} - \frac{32}{1814400} - \frac{393216}{1008000} - \frac{629856}{1008000} - \frac{55296}{1008000} - \frac{96}{1008000}$$

$= -0.00038$

Thus:

$C_0 = C_1 = C_2 = C_3 = C_4 = C_5 = C_6 = C_7 = C_8 = 0,\ C_9 = -0.00038$

Hence method (2.28) is of order 8 and the principal error constant

$C_{p+1} = -0.00038$

3.4.2 Consistency for the method (2.28)

(i) $P > 1$

(ii) $\rho(r) = r^4 + \frac{32r^3}{5} - \frac{32r}{5} - 1$ 3.32

$\rho(1) = 1 + \frac{32}{5} - \frac{32}{5} - 1 = 0$

(iii) $\rho'(r)\ 4r^3 + \frac{96r^2}{5} - \frac{32}{5}$ 3.33

$\rho'(1) = 4 + \frac{96}{5} - \frac{32}{5} = \frac{84}{5}$

$\delta(r) = \frac{6r^4}{25} + \frac{96r^3}{25} + \frac{216r^2}{25} + \frac{96r}{25} + \frac{6}{25}$ 3.34

$\delta(1) = \frac{6r^4}{25} + \frac{96}{25} + \frac{216}{25} + \frac{96}{25} + \frac{6}{25} = \frac{84}{5}$

Thus:

$\rho'(1) = \delta(1)$

This implies that the scheme (2.28) is consistent.

3.4.3 Zero stability

Solving the first characteristic polynomial equation

$$r^4 + 32r^3 - 32r - 1 = 0 \qquad (3.35)$$

Giving $r = 1, -1, 0.16$ or 6.04

This shows that the scheme (2.28) is not zero stable

3.4.4 Region of absolute stability of method (2.28)

By adopting the same process described in section 3.1.4.

Hence:

$$h(r) = \frac{5}{6}\left\{ (5r^4 + 32r^3 - 32r - 5) / (r^4 + 16r^3 + 36r^2 + 16r + 1) \right\} \quad (3.36)$$

$$= \frac{5}{6}\left\{ (5e^{4i\theta} + 32e^{3i\theta} - 32e^{i\theta} - 5) / (e^{4i\theta} + 16e^{3i\theta} + 36e^{2i\theta} + e^{i\theta} + 5) \right\}$$

$$= \frac{5}{6}\left\{ (5\cos4\theta + 5i\sin4\theta + 32\cos3\theta + 32i\sin3\theta - 32\cos\theta - 32i\sin\theta - 5) / (\cos4\theta + i\sin4\theta + 16\cos3\theta + 16i\sin3\theta + 36\cos2\theta + 36i\sin2\theta + 16\cos\theta + 16i\sin\theta + 1) \right\}$$

$$N(\theta) = 5\left\{ 5\cos4\theta + 32\cos3\theta - 32\cos\theta - 5)(\cos4\theta + 16\cos3\theta + 36\cos2\theta + 16\cos\theta + 1) + (5\sin4\theta + 32\sin3\theta - 32\sin\theta)(\sin4\theta + 16\sin3\theta + 36\sin2\theta + 16\sin\theta \right\} = 0$$

$$D(\theta) = 6\left\{ (\cos4\theta + 16\sin3\theta + 36\cos2\theta + 16\cos\theta + 1)^2 + (\sin4\theta + 16\sin3\theta + 36\sin2\theta + 16\sin\theta)^2 \right\}$$

$$= (1810 + 2368\cos\theta + 650\cos2\theta + 64\cos3\theta - 2\cos4\theta)$$

$$iN(\theta) = 5\left\{ (5\cos4\theta + 32\cos3\theta - 32\cos\theta - 5)(\sin4\theta + 16\sin3\theta + 36\sin2\theta + (16\sin\theta) + (\cos4\theta + 16\cos3\theta + 36\cos2\theta + 16\cos\theta + 1)(5\sin4\theta + 32\mathrm{Sin}3\theta - 32\mathrm{Sin}\theta \right\}$$

$$X(\theta) = \frac{N(\theta)}{D(\theta)} = 0$$

$$i.y(\theta) = \frac{(N\theta)}{D(\theta)}$$

$$= \frac{5}{6}\left\{ 10\sin8\theta + 112\sin7\theta + 1204\sin6\theta + 1200\sin5\theta - 1200\sin3\theta - 1204\sin2\theta - 112\sin\theta) / (1810 + 2368\cos\theta + 650\cos2\theta + 64\cos3\theta - 2\cos4\theta) \right\}$$

Table 4: Interval of absolute stability of method (2.28)

θ	0^0	30^0	60^0	90^0	120^0	150^0	180^0
X(θ)	0	0	0	0	0	0	0
Y(θ)	0	-0.4521	-0.3827	0.8341	-0.0119	3.3368	0

Evaluation of x (θ) and iy (θ) within the range $0 \le \theta \le 180^0$ at intervals of 30^0, shows that the interval of absolute stability of method (2.28) is at origin that is x (θ) = (0,0)

Table 5: Summary of the analysis of basic property of the methods.

S/n	K	Main scheme	Order	Error Constant	R.A.S
1.	2	$y_{n+2} - 2y_{n+1} + y_n = \frac{h}{2}(f_{n+2} - f_n)$ (2.11)	3	-1/12	0,0
2.	3	$7y_{n+3} - 18y_{n+2} + 9y_{n+1} + 2y_n = 3h(f_{n+3} - 3f_{n+1})$ (2.16)	4	-3/10	-5.33,0
3.	4	$17y_{n+4} - 64y_{n+3} + 64y_{n+1} - 17y_n = 6h(f_{n+4} - 12f_{n+2} + f_n)$ (2.22)	6	-8/596	0,0
4.	4	$5y_{n+4} + 33y_{n+3} - 33y_{n+1} - 5y_n = \frac{6h}{5}(f_{n+4} + 16f_{n+3} + 36f_{n+2} + 16f_{n+1} + f_n)$ (2.28)	8	-0.00038	0,0

S/N	K	Predictors	Order	Error Constant	R.A.S
1.	2	$y_{n+2} = -4y_{n+1} + 5y_n + 2h(2f_{n+1} + f_n)$ (2.36)	3	1/6	0,4
2.	3	$y_{n+3} = 3y_{n+2} + 9_{n+1} - 11y_n + 2h(f_{n+2} - 5f_{n+1} - 2f_n)$ (2.43)	4	7/30	0,2
3.	4	$y_{n+4} = -28y_{n+3} + 28y_{n+1} + y_n + 12h(f_{n+3} + 3f_{n+2} + f_{n+1})$ (2.50)	6	1/35	0,0

CHAPTER FOUR

4.1 NUMERICAL EXPERIMENT

To access the practical feasibility and accuracy of the methods, three sample problems, one linear and two non-linear, were solved using the Fortran computer programme developed for the new methods. The result of problem (1) also solved by Lambert (1973, pg 129) was compared with the results of the new methods (2.22) and (2.28).

Problem (1) $y' = e^{10(x-y)}$, $y(0) = 0.1$

Theoretical solution $y = {}^1/_{10} \ln (e^{10x} + e - 1)$

Problem (2) $y' = (1-x)y^2 - y$, $y(0) = 1$

Theoretical solution $y = 1/(e^{2x} - X)$

Problem (3) $y' = -10 (y-1)^2$, $y(0) = 2$

Theoretical solution $y = 1 + 1/(1+10x)$

4.2 RESULTS OF PROBLEMS (1) - (3) FOR SCHEMES (2.22) AND (2.28) ARE SHOWN IN THE TABLES BELOW:

TABLE 6A: RESULTS FOR PROBLEM (1) USING SCHEME (2.22)

EXACT SOLUTION: Y=DLOG (DEXP (10.D0*X)+DEXP (1.D0)-1.D0)

X	YEX	YC	ER
0.1	0.1489880126D+01	0.1489880126D+01	0.7105427358D-14
0.2	0.2209080454D+01	0.2209080454D+01	0.1065814104D-13
0.3	0.3082085127D+01	0.3082085127D+01	0.1465494393D-13
0.4	0.4030986355D+01	0.4030986355D+01	0.1953992523D-13
0.5	0.5011511183D+01	0.5011511183D+01	0.1065814104D-13
0.6	0.6029145431D+01	0.6029145431D+01	0.1154631946D-13
0.7	0.7026527018D+01	0.7026527018D+01	0.1332267630D-13
0.8	0.8025562030D+01	0.8025562030D+01	0.1421085472D-13
0.9	0.9025206796D+01	0.9025206796D+01	0.1776356839D-13
1.0	0.1002507608D+01	0.1002507608D+01	0.1598721155D-13

TABLE (6B) RESULTS FOR PROBLEM (1) USING SCHEME (2.28)

EXACT SOLUTION: Y=DLOG (DEXP (10.D0*X)+DEXP (1.D0)-1.D0)

X	YEX	YC	ER
0.1	0.1489880126D+00	0.1489880126D+00	0.6938893904D-15
0.2	0.2209080454D+00	0.2209080454D+00	0.9992007222D-15
0.3	0.3082085127D+00	0.3082085127D+00	0.1443289932D-14
0.4	0.4030986355D+00	0.4030986355D+00	0.1887379142D-14
0.5	0.5011511183D+00	0.5011511183D+00	0.2220446049D-15
0.6	0.6029145431D+00	0.6029145431D+00	0.3330669074D-15
0.7	0.7026527018D+00	0.7026527018D+00	0.5551115123D-15
0.8	0.8025562030D+00	0.8025562030D+00	0.6661338148D-15
0.9	0.9025206796D+00	0.9025206796D+00	0.7771561172D-15
1.0	0.1002507608D+01	0.1002507608D+01	0.6661338148D-15

TABLE 7A RESULT FOR PROBLEM USING SCHEME (2.22)

EXACT SOLUTION: Y=1/(EXP (2X)-X)

X	YEX	YC	ER
0.1	0.8917402715D+00	0.8917402715D+00	0.3452793607D-13
0.2	0.7740988401D+00	0.7740988401D+00	0.3008704397D-13
0.3	0.6569789426D+00	0.6569789426D+00	0.2542410726D-13
0.4	0.5477828431D+00	0.5477828431D+00	0.2120525977D-13
0.5	0.4507993471D+00	0.4507993471D+00	0.6106226635D-14
0.6	0.3657297906D+00	0.3657297906D+00	0.4940492460D-14
0.7	0.2964696330D+00	0.2964696330D+00	0.4052314040D-14
0.8	0.2395003293D+00	0.2395003293D+00	0.3247402347D-14
0.9	0.1931444741D+00	0.1931444741D+00	0.2609024108D-14
1.0	0.1556760929D+00	0.1556760929D+00	0.2109423747D-14

TABLE (7B) RESULTS FOR PROBLEM 2 USING SCHEME (2.28)

EXACT SOLUTION: Y=1/(EXP (2X)-X)

X	YEX	YC	ER
0.1	0.8917402715D+00	0.8917402715D+00	0.4107825191D-14
0.2	0.7740988401D+00	0.7740988401D+00	0.3552713679D-14
0.3	0.6569789426D+00	0.6569789426D+00	0.2997602166D-14
0.4	0.5477828431D+00	0.5477828431D+00	0.2664535259D-14
0.5	0.4507993471D+00	0.4507993471D+00	0.1665334537D-15
0.6	0.3657297906D+00	0.3657297906D+00	0.1665334537D-15
0.7	0.2964696330D+00	0.2964696330D+00	0.1110223025D-15
0.8	0.2395003293D+00	0.2395003293D+00	0.8326672685D-16
0.9	0.1931444741D+00	0.1931444741D+00	0.8326672685D-16
1.0	0.1556760929D+00	0.1556760929D+00	0.8326672685D-16

TABLE 8: RESULTS FOR PROBLEMS 3 USING METHOD (2.22)

X	YEX	YC	ER
0.1	0.1442688366D+01	0.1442688366D+01	0.6661338148D-14
0.2	0.1386863180D+01	0.1386863180D+01	0.6661338148D-14
0.3	0.1333860575D+01	0.1333860575D+01	0.6217248938D-14
0.4	0.1284747249D+01	0.1284747249D+01	0.5995204333D-14
0.5	0.1240253073D+01	0.1240253073D+01	0.1554312234D-14
0.6	0.1199837944D+01	0.1199837944D+01	0.1332267630D-14
0.7	0.1165540818D+01	0.1165540818D+01	0.1110223025D-14
0.8	0.1136128524D+01	0.1136128524D+01	0.1110223025D-14
0.9	0.1111245354D+01	0.1111245354D+01	0.1110223025D-14
1.0	0.1090434469D+01	0.1090434469D+01	0.6661338148D-15

Table 9: Comparison of the result of problem (1) solved by Lambert

(1973, pg 129) with the result of the new scheme (2.22) and (2.28)

X	Lambert (1973)	New Method (2.22)	New method (2.28)
0.1	1.8×10^{-6}	0.7105427358D-14	0.6938893904D-15
0.2	$6.5X10^{-6}$	0.1065814104D-13	0.9992007222D-15
0.3	$1.9X10^{-5}$	0.1465494393D-13	0.1443289932D-14
0.4	$5.4X10^{-5}$	0.1953992523D-13	0.1887379142D-14
0.5	$1.5X10^{-4}$	0.1065814104D-13	0.2220446049D-15
0.6	$4.1X10^{-4}$	0.1154631946D-13	0.3330669074D-15
0.7	$1.1X10^{-3}$	0.1332267630D-13	0.5551115123D-15
0.8	$2.9X10^{-3}$	0.1421085472D-13	0.6661338148D-15
0.9	$8.2X10^{-3}$	0.1776356839D-13	0.7771561172D-15
1.0	$2.2X10^{-2}$	0.1598721155D-13	0.6661338148D-15

4.3 DISCUSSION OF NUMERICAL RESULTS

Three numerical examples (one linear and two non-linear ODE's) were solved by the new methods (2.22) and (2.28) to test their accuracy and applicability.

Table 6 presents solution to a linear ODE's while tables 7 and 8 shows the solutions to non – linear ODE problems. This implies that the new methods are suitable for both linear and non-linear ODE problems.

Table 9 presents the comparison of the result solved by Lambert (1973, pg 129) with the results of the problem (1) using the proposed methods. The new methods are found to be more accurate than the method proposed by Lambert (1973).

It was observed from the tables that, as the number of iterations increases the error becomes smaller.

CHAPTER FIVE

5.1 CONCLUSSION

In this work, a new class of continuous linear multistep method for initial value problems of first order ordinary differential equation was developed for step number $k > 1$ based on collocation and interpolation at the grid points. Four discrete schemes were obtained from the continuous scheme and analysed.

Theoretical analyses show that the methods are consistent but not zero stable. Computer implementation shows that the methods are highly accurate and the results are as shown in tables 6 - 9

It should be noted that the interval of absolute stability of each method for even k is at the origin. This is not surprising when we look at Simpson method with step-number $K = 2$ whose interval of stability is also at the origin.

This implies that for all even step-numbers $K \geq 2$, the interval of absolute stability is expected to be at the origin. For odd step-numbers $K \geq 3$ the situation may be different, as the method developed for $k = 3$ has better interval of absolute stability.

5.2 CONTRIBUTION TO KNOWLEDGE

In this study, the corrector – predictor schemes for solving first order ordinary differential equation has been developed. These new methods have their orders of accuracy equal to the orders of accuracy of the predictors for corresponding step numbers (see page 40 of the thesis). Thus the predictors each of which is self-starting could be used without the correctors to produce the same degree of accuracy, which is shown on pages (42 – 44). This is a major contribution to knowledge as it has now happened like this previously for first order ordinary differential equation in the existing literature.

5.3 RECOMMENDATION

The proposed methods are recommended for numerical solution of single first order ODE's as well as system of first order ODEs. The step number K can also be increased to K > 4, Thus, comparison in terms of accuracy and the interval of stability could be made with the new methods.

It is also noted that collocation and interpolation points in this work were taken at grid points only. Further research work, in which collocations are taking at both the grid and off grid points could be carried out and the results compared with new methods.

It is also recommended that methods based on odd step-numbers K > 3 should be proposed and the results should be compared with the even step-number methods.

REFERENCES

1. Ademiluyi R.A. (1987): "New hybrid methods for systems of stiff ordinary differential equations." A Ph. D Thesis University of Benin. Unpublished.

2. Ademiluyi R.A. and Kayode G.J (2001): Maximum order second derivative hybrid multistep methods for integration of initial value problems in ordinary differential equation J. Nig. Ass Math. Pry vol.5 pp. 251 – 262.

3. Awoyemi D.O. (1992): On some continuous Linear multistep methods for initial value problems. Ph.D Thesis (Unpublished). University of Ilorin Nigeria.

4. Awoyemi D.O. (1999): 'A new sixth order algorithm for general second order ordinary differential equations' intern J. Computer Math. Vol 77 pp. 177-124.

5. Awoyemi D.O. (2001) "A Family of continuous Linear Multistep Hybrid methods for a system of first orders ordinary differential equation. NJTE Vol. 18 Nos. 1 & 2, pp 124 - 131.

6. Brugnano L. and Magherini C. (2003): Journal of Computational and Applied Mathematics.

7. Collatz. L. (1960): The numerical treatment of differential equations. Springer verlag. Berlin.

8. Fatunla S.O. (1988): Numerical methods for 1VPs in ordinary differential equations. Academic press Inc. Harcourt Brace Jovanorich publishers, New York.

9. Frazer R.A. Duncan, W.J. and collar, A.R. (1938): Elementary matrices and some applications Cambridge University Press, England.

10. Gear, C.W. (1991): Numerical initial value problems in ordinary differential equations. Prentice – Hall, Englewood Cliffs, New Jersey.

11. Henrici, P. (1962): Discrete variable methods in ordinary differential equations. John Wiley and Sons N.Y.

12. Jacques I and Judd, C.J. (1987): Numerical Analysis: Chapman and Hall, N. York.

13. Kayode S.J. (2004): A class of linear multistep collocation methods for direct solution of second order ordinary differential equations. A Ph.D Thesis (Unpublished) Fed. University of Tech. Akure.

14. Kantorovich, L.V. (1934): "On a new method of approximate solution of partial differential equations" Dokl. Akad. Nauk. USSR, Vol. 4, pp 532 – 536.

15. Kockler, N. (1994): Numerical methods and scientific computing Clarendon Press, Oxford, London.

16. Lambert J.D. (1973): Computational methods in ordinary differential equations. John Wiley and Sons New York.

17. Lambert J.D and Watson, A. (1976). "Symmetric multistep method for pardic initial value problems". Journal of the institute of mathematics Application "Vol. 18 pp 189 – 702.

18. Lanczos, C. (1973) "Leqiudre versus Chebyshey polynomials" in Millen topics in numerical analysis. Academic press, London.

19. Lambert J.D. (1991): Numerical methods for ordinary differential systems of initial value problems. John Wiley and Sons. New York.

20. Milne W.E and R.R. Reymolds, (1960) "Stability of a numerical solution of differential equations" J. Assoc. Comput March. 7 pp 46 – 56

21. Oladele J.O. (1991) "Some New collocation formulae for the continuous numerical solution of initial value problem" M.Sc. Degree Dissertation (Unpublished), University of Ilorin, Ilorin.

22. Onumanyi, P, Awoyemi, D. O., Jator S. N. and Sirisean U.W. (1994) New Linera Multistep Method with Continuous Coefficient for first order Difference Approximation for solving Differential Equations, Comp. Maths. 72, 15 – 27.

APPENDIX I: GRAPHS OF ERRORS OF PROBLEMS 1 AND 2

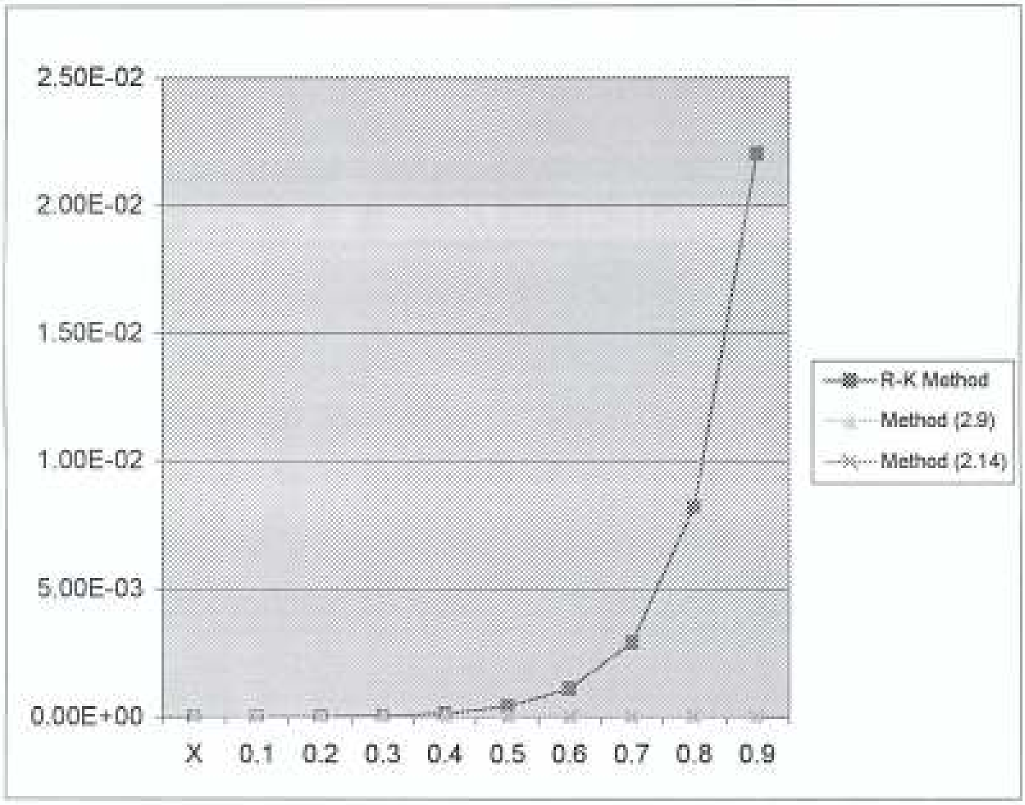

Fig. 1: Comparison of Errors in Problem 1

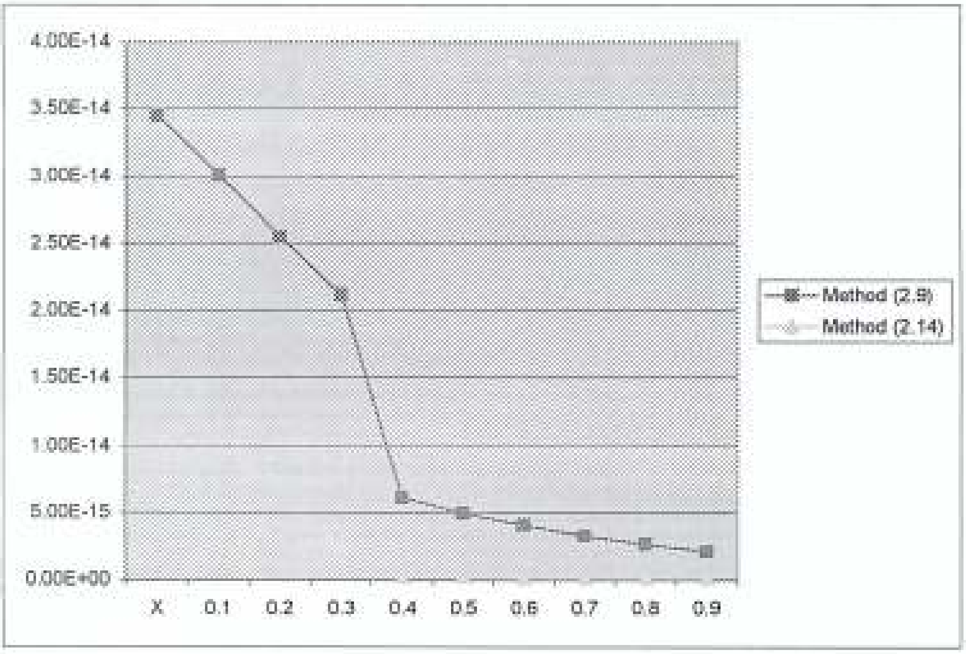

Fig. 2: Comparison of Errors in Problem 2

APPENDIX II COMPUTER PROGRAMS FOR THE SOLUTION OF SAMPLE PROBLEMS

```
C     NAME OF FILE: AYO3.FOR
C     K=4 (Prob. 2)
C     SOLUTION OF FIRST ORDER INITIAL VALUE PROBLEMS
C     OF THE FORM Y'=F (X, Y)
      IMPLICIT DOUBLE PRECISION (A-H, O-Z)
      DIMENSION Y4C (15,40), YEX (15,40), EC (15,40), TT (15,40)
      F (X, Y)=(1.D0-X)*Y*Y-Y
      Y (X)=1.D0/(DEXP (2.D0*X)-X)
      OPEN (6,FILE='AYO3.OUT')
      N=10
      NSTEP=40
      A=0.D0
      B=A+. 1D0
      D=1
      DIST=B-A
      H=DIST/FLOAT (NSTEP)
      C=A+H
      DX=H/FLOAT (N)
      XN=A
      X=XN
      YN=1.D0
      XN1=XN+H
      XN2=XN+2.D0*H
      XN3=XN+3.D0*H
      XN4=XN+4.D0*H
      WRITE (6,9)
    9 FORMAT (4X, 23HPROBLEM: Y'=(1-x)*Y*Y-Y/)
      WRITE (6,11)
   11 FORMAT (4X, 7H Y (0)=1/)
      WRITE (6,8)
    8 FORMAT (4X,  'H=. 1/40'/)
      WRITE (6,7)
```

```
  7 FORMAT (4X,'EXACT SOLUTION: Y=1/(EXP (2X)-X)')
    WRITE (6,5)
  5 FORMAT (7X,'X', 15X,'YEX', 20X,'YC', 20X,'ER'/)
C     CALCULATE PREDICTORS
    K=0
    DO 1 I=1,11
    DO 2 J=1,NSTEP
    F0=F(XN,YN)
    PDFX=-YN*YN
    PDFY=2.D0*(1.D0-XN)*YN-1.D0
    YN1=YN+H*F0+(H*H/2.D0)*(PDFX+F0*PDFY)
    YN1=Y (XN1)
    F1=F (XN1, YN1)
    YN2=-4.D0*YN1+5.D0*YN+2.D0*H*(2.D0*F1+F0)
    F2=F (XN2, YN2)
    YN3=3.D0*YN2+9.D0*YN1-11.D0*YN+2.D0*H*(F2-5.D0*F1-2.D0*F0)
    F3=F (XN3, YN3)
    K=K+1
    IF (K.GE.2) THEN
    YN4=YC
    ELSE
    YN4=-28.D0*YN3+28.D0*YN1+YN+(12.D0*H)*(F3+3.D0*F2+F1)
    ENDIF
    F4=F (XN4, YN4)
C     CALCULATE COEFFICIENTS OF CONTINUOUS METHOD
    DO 3 K=1,10
    TT (I, J)=XN+DX*FLOAT (K)
    X=TT (I, J)
    T=(X-XN3)/H
    A0T =(1.D0/144.D0)*(66.D0*T+137.D0*T*T+48.D0*T**3-58.D0*T**4-
   142.D0*T**5-7.D0*T**6)
    A1T=(1.D0/34.D0)*(-63.D0*T-123.D0*T*T-34.D0*T**3+54.D0*T**4+
   133.D0*T*T*T *T* T+ 5.D0*T*T*T*T*T*T)
    A2T=(1.D0/16.D0)*(-18.D0*T+15.D0*T*T+16.D0*T**3-6.D0*T**4-
```

```
     16.D0*T**5-T**6)
      A3T=(1.D0/306.D0)*(306.D0+771.D0*T+529.D0*T*T-102.D0*T**3-
     1248.D0*T**4-93.D0*T**5-11.D0*T**6)
      B4T=(H/544.D0)*(18.D0*T+57.D0*T*T+68.D0*T**3+38.D0*T**4+10.D0
     1*T**5+T**6)
      B2T=(H/68.D0)*(-180.D0*T-264.D0*T*T-17.D0*T**3+113.D0*T**4+53
     1.D0*T**5+7.D0*T**6)
      B0T=(H/1632.D0)*(258.D0*T+545.D0*T*T+204.D0*T**3-226.D0*T**4-
     1174.D0*T**5-31.D0*T**6)
      Y4C (I, J)=A0T*YN+A1T*YN1+A2T*YN2+A3T*YN3+B0T*F0+B2T*F2+B4T*F4
      YC=Y4C (I, J)
C     CALCULATE EXACT SOLUTION
      YEX (I, J)=Y (X)
      YE=YEX (I, J)
      EC (I, J)=DABS (YC-YE)
      IF (X.GE.C) THEN
      ER=EC (I, J)
      GO TO 3
         ELSE
      ENDIF
    3 CONTINUE
      IF (C.GE.B) THEN
      WRITE (6,10) X, YE, YC, ER
   10 FORMAT (5X,F5.1, 3X, 3D20.10)
      GO TO 4
         ELSE
C     CHANGE VARIABLE
      C=C+H
      XN=XN1
      XN1=XN2
      XN2=XN3
      XN3=XN4
      XN4=XN4+H
      YN=YN1
```

```
      YN1=YN2
      YN2=YN3
      YN3=YN4
      ENDIF
      GO TO 2
    4 IF (B.GE.D) GO TO 1
      B=B+DIST
    2 CONTINUE
    1 CONTINUE
      STOP
      END
C     NAME OF FILE: AYO4.FOR
C     K=4 (Prob. 1)
C     SOLUTION OF FIRST ORDER INITIAL VALUE PROBLEMS
C     OF THE FORM Y'=F (X, Y)
      IMPLICIT DOUBLE PRECISION (A-H, O-Z)
      DIMENSION Y4C (15,40), YEX (15,40), EC (15,40), TT (15,40)
      F (X, Y)=DEXP (10.D0*(X-Y))
      Y (X)=(DLOG (DEXP (10.D0*X)+DEXP (1.D0)-1.D0))/10.D0
      OPEN (6,FILE='AYO4.OUT')
      N=10
      NSTEP=40
      A=0.D0
      B=A+. 1D0
      D=1
      DIST=B-A
      H=DIST/FLOAT (NSTEP)
      C=A+H
      DX=H/FLOAT (N)
      XN=A
      X=XN
      YN=. 1D0
      XN1=XN+H
      XN2=XN+2.D0*H
      XN3=XN+3.D0*H
      XN4=XN+4.D0*H
      WRITE (6,9)
    9 FORMAT (4X, 34HPROBLEM: Y'=(DEXP (10.D0*(X-Y)))/10/)
      WRITE (6,11)
   11 FORMAT (4X, 9H Y (0)=0.1/)
      WRITE (6,8)
    8 FORMAT (4X,  'H=. 01'/)
      WRITE (6,7)
    7 FORMAT (4X,'EXACT SOLUTION: Y=DLOG (DEXP (10.D0*X)+DEXP (1.D0)-
1.D0)')
      WRITE (6,5)
    5 FORMAT (7X,'X', 15X,'YEX', 20X,'YC', 20X,'ER'/)
```

```
C     CALCULATE PREDICTORS
      K=0
      DO 1 I=1,11
      DO 2 J=1,NSTEP
      F0=F (XN, YN)
      PDFX=10.D0*DEXP (10.D0*(XN-YN))
      PDFY=-10.D0*DEXP (10.D0*(XN-YN))
      YN1=YN+H*F0+(H*H/2.D0)*(PDFX+F0*PDFY)
      YN1=Y (XN1)
      F1=F (XN1, YN1)
      YN2=-4.D0*YN1+5.D0*YN+2.D0*H*(2.D0*F1+F0)
      F2=F (XN2, YN2)
      YN3=3.D0*YN2+9.D0*YN1-11.D0*YN+2.D0*H*(F2-5.D0*F1-2.D0*F0)
      F3=F (XN3, YN3)
      K=K+1
      IF (K.GE.2) THEN
      YN4=YC
      ELSE
      YN4=-28.D0*YN3+28.D0*YN1+YN+(12.D0*H)*(F3+3.D0*F2+F1)
      ENDIF
      F4=F (XN4, YN4)
C     CALCULATE COEFFICIENTS OF CONTINUOUS METHOD
      DO 3 K=1,10
      TT (I, J)=XN+DX*FLOAT (K)
      X=TT (I, J)
      T=(X-XN3)/H
      A0T =(1.D0/144.D0)*(66.D0*T+137.D0*T*T+48.D0*T**3-58.D0*T**4-
     142.D0*T**5-7.D0*T**6)
      A1T=(1.D0/34.D0)*(-63.D0*T-123.D0*T*T-34.D0*T**3+54.D0*T**4+
     133.D0*T*T*T *T* T+ 5.D0*T*T*T*T*T*T)
      A2T=(1.D0/16.D0)*(-18.D0*T+15.D0*T*T+16.D0*T**3-6.D0*T**4-
     16.D0*T**5-T**6)
      A3T=(1.D0/306.D0)*(306.D0+771.D0*T+529.D0*T*T-102.D0*T**3-
     1248.D0*T**4-93.D0*T**5-11.D0*T**6)
      B4T=(H/544.D0)*(18.D0*T+57.D0*T*T+68.D0*T**3+38.D0*T**4+10.D0
     1*T**5+T**6)
      B2T=(H/68.D0)*(-180.D0*T-264.D0*T*T-17.D0*T**3+113.D0*T**4+53
     1.D0*T**5+7.D0*T**6)
      B0T=(H/1632.D0)*(258.D0*T+545.D0*T*T+204.D0*T**3-226.D0*T**4-
     1174.D0*T**5-31.D0*T**6)
      Y4C (I, J)=A0T*YN+A1T*YN1+A2T*YN2+A3T*YN3+B0T*F0+B2T*F2+B4T*F4
      YC=Y4C (I, J)
C     CALCULATE EXACT SOLUTION
      YEX (I, J)=Y (X)
      YE=YEX (I, J)
      EC (I, J)=DABS (YC-YE)
      IF (X.GE.C) THEN
      ER=EC (I, J)
      GO TO 3
    ELSE
```

```
      ENDIF
    3 CONTINUE
      IF (C.GE.B) THEN
      WRITE (6,10) X, YE, YC, ER
   10 FORMAT (5X,F5.1, 3X, 3D20.10)
      GO TO 4
         ELSE
C     CHANGE VARIABLE
      C=C+H
      XN=XN1
      XN1=XN2
      XN2=XN3
      XN3=XN4
      XN4=XN4+H
      YN=YN1
      YN1=YN2
      YN2=YN3
      YN3=YN4
      ENDIF
      GO TO 2
    4 IF (B.GE.D) GO TO 1
      B=B+DIST
    2 CONTINUE
    1 CONTINUE
      STOP
      END
```