DYNAMIC RESPONSE TO A MOVING LOAD OF A HIGHLY PRESTRESSED ISOTROPIC RECTANGULAR PLATE ON A BI-PARAMETRIC SUBGRADE

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CERTIFICATION

(A) ***BY THE STUDENT:*** This work has not been presented elsewhere for the award of a degree, or any other purpose.

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The thesis is dedicated to the memory of my beloved and only sister Miss Adunwole Bosede Ogunbamike.

ABSTRACT

In this thesis, the dynamic response of a highly prestressed isotropic rectangular plate resting on a bi-parametric subgrade under the action of a moving load is investigated. In particular, the bi-parametric subgrade is the so called Pasternak foundation model. The equation of motion of the dynamical system which is a fourth order non-homogeneous partial differential equation is presented in a non-dimensionalized form. As a result of this, a small parameter ε (the ratio of bending stiffness to the axial prestress) multiplies the highest derivative in the governing partial differential equation. For an analytical solution to be obtained, the equation was subjected to Laplace transformation while the resulting partial differential equation was solved using the singular perturbation technique, specifically the Method of Matched Asymptotic Expansion (MMAE). The methods of integral transformations and the Cauchy residue theory were then used to solve the resulting partial differential equations to obtain a uniformly valid analytical solution in the entire domain of definition of the rectangular plate.

Analysis of analytical solutions and numerical results in plotted curves were presented. The results show that the prestress, shear modulus and foundation stiffness affect the response to $0(\varepsilon^1)$ of the rectangular plate. Also, the critical velocities of the dynamical system increase with prestress, shear modulus and foundation stiffness. Thus, resonance is reached earlier for lower values of prestress, shear modulus and foundation stiffness.

NOMENCLATURE

D is the bending stiffness

W is the deflection of the plate

K is the foundation stiffness

G is the shear modulus

m is the mass of the plate

ε is the ratio of bending stiffness to the axial prestress

$N_{\bar{x}}$ is the axial prestress in the $\bar{x}$ direction

$N_{\bar{y}}$ is the axial prestress in the $\bar{y}$ direction

N_o is the prestress in preferred direction

ω_o characteristic frequency

u is the velocity of the moving load

g is the acceleration due to gravity

$\delta(\bullet)$ is the dirac delta function

M is the mass of the moving load

x, y outer variable

X, Y inner variable

V^o outer solution

φ^u inner solution

L is the characteristic length

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CHAPTER ONE

1.0 INTRODUCTION

The flexural motions of structural members carrying moving loads are of considerable practical importance and have been of interest in Engineering and Mathematical Physics Fryba [1,2]. Simple examples of structural members are bridges, railroad rails, decking slab, elevated roadways to moving vehicles, girders, belt drive (carrying machine chain) and even floppy disks / cassette player heads carrying tape. Moving loads on the other hand include moving trains, trucks, cars, bicycles, cranes etc.

In most analytical studies in Engineering and Mathematical Physics, structural members are commonly modelled as a beam or as a plate. These structures may be elastic, viscoelatic or inelastic and the moving loads may either be of constant or variable magnitude. While constant stationary loads produce reaction stresses and deformations that are constant, moving loads produce effects that are variable functions of the position of the load (which is also a function of time). Thus, when structural members are under the passage of moving loads, the interaction between the passing load and the structure makes the dynamic response analysis very complex [3]. Particularly, in our time, modern means of transport are ever faster and heavier, while the structure over which they move are ever more slender and lighter. The dynamic stresses they produce are larger by far than the static ones. By virtue of the relevance in the analysis and design of railway tracks, bridges, elevated road ways, decking slab etc, the dynamic response of structural members under the passage of moving loads has been extensively investigated and a number of experimental and numerical studies have been published in recent years.

Basically, while a beam may be considered as a slender bar acted upon by forces and moments producing primarily bending, a plate is an initially flat structural element for which thickness is much smaller than the other dimensions. Generally, plates are considered as two-dimensional structure while a beam is considered as one-dimensional body whose physical properties (stiffness, mass, length etc) are described with reference to a single dimension, the position along the elastic axis. A typical beam problem is governed by a fourth order partial differential equation of two independent variables, namely, distance along the elastic axis and time while a plate problem is governed by a fourth order partial differential equation of three independent variables, distances along the two elastic axes i.e. along x and y directions and time.

In recent years, the problem of the response of elastic structural members such as beams and plates to travelling load has been the objective of numerous investigators in Bridge Engineering, Mechanical Engineering and Applied Mathematics. However, the vibration of plate under the action of moving loads has so far received but scanty attention. This is due at least in part, to the great amount of computational labour which is required to set up and to solve necessary equations. Infact, when the effect of small bending rigidity is considered, plate moving load problems have rarely been solved in literature.

Furthermore, in most of the previous investigations, only plates not resting on an elastic foundation were considered. For practical application, it is useful to consider plates supported by an elastic foundation. For instance, an analysis involving such a foundation can be used to determine the behaviour of plates of road ways or runways, as such plates, concrete or reinforced concrete, resting on various prepared foundations can very roughly be approximated by the Winkler foundation with foundation constant K. However, since the characteristic feature of the well-known Winkler foundation model is the discontinuous behaviour of the surface displacement, in reality, the surface displacements continue beyond the load (force) region. A more realistic elastic foundation model, known as the Pasternak foundation model is considered in this thesis. For this model, a second foundation constant, the "Shear modulus" G, enters the analysis.

In many practical applications of plate problems it has been observed that the elastic properties of structural materials are different in various directions. Materials that exhibit such natural orthotropy include plywood, delta-wood, fibre reinforced plastics and two-way reinforced-concrete slabs. Similarly, in such practical problems, often, the in-plane loads (prestress) are large compared with the flexural rigidity of the plate. When the prestress becomes so large that its ratio of flexural rigidity becomes very small, the plate problem transforms to that of when a small parameter multiplies the highest derivative in the governing differential equation. Explicit exact analytical solutions to this class of problems are not possible even when the applied load is stationary or the (dynamical) system is executing free vibration. It becomes more difficult when the applied load is the moving type.

According to Nayfey [4], most of the problems facing Engineers, Physicist and Applied Mathematicians today exhibit certain essential features which preclude exact analytical solutions. Some of these features are non-linearities, variable coefficients, complex boundary shapes, non-linear boundary conditions at known or in some cases, unknown boundaries, small parameter multiplying the highest derivatives in the governing differential equation etc. Even if the exact solution of a problem can be found explicitly, it may be useless for mathematical and physical interpretation or numerical evaluation. Thus in order to obtain information about solutions of equations, one is forced to resort to approximation methods, numerical solutions or combination of both. Foremost among the approximation methods are perturbation (asymptotic) methods. According to these techniques, the solution is presented by the first few terms of an asymptotic expansion, usually not more than two terms.

The expansion may be carried out in terms of parameter (small or large) which appears naturally in the equations, or which may be artificially introduced for convenience. Such expansions are called parameter perturbation.

An interesting feature of a dynamical system in which a small parameter multiplies the highest derivative is that straight forward perturbation expansion does not yield a uniformly valid solution in the entire domain of definition of the problem. The reason for this is clear. Sharp changes in dependent variables take place in some regions of the domain of the independent variables. It thus becomes necessary to introduce boundary or initial layers where the solution of the given problem undergoes a rapid transformation from a form that satisfies all the data given for the problem to a form represented by the perturbation series. The determination of the boundary layers and the approximate terms of the given equations in these regions form the subject of boundary layer theory. The small interval across which the independent variable changes very rapidly is called the boundary layer in fluid mechanics, the edge layer in solid mechanics and the skin layer in electrodynamics, Erich [5]. The solution in this region is usually termed the inner solution and the solution valid away from this region is termed the outer solution. The procedure whereby solution valid in the boundary layers are identified with the perturbation series solution in the so called outer region, is often called matching process. By combining the perturbation and boundary layer solutions, a fairly good approximate description of the solution of the given problem can often be found for problems where the exact solution is difficult or impossible to determine or where the solution is not easy to interpret or evaluate.

In most of the previous investigations in literature on vibration of rectangular plate under moving loads, the cases when prestress is very high are rarely addressed. Also to the best of the author's knowledge, the cases when the plates which are highly prestressed are placed on a bi-parametric subgrade are outstanding. Thus, this thesis is concerned with the problem of assessing the effect of prestress, Pasternak foundation and cross-sectional dimensions of the plate on the dynamic response to a moving load of a highly prestress elastic rectangular plate.

1.1 RELATED LITERATURE

Work on dynamic loading of one dimensional solids such as beams have received more attention in literature than we have in moving load plate problems. In particular, the problem of a beam carrying moving masses was first attacked by Jeffcott [6] in 1929. He was closely followed by Steuding [7] and Odman [8]. In their papers, the solutions were presented in approximate form involving rather laborious permutation techniques. Timoshenko [9] also studied simply supported finite beams on elastic foundations subjected to time dependent point loads moving with uniform velocity across the beam. He used energy methods and obtained solution in series form. In a more recent development, Milomir et al [10] developed a method based on Fourier analysis to solve the problem of response of beams to an arbitrary number of concentrated moving masses. The method leads to an approximate rapidly converging solution readily amenable to design analysis and calculation. This method is not connected with any previously developed techniques in this subject. Among several authors that have worked on this subject are Steele [11], Bolotin [12], Stanisic et al [13] and Kenny [14]. One problem in these previous works is that the methods of obtaining the solutions are inadequate to solve moving load problems involving end conditions other than simple ones. This shortcoming was recently addressed by Sadiku and Leipholz [15]. The problem of elastic beam under the actions of moving concentrated masses was studied. A method capable of solving this problem for all variants of classical boundary conditions was developed. The technique involves obtaining the Green's function of the moving force problem. Thus, the differential equation governing the moving mass problem is then transformed into an intergro-differential equation using Green's function. The intergro-differential equation admits an iterative solution process to any desired degree of accuracy.

In a similar manner, the problem of the response of an elastic plate to a moving load has obtained important attention in the literature in connection with the development of modern traffic over long and wide highway bridges, where, a number of various vehicles moving with arbitrary velocities are involved. Whereas, moving force plate problems have been considered by several authors, the response of the elastic plate to a moving mass has been an outstanding problem in Engineering and Applied Mathematics for many years. This is due, at least in part, to the great amount of computational labour which is required both to set up and to solve the necessary equations. Among the earliest work on moving load plate problems is the work of Holl [16]. He solved the problem of a rectangular plate under the action of uniform moving loads. He indicated that a critical velocity existed for each vibrational mode. In their investigations Wu et al [17] presented a general numerical analysis theory which is capable of solving the dynamic response of non-uniform rectangular flat plates with various boundary conditions subjected to moving mass. Outstanding among recent investigations into the response of the elastic plate to a moving mass is the work of Stanisic et al [18]. They studied the two dimensional problems of flexural vibration of plate under the actions of moving masses. Only the inertia term that measures the effect of local acceleration in the direction of the deflection was considered. The method of solution was based on the Fourier Sine transform technique suitable only for simply supported boundary conditions. The solutions so obtained were shown to converge very rapidly. Also, Aiyesimi [19] studied the dynamic response of an elastic isotropic rectangular non-Minldin plate resting on a viscoelastic foundation and under the action of a force moving with variable velocity. His method was based on the finite integral Fourier transform suitable only for simply supported end conditions. It was shown that the maximum steady amplitude was attained when the influence of the foundation was small. It was also shown that there was a slight drop in the maximum amplitude for the static load case before a steady state was attained. The work in stanisic et al [18, 20] was taken up much later by Gbadeyan and Oni [21] who studied the dynamic analysis of an elastic plate continuously supported by an elastic Pasternak foundation traversed by an arbitrary number of concentrated masses. All the components of the inertia terms were considered and the rectangular plate was assumed to be simply supported. The deflection of the plate was calculated for several values of the foundation moduli and shown graphically as a function of time. As in the previous paper, the method of solution is suitable for simply supported boundary conditions. More recently, Oni [22] developed a versatile solution technique for solving plate moving load problems for all variants of classical boundary conditions. This is a two dimensional analogue of the versatile technique discussed in Gbadeyan and Oni [23]. The technique involves the use of the modified generalized two-dimensional integral transform to reduce the fourth order differential equation governing the motion of the plate to second order ordinary differential equation which is then treated using the modified asymptotic method of Struble [24].

Recently, the differential quadrature method was shown [39] an efficient way of obtaining accurate solutions to the problem of rectangular plates resting on an elastic foundation and carrying any number of sprung masses. There was an excellent agreement between the method and known solutions published in literature. Similarly, the problem of nonlinear transient dynamic response of clamped rectangular plate on two-parameter foundations was tackled by Civalek et al [40] using the algorithm of singular convolution. In particular, the problem was discretized in space and time domain using discrete singular convolution (DSC) and harmonic differential quadrature (HDQ) methods respectively.

In all the aforementioned studies, no consideration has been given to bending effects at the boundaries.

In particular, when a plate structure is highly prestressed, a small parameter multiplies the highest derivative in the governing differential equation. Thus the methods of solution of all the aforementioned authors break down. This is so because when dealing with a highly prestressed rectangular plate of moderate thickness, bending effects must be duly taken into account. In particular, the domain far from the boundaries can generally be regarded as obeying the reduced order theory, whereas close to the boundary, bending effects become significant and may even dominate the deformation pattern thus; a solution valid in the domain far from the boundary breaks down near and at the boundaries.

Closed form solutions to this class of plate dynamical problems in which a small parameter multiplies the highest derivative in the governing differential equation is not common in literature when the plate is subjected to a moving load. However, this class of plate problems has been solved when the plate is executing free vibration or when a static load is acting on such plate, Hutter and Olunloyo [25].

Singular perturbation has to date seen relatively little use in solid mechanics but it is nonetheless being successfully used, Cole [26]. In particular, Hutter and Olunloyo [27] has employed it in investigating rectangular membranes with small bending stiffness while Hutter and Olunloyo [28] treated, among other things, the vibration of a thick strip-like membrane under anisotropic prestress. Similarly, Schneider [29] considered the vibrations of isotropically prestressed rectangular plates with built-in edges. In his paper, he essentially constructs outer (core) and inner (boundary layer) solutions which are valid in partly disjoint domains. These solutions are then matched in an intermediate domain where both asymptotic expansions are valid. Much later, Olunloyo and Hutter [30] studied the response of thin, isotropic, prestressed rectangular plate for the case when the ratio of bending rigidity to the applied in plane loading is small. He used the Method of Composite Expansion (MCE) to construct solutions for various boundary conditions. Oyediran and Gbadeyan [31] considered the case when the clamped highly prestressed rectangular plate exhibits natural material orthotropy. The problem was solved using the Method of Matched Asymptotic Expansions (MMAE). In a more recent article, Gbadeyan and Oyediran [32] compared the two singular perturbation techniques (MCE and MMAE) for initially stressed thin rectangular plate. They found that the results of the MMAE agree with those obtained using generalized MCE and specialized version of MCE when the effect of shearing deformation is $0(\varepsilon)$. Another work worthy of mention is the work of Olunloyo and Hutter [33] who investigated the dynamic response of prestressed rectangular membrane to certain external time dependent forces when the effect of bending rigidity is small using the MCE. It is remarked at this juncture that, until recently to the best of the authors knowledge while the effect of small bending rigidity has been investigated for free and some forced vibration of plate problems, aside the work of Oni [34], calculations for this class of problems for moving load plate problems do not exist in literature.

After an earlier work by Oni [34] where he studied the dynamic response to a moving load (using the Method of Matched Asymptotic Expansion MMAE) of a fully clamped prestressed orthotropic rectangular plate, Oni and Tolorunsagba [35] took up the problem of assessing the rotatory inertia influence on the highly prestressed orthotropic rectangular plate when it is under the action of moving load. The method of composite expansion (MCE), an alternate singular perturbation technique is employed in conjunction with the method of integral transformation and Cauchy residue theorem to obtain an approximately uniformly valid solution in the entire domain of definition of the rectangular plate. Analysis showed that the critical velocities of the dynamical system increase with an increase in prestress and rotatory inertia values. Thus, resonance is reached earlier for lower values of prestress and rotatory inertia. Also, for high values of rotatory inertia correction factor, the critical velocity approaches a constant value indicating that resonant effect is remote for higher values of rotatory inertia correction factor.

However, in the work of Oni [34] and Oni and Tolorunsagba [35], only plates not resting on foundation were considered. For practical application, it is pertinent to considered plates supported by an elastic foundation.

Thus, in this work the dynamic response to a moving load of a highly prestressed isotropic rectangular plate resting on a Pasternak-type foundation is considered.

1.2 OBJECTIVES OF THE RESEARCH

The specific objectives of this project work are to:

(i) obtain uniformly valid analytical solutions that cover the whole domain of interest.

(ii) establish the resonance conditions of the dynamical system.

(iii) determine the effect of prestress and elastic moduli on the resonance condition.

(iv) determine the effects of elastic foundation on the response of a highly prestressed plate.

1.3 GOVERNING DIFFERENTIAL EQUATION.

The dynamic response of isotropic rectangular plate resting on a Pasternak Foundation under the action of a moving load is governed by the fourth order partial differential equation [36]

$$D\left[\frac{\partial^4 W(x,y;t)}{\partial \bar{x}^4}+\frac{\partial^4 W(x,y;t)}{\partial \bar{x}^2 \partial \bar{y}^2}+\frac{\partial^4 W(x,y;t)}{\partial \bar{y}^4}\right]+m\frac{\partial^2 W(x,y;t)}{\partial \bar{t}^2}=P(\bar{x},\bar{y};\bar{t})-P_{FM}(\bar{x},\bar{y};\bar{t})$$

Where (1.1)

D is the bending stiffness

$\bar{x}$ is the position coordinate in the x – direction

$\bar{y}$ is the position coordinate in the y – direction

$\bar{t}$ is the time coordinate

W is the deflection of the plate

m is the mass of the plate per unit area

$P(\bar{x},\bar{y};\bar{t})$ is the applied dynamic load

If the rectangular plate is prestress, two additional terms given by

$$-\left[N_{\bar{x}}\frac{\partial^2 W(x,y;t)}{\partial \bar{x}^2}+N_{\bar{y}}\frac{\partial^2 W(x,y;t)}{\partial \bar{y}^2}\right] \qquad (1.2)$$

are added to the left hand side of the equation

Also for a rectangular plate, the foundation for which the relation

$$P_{PM}(\bar{x},\bar{y};t) = KW(x,y;t) + G\left[\frac{\partial^2 W(x,y;t)}{\partial \bar{x}^2} + \frac{\partial^2 W(x,y;t)}{\partial \bar{y}^2}\right] \qquad (1.3)$$

holds is known as Pasternak foundation.

1.4 FEATURES OF THE THESIS

In the next chapter of this thesis, the initial–boundary value problem of the transverse vibration of a highly prestressed isotopic rectangular plate resting on a bi-parametric subgrade is developed. The leading order solution via the systematic matched asymptotic expansion is also obtained. In chapter three, the first order correction to the approximate solution is obtained and analysis of the entire uniformly valid solution to the first order is carried out while the last chapter deals with general conclusions.

CHAPTER TWO

2.0 GOVERNING EQUATION OF MOTION

The equation governing the dynamic transverse displacement $W(x,y;t)$ of an isotropic rectangular plate of span L_y along the y-axis and span L_x along the x-axis and transverse by several moving loads moving with velocity along a straight line parallel to x-axis when it is resting on a Pasternak foundation is governed by the fourth order partial differential equation given by

$$D\left[\frac{\partial^4 W(\bar{x},\bar{y};t)}{\partial\bar{x}^4}+\frac{\partial^4 W(\bar{x},\bar{y};t)}{\partial\bar{x}^2\partial\bar{y}^2}+\frac{\partial^4 W(\bar{x},\bar{y};t)}{\partial\bar{y}^4}\right]-N_{\bar{x}}\frac{\partial^2 W(\bar{x},\bar{y};t)}{\partial\bar{x}^2}-N_{\bar{y}}\frac{\partial^2 W(\bar{x},\bar{y};t)}{\partial\bar{y}^2}$$

$$+\frac{m\partial^2 W(\bar{x},\bar{y};t)}{\partial\bar{t}^2}+KW(\bar{x},\bar{y};t)+G\left[\frac{\partial^2 W(\bar{x},\bar{y};t)}{\partial\bar{x}^2}+\frac{\partial^2 W(\bar{x},\bar{y};t)}{\partial\bar{y}^2}\right]=P(\bar{x},\bar{y};\bar{t}) \qquad (2.1)$$

where all parameters are defined in the previous section.

To complete the problem, we shall specify that the boundary condition is fully clamped, thus both the deflection and slope varnish at the boundaries.

Thus,

$$\left.\begin{array}{ll}\bar{x}=0, & 0\le\bar{y}\le B\\ \bar{x}=L, & 0\le\bar{y}\le B\end{array}\right\} W(\bar{x},\bar{y},\bar{t})=0; \qquad \frac{\partial W(\bar{x},\bar{y};t)}{\partial\bar{x}}=0$$

$$\left.\begin{array}{ll}\bar{y}=0, & 0\le\bar{x}\le L\\ \bar{y}=L, & 0\le\bar{x}\le L\end{array}\right\} W(\bar{x},\bar{y},\bar{t})=0; \qquad \frac{\partial W(\bar{x},\bar{y};t)}{\partial\bar{y}}=0 \qquad (2.2)$$

and for simplicity, the initial conditions are taken to be

$$W(\bar{x},\bar{y};0)=0; \qquad \frac{\partial W(\bar{x},\bar{y};o)}{\partial\bar{t}}=0 \qquad (2.3)$$

At this juncture, it is pertinent to present equation (2.1) in a non-dimensionalized form for the purpose of solution. In this respect, we introduce N$_o$, a reference prestress and L is a characteristic length with respect to which the deflection and the two coordinates are normalized viz:

$W = \overline{V}L,\ \overline{x} = xL,\ \overline{y} = yL$

On the order hand, time is normalized with respect to a characteristic frequency ω_o such that

$\overline{t} = \frac{t}{\omega_o}$ and $\frac{N_o}{m\omega_0^2 L^2} = 1$

The governing equation (2.1) in non-dimensional form becomes,

$$\varepsilon^2\left[\frac{\partial^4 \overline{V}(x,y;t)}{\partial x^4} + \frac{\partial^4 \overline{V}(x,y;t)}{\partial x^2 \partial y^2} + \frac{\partial^4 \overline{V}(x,y;t)}{\partial y^4}\right] - \frac{\beta_1^2 \partial^2 \overline{V}(x,y,t;)}{\partial x^2} - \frac{\beta_2^2 \partial^2 \overline{V}(x,y,t)}{\partial y^2}$$

$$+ \frac{\partial^2 \overline{V}(x,y,t)}{\partial t^2} + \eta \overline{V} + \sigma_1^2\left[\frac{\partial^2 \overline{V}(x,y,t)}{\partial x^2} + \frac{\partial^2 \overline{V}(x,y,t)}{\partial y^2}\right] = p_o(x,y;t) \tag{2.4}$$

Where $\varepsilon << 1$ is defined by the relation

$$\varepsilon^2 = \frac{D}{N_o L^2},\quad \eta = \frac{K}{m\omega_0^2},\quad \sigma_1^2 = \frac{G}{N_o} \tag{2.5a}$$

and

β_1^2, β_2^2 measure the prestress ratio and are defined as

$$\beta_1^2 = \frac{N_x}{N_o},\quad \beta_2^2 = \frac{N_y}{N_o} \tag{2.5b}$$

respectively. These coefficients are all 0(1).

In the same manner, the boundary conditions and the initial conditions in non-dimensionalized form become:

$$\left.\begin{matrix} x=0, & 0\le y\le b \\ x=1, & 0\le y\le b \end{matrix}\right\} \quad \overline{V}(x,y,t)=0; \qquad \frac{\partial\overline{V}(x,y;t)}{\partial x}=0$$

$$\left.\begin{matrix} y=0, & 0\le x\le 1 \\ y=b, & 0\le x\le 1 \end{matrix}\right\} \quad \overline{V}(x,y,t)=0; \qquad \frac{\partial\overline{V}(x,y;t)}{\partial y}=0 \tag{2.6}$$

and

$$\overline{V}(x,y,0)=0; \qquad \frac{\partial\overline{V}(x,y,0)}{\partial t}=0 \tag{2.7}$$

When the inertia effects of the moving load is regarded as negligible on the response of the plate, the transverse load can be expressed as

$$p_o(x,y;t)=Mg\delta(x-ut)\delta(y-y_o) \tag{2.8}$$

where M is the mass of the moving load, g is the acceleration due to gravity and $\delta(\bullet)$ is the dirac delta function defined as

$$\delta(x-ut)=\begin{cases} o, & x\neq ut \\ \infty, & x=ut \end{cases} \tag{2.9a}$$

with the properties

$$\delta(-x)=\delta(x)$$

$$\int \partial(x-ut)f(x)dx=\begin{cases} o, & x\neq ut \\ f(ut), & a\le ut<b \\ \infty, & x=ut \end{cases} \tag{2.9b}$$

when equation (2.8) is incorporated into (2.4); we have

$$\varepsilon^2\left[\frac{\partial^4\overline{V}(x,y;t)}{\partial x^4}+\frac{\partial^4\overline{V}(x,y;t)}{\partial x^2\partial y^2}+\frac{\partial^4\overline{V}(x,y;t)}{\partial y^4}\right]-\frac{\beta_1^2\partial^2\overline{V}(x,y;t)}{\partial x^2}-\frac{\beta_2^2\partial^2\overline{V}(x,y;t)}{\partial y^2}+\frac{\partial^2\overline{V}(x,y;t)}{\partial t^2}$$

$$+\eta V(x,y;t)+\sigma_1^2\left[\frac{\partial^2\overline{V}(x,y;t)}{\partial x^2}+;t)\frac{\partial^2\overline{V}(x,y;t)}{\partial y^2}\right]=p_a\delta(x-ut)\delta(y-y_a) \qquad (2.10)$$

where $P_a = Mg$

Equation (2.10) together with boundary conditions (2.6) and initial conditions (2.7) define completely the equation of a fully clamped highly prestressed isotropic rectangular plate occupying the domain 0≤ x ≤ 1, 0≤ y ≤ b in a non–dimensionalized form. It is observed that a small parameter multiplies the highest derivatives in (2.10) and as such the problem is amenable to singular perturbations.

However, equation (2.10) is considerably simplified by introducing the Laplace transform

$$(\tilde{\bullet})=\int_0^\infty(\bullet)e^{-st}dt \qquad (2.11)$$

with the inverse

$$(\bullet)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}(\tilde{\bullet})e^{st}dt \qquad (2.12)$$

in conjunction with the initial conditions defined in (2.3).

Taking t as the principal variable, the Laplace of (2.10) is given as

$$\varepsilon^2\left[\frac{\partial^4 V(x,y;t)}{\partial x^4}+\frac{\partial^4 V(x,y;t)}{\partial x^2\partial y^2}+\frac{\partial^4 V(x,y;t)}{\partial y^4}\right]-\frac{\beta_1^2\partial^2 V(x,y;t)}{\partial x^2}-\frac{\beta_2^2\partial^2 V(x,y;t)}{\partial y^2}+s^2V(x,y;t)$$

$$+\eta V(x,y;t)+\sigma_1^2\left[\frac{\partial^2 V(x,y;t)}{\partial x^2}+\frac{\partial^2 V(x,y;t)}{\partial y^2}\right]=\frac{p_a\delta(y-y_a)}{u}e^{-\frac{s}{u}x} \qquad (2.13)$$

Subject to

$$\left.\begin{array}{ll} x=0, & 0\le y\le b \\ x=1, & 0\le y\le \mathrm{b} \end{array}\right\} \quad V(x,y)=0; \qquad \frac{\partial V(x,y)}{\partial x}=0$$

$$\left.\begin{array}{ll} y=0, & 0\le x\le 1 \\ y=b, & 0\le x\le 1 \end{array}\right\} \quad V(x,y)=0; \qquad \frac{\partial V(x,y)}{\partial y}=0 \qquad (2.14)$$

and initial conditions

$$V(x,y;0)=0; \qquad \frac{\partial V(x,y;0)}{\partial t}=0$$

2.1 METHOD OF SOLUTION

In equation (2.13), an exact uniformly valid solution in the entire domain is not feasible since a small parameter ε, multiplies the highest derivative in the governing differential equation. This is due to the bending effects at the boundaries. Consequently, solution valid away from the boundaries breaks down near as well as at the boundaries. Thus, only approximate solutions are possible.

The two but equivalent approaches that could be used to tackle this type of problem are the method of composite expansion (MCE) and the Method of Matched Asymptotic Expansion (MMAE). In MCE, it is assumed that each dependent variable is the sum of

(i) parts characterized by the original independent variables and

(ii) parts characterized by magnified independent variables one for each sharp-change region.

In MMAE, an approximate solution to the given problem is sought not as a single expansion in terms of a single scale but as two separate expansions which are valid in part of the domain. This provides two separate solutions, one valid at and near the boundaries and the other valid away from the boundaries. These two solutions are then blended together to obtain a uniformly valid solution in the entire domain of interest. This method of matched asymptotic expansion is preferred to the method of composite expansion in this study.

The Method of Matched Asymptotic Expansion (MMAE) developed by Bretheton (1962) requires that the asymptotic solution of equation (2.13) be of the form [18]

$$V = V_0 + \varepsilon V_1 + \varepsilon^2 V_2 + \cdots \tag{2.15}$$

when the expansion (2.15) is incorporated into (2.13) and we equate coefficients of the power of ε, one obtains the recurrence relation

$$\beta^2 \frac{\partial^2 V_v^o(x,y;t)}{\partial x^2} + \beta_2^2 \frac{\partial^2 V_v^o(x,y;t)}{\partial y^2} + s^2 V_v^o(x,y;t) - \sigma_1^2 \left[\frac{\partial^2 V_v^o(x,y;t)}{\partial x^2} + \frac{\partial^2 V_v^o(x,y;t)}{\partial y^2} \right]$$

$$-\eta V_v^o(x,y;t) = \begin{cases} \dfrac{P_o}{u} \delta(y - y_o) e^{-\frac{1}{\varepsilon}x} & v = 0 \\ 0 & v = 1 \\ \nabla^4 V_{v-2}^0 & v \geq 2 \end{cases} \tag{2.16}$$

where $\nabla^4 V_{v-2}^o = \dfrac{\partial^4 V_{v-2}^o}{\partial x^4} + \dfrac{\partial^4 V_{v-2}^o}{\partial x^2 \partial y^2} + \dfrac{\partial^4 V_{v-2}^o}{\partial y^4}$

where the subscripts denote the order in ε.

It is remarked here that equation (2.16) are not uniformly valid in the entire domain of the rectangular plate under consideration. In fact, solutions obtained for

V_o, V_ν, $\nu \geq 1$ are not valid near the boundaries. The reason for this is simple. The order of the partial differential equation (2.13) has been reduced but the number of boundary conditions is not reduced. These solutions are termed outer solutions and the equation (2.16) outer problem.

2.2 EXPRESSION NEAR THE BOUNDARY

It is observed that equation (2.16) is not valid near and at the boundaries. In order to obtain an expression that is valid at the boundary, near x = 0, we set the inner variable as $X = \frac{x}{\varepsilon}$ and write the eigenfunction valid near x=0

as $$V = \psi^i = \psi^i_o(X,y) + \varepsilon\psi^i_1(X,y) + \varepsilon^2\psi^i_2(X,y) + O(\varepsilon^3) \qquad (2.17a)$$

where superscript i denotes the inner solution. Equation (2.17a) is also valid near x = 1, where we set the inner variable $X = \frac{(1-x)}{\varepsilon}$.

An expression similar to (2.17a) can be written down for the eigenfunction near y=0 and y=b where we set the inner variable as $Y = \frac{y}{\varepsilon}$ and $Y = \frac{(b-y)}{\varepsilon}$ respectively, as

$$V = \psi^i = \psi^i_o(x,Y) + \varepsilon\psi^i_1(x,Y) + \varepsilon^2\psi^i_2(x,Y) + O(\varepsilon^3) \qquad (2.17b)$$

Using equation (2.17a) in (2.13) near either x = 0 or x = 1, the differential equation on ψ^i gives

$$\frac{\partial^4\psi^i_\nu}{\partial X^4} - \left(\beta_1^2 - \sigma_1^2\right)\frac{\partial^2\psi^i_\nu}{\partial X^2} = \frac{\beta_2\partial^2\psi^i_{\nu-2}}{\partial y^2} - \frac{\partial^4\psi^i_{\nu-2}}{\partial y^2\partial x^2} - S^2\psi^i_{\nu-2} - \eta\psi^i_{\nu-2}$$

$$-\sigma_1^2 \frac{\partial^2 \psi^i_{v-2}}{\partial y^2} - \frac{\partial^4 \psi_{v-4}}{\partial y^4}, \quad v = 0,1,3,4.... \tag{2.18}$$

$$\frac{\partial^4 \psi^i_v}{\partial X^4} - \left(\beta_1^2 - \sigma_1^2\right)\frac{\partial^2 \psi^i_v}{\partial X^2} = \frac{\beta_2 \partial^2 \psi^i_{v-2}}{\partial y^2} - \frac{\partial^4 \psi^i_{v-2}}{\partial y^2 \partial x^2} - S^2 \psi^i_{v-2} - \eta \psi^i_{v-2}$$

$$-\sigma_1^2 \frac{\partial^2 \psi^i_{v-2}}{\partial y^2} + \frac{p_a \delta\left(y - y_a\right)}{u} e^{-\frac{1}{2}x}, \quad v = 2 \tag{2.19}$$

Subject to boundary condition

$$\psi^i_v = \frac{\partial \psi^i_v}{\partial X} = 0 \qquad v = 0, 1, 2, 3, ... \tag{2.20}$$

The differential equation near y = 0, or y = b can similarly be written as

$$\frac{\partial^4 \psi^i_v}{\partial Y^4} - \left(\beta_2^2 - \sigma_1^2\right)\frac{\partial^2 \psi^i_v}{\partial Y^2} = \beta_1^2 \frac{\partial^2 \psi^i_{v-2}}{\partial x^2} - \frac{\partial^4 \psi^i_{v-2}}{\partial x^2 \partial y^2} - S\psi^i_{v-2} - \eta \psi^i_{v-2}$$

$$-\sigma_1^2 \frac{\partial^2 \psi^i_v}{\partial x^2} - \frac{\partial^4 \psi^i_{v-2}}{\partial x^4} \qquad v = 0, 1, 3, 4 .... \tag{2.21}$$

$$\frac{\partial^4 \psi^i_v}{\partial Y^4} - \left(\beta_2^2 - \sigma_1^2\right)\frac{\partial^2 \psi^i_v}{\partial Y^2} = \beta_1^2 \frac{\partial^2 \psi^i_{v-2}}{\partial x^2} - \frac{\partial^4 \psi^i_{v-2}}{\partial x^2 \partial y^2} - s^2 \psi^i_{v-2} - \eta \psi^i_{v-2}$$

$$-\sigma_1^2 \frac{\partial^2 \psi^i_v}{\partial x^2} + \frac{P_a \partial\left(y - y_a\right)}{u} e^{-\frac{1}{2}x} \qquad v = 2 \tag{2.22}$$

Subject to boundary condition

$$\psi^i_v = \frac{\partial \psi^i_v}{\partial Y} = 0 \qquad v = 0, 1, 2, 3, ... \tag{2.23}$$

2.3 LEADING ORDER SOLUTION (OUTER SOLUTION)

The leading order solution is obtained by setting v = 0 in the recurrence equation (2.16),

$$\left(\beta_1^2-\sigma_1^2\right)\frac{\partial^2 V_o(x,y;t)}{\partial^2 x}+\left(\beta_2^2-\sigma_1^2\right)\frac{\partial^2 V_o(x,y;t)}{\partial y^2}-S^2 V_o(x,y;t)-\eta V_o(x,y;t)=\frac{-P_o\delta\left(y-y_o\right)}{u}e^{-\frac{s}{u}x} \tag{2.24}$$

The method of finite Fourier sine transform defined as

$$V(n,y)=\int_0^1 V(x,y)\sin n\pi x dx \tag{2.25}$$

with the inverse

$$V(x,y)=2\sum_{n=1}^{\infty}V\ (n,y)\sin n\pi x \tag{2.26}$$

is used to obtain the solution of equation (2.24). Equations (2.25) and (2.26) transform (2.24) into

$$-n^2\pi^2\left(\beta_1^2-\sigma_1^2\right)V_o(n,y)+\left(\beta_2^2-\sigma_1^2\right)V_{o,yy}(n,y)-s^2V_o(n,y)-\eta V_o(n,y)$$

$$=\frac{P_o n\pi u\left[(-1)^n e^{-\frac{s}{u}}-1\right]}{s^2+n^2\pi^2u^2}\delta\left(y-y_o\right)$$

So that

$$V_{o,yy}(n,y)-\left[\frac{\left(\beta_1^2-\sigma_1^2\right)n^2\pi^2+s^2+\eta}{\beta_2^2-\sigma_1^2}\right]V_o(n,y)=\frac{P_o n\pi u\left[(-1)^n e^{-\frac{s}{u}}-1\right]}{(\beta_2^2-\sigma_1^2)(s^2+n^2\pi^2u^2)}\delta\left(y-y_o\right) \tag{2.27}$$

So that the transform of (2.24) with respect to x is

$$V_{o,yy}(n,y)+\alpha^2V_o(n.y)=T\delta\left(y-y_o\right) \tag{2.28}$$

$$\text{where } \alpha^2=-\left[\frac{\left(\beta_1^2-\sigma_1^2\right)n^2\pi^2+s^2+\eta}{\beta_2^2-\sigma_1^2}\right] \tag{2.29}$$

and

$$T = \frac{P_a n\pi u\left[(-1)^n e^{-\frac{s}{u}} - 1\right]}{\left(\beta_2^2 - \sigma_1^2\right)\left(s^2 + n^2\pi^2 u^2\right)} \qquad (2.30)$$

The complimentary solution of (2.28) gives

$$V_{nc}(n.y) = C_2 \cos\alpha y + D_2 \sin\alpha y \qquad (2.31)$$

Where C_2 and D_2 are arbitrary constants to be determined

Using the methods of variation of parameters, the particular solution of (2.28) can be shown to be

$$V_{np}(n, y) = \frac{-T}{\alpha}\sin\alpha y_n \cos\beta_y + \frac{T}{\alpha}\cos\alpha y_n \sin\beta_y \qquad (2.32)$$

Consequently, the general solution of the ordinary differential equation (2.28) can be obtained as

$$V_n(n, y) = C_2 \cos\alpha y + D_2 \sin\alpha y + \Gamma_1 \sin\alpha(y - y_n) \qquad (2.33)$$

where

$$\Gamma_1 = \frac{P_a n\pi u\left[(-1)^n e^{-s/u} - 1\right]}{\alpha\left(\beta_2^2 - \sigma_1^2\right)\left(s^2 + n^2\pi^2 u^2\right)} \qquad (2.34)$$

Similarly the finite Fourier sine transform with respect to y is defined as

$$V(m, x) = \int_0^b V(x, y)\sin\frac{m\pi}{b} y\,dy \qquad (2.35)$$

with the inverse

$$V(x, y) = \frac{2}{b}\sum_{m=1}^{\infty} V(m, x)\sin\frac{m\pi}{b} y \qquad (2.36)$$

Using equations (2.35) and (2.36) on (2.24), we obtain

$$(\beta_1^2-\sigma_1^2)V_{o,xx}(m,x)-\frac{m^2\pi^2}{b^2}(\beta_2^2-\sigma_1^2)V_o(m,x)-s^2V_o(m,x)-\eta V_o(m,x)=\frac{-P_o}{u}e^{-\frac{s}{u}x}\sin\frac{m\pi}{b}y_o$$

So that

$$V_{o,xx}(m,x)-\left[\frac{(\beta_1^2-\sigma_1^2)m^2\pi^2+(s^2+\eta)b^2}{(\beta_1^2-\sigma_1^2)b^2}\right]V_o(m,x)=\frac{-P_o}{(\beta_1^2-\sigma_1^2)u}e^{-\frac{s}{u}x}\sin\frac{m\pi}{b}y_o \qquad (2.37)$$

Equation (2.37) can be written as

$$V_{o,xx}(m,x)+\sigma^2V_o(m,x)=\frac{-P_o}{(\beta_1^2-\sigma_1^2)u}e^{\frac{-s}{u}x}\sin\frac{m\pi}{b}y_o \qquad (2.38)$$

$$\text{where } \sigma^2=-\left[\frac{(\beta_2^2-\sigma_1^2)m^2\pi^2+(s^2+\eta)b^2}{(\beta_1^2-\sigma_1^2)b^2}\right] \qquad (2.39)$$

The complimentary solution to the equation (2.38) is

$$V_{oc}(m,x)=E_3\cos\alpha x+F_3\sin\alpha x \qquad (2.40)$$

Where E$_3$ and F$_3$ are arbitrary constants to be determined.

To solve the particular integral (2.38), we use operator method to have

$$(D^2+\sigma^2)V_{op}(m,x)=\frac{-P_o}{(\beta_1^2-\sigma_1^2)u}e^{-\frac{s}{u}x}\sin\frac{m\pi}{b}y_o$$

which gives

$$V_{op}(m,x)=\frac{-P_o}{(\beta_1^2-\sigma_1^2)u}\sin\frac{m\pi}{b}y_o\ X\ \frac{1}{\frac{s^2}{u^2}+\sigma^2}e^{-\frac{s}{u}x}$$

The particular solution is obtained as

$$V_{op}=\frac{P_o u\sin\frac{m\pi}{b}y_o}{(\beta_1^2-\sigma_1^2)(s^2+\sigma^2u^2)}e^{\frac{-s}{u}x} \qquad (2.41)$$

Hence the general solution of (2.38) can be shown as

$$V_{\alpha}(m,x) = E_3 \cos\alpha x + F_3 \sin\alpha x - \Gamma_2 e^{\frac{-s}{u}x} \tag{2.42}$$

where

$$\Gamma_2 = \frac{P_a u \sin\frac{m\pi}{b}y}{\left(\beta_1^2 - \sigma_1^2\right)\left(s^2 + \sigma^2 u^2\right)} \tag{2.43}$$

The inversion of (2.33) and (2.42) gives the general solution of the equation (2.24). Thus,

$$V_o(x,y) = 2\left[C_2 \cos\alpha y + D_2 \sin\alpha y + \Gamma_1 \sin\alpha(y - y_o)\right]\sin n\pi x$$

$$+\frac{2}{b}\left[E_3 \cos\alpha x + F_3 \sin\alpha x - \Gamma_2 e^{\frac{-s}{u}x}\right]\sin\frac{m\pi}{b}y \tag{2.44}$$

Where C_2, D_2, E_3 and F_3 are arbitrary constants yet to be determined by matching.

2.3 LEADING ORDER SOLUTION (INNER PROBLEM)

The differential equation governing the inner solution is (2.18) and (2.20), when we substitute v = 0 and neglecting terms with negative subscripts, we have the leading order inner problem near x = 0 or x = 1 given as

$$\frac{\partial^4 \psi_o^i}{\partial X^4} - \left(\beta_1^2 - \sigma_1^2\right)\frac{\partial^2 \psi_o^i}{\partial X^2} = 0 \tag{2.45}$$

subjected to

$$\psi_o^i = \frac{\partial \psi_o^i}{\partial X} = 0 \tag{2.46}$$

solving equation (2.45) together with (2.46) gives

$$\psi_o^i = \begin{cases} \bar{q}_o(y)\left\{X + \dfrac{1}{\sqrt{\beta_1^2 - \sigma_1^2}} e^{-\sqrt{\beta_1^2-\sigma_1^2}\,X} - \dfrac{1}{\sqrt{\beta_1^2 - \sigma_1^2}}\right\} & near\ \ \mathrm{x} = 0 \\ \bar{\bar{q}}_o(y)\left\{X + \dfrac{1}{\sqrt{\beta_1^2 - \sigma_1^2}} e^{-\sqrt{\beta_1^2-\sigma_1^2}\,X} - \dfrac{1}{\sqrt{\beta_1^2 - \sigma_1^2}}\right\} & near\ \ \mathrm{x} = 1 \end{cases} \qquad (2.47)$$

Similarly, the leading order inner problem near y = 0 and y = b obtained from (2.21) and (2.22) yields

$$\psi_o^i = \begin{cases} \bar{r}_o(x)\left\{Y + \dfrac{1}{\sqrt{\beta_2^2 - \sigma_1^2}} e^{-\sqrt{\beta_2^2-\sigma_1^2}\,Y} - \dfrac{1}{\sqrt{\beta_2^2 - \sigma_1^2}}\right\} & near\ \ \mathrm{y} = 0 \\ \bar{\bar{r}}_o(x)\left\{Y + \dfrac{1}{\sqrt{\beta_2^2 - \sigma_1^2}} e^{-\sqrt{\beta_2^2-\sigma_1^2}\,Y} - \dfrac{1}{\sqrt{\beta_2^2 - \sigma_1^2}}\right\} & near\ \ \mathrm{y} = \mathrm{b} \end{cases} \qquad (2.48)$$

In (2.47) and (2.48), exponentially growing terms have been neglected while the functions $\bar{q}_o(y)$, $\bar{\bar{q}}_o(y)$, $\bar{r}_o(x)$, and $\bar{\bar{r}}_o(x)$ are to be determined by matching. The unknown in (2.44), (2.47) and (2.48) will be determined by matching inner and outer solutions.

To this end, Van Dyke's matching principle which requires m-term inner expansion of (the n – term outer expansion) equals the n–term outer expansion of (the m– term inner expansion) is adopted.

According to the principle, we match one term outer expansion written in inner variable (2.47) with one term inner expansion written in outer variable (2.44) for the boundary condition x = 0 (1-1 matching).

We proceed systematically as follows:

One term outer expansion:

$$V_o(x,y) = 2\left[C_2\cos\alpha y + D_2\sin\alpha y + \Gamma_1\sin\alpha(y-y_o)\right]\sin n\pi x$$

$$+\frac{2}{b}\left[E_1\cos\sigma x + F_1\sin\sigma x - \Gamma_2 e^{-\frac{\sigma}{u}x}\right]\sin\frac{m\pi}{b}y \tag{2.49a}$$

Rewritten in inner variable:

For inner variable $X = \frac{x}{\varepsilon}$ which implies that $x = \varepsilon X$ is given by

$$= 2\left[C_2\cos\alpha y + D_2\sin\alpha y + \Gamma_1\sin\alpha(y-y_o)\right]\sin n\pi\varepsilon X$$

$$+\frac{2}{b}E_1\sin\frac{m\pi}{b}y\cos\sigma\varepsilon X + \frac{2}{b}F_3\sin\frac{m\pi}{b}y\sin\sigma\varepsilon X - \frac{2}{b}\Gamma_2\sin\frac{m\pi}{b}ye^{-\frac{\sigma}{u}\varepsilon X} \tag{2.49b}$$

Expanded for small ε:

$$2\left[C_2\cos\alpha y + D_2\sin\alpha y + \Gamma_1\sin\alpha(y-y_o)\right]\left[n\pi\varepsilon X - \frac{n^3\pi^3\varepsilon^3X^3}{3!} + \frac{n^5\pi^5\varepsilon^5X^5}{5!} - \cdots\cdots\right]$$

$$+\frac{2}{b}E_3\sin\frac{m\pi}{b}y\left[1 - \frac{\sigma^2\varepsilon^2X^2}{2!} + \frac{\sigma^4\varepsilon^4X^4}{4!} - \cdots\cdots\right]$$

$$+\frac{2}{b}F_3\sin\frac{m\pi}{b}y\left[\sigma\varepsilon X - \frac{\sigma^3\varepsilon^3X^3}{3!} + \frac{\sigma^5\varepsilon^5X^5}{5!} - \cdots\cdots\right]$$

$$-\frac{2}{b}\Gamma_2\sin\frac{m\pi}{b}y\left[1 - \frac{\sigma\varepsilon X}{u} + \frac{\sigma^2\varepsilon^2X^2}{2!u^2} - \cdots\cdots\right] \tag{2.49c}$$

One term inner expansion:

$$\frac{2}{b}E_3\sin\frac{m\pi}{b}y - \frac{2}{b}\Gamma_2\sin\frac{m\pi}{b}y \tag{2.49d}$$

One term inner expansion:

$$\psi_e^i = \bar{q}_o(y)\left\{X + \frac{1}{\sqrt{\beta_1^2 - \sigma_1^2}}e^{-\sqrt{\beta_1^2-\sigma_1^2}\,x} - \frac{1}{\sqrt{\beta_1^2 - \sigma_1^2}}\right\} \tag{2.49e}$$

Rewritten in outer variable:

$$= \bar{q}_o(y)\left\{\frac{x}{\varepsilon} + \frac{1}{\sqrt{\beta_1^2 - \sigma_1^2}}e^{-\sqrt{\beta_1^2-\sigma_1^2}\frac{x}{\varepsilon}} - \frac{1}{\sqrt{\beta_1^2 - \sigma_1^2}}\right\} \tag{2.49f}$$

Expanded for small ε:

As $\varepsilon \to 0$, $e^{-\sqrt{\beta_1^2-\sigma_1^2}\frac{x}{\varepsilon}}$ becomes $e^{-\infty} = 0$

$$\bar{q}_o(y)\frac{x}{\varepsilon} + \frac{\bar{q}_o(y)}{\sqrt{\beta_1^2 - \sigma_1^2}}e^{-\sqrt{\beta_1^2-\sigma_1^2}\frac{x}{\varepsilon}} - \frac{\bar{q}_o(y)}{\sqrt{\beta_1^2 - \sigma_1^2}} \tag{2.49g}$$

One term outer expansion:

$$\bar{q}_o(y)\frac{x}{\varepsilon} = \bar{q}_o(y)X \tag{2.49h}$$

Equating (2.49d) and (2.49h) according to matching principle, we obtain

$$\bar{q}_o(y) = 0$$

and $E_3 = \Gamma_2$

which implies that

$$E_3 = \frac{P_n \sin\frac{m\pi}{b}y}{\left(\beta_1^2 - \sigma_1^2\right)\left(s^2 + \sigma^2 u^2\right)} \tag{2.50}$$

Using the same matching procedure for the boundary condition x = 1, we proceed as follows:

One term outer expansion:

$$V_n(x,y) = 2\left[C_2 \cos\alpha y + D_2 \sin\alpha y + \Gamma_1 \sin\alpha\left(y - y_o\right)\right]\sin n\pi x$$
$$+\frac{2}{b}\left[E_3 \cos\sigma x + F_3 \sin\sigma x - \Gamma_2 e^{-\frac{s}{u}x}\right]\sin\frac{m\pi}{b}y \tag{2.51a}$$

Rewritten in inner variable:

For inner variable $X = \frac{(1-x)}{\varepsilon}$ which implies that $x = 1 - \varepsilon X$ is given by

$$2\left[C_2 \cos\alpha y + D_2 \sin\alpha y + \Gamma_1 \sin\alpha\left(y - y_o\right)\right]\sin n\pi(1 - \varepsilon X)$$
$$+\frac{2}{b}E_3 \sin\frac{m\pi}{b}y\cos\sigma(1-\varepsilon X) + \frac{2}{b}F_3 \sin\frac{m\pi}{b}y\sin\sigma(1-\varepsilon X) - \frac{2}{b}\Gamma_2 \sin\frac{m\pi}{b}ye^{-\frac{s}{u}(1-\varepsilon X)} \tag{2.51b}$$

Expanded for small ε:

$$= -2C_2\cos\alpha y(-1)^n\left[n\pi\varepsilon X - \frac{n^3\pi^3\varepsilon^3X^3}{3!} + \frac{n^5\pi^5\varepsilon^3X^5}{5!} - \dots\dots\dots\dots\right]$$
$$-2D_2\sin\alpha y(-1)^n\left[n\pi\varepsilon X - \frac{n^3\pi^3\varepsilon^3X^3}{3!} + \frac{n^5\pi^5\varepsilon^3X^5}{5!} - \dots\dots\dots\dots\right]$$
$$+\frac{2}{b}E_3\sin\frac{m\pi}{b}y\cos\sigma\left[1 - \frac{\sigma^2\varepsilon^2X^3}{2!} + \frac{\sigma^4\varepsilon^4X^4}{4!} - \dots\dots\dots\right]$$

$$+\frac{2}{b}E_3\sin\frac{m\pi}{b}y\sin\sigma\left[\sigma\varepsilon X-\frac{\sigma^3\varepsilon^3X^3}{3!}+\frac{\sigma^5\varepsilon^5X^5}{5!}-\ldots\ldots\ldots\right]$$

$$+\frac{2}{b}F_3\sin\frac{m\pi}{b}y\sin\sigma\left[1-\frac{\sigma^2\varepsilon^2X^2}{2!}+\frac{\sigma^4\varepsilon^4X^4}{4!}-\ldots\ldots\ldots\right]$$

$$-\frac{2}{b}F_3\sin\frac{m\pi}{b}y\cos\sigma\left[\sigma\varepsilon X-\frac{\sigma^3\varepsilon^3X^3}{3!}+\frac{\sigma^5\varepsilon^5X^5}{5!}-\ldots\ldots\ldots\right]$$

$$-2\Gamma_2(\sin\alpha y\cos\alpha y_0-\cos\alpha y\sin\alpha y_0)(-1)^n\left[n\pi\varepsilon X-\frac{n^3\pi^3\varepsilon^3X^3}{3!}+\frac{n^5\pi^5\varepsilon^3X^5}{5!}-\ldots\ldots\ldots\ldots\right]$$

$$-\frac{2}{b}\Gamma_2\sin\frac{m\pi}{b}ye^{-\frac{1}{u}}\left[1-\frac{\sigma\varepsilon X}{u}+\frac{\sigma^2\varepsilon^2X^2}{2!u^2}-\ldots\ldots\ldots\right] \qquad (2.51c)$$

One term inner expansion:

$$\frac{2}{b}E_3\sin\frac{m\pi}{b}y\cos\sigma+\frac{2}{b}F_3\sin\frac{m\pi}{b}y\sin\sigma-\frac{2}{b}\Gamma_2\sin\frac{m\pi}{b}ye^{-\frac{1}{u}} \qquad (2.51d)$$

One term inner expansion:

$$\psi_o^i=\bar{\bar{q}}_o(y)\left\{X+\frac{1}{\sqrt{\beta_1^2-\sigma_1^2}}e^{-\sqrt{\beta_1^2-\sigma_1^2}X}-\frac{1}{\sqrt{\beta_1^2-\sigma_1^2}}\right\} \qquad (2.51e)$$

Rewritten in outer variable:

$$=\bar{\bar{q}}_o(y)\left\{\frac{1-x}{\varepsilon}+\frac{1}{\sqrt{\beta_1^2-\sigma_1^2}}e^{-\sqrt{\beta_1^2-\sigma_1^2}\frac{(1-x)}{\varepsilon}}-\frac{1}{\sqrt{\beta_1^2-\sigma_1^2}}\right\} \qquad (2.51f)$$

Expanded for small ε:

As $\varepsilon \to 0$, $e^{-\sqrt{\beta_1^2-\sigma_1^2}\frac{(1-x)}{\varepsilon}}$ becomes $e^{-\infty}=0$

$$\bar{\bar{q}}_o(y)\frac{1-x}{\varepsilon}+\frac{\bar{\bar{q}}_o(y)}{\sqrt{\beta_1^2-\sigma_1^2}}e^{-\sqrt{\beta_1^2-\sigma_1^2}\frac{(1-x)}{\varepsilon}}-\frac{\bar{\bar{q}}_o(y)}{\sqrt{\beta_1^2-\sigma_1^2}} \tag{2.51g}$$

One term outer expansion:

$$\bar{\bar{q}}_o(y)\frac{(1-x)}{\varepsilon}=\bar{\bar{q}}_o(y)X \tag{2.51h}$$

Equating (2.51d) and (2.51h) according to matching principle, we obtain

$$\frac{2}{b}E_3\sin\frac{m\pi}{b}y\cos\sigma+\frac{2}{b}F_3\sin\frac{m\pi}{b}y\sin\sigma-\frac{2}{b}\Gamma_2\sin\frac{m\pi}{b}ye^{-\frac{t}{s}} = \bar{\bar{q}}_o(y)\frac{1-x}{\varepsilon}$$

But $\bar{\bar{q}}_o(y)=0$

which implies that

$$E_3\cos\sigma+F_3\sin\sigma-\Gamma_2e^{-\frac{t}{s}}=0$$

and

$$F_3=\frac{P_o u\sin\frac{m\pi}{b}y_o}{\left(\beta_1^2-\sigma_1^2\right)\left(s^2+\sigma^2u^2\right)}\left[\frac{e^{-\frac{t}{s}}}{\sin\sigma}-\cot\sigma\right] \tag{2.52}$$

Also using (1-1 matching) for the boundary condition y=0 and we proceed as follows:

One term outer expansion:

$$V_o(x,y)=2C_2\sin n\pi x\cos\alpha y+2D_2\sin n\pi x\sin\alpha y$$

$$+2\Gamma_1\sin n\pi x\cos\alpha y_0\sin\alpha y-2\Gamma_1\sin n\pi x\sin\alpha y\cos\alpha y$$

$$+\frac{2}{b}E_3\cos\alpha x\sin\frac{m\pi}{b}y+\frac{2}{b}F_3\sin\alpha x\sin\frac{m\pi}{b}y-\frac{2}{b}\Gamma_2 e^{-\frac{\lambda}{u}x}\sin\frac{m\pi}{b}y \tag{2.53a}$$

Rewritten in inner variable:

For inner variable $Y=\frac{y}{\varepsilon}$ which implies that $y=\varepsilon Y$ is given by

$$=\left[2C_2\sin n\pi x-2\Gamma_1\sin n\pi x\sin\alpha y_0\right]\cos\alpha\varepsilon Y$$

$$+\left[2D_2\sin n\pi x+2\Gamma_1\sin n\pi x\cos\alpha y_0\right]\sin\alpha\varepsilon Y$$

$$+\left[\frac{2}{b}E_3\cos\alpha x+\frac{2}{b}F_3\sin\alpha x-\frac{2}{b}\Gamma_2 e^{-\frac{\lambda}{u}\alpha x}\right]\sin\frac{m\pi}{b}\varepsilon Y \tag{2.53b}$$

Expanded for small ε:

$$=\left[2C_2\sin n\pi x-2\Gamma_1\sin n\pi x\sin\alpha y_0\right]\left[1-\frac{\alpha^2\varepsilon^2Y^2}{2!}+\frac{\alpha^4\varepsilon^4Y^4}{4!}-\dots\dots\dots\right]$$

$$+\left[2D_2\sin n\pi x-2\Gamma_1\sin n\pi x\cos\alpha y_0\right]\left[\alpha\varepsilon Y-\frac{\alpha^3\varepsilon^3Y^3}{3!}+\frac{\alpha^5\varepsilon^5Y^5}{5!}-\dots\dots\dots\right]$$

$$+\frac{2}{b}E_3\sin\frac{m\pi}{b}y\left[\frac{m\pi\varepsilon Y}{b}-\frac{m^3\pi^3\varepsilon^3Y^3}{3!b^3}+\frac{m^5\pi^5\varepsilon^5Y^5}{5!b^5}-\dots\dots\dots\right]$$

$$+\frac{2}{b}F_1\sin\frac{m\pi}{b}y\left[\frac{m\pi\varepsilon Y}{b}-\frac{m^3\pi^3\varepsilon^3Y^3}{3!b^3}+\frac{m^5\pi^5\varepsilon^5Y^5}{5!b^5}-\dots\dots\dots\right]$$

$$+\frac{2}{b}\Gamma_2 e^{-\frac{\lambda}{u}x}\left[\frac{m\pi\varepsilon Y}{b}-\frac{m^3\pi^3\varepsilon^3Y^3}{3!b^3}+\frac{m^5\pi^5\varepsilon^5Y^5}{5!b^5}-\dots\dots\dots\right] \tag{2.53c}$$

One term inner expansion:

$$= 2C_2 \sin n\pi x - 2\Gamma_1 \sin n\pi x \sin \alpha y_0 \qquad (2.53d)$$

One term inner expansion:

$$\psi_o^i = \bar{r}_o(x)\left\{Y + \frac{1}{\sqrt{\beta_2^2 - \sigma_1^2}} e^{-\sqrt{\beta_2^2 - \sigma_1^2}\,Y} - \frac{1}{\sqrt{\beta_2^2 - \sigma_1^2}}\right\} \qquad (2.53e)$$

Rewritten in outer variable:

$$= \bar{r}_o(x)\left\{\frac{y}{\varepsilon} + \frac{1}{\sqrt{\beta_2^2 - \sigma_1^2}} e^{-\sqrt{\beta_2^2 - \sigma_1^2}\,\frac{y}{\varepsilon}} - \frac{1}{\sqrt{\beta_2^2 - \sigma_1^2}}\right\} \qquad (2.53f)$$

Expanded for small ε:

As $\varepsilon \to 0$, $e^{-\sqrt{\beta_2^2 - \sigma_1^2}\,\frac{x}{\varepsilon}}$ becomes $e^{-\infty} = 0$

$$\bar{r}_o(x)\frac{y}{\varepsilon} + \frac{\bar{r}_o(x)}{\sqrt{\beta_2^2 - \sigma_1^2}} e^{-\sqrt{\beta_2^2 - \sigma_1^2}\,\frac{x}{\varepsilon}} - \frac{\bar{r}_o(x)}{\sqrt{\beta_2^2 - \sigma_1^2}} \qquad (2.53g)$$

One term outer expansion:

$$\bar{r}_o(x)\frac{y}{\varepsilon} = \bar{r}_o(x)Y \qquad (2.53h)$$

Equating (2.53d) and (2.53h) according to matching principle, we obtain

$$2C_2 \sin n\pi x - 2\Gamma_1 \sin n\pi x \sin \alpha y_0 = \bar{r}_o(x)\frac{y}{\varepsilon} \qquad (2.53d)$$

which implies that $\bar{r}_o(x) = 0$

so that

$$C_3 = \frac{P_o n\pi u\left[(-1)^n e^{-\frac{s}{u}} - 1\right]}{\alpha\left(\beta_2^2 - \sigma_1^2\right)\left(s^2 + n^2\pi^2u^2\right)}\sin\alpha y \tag{2.54}$$

In a similar manner, the (1-1 matching) for the boundary condition y = b can be represented as follows:

One term outer expansion:

$$V_o(x,y) = 2C_2 \sin n\pi x \cos\alpha y + 2D_2 \sin n\pi x \sin\alpha y$$

$$+ 2\Gamma_1 \sin n\pi x \cos\alpha y_0 \sin\alpha y - 2\Gamma_1 \sin n\pi x \sin\alpha y \cos\alpha y$$

$$+ \frac{2}{b}E_3 \cos\sigma x \sin\frac{m\pi}{b}y + \frac{2}{b}F_3 \sin\sigma x \sin\frac{m\pi}{b}y - \frac{2}{b}\Gamma_2 e^{-\frac{s}{u}x} \sin\frac{m\pi}{b}y \tag{2.55a}$$

Rewritten in inner variable:

For inner variable $Y = \dfrac{b-y}{\varepsilon}$ which implies that $y = b - \varepsilon Y$ is given by

$$= \left[2C_2 \sin n\pi x - 2\Gamma_1 \sin n\pi x \sin\alpha y_0\right]\cos\alpha(b - \varepsilon Y)$$

$$+ \left[2D_2 \sin n\pi x + 2\Gamma_1 \sin n\pi x \cos\alpha y_0\right]\sin\alpha(b - \varepsilon Y)$$

$$+ \left[\frac{2}{b}E_3 \cos\sigma x + \frac{2}{b}F_3 \sin\sigma x - \frac{2}{b}\Gamma_2 e^{-\frac{s}{u}\varepsilon Y}\right]\sin\frac{m\pi}{b}(b - \varepsilon Y) \tag{2.55b}$$

Expanded for small ε:

$$= \left[2C_2 \sin n\pi x - 2\Gamma_1 \sin n\pi x \sin\alpha y_0\right]\cos\alpha b\left[1 - \frac{\sigma^2\varepsilon^2X^2}{2!} + \frac{\sigma^4\varepsilon^4X^4}{4!} - \dots\dots\dots\dots\right]$$

$$+\left[2C_2\sin n\pi x-2\Gamma_1\sin n\pi x\sin\alpha y_0\right]\sin\alpha b\left[\alpha\varepsilon Y-\frac{\alpha^3\varepsilon^3Y^3}{3!}+\frac{\alpha^5\varepsilon^5Y^5}{5!}-\ldots\ldots\ldots\right]$$

$$+\left[2D_2\sin n\pi x+2\Gamma_1\sin n\pi x\cos\alpha y_0\right]\sin\alpha b\left[1-\frac{\sigma^2\varepsilon^2X^2}{2!}+\frac{\sigma^4\varepsilon^4X^4}{4!}-\ldots\ldots\ldots\right]$$

$$-\left[2D_2\sin n\pi x+2\Gamma_1\sin n\pi x\cos\alpha y_0\right]\cos\alpha b\left[\alpha\varepsilon Y-\frac{\alpha^3\varepsilon^3Y^3}{3!}+\frac{\alpha^5\varepsilon^5Y^5}{5!}-\ldots\ldots\ldots\right]$$

$$-\frac{2}{b}E_3\cos\sigma x(-1)^n\left[\frac{m\pi\varepsilon Y}{b}-\frac{m^3\pi^3\varepsilon^3Y^3}{3!b^3}+\frac{m^5\pi^5\varepsilon^5Y^5}{5!b^5}-\ldots\ldots\ldots\right]$$

$$-\frac{2}{b}F_3\sin\sigma x(-1)^n\left[\frac{m\pi\varepsilon Y}{b}-\frac{m^3\pi^3\varepsilon^3Y^3}{3!b^3}+\frac{m^5\pi^5\varepsilon^5Y^5}{5!b^5}-\ldots\ldots\ldots\right]$$

$$+\frac{2}{b}\Gamma_2e^{-\frac{t}{\kappa}x}(-1)^n\left[\frac{m\pi\varepsilon Y}{b}-\frac{m^3\pi^3\varepsilon^3Y^3}{3!b^3}+\frac{m^5\pi^5\varepsilon^5Y^5}{5!b^5}-\ldots\ldots\ldots\right] \tag{2.55c}$$

One term inner expansion:

$$=\left[2C_2\sin n\pi x-2\Gamma_1\sin n\pi x\sin\alpha y_0\right]\cos\alpha b$$

$$+\left[2D_2\sin n\pi x+2\Gamma_1\sin n\pi x\cos\alpha y_0\right]\sin\alpha b \tag{2.55d}$$

One term inner expansion:

$$\psi_n^i=\bar{\bar{r}}_n(x)\left\{Y+\frac{1}{\sqrt{\beta_2^2-\sigma_1^2}}e^{-\sqrt{\beta_2^2-\sigma_1^2}\,Y}-\frac{1}{\sqrt{\beta_2^2-\sigma_1^2}}\right\} \tag{2.55e}$$

Rewritten in outer variable:

$$= \bar{\bar{r}}_o(x)\left\{\frac{b-y}{\varepsilon} + \frac{1}{\sqrt{\beta_2^2 - \sigma_1^2}} e^{-\sqrt{\beta_2^2-\sigma_1^2}\frac{(b-y)}{\varepsilon}} - \frac{1}{\sqrt{\beta_2^2 - \sigma_1^2}}\right\} \tag{2.55f}$$

Expanded for small ε:

As $\varepsilon \to 0$, $e^{-\sqrt{\beta_2^2-\sigma_1^2}\frac{(b-x)}{\varepsilon}}$ becomes $e^{-\infty} = 0$

$$\bar{\bar{r}}_o(x)\frac{b-y}{\varepsilon} + \frac{\bar{\bar{r}}_o(x)}{\sqrt{\beta_2^2 - \sigma_1^2}} e^{-\sqrt{\beta_2^2-\sigma_1^2}\frac{(b-x)}{\varepsilon}} - \frac{\bar{\bar{r}}_o(x)}{\sqrt{\beta_2^2 - \sigma_1^2}} \tag{2.55g}$$

One term outer expansion:

$$\bar{\bar{r}}_o(x)\frac{b-y}{\varepsilon} = \bar{\bar{r}}_o(x)Y \tag{2.55h}$$

Equating (2.55d) and (2.55h) according to matching principle, we obtain

$$2C_2 \sin n\pi x \cos \alpha b + 2D_2 \sin n\pi x \sin \alpha b - 2\Gamma_1 \sin n\pi x \sin \alpha y_o \cos \alpha b$$

$$+ 2\Gamma_1 \sin n\pi x co \alpha y_0 \sin \alpha b = \bar{\bar{r}}_0(x)\frac{b-y}{\varepsilon}$$

which implies that $\bar{\bar{r}}_o(x) = 0$

so that

$$D_2 = \frac{P_o n\pi u\left[1-(-1)^n e^{-\frac{s}{u}} - 1\right]}{\alpha\left(\beta_2^2 - \sigma_1^2\right)\left(s^2 + n^2\pi^2 u^2\right)} \cos \alpha y_o \tag{2.56}$$

Also

$$\bar{q}_0(y) = \bar{\bar{q}}_0(y) = \bar{r}_0(x) = \bar{\bar{r}}_0(x) = 0 \tag{2.57}$$

Substituting (2.50), (2.52), (2.54) and (2.56) into (2.44) yields

$$V_0(x,y)=\frac{2P_0 u\sin\frac{m\pi}{b}y_0}{b(\beta_1^2-\sigma_1^2)(s^2+\sigma^2u^2)}\left[\cos\sigma x-e^{-\frac{s}{u}x}+e^{-\frac{s}{u}}\frac{\sin\sigma x}{\sin\sigma}-\frac{\cos\sigma\sin\sigma x}{\sin\sigma}\right]\sin\frac{m\pi}{b}y \tag{2.58}$$

and

$$\psi_o^i=0 \tag{2.59}$$

2.5 LAPLACE INVERSION OF LEADING ORDER SOLUTION

The Laplace inversion of equation (2.58) cannot be easily obtained due to the complexity of α and σ. Hence, equation (2.58) can be rewritten in a simplified form as

$$V_o(x,y)=\frac{2P_o u\sin\dfrac{m\pi}{b}y_o\sin\dfrac{m\pi}{b}y}{b\left(\beta_1^2-\sigma_1^2\right)}$$

$$\left[\frac{\cosh\sigma^r x}{s^2+\sigma^2u^2}-\frac{e^{-\frac{s}{u}x}}{s^2+\sigma^2u^2}+\frac{e^{-\frac{s}{u}x}\sinh\sigma^r x}{\left(s^2+\sigma^2u^2\right)\sinh\sigma^r}-\frac{\cosh\sigma^r\sinh\sigma^r x}{\left(s^2+\sigma^2u^2\right)\sinh\sigma^r}\right] \tag{2.60}$$

where

$$\sigma^2=-\left[\frac{\left(\beta_2^2-\sigma_1^2\right)m^2\pi^2+\eta b^2+s^2b^2}{b^2\left(\beta_1^2-\sigma_1^2\right)}\right] \tag{2.61}$$

and

$$\sigma^c = \frac{1}{\left(\beta_2^2 - \sigma_1^2\right)^{\frac{1}{2}}}\left[\frac{\left(\beta_2^2 - \sigma_1^2\right)m^2\pi^2 + \eta b^2}{b^2} + s^2\right]^{\frac{1}{2}} \tag{2.62}$$

the Laplace inversion of (2.60) is defined as

$$V_s(x,y) = \frac{2P_o u \sin\frac{m\pi}{b} y_o \sin\frac{m\pi}{b} y}{b\left(\beta_1^2 - \sigma_1^2\right)}\left\{F_1(x,y;t) - F_2(x,y;t) + F_3(x,y;t) - F_4(x,y;t)\right\} \tag{2.63}$$

where

$$F_1(x,y;t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{e^{st}\cosh\sigma^c x}{s^2 + \sigma^2 u^2}ds \tag{2.64a}$$

$$F_2(x,y;t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{e^{s\left(t-\frac{x}{u}\right)}\cosh\sigma^c x}{s^2 + \sigma^2 u^2}ds \tag{2.64b}$$

$$F_3(x,y;t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{e^{s\left(t-\frac{1}{u}\right)}\sinh\sigma^c x}{\left(s^2 + \sigma^2 u^2\right)\sinh\sigma^c}ds \tag{2.64c}$$

$$F_4(x,y;t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{e^{st}\cosh\sigma^c \sinh\sigma^c x}{\left(s^2 + \sigma^2 u^2\right)\sinh\sigma^c}ds \tag{2.64d}$$

In order to evaluate equation (2.64a) to (2.64d), we used Cauchy residue theorem defined as

$$\sum_{k=1}^{n}\underset{s=s_k}{\mathrm{Re}}\left[e^{st}f(s)\right] = \frac{1}{2\pi}\int_{a-\infty}^{a+\infty}e^{st}f(s)ds \tag{2.65a}$$

where the residue Res f(s) of a pole of order n at $s = s_k$ is defined as

$$\mathrm{Re}\left[e^{st}f(s)\right] = \begin{cases}\lim\limits_{s\to s_k}\left[(s-s_k)e^{st}f(s)\right] \text{for simple pole} \\ \frac{1}{(n-1)!}\frac{d^{n-1}}{ds^{n-1}}\left[(s-s_k)^n e^{st}f(s)\right]_{s=s_k} \text{ for multiple pole}\end{cases} \tag{2.65b}$$

To this end, we rewrite the integrand (2.64a) in form

$$\phi_n = \frac{e^{st}\cosh\sigma^c x}{A_1\left[s^2 - u^2\frac{\left[\left(\beta_2^2-\sigma_1^2\right)m^2\pi^2+\eta b^2\right]}{b^2\left(\beta_1^2-\sigma_1^2-u^2\right)}\right]} \tag{2.66}$$

It is clear to show that the singularities in the above expressions (2.64) are poles. In particular the denominators of the integrands of $F_1(x,y;t)$ and $F_2(x,y;t)$ have simple poles at

$$s = \pm u\sqrt{\frac{\left(\beta_2^2-\sigma_1^2\right)m^2\pi^2+\eta b^2}{b^2\left(\beta_1^2-\sigma_1^2-u^2\right)}} \tag{2.67}$$

Thus, incorporating (2.66) into (2.64a), we obtain

$$F_1(x,y;t) = \frac{1}{2\pi i A_1}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{st}\cosh\Omega_3 x}{s^2-\Omega_1^2}ds \tag{2.68}$$

$$A_1 = 1\text{-}D \tag{2.69}$$

$$D = \frac{u^2}{\beta_1^2-\sigma_1^2} \tag{2.70}$$

The contribution towards $F_1(x,y;t)$ due to simple poles at $s = +\Omega_1$

$$F_{11}(x,y;t) = \frac{1}{A}\lim_{s\to\Omega_1}\left[\left(s-\Omega_1\right)e^{st}\frac{\cosh\Omega_3 x}{s^2-\Omega_1^2}\right]$$

which gives

$$F_{11}(x,y;t) = \frac{1}{2A_1\Omega_1}e^{\Omega_1 t}\cosh\Omega_3 x \tag{2.71}$$

where

$$\Omega_1 = u\sqrt{\frac{\left(\beta_2^2-\sigma_1^2\right)m^2\pi^2+\eta b^2}{b^2\left(\beta_1^2-\sigma_1^2-u^2\right)}} \tag{2.72}$$

$$\Omega_2 = \frac{\left(\beta_2^2 - \sigma_1^2\right)m^2\pi^2 + \eta b^2}{b^2} \tag{2.73}$$

$$\Omega_3 = \frac{1}{\left(\beta_1^2 - \sigma_1^2\right)^{\frac{1}{2}}}\left(\Omega_1^2 + \Omega_2\right)^{\frac{1}{2}} \tag{2.74}$$

In a similar manner, the contribution towards $F_1(x,y;t)$ due to simple pole at $s = -\Omega_1$ is

$$F_{12}(x,y;t) = -\frac{1}{2A_1\Omega_1}e^{-\Omega_1 t}\cosh\Omega_3 x \tag{2.75}$$

Hence

$$F_1(x,y;t) = F_{11} + F_{12}$$

so that

$$F_1(x,y;t) = -\frac{\cosh\Omega_3 x}{2A_1\Omega_1}\left(e^{\Omega_1 t} - e^{-\Omega_1 t}\right) \tag{2.76}$$

In order to evaluate $F_2(x,y;t)$, its integrand is rewritten in the form

$$F_2(x,y;t) = \frac{1}{2\pi A_1}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{e^{s\left(t-\frac{x}{u}\right)}}{s^2 - \Omega_1^2}\cosh\sigma^r x\,ds \tag{2.77}$$

Also, it is straight forward to show that the Laplace inversion of F$_2$(x, y; t) is

$$F_2(x,y;t) = -\frac{1}{2A_1\Omega_1}\left[e^{\Omega_1\left(t-\frac{x}{u}\right)} - e^{-\Omega_1\left(t-\frac{x}{u}\right)}\right] \tag{2.78}$$

Furthermore, to evaluate F$_3$(x, y; t), its integrand is rewritten to take the form

$$F_3(x,y;t) = \frac{1}{2\pi i A_1}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{e^{s\left(t-\frac{x}{u}\right)}\sinh\sigma^r x}{\left(s^2 - \Omega_1^2\right)\sinh\sigma^r}ds \tag{2.79}$$

where poles from the first denominator are

$s = \Omega_1$ *and* $s = -\Omega_1$

while those emanating from $\sinh \sigma^c$ are obtained by setting it to zero, i.e.

$$\sinh \sigma^c = 0 \tag{2.80}$$

which implies

$$\sigma^c = \pm i\nu\pi \tag{2.81}$$

There are four distinct poles are

$$s = \pm\Omega_1 \tag{2.82}$$

$$\begin{matrix} \sigma^c = +i\nu\pi \\ \sigma^c = -i\nu\pi \end{matrix} \quad \begin{cases} \sigma^{c^2} = -\nu^2\pi^2 \\ \sigma^{c^2} = -\nu^2\pi^2 \end{cases} \tag{2.83}$$

Explicitly, equation (2.83) with (2.62) implies

$$s^2 = -\nu^2\pi^2\left(\beta_1^2 - \sigma_1^2\right) - \frac{\left(\beta_2^2 - \sigma_1^2\right)m^2\pi^2 + \eta b^2}{b^2}$$

so that

$$s = \pm i\left[\nu^2\pi^2\left(\beta_1^2 - \sigma_1^2\right) + \frac{\left(\beta_2^2 - \sigma_1^2\right)m^2\pi^2 + \eta b^2}{b^2}\right]^{\frac{1}{2}} \tag{2.84}$$

Therefore the four poles are

$$s = \Omega_1 \tag{2.85a}$$

$$s = -\Omega_1 \tag{2.85b}$$

$$s = i\sqrt{\Omega_4} \tag{2.85c}$$

$$s = -i\sqrt{\Omega_4} \tag{2.85d}$$

where

$$\Omega_4 = v^2\pi^2\left(\beta_1^2 - \sigma_1^2\right) + \frac{\left(\beta_2^2 - \sigma_1^2\right)m^2\pi^2 + \eta b^2}{b^2} \tag{2.86}$$

Thus, the contribution towards F$_3$(x, y; t) due to simple poles at $s = \Omega_1$ is given by

$$F_{31} = \frac{1}{A_1}\lim_{s \to \Omega_1}\left\{(s - \Omega_1)\frac{e^{s\left(t - \frac{1}{u}\right)}\sinh\Omega_3 x}{\left(s^2 - \Omega_1^2\right)\sinh\Omega_3}\right\} \tag{2.87}$$

so that

$$F_{31} = \frac{1}{2A_1\Omega_1}\frac{\sinh\Omega_3 x}{\sinh\Omega_3}e^{\Omega_1\left(t - \frac{1}{u}\right)} \tag{2.88}$$

Similarly, contribution due to simple pole at $s = -\Omega_1$ is

$$F_{32} = \frac{1}{A_1}\lim_{s \to -\Omega_1}\left\{(s + \Omega_1)\frac{e^{-s\left(t - \frac{1}{u}\right)}\sinh\Omega_3 x}{\left(s^2 + \Omega_1^2\right)\sinh\Omega_3}\right\} \tag{2.89}$$

which implies that

$$F_{32} = -\frac{1}{2A_1\Omega_1}\frac{\sinh\Omega_3 x}{\sinh\Omega_3}e^{-\Omega_1\left(t - \frac{1}{u}\right)} \tag{2.90}$$

In a similar manner, the contribution due to $s = i\sqrt{\Omega_4}$ is given by

$$F_{33}(x, y; t) = \frac{1}{A_1}\lim_{s \to i\sqrt{\Omega_4}}\left\{\left(s - i\sqrt{\Omega_4}\right)\frac{e^{s_1\left(t - \frac{1}{u}\right)}\sinh\sigma^e x}{\left(s^2 - \Omega_1^2\right)\sinh\sigma^e}\right\} \tag{2.91}$$

$$= \frac{1}{A_1}\lim_{s \to i\sqrt{\Omega_4}}\left[\frac{s - i\sqrt{\Omega_4}}{\sinh\sigma^e}\right]\lim_{s \to i\sqrt{\Omega_4}}\left\{\frac{e^{s\left(t - \frac{1}{u}\right)}\sinh\sigma^e x}{s^2 - \Omega_1^2}\right\}$$

$= F_{33a} \; . \; F_{33b}$

where

$$F_{33a} = \lim_{s \to i\sqrt{\Omega_4}} \left[\frac{s - i\sqrt{\Omega_4}}{\sinh \sigma^c} \right] \tag{2.92a}$$

and

$$F_{33b} = \lim_{s \to i\sqrt{\Omega_4}} \left[\frac{e^{s\left(t - \frac{1}{u}\right)} \sinh \sigma^c x}{s^2 - \Omega_1^2} \right] \tag{2.92b}$$

In order not to obtain an indeterminate term in F$_{33a}$, we adopt L' Hospital rule to obtain

$$F_{33a} = \frac{f(s)}{g(s)} = \frac{f'(s)}{g'(s)} \tag{2.93}$$

with

$$\frac{f(s)}{g(s)} = \frac{1}{A_1} \lim_{s \to i\sqrt{\Omega_4}} \left[\frac{s - i\sqrt{\Omega_4}}{\sinh \sigma^c} \right]$$

and

$$\frac{f'(s)}{g'(s)} = \frac{1}{\frac{1}{2(\beta_1^2 - \sigma_1^2)^{\frac{1}{2}}} (s^2 + \Omega_2)^{\frac{1}{2}} \; X \; 2s \cosh \frac{1}{(\beta_1^2 - \sigma_1^2)^{\frac{1}{2}}} (s^2 + \Omega_2)^{\frac{1}{2}}}$$

$$= \frac{1}{A_1} \lim_{s \to i\sqrt{\Omega_4}} \frac{(\beta_1^2 - \sigma_1^2)^{\frac{1}{2}} (s^2 + \Omega_2)^{\frac{1}{2}}}{s \cosh \frac{1}{(\beta_1^2 - \sigma_1^2)^{\frac{1}{2}}} (s^2 + \Omega_2)^{\frac{1}{2}}} \tag{2.94}$$

Therefore

$$F_{33a}(x,y;t)=\frac{\pm(-1)\left(\beta_1^2-\sigma_1^2\right)\nu\pi}{A_1\sqrt{\Omega_4}} \tag{2.95}$$

Similarly, evaluating the second part $F_{33b}(x,y;t)$ of the limiting process gives

$$F_{33b}(x,y;t)=\lim_{s\to i\sqrt{\Omega_4}}\left[\frac{e^{s\left(t-\frac{1}{u}\right)}\sinh\frac{1}{(\beta_1^2-\sigma_1^2)^{\frac{1}{2}}(s^2+\Omega_2)^{\frac{1}{2}}}x}{s^2+\Omega_1^2}\right] \tag{2.96}$$

which gives

$$F_{33b}(x,y;t)=\frac{-e^{i\sqrt{\Omega_4}\left(t-\frac{1}{u}\right)}}{\Omega_4+\Omega_1^2}\sin\nu\pi x \tag{2.97}$$

Therefore $F_{33}(x,y;t)$ gives

$$F_{33}(x,y;t)=\frac{(-1)^{\nu+1}\left(\beta_1^2-\sigma_1^2\right)\nu\pi}{A_1\sqrt{\Omega_4}\left(\Omega_4+\Omega_1^2\right)}e^{i\sqrt{\Omega_4}\left(t-\frac{1}{u}\right)}\sin\nu\pi x \tag{2.98}$$

Similarly, the contribution due to simple pole at $s=-i\sqrt{\Omega_4}$ can be evaluated as

$$F_{34}(x,y;t)=\frac{1}{A_1}\lim_{s\to i\sqrt{\Omega_4}}\left\{\left(s+i\sqrt{\Omega_4}\right)\frac{e^{s\left(t-\frac{1}{u}\right)}\sinh\sigma^c x}{\left(s^2-\Omega_1^2\right)\sinh\sigma^c}\right\}$$

$$=\frac{1}{A_1}\lim_{s\to -i\sqrt{\Omega_4}}\frac{\left(s+i\sqrt{\Omega_4}\right)}{\sinh\sigma^c}\lim_{s\to -i\sqrt{\Omega_4}}\left\{\frac{e^{s\left(t-\frac{1}{u}\right)}\sinh\sigma^c x}{s^2-\Omega_1^2}\right\} \tag{2.99}$$

$$=F_{34a}\cdot F_{34b}$$

where

$$F_{34a} = \frac{1}{A_1} \lim_{s \to -i\sqrt{\Omega_4}} \frac{\left(s + i\sqrt{\Omega_4}\right)}{\sinh \sigma^r} \tag{2.100a}$$

and

$$F_{34b} = \lim_{s \to -i\sqrt{\Omega_4}} \left\{ \frac{e^{s\left(t - \frac{1}{n}\right)} \sinh \sigma^r x}{s^2 - \Omega_1^2} \right\} \tag{2.100b}$$

Using L'Hospital rule and follow similar procedure in $F_{33}(x, y; t)$, we have

$$F_{34a}(x, y; t) = \frac{(-1)^{\nu+1}\left(\beta_1^2 - \sigma_1^2\right)\nu\pi}{A_1\sqrt{\Omega_4}} \tag{2.101}$$

and

$$F_{34b}(x, y; t) = \frac{-e^{-i\sqrt{\Omega_4}\left(t - \frac{1}{n}\right)} \sin \nu\pi x}{\Omega_4 + \Omega_1^2} \tag{2.102}$$

which gives

$$F_{34}(x, y; t) = \frac{-(-1)^{\nu+1}\left(\beta_1^2 - \sigma_1^2\right)\nu\pi}{A_1\sqrt{\Omega_4}\left(\Omega_4 + \Omega_1^2\right)} e^{-i\sqrt{\Omega_4}\left(t - \frac{1}{n}\right)} \sin \nu\pi x \tag{2.103}$$

Combining the results in (2.88), (2.90), (2.98) and (2.103) gives

$$F_3(x, y; t) = \frac{\sinh \Omega_4 x}{2A_1 \sinh \Omega_3}\left[e^{\Omega_1\left(t - \frac{1}{n}\right)} - e^{-\Omega_1\left(t - \frac{1}{n}\right)}\right]$$

$$+ \frac{(-1)^{\nu+1}\left(\beta_1^2 - \sigma_1^2\right)\nu\pi \sin \nu\pi x}{A_1\sqrt{\Omega_4}\left(\Omega_4 + \Omega_1^2\right)}\left[e^{i\sqrt{\Omega_4}\left(t - \frac{1}{n}\right)} - e^{-i\sqrt{\Omega_4}\left(t - \frac{1}{n}\right)}\right] \tag{2.104}$$

The contributions toward $F_4(x, y; t)$ are obtained in a similar manner as we have in

$F_3(x, y; t)$

Thus,

$$F_4(x,y;t)=\frac{\sinh\Omega_3 x}{2A_1\Omega_1\sinh\Omega_3}\left(e^{\Omega_1 t}-e^{-\Omega_1 t}\right)+\frac{(-1)^{2v+1}\left(\beta_1^2-\sigma_1^2\right)v\pi\sin v\pi x}{A_1\sqrt{\Omega_4}\left(\Omega_4+\Omega_1^2\right)} \tag{2.105}$$

Substituting (2.76), (2.78), (2.104) and (2.105) into (2.63) yields

$$\begin{aligned}V_a(x,y;t)=&\frac{2P_a u\sin\frac{m\pi}{b}y_a\sin\frac{m\pi}{b}y}{bA_1\left(\beta_1^2-\sigma_1^2\right)}\left\{\frac{\cosh\Omega_3 x}{2\Omega_1}\left(e^{\Omega_1 t}-e^{-\Omega_1 t}\right)\right.\\ &+\frac{(-1)^{v+1}\left(\beta_1^2-\sigma_1^2\right)v\pi\sin v\pi x}{\sqrt{\Omega_4}\left(\Omega_4+\Omega_1^2\right)}\left[e^{i\sqrt{\Omega_4}\left(t-\frac{1}{u}\right)}-e^{-i\sqrt{\Omega_4}\left(t-\frac{1}{u}\right)}\right]-\frac{\sinh\Omega_3 x\cosh\Omega_3}{2\Omega_1\sinh\Omega_3}\left(e^{\Omega_1 t}-e^{-\Omega_1 t}\right)\\ &\left.-\frac{(-1)^{2v+1}\left(\beta_1^2-\sigma_1^2\right)v\pi\sin v\pi x}{\sqrt{\Omega_4}\left(\Omega_4+\Omega_1^2\right)}\left(e^{i\sqrt{\Omega_4}t}-e^{-i\sqrt{\Omega_4}t}\right)\right\}\end{aligned} \tag{2.106}$$

The combination of the results (2.59) and (2.106) yield the desired leading order solution of (2.1) which represents the uniformly valid solution of the entire domain of definition of the given plate.

It is pertinent to show that, for the isotropic prestressed plate resting on a Pasternak foundation, the leading order solution is affected by the foundation parameter.

CHAPTER THREE

3.0 FIRST ORDER CORRECTION (OUTER SOLUTION)

The next corrections in outer solution and eigenvalue are obtained from (2.16) by setting $v = 1$. For the outer solution, the governing equation for V_1^0 is given as

$$\left(\beta_1^2 - \sigma_1^2\right)\frac{\partial^2 V_1(x,y;t)}{\partial^2 x} + \frac{\left(\beta_2^2 - \sigma_1^2\right)\partial^2 V_1(x,y;t)}{\partial y^2} - s^2 V_1(x,y;t) - \eta V_1(x,y;t) = 0 \qquad (3.1)$$

Using the method of finite Fourier sine transform (2.25) – (2.26) on (3.1) with respect to x gives

$$V_{1,yy}(n,y) - \left[\frac{\left(\beta_1^2 - \sigma_1^2\right)n^2\pi^2 + s^2 + \eta}{\beta_2^2 - \sigma_1^2}\right]V_1(n,y) = 0 \qquad (3.2)$$

so that

$$V_{1,yy}(n,y) + \alpha^2 V_1(n,y) = 0 \qquad (3.3)$$

The homogeneous solution of (3.3) gives

$$V_1(n,y) = C_3 \cos\alpha y + D_3 \sin\alpha y \qquad (3.4)$$

where C_3 and D_3 are arbitrary constants.

Similarly, if equation (3.1) is subjected to finite Fourier sine transform with respect to y, one obtains

$$\left(\beta_1^2 - \sigma_1^2\right)V_{1,xx}(m,x) + \frac{m^2\pi^2}{b^2}\left(\beta_2^2 - \sigma_1^2\right)V_1(m,x) - s^2 V_1(m,x) - \eta V_1(m,x) = 0$$

which implies that

$$V_{1,xx}(m,x)-\left[\frac{\left(\beta_2^2-\sigma_1^2\right)m^2\pi^2+(s^2+\eta)b^2}{(\beta_1^2-\sigma_1^2)b^2}\right]V_1(m,x)=0 \tag{3.5}$$

so that

$$V_{1,xx}(m,x)+\sigma^2 V_1(m,x)=0 \tag{3.6}$$

The homogeneous solution of (3.6) gives

$$V_1(m,x)=E_4\cos\sigma x+F_4\sin\sigma x \tag{3.7}$$

Substituting equation (3.4) in (2.26) together with equation (3.7) in (2.36) gives

$$V_1(x,y)=2\left[C_3\cos\alpha y+D_3\sin\alpha y\right]\sin n\pi x+\frac{2}{b}\left[E_4\cos\sigma x+F_4\sin\sigma x\right]\sin\frac{m\pi}{b}y \tag{3.8}$$

Where C_3, D_3, E_4 and F_4 are unknown constants to be determined by matching.

3.1 FIRST ORDER CORRECTION (INNER SOLUTION)

The first order correction is obtained by setting $\nu=1$ in the differential equation (2.18). Doing this and neglecting terms with negative subscripts, we have

$$\frac{\partial^4\psi_1^i}{\partial X^4}-\left(\beta_1^2-\sigma_1^2\right)\frac{\partial^2\psi_1^i}{\partial X^2}=0 \tag{3.9}$$

with the boundary condition

$$\psi_1^i=\frac{\partial\psi_1^i}{\partial X}=0 \tag{3.10}$$

Following usual argument in equation (2.47) and (2.48), the first order correction of the inner problem can be written as:

$$\psi_1^i = \begin{cases} \vec{q}_1(y)\left\{X + \dfrac{1}{\sqrt{\beta_1^2-\sigma_1^2}}e^{-\sqrt{\beta_1^2-\sigma_1^2}\,x} - \dfrac{1}{\sqrt{\beta_1^2-\sigma_1^2}}\right\} near\ x = 0 \\ \vec{\vec{q}}_1(y)\left\{X + \dfrac{1}{\sqrt{\beta_1^2-\sigma_1^2}}e^{-\sqrt{\beta_1^2-\sigma_1^2}\,x} - \dfrac{1}{\sqrt{\beta_1^2-\sigma_1^2}}\right\} near\ x = 1 \\ \vec{r}_1(x)\left\{Y + \dfrac{1}{\sqrt{\beta_2^2-\sigma_1^2}}e^{-\sqrt{\beta_2^2-\sigma_1^2}\,Y} - \dfrac{1}{\sqrt{\beta_2^2-\sigma_1^2}}\right\} near\ y = 0 \\ \vec{\vec{r}}_1(x)\left\{Y + \dfrac{1}{\sqrt{\beta_2^2-\sigma_1^2}}e^{-\sqrt{\beta_2^2-\sigma_1^2}\,Y} - \dfrac{1}{\sqrt{\beta_2^2-\sigma_1^2}}\right\} near\ y = b \end{cases} \qquad (3.11)$$

Where exponentially growing terms have been neglected as unmatchable.

The functions $\vec{q}_1(y)$, $\vec{\vec{q}}_1(y)$, $\vec{r}_1(x)$ and $\vec{\vec{r}}_1(x)$ will be determined by matching.

3.2 MATCHING PROCEDURE

By matching one term outer expansion written in inner variable with two terms inner expansion written in outer variable, we proceed systematically as follows:

Matching (1 – 2 matching) procedure for the boundary condition x = 0.

One term outer expansion:

$$\bar{V}_0(x,y) = 2C_3\cos\alpha y\sin n\pi x + 2D_2\sin\alpha y\sin n\pi x$$

$$+\frac{2}{b}E_3\cos\alpha x\sin\frac{m\pi}{b}y + \frac{2}{b}F_3\sin\alpha x\sin\frac{m\pi}{b}y$$

$$+2\Gamma_1(\sin\alpha y\cos\alpha y_0 - \cos\alpha y\sin\alpha y_0)\sin n\pi x - \frac{2}{b}\Gamma_2 e^{-\frac{\lambda}{\varepsilon}x}\sin\frac{m\pi}{b}y \qquad (3.12a)$$

Rewritten in inner variable:

For inner variable $X = \frac{x}{\varepsilon}$ which implies that $x = \varepsilon X$ is given by

$$2C_2 \cos\alpha y \sin n\pi\varepsilon X + 2D_2 \sin\alpha y \sin n\pi\varepsilon X$$

$$+\frac{2}{b}E_3 \sin\frac{m\pi}{b} y\cos\sigma\varepsilon X + \frac{2}{b}F_3 \sin\frac{m\pi}{b} y\sin\sigma\varepsilon X$$

$$+2\Gamma_1(\sin\alpha y\cos\alpha y_0 - \cos\alpha y \sin\alpha y_0)\sin n\pi\varepsilon X - \frac{2}{b}\Gamma_2 \sin\frac{m\pi}{b} y e^{-\frac{s}{u}\varepsilon X} \qquad (3.12\text{b})$$

Expanded for small ε:

$$+2C_2\cos\alpha y\left[n\pi\varepsilon X - \frac{n^3\pi^3\varepsilon^3X^3}{3!} + \frac{n^5\pi^5\varepsilon^5X^5}{5!} - \ldots\ldots\ldots\ldots\right]$$

$$+2D_2\sin\alpha y\left[n\pi\varepsilon X - \frac{n^3\pi^3\varepsilon^3X^3}{3!} + \frac{n^5\pi^5\varepsilon^5X^5}{5!} - \ldots\ldots\ldots\ldots\right]$$

$$+\frac{2}{b}E_3\sin\frac{m\pi}{b}y\left[1 - \frac{\sigma^2\varepsilon^2X^2}{2!} + \frac{\sigma^4\varepsilon^4X^4}{4!} - \ldots\ldots\ldots\ldots\right]$$

$$+\frac{2}{b}F_3\sin\frac{m\pi}{b}y\left[\sigma\varepsilon X - \frac{\sigma^3\varepsilon^3X^3}{3!} + \frac{\sigma^5\varepsilon^5X^5}{5!} - \ldots\ldots\ldots\ldots\right]$$

$$+2\Gamma_1(\sin\alpha y\cos\alpha y_0 - \cos\alpha y\sin\alpha y_0)\left[n\pi\varepsilon X - \frac{n^3\pi^3\varepsilon^3X^3}{3!} + \frac{n^5\pi^5\varepsilon^5X^5}{5!} - \ldots\ldots\ldots\ldots\right]$$

$$-\frac{2}{b}\Gamma_2\sin\frac{m\pi}{b}y\left[1 - \frac{s\varepsilon X}{u} + \frac{s^2\varepsilon^2X^2}{2!u} - \ldots\ldots\ldots\ldots\right] \qquad (3.12\text{c})$$

Two terms inner expansion:

$$\frac{2}{b}E_1\sin\frac{m\pi}{b}y-\frac{2}{b}\Gamma_2\sin\frac{m\pi}{b}y+\varepsilon X\left\{2C_2n\pi\cos\alpha y+2D_2n\pi\sin\alpha y+\frac{2}{b}F_3\sigma\sin\frac{m\pi}{b}y\right.$$

$$\left.+2\Gamma_1n\pi(\sin\alpha y\cos\alpha y_0-\cos\alpha y\sin\alpha y_0)+\frac{2}{b}\Gamma_2\frac{s}{u}\sin\frac{m\pi}{b}y\right\} \tag{3.12d}$$

Two terms inner expansion :

$$\psi^i=\psi_0^i+\varepsilon\psi_1^i$$

$$=\bar{q}_o(y)\left\{X+\frac{1}{\sqrt{\beta_1^2-\sigma_1^2}}e^{-\sqrt{\beta_1^2-\sigma_1^2}X}-\frac{1}{\sqrt{\beta_1^2-\sigma_1^2}}\right\}$$

$$+\varepsilon\bar{q}_1(y)\left\{X+\frac{1}{\sqrt{\beta_1^2-\sigma_1^2}}e^{-\sqrt{\beta_1^2-\sigma_1^2}X}-\frac{1}{\sqrt{\beta_1^2-\sigma_1^2}}\right\} \tag{3.12e}$$

Rewritten in outer variable:

$$\bar{q}_o(y)\left\{\frac{x}{\varepsilon}+\frac{1}{\sqrt{\beta_1^2-\sigma_1^2}}e^{-\sqrt{\beta_1^2-\sigma_1^2}\frac{x}{\varepsilon}}-\frac{1}{\sqrt{\beta_1^2-\sigma_1^2}}\right\}$$

$$+\varepsilon\bar{q}_1(y)\left\{\frac{x}{\varepsilon}+\frac{1}{\sqrt{\beta_1^2-\sigma_1^2}}e^{-\sqrt{\beta_1^2-\sigma_1^2}\frac{x}{\varepsilon}}-\frac{1}{\sqrt{\beta_1^2-\sigma_1^2}}\right\} \tag{3.12f}$$

Expanded for small ε:

As $\varepsilon\to0$, $e^{-\sqrt{\beta_1^2-\sigma_1^2}\frac{x}{\varepsilon}}$ becomes $e^{-\infty}=0$

$$\bar{q}_o(y)\frac{x}{\varepsilon}+\frac{\bar{q}_o(y)}{\sqrt{\beta_1^2-\sigma_1^2}}e^{-\sqrt{\beta_1^2-\sigma_1^2}\frac{x}{\varepsilon}}-\frac{\bar{q}_o(y)}{\sqrt{\beta_1^2-\sigma_1^2}}$$

$$+\bar{q}_1(y)x+\frac{\varepsilon\bar{q}_1(y)}{\sqrt{\beta_1^2-\sigma_1^2}}e^{-\sqrt{\beta_1^2-\sigma_1^2}\frac{x}{\varepsilon}}-\frac{\varepsilon\bar{q}_1(y)}{\sqrt{\beta_1^2-\sigma_1^2}} \tag{3.12g}$$

One term outer expansion:

$$\bar{q}_1(y)x \tag{3.12h}$$

Equating (3.12d) and (3.12h) according to matching principle, we obtain

$$\frac{2}{b}E_3\sin\frac{m\pi}{b}y-\frac{2}{b}\Gamma_2\sin\frac{m\pi}{b}y+\varepsilon X\{2C_2n\pi\cos\alpha y+2D_2n\pi\sin\alpha y+\frac{2}{b}F_3\sigma\sin\frac{m\pi}{b}y$$

$$+2\Gamma_1n\pi(\sin\alpha y\cos\alpha y_0-\cos\alpha y\sin\alpha y_0)+\frac{2}{b}\Gamma_2\frac{s}{u}\sin\frac{m\pi}{b}y\Big\}=\bar{q}_1(y)x$$

which implies that

$$\frac{2}{b}E_3\sin\frac{m\pi}{b}y-\frac{2}{b}\Gamma_2\sin\frac{m\pi}{b}y=0$$

and

$$2C_2n\pi\sin\alpha y+2D_2n\pi\sin\alpha y+\frac{2}{b}F_3\sigma\sin\frac{m\pi}{b}y$$

$$+2\Gamma_1n\pi(\sin\alpha y\cos\alpha y_0-\cos\alpha y\sin\alpha y_0)+\frac{2}{b}\Gamma_2\frac{s}{u}\sin\frac{m\pi}{b}y=\bar{q}_1(y)$$

so that

$$\bar{q}_1(y)=\frac{2P_o u\sin\frac{m\pi}{b}y_o}{b(\beta_1^2-\sigma_1^2)(s^2+\sigma^2u^2)}\left[\frac{e^{-\frac{s}{u}}}{\sin\sigma}-\cot\sigma+\frac{s}{u\sigma}\right]\sigma\sin\frac{m\pi}{b}y \tag{3.13}$$

Matching (1-2 matching) procedure for the boundary conditions x = 1.

One term outer expansion:

$$\bar{V}_0(x,y) = 2C_2\cos\alpha y\sin n\pi x + 2D_2\sin\alpha y\sin n\pi x$$

$$+\frac{2}{b}E_3\cos\sigma x\sin\frac{m\pi}{b}y + \frac{2}{b}F_3\sin\sigma x\sin\frac{m\pi}{b}y$$

$$+2\Gamma_1(\sin\alpha y\cos\alpha y_0 - \cos\alpha y\sin\alpha y_0)\sin n\pi x - \frac{2}{b}\Gamma_2 e^{-\frac{1}{\varepsilon}x}\sin\frac{m\pi}{b}y \qquad (3.14a)$$

Rewritten in inner variable:

For inner variable $X = \frac{1-x}{\varepsilon}$ which implies that $x = 1-\varepsilon X$ is given by

$$\bar{V}_0(x,y) = 2C_2\cos\alpha y\sin n\pi(1-\varepsilon X) + 2D_2\sin\alpha y\sin n\pi(1-\varepsilon X)$$

$$+\frac{2}{b}E_3\sin\frac{m\pi}{b}y\cos\sigma(1-\varepsilon X) + \frac{2}{b}F_3\sin\frac{m\pi}{b}y\sin\sigma(1-\varepsilon X)$$

$$+2\Gamma_1(\sin\alpha y\cos\alpha y_0 - \cos\alpha y\sin\alpha y_0)\sin n\pi(1-\varepsilon X) - \frac{2}{b}\Gamma_2\sin\frac{m\pi}{b}ye^{-\frac{1}{\varepsilon}(1-\varepsilon X)} \qquad (3.14b)$$

Expanded for small ε:

$$-2C_2\cos\alpha y(-1)^n\left[n\pi\varepsilon X - \frac{n^3\pi^3\varepsilon^3X^3}{3!} + \frac{n^5\pi^5\varepsilon^5X^5}{5!} - \cdots\cdots\right]$$

$$-2D_2\sin\alpha y(-1)^n\left[n\pi\varepsilon X - \frac{n^3\pi^3\varepsilon^3X^3}{3!} + \frac{n^5\pi^5\varepsilon^5X^5}{5!} - \cdots\cdots\right]$$

$$+\frac{2}{b}E_3\sin\frac{m\pi}{b}y\cos\sigma\left[1 - \frac{\sigma^2\varepsilon^2X^2}{2!} + \frac{\sigma^4\varepsilon^4X^4}{4!} - \cdots\cdots\right]$$

$$+\frac{2}{b}E_3\sin\frac{m\pi}{b}y\sin\sigma\left[\sigma\varepsilon X - \frac{\sigma^3\varepsilon^3X^3}{3!} + \frac{\sigma^5\varepsilon^5X^5}{5!} - \cdots\cdots\right]$$

$$+\frac{2}{b}F_3\sin\frac{m\pi}{b}y\sin\sigma\left[1-\frac{\sigma^2\varepsilon^2X^2}{2!}+\frac{\sigma^4\varepsilon^4X^4}{4!}-\dots\dots\dots\right]$$

$$-\frac{2}{b}F_3\sin\frac{m\pi}{b}y\cos\sigma\left[\sigma\varepsilon X-\frac{\sigma^3\varepsilon^3X^3}{3!}+\frac{\sigma^5\varepsilon^5X^5}{5!}-\dots\dots\dots\right]$$

$$-2\Gamma_1(\sin\alpha y\cos\alpha y_0-\cos\alpha y\sin\alpha y_0)(-1)^n\left[n\pi\varepsilon X-\frac{n^3\pi^3\varepsilon^3X^3}{3!}+\frac{n^5\pi^5\varepsilon^5X^5}{5!}-\dots\dots\dots\right]$$

$$-\frac{2}{b}\Gamma_2\sin\frac{m\pi}{b}ye^{-\frac{s}{u}}\left[1+\frac{s\varepsilon X}{u}+\frac{s^2\varepsilon^2X^2}{2!u}-\dots\dots\dots\right] \qquad (3.14c)$$

Two terms inner expansion:

$$\frac{2}{b}E_3\sin\frac{m\pi}{b}y\cos\sigma+\frac{2}{b}F_3\sin\frac{m\pi}{b}y\sin\sigma-\frac{2}{b}\Gamma_2\sin\frac{m\pi}{b}ye^{-\frac{s}{u}}+\varepsilon X\Big\{-2(-1)^nC_2n\pi\cos\alpha y$$

$$-2(-1)^nD_2n\pi\sin\alpha y+\frac{2}{b}E_3\sigma\sin\frac{m\pi}{b}y\sin\sigma-\frac{2}{b}F_3\sigma\sin\frac{m\pi}{b}y\cos\sigma$$

$$-2(-1)^nn\pi\Gamma_1(\sin\alpha y\cos\alpha y_0-\cos\alpha y\sin\alpha y_0)-\frac{2}{b}\Gamma_2\frac{s}{u}\sin\frac{m\pi}{b}ye^{-\frac{s}{u}}\Big\} \qquad (3.14d)$$

Two terms inner expansion:

$$\psi^i=\psi_0^i+\varepsilon\psi_1^i$$

$$=\bar{\bar{q}}_0(y)\left\{X+\frac{1}{\sqrt{\beta_1^2-\sigma_1^2}}e^{-\sqrt{\beta_1^2-\sigma_1^2}X}-\frac{1}{\sqrt{\beta_1^2-\sigma_1^2}}\right\}$$

$$+\varepsilon\bar{\bar{q}}_1(y)\left\{X+\frac{1}{\sqrt{\beta_1^2-\sigma_1^2}}e^{-\sqrt{\beta_1^2-\sigma_1^2}X}-\frac{1}{\sqrt{\beta_1^2-\sigma_1^2}}\right\} \qquad (3.14e)$$

Rewritten in outer variable:

$$\bar{\bar{q}}_o(y)\left\{\frac{1-x}{\varepsilon}+\frac{1}{\sqrt{\beta_1^2-\sigma_1^2}}e^{-\sqrt{\beta_1^2-\sigma_1^2}\frac{(1-x)}{\varepsilon}}-\frac{1}{\sqrt{\beta_1^2-\sigma_1^2}}\right\}$$

$$+\varepsilon\bar{\bar{q}}_1(y)\left\{\frac{1-x}{\varepsilon}+\frac{1}{\sqrt{\beta_1^2-\sigma_1^2}}e^{-\sqrt{\beta_1^2-\sigma_1^2}\frac{(1-x)}{\varepsilon}}-\frac{1}{\sqrt{\beta_1^2-\sigma_1^2}}\right\} \quad (3.14f)$$

Expanded for small ε:

As $\varepsilon \to 0$, $e^{-\sqrt{\beta_1^2-\sigma_1^2}\frac{(1-x)}{\varepsilon}}$ becomes $e^{-\infty}=0$

$$\bar{\bar{q}}_o(y)\frac{1-x}{\varepsilon}+\frac{\bar{\bar{q}}_o(y)}{\sqrt{\beta_1^2-\sigma_1^2}}e^{-\sqrt{\beta_1^2-\sigma_1^2}\frac{(1-x)}{\varepsilon}}-\frac{\bar{\bar{q}}_o(y)}{\sqrt{\beta_1^2-\sigma_1^2}}$$

$$+\bar{\bar{q}}_1(y)(1-x)+\frac{\varepsilon\bar{\bar{q}}_1(y)}{\sqrt{\beta_1^2-\sigma_1^2}}e^{-\sqrt{\beta_1^2-\sigma_1^2}\frac{(1-x)}{\varepsilon}}-\frac{\varepsilon\bar{\bar{q}}_1(y)}{\sqrt{\beta_1^2-\sigma_1^2}} \quad (3.14g)$$

One term outer expansion:

$$\bar{\bar{q}}_1(y)(1-x) \quad (3.14h)$$

Equating (3.14d) and (3.14h) according to matching principle, we obtain

$$\frac{2}{b}E_3\sin\frac{m\pi}{b}y\cos\sigma+\frac{2}{b}F_3\sin\frac{m\pi}{b}y\sin\sigma-\frac{2}{b}\Gamma_2\sin\frac{m\pi}{b}ye^{-\div}+\varepsilon X\Big\{-2(-1)^n C_1 n\pi\cos\alpha y$$

$$-2(-1)^n D_2 n\pi\sin\alpha y+\frac{2}{b}E_3\sigma\sin\frac{m\pi}{b}y\sin\sigma-\frac{2}{b}F_3\sigma\sin\frac{m\pi}{b}y\cos\sigma$$

$$-2(-1)^n n\pi\Gamma_1(\sin\alpha y\cos\alpha y_0-\cos\alpha y\sin\alpha y_0)-\frac{2}{b}\Gamma_2\frac{s}{u}\sin\frac{m\pi}{b}ye^{-\div}\Big\}=\bar{\bar{q}}_0(y)(1-x)$$

which implies that

$$\frac{2}{b}E_3 \sin\frac{m\pi}{b}y\cos\sigma + \frac{2}{b}F_3 \sin\frac{m\pi}{b}y\sin\sigma - \frac{2}{b}\Gamma_2 \sin\frac{m\pi}{b}ye^{-\frac{s}{u}} = 0$$

and

$$-2(-1)^n C_2 n\pi\cos\alpha y - 2(-1)^n D_2 n\pi\sin\alpha y + \frac{2}{b}E_3\sigma\sin\frac{m\pi}{b}y\sin\sigma - \frac{2}{b}F_3\sigma\sin\frac{m\pi}{b}y\cos\sigma$$

$$-2(-1)^n n\pi\Gamma_1(\sin\alpha y\cos\alpha y_0 - \cos\alpha y\sin\alpha y_0) - \frac{2}{b}\Gamma_2\frac{s}{u}\sin\frac{m\pi}{b}ye^{-\frac{s}{u}}\Big\} = \bar{\bar{q}}_0(y)$$

So that

$$\bar{\bar{q}}_1(y) = \frac{2P_0 u\sin\frac{m\pi}{b}y_0}{b(\beta_1^2 - \sigma_1^2)(s^2 + \sigma^2 u^2)}\left[\sigma\sin\sigma - \sigma e^{-\frac{s}{u}}\frac{\cos\sigma}{\sin\sigma} + \sigma\cos\sigma\cot\sigma - \frac{s}{u}e^{-\frac{s}{u}}\right]\sin\frac{m\pi}{b}y \tag{3.15}$$

Matching (1 – 2 matching) procedure for the boundary condition y = 0.

One term outer expansion:

$$\vec{V}_0(x,y) = 2C_2\sin n\pi x\cos\alpha y + 2D_2\sin n\pi x\sin\alpha y$$

$$+\frac{2}{b}E_3\cos\sigma x\sin\frac{m\pi}{b}y + \frac{2}{b}F_3\sin\sigma x\sin\frac{m\pi}{b}y$$

$$+2\Gamma_1(\sin\alpha y\cos\alpha y_0 - \cos\alpha y\sin\alpha y_0)\sin n\pi x - \frac{2}{b}\Gamma_2 e^{-\frac{s}{u}x}\sin\frac{m\pi}{b}y \tag{3.16a}$$

Rewritten in inner variable:

For inner variable $X = \frac{x}{\varepsilon}$ which implies that $x = \varepsilon X$ is given by

$$\left[2C_2\sin n\pi x - 2\Gamma_1\sin n\pi x\sin\alpha y_0\right]\cos\alpha\varepsilon Y.$$

$$+\left[2D_2 \sin n\pi x + 2\Gamma_1 \sin n\pi x \cos \alpha y_0\right] \sin \alpha\varepsilon Y$$

$$+\left[\frac{2}{b} E_3 \cos \sigma x + \frac{2}{b} F_3 \sin \sigma x - \frac{2}{b}\Gamma_2 e^{-\frac{1}{2}\varepsilon Y}\right] \sin \frac{m\pi}{b}\varepsilon Y \qquad (3.16b)$$

Expanded for small ε:

$$[2C_2 \sin n\pi x - 2\Gamma_1 \sin n\pi x \sin \alpha y_0]\left[1 - \frac{\alpha^2\pi^2\varepsilon^2 Y^2}{2!} + \frac{\alpha^4\pi^4\varepsilon^4 Y^4}{4!} - \dots\dots\dots\right]$$

$$+[2D_2 \sin n\pi x + 2\Gamma_1 \sin n\pi x \cos \alpha y_0]\left[\alpha\pi\varepsilon Y - \frac{\alpha^3\pi^3\varepsilon^3 Y^3}{3!} + \frac{\alpha^5\pi^5\varepsilon^5 Y^5}{5!} - \dots\dots\dots\right]$$

$$+\frac{2}{b} E_3 \cos \sigma x\left[\frac{m\pi\varepsilon Y}{b} - \frac{m^3\pi^3\varepsilon^3 Y^3}{3!b^3} + \frac{m^5\pi^5\varepsilon^5 Y^5}{5!b^5} - \dots\dots\dots\right]$$

$$+\frac{2}{b} F_3 \sin \sigma x\left[\frac{m\pi\varepsilon Y}{b} - \frac{m^3\pi^3\varepsilon^3 Y^3}{3!b^3} + \frac{m^5\pi^5\varepsilon^5 Y^5}{5!b^5} - \dots\dots\dots\right]$$

$$-\frac{2}{b}\Gamma_2 e^{-\frac{1}{2}\varepsilon Y}\left[\frac{m\pi\varepsilon Y}{b} - \frac{m^3\pi^3\varepsilon^3 Y^3}{3!b^3} + \frac{m^5\pi^5\varepsilon^5 Y^5}{5!b^5} - \dots\dots\dots\right] \qquad (3.16c)$$

Two terms inner expansion:

$$2C_2 \sin n\pi x - 2\Gamma_1 \sin n\pi x \sin \alpha y_0 + \varepsilon Y\{2D_2 \sin n\pi x + 2\Gamma_1 \sin n\pi x \cos \alpha y_0$$

$$+\frac{2}{b^2} E_3 m\pi \cos \sigma x + \frac{2}{b^2} F_3 m\pi \sin \sigma x - \frac{2}{b^2}\Gamma_2 m\pi e^{-\frac{1}{2}\varepsilon Y}\} \qquad (3.16d)$$

Two terms inner expansion:

$$\psi' = \psi'_0 + \varepsilon\psi'_1$$

$$= \bar{r}_u(x)\left\{Y + \frac{1}{\sqrt{\beta_2^2 - \sigma_1^2}} e^{-\sqrt{\beta_2^2 - \sigma_1^2}\,Y} - \frac{1}{\sqrt{\beta_2^2 - \sigma_1^2}}\right\},$$

$$+\varepsilon\bar{r}_1(x)\left\{Y+\frac{1}{\sqrt{\beta_2^2-\sigma_1^2}}e^{-\sqrt{\beta_2^2-\sigma_1^2}Y}-\frac{1}{\sqrt{\beta_2^2-\sigma_1^2}}\right\} \tag{3.16e}$$

Rewritten in outer variable:

$$\bar{r}_o(x)\left\{\frac{y}{\varepsilon}+\frac{1}{\sqrt{\beta_2^2-\sigma_1^2}}e^{-\sqrt{\beta_2^2-\sigma_1^2}\frac{y}{\varepsilon}}-\frac{1}{\sqrt{\beta_2^2-\sigma_1^2}}\right\}$$

$$+\varepsilon\bar{r}_1(x)\left\{\frac{y}{\varepsilon}+\frac{1}{\sqrt{\beta_2^2-\sigma_1^2}}e^{-\sqrt{\beta_2^2-\sigma_1^2}\frac{y}{\varepsilon}}-\frac{1}{\sqrt{\beta_2^2-\sigma_1^2}}\right\} \tag{3.16f}$$

Expanded for small ε:

As $\varepsilon \to 0$, $e^{-\sqrt{\beta_2^2-\sigma_1^2}\frac{y}{\varepsilon}}$ becomes $e^{-\infty}=0$

$$\bar{r}_o(x)\frac{y}{\varepsilon}+\frac{\bar{r}_o(x)}{\sqrt{\beta_2^2-\sigma_1^2}}e^{-\sqrt{\beta_2^2-\sigma_1^2}\frac{y}{\varepsilon}}-\frac{\bar{r}_o(x)}{\sqrt{\beta_2^2-\sigma_1^2}}$$

$$+\bar{r}_1(x)y+\frac{\varepsilon\bar{r}(x)}{\sqrt{\beta_2^2-\sigma_1^2}}e^{-\sqrt{\beta_2^2-\sigma_1^2}\frac{y}{\varepsilon}}-\frac{\varepsilon\bar{r}_1(x)}{\sqrt{\beta_2^2-\sigma_1^2}} \tag{3.16g}$$

One term outer expansion:

$$\bar{r}_1(x)y \tag{3.16h}$$

Equating (3.16d) and (3.16h) according to matching principle, we obtain

$$2\mathrm{C}_2\sin n\pi x-2\Gamma_1\sin n\pi x\sin\alpha y_0+\varepsilon Y\{2\mathrm{D}_2\sin n\pi x+2\Gamma_1\sin n\pi x\cos\alpha y_0$$

$$+\frac{2}{b^2}E_3 m\pi\cos\sigma x+\frac{2}{b^2}F_3 m\pi\sin\alpha x-\frac{2}{b^2}\Gamma_2 m\pi e^{-\frac{1}{2}\varepsilon Y}\}=\bar{r}_1(x)y$$

so that

$$2C_2 \sin n\pi x - 2\Gamma_1 \sin n\pi x \sin \alpha y_0 = 0$$

and

$$\{2D_2\alpha \sin n\pi x + 2\Gamma_1 \alpha \sin n\pi x \cos \alpha y_0 + \frac{2}{b^2} E_3 m\pi \cos \sigma x + \frac{2}{b^2} F_3 m\pi \sin \sigma x - \frac{2}{b^2}\Gamma_2 m\pi e^{-\frac{1}{\varepsilon}\varepsilon X}\} = \bar{r}_1(x)$$

which implies that

$$\bar{r}_1(x) = \frac{2m\pi P_0 u \sin \frac{m\pi}{b} y_0}{b^2\left(\beta_1^2 - \sigma_1^2\right)\left(s^2 + \sigma^2 u^2\right)}\left[\cos \sigma x + e^{-\frac{1}{\varepsilon}} \frac{\sin \sigma x}{\sin \sigma} - \cot \sigma \sin \sigma x - e^{-\frac{1}{\varepsilon}x}\right] \qquad (3.17)$$

Matching (1-2 matching) procedure for the boundary condition y = b.

One term outer expansion:

$$\tilde{V}_0(x, y) = 2C_2 \sin n\pi x \cos \alpha y + 2D_2 \sin n\pi x \sin \alpha y$$

$$+ \left[\frac{2}{b} E_3 \cos \sigma x \sin \frac{m\pi}{b} y + \frac{2}{b} F_3 \sin \sigma x - \frac{2}{b}\Gamma_2 e^{-\frac{1}{\varepsilon}x}\right] \sin \frac{m\pi}{b} y$$

$$+ 2\Gamma_1 \sin n\pi x \cos \alpha y_0 \sin \alpha y - 2\Gamma_1 \sin n\pi x \sin \alpha y_0 \cos \alpha y \qquad (3.18a)$$

Rewritten in inner variable:

For inner variable $Y = \frac{b - y}{\varepsilon}$ which implies that $y = b - \varepsilon Y$ is given by

$$\left[2C_2 \sin n\pi x - 2\Gamma_1 \sin n\pi x \sin \alpha y_0\right] \cos \alpha(b - \varepsilon Y)$$

$$\left[2D_2 \sin n\pi x + 2\Gamma_1 \sin n\pi x \cos \alpha y_0\right] \sin \alpha(b - \varepsilon Y)$$

$$\left[\frac{2}{b} E_3 \cos \sigma x + \frac{2}{b} F_3 \sin \sigma - \frac{2}{b}\Gamma_2 e^{-\frac{1}{\varepsilon}\varepsilon x}\right] \sin \frac{m\pi}{b}(b - \varepsilon Y) \qquad (3.18b)$$

Expanded for small ε:

$$[2C_2\sin n\pi x - 2\Gamma_1\sin n\pi x\sin\alpha y_0]\cos\alpha b\left[1-\frac{\alpha^2\pi^2\varepsilon^2Y^2}{2!}+\frac{\alpha^4\pi^4\varepsilon^4Y^4}{4!}-\ldots\ldots\ldots\ldots\right]$$

$$+[2C_2\sin n\pi x - 2\Gamma_1\sin n\pi x\sin\alpha y_0]\sin\alpha b\left[\alpha\pi\varepsilon Y-\frac{\alpha^3\pi^3\varepsilon^3Y^3}{3!}+\frac{\alpha^5\pi^5\varepsilon^5Y^5}{5!}-\ldots\ldots\ldots\ldots\right]$$

$$+[2D_2\sin n\pi x + 2\Gamma_1\sin n\pi x\cos\alpha y_0]\sin\alpha b\left[1-\frac{\alpha^2\pi^2\varepsilon^2Y^2}{2!}+\frac{\alpha^4\pi^4\varepsilon^4Y^4}{4!}-\ldots\ldots\ldots\ldots\right]$$

$$-[2D_2\sin n\pi x + 2\Gamma_1\sin n\pi x\cos\alpha y_0]\cos\alpha b\left[\alpha\pi\varepsilon Y-\frac{\alpha^3\pi^3\varepsilon^3Y^3}{3!}+\frac{\alpha^5\pi^5\varepsilon^5Y^5}{5!}-\ldots\ldots\ldots\ldots\right]$$

$$-(-1)^n\left[\frac{2}{b}E_3\cos\sigma+\frac{2}{b}F_3\sin\sigma-\frac{2}{b}\Gamma_2e^{\frac{\varepsilon}{a}Y}\right]\left[\frac{m\pi\varepsilon Y}{b}-\frac{m^3\pi^3\varepsilon^3Y^3}{3!b^3}+\frac{m^5\pi^5\varepsilon^5Y^5}{5!b^5}-\ldots\right]\quad(3.18c)$$

Two terms inner expansion:

$$[2C_2\sin n\pi x - 2\Gamma_1\sin n\pi x\sin\alpha y_0]\cos\alpha b+[2D_2\sin n\pi x + 2\Gamma_1\sin n\pi x\cos\alpha y_0]\sin\alpha b$$

$$+\varepsilon Y\{(2C_2\sin n\pi x - 2\Gamma_1\sin n\pi x\sin\alpha y_0)\alpha\sin\alpha b-(2D_2\sin n\pi x + 2\Gamma_1\sin n\pi x\cos\alpha y_0)\alpha\cos\alpha b$$

$$-(-1)^n\frac{m\pi}{b}\left[\frac{2}{b}E_3\cos\sigma x+\frac{2}{b}F_3\sin\sigma x-\frac{2}{b}\Gamma_2e^{-\frac{\varepsilon}{a}Y}\right]\}\quad(3.18d)$$

Two terms inner expansion:

$$\psi^i=\psi_0^i+\varepsilon\psi_1^i$$

$$=\bar{\bar{F}}_0(x)\left\{Y+\frac{1}{\sqrt{\beta_2^2-\sigma_1^2}}e^{-\sqrt{\beta_2^2-\sigma_1^2}\,Y}-\frac{1}{\sqrt{\beta_2^2-\sigma_1^2}}\right\}$$

$$+\varepsilon\bar{\bar{\sigma}}_1(x)\left\{Y+\frac{1}{\sqrt{\beta_2^2-\sigma_1^2}}e^{-\sqrt{\beta_2^2-\sigma_1^2}Y}-\frac{1}{\sqrt{\beta_2^2-\sigma_1^2}}\right\} \qquad (3.18e)$$

Rewritten in outer variable:

$$\bar{\bar{r}}_o(x)\left\{\frac{b-y}{\varepsilon}+\frac{1}{\sqrt{\beta_2^2-\sigma_1^2}}e^{-\sqrt{\beta_2^2-\sigma_1^2}\frac{(b-y)}{\varepsilon}}-\frac{1}{\sqrt{\beta_2^2-\sigma_1^2}}\right\}$$

$$+\varepsilon\bar{\bar{\sigma}}_1(x)\left\{\frac{b-y}{\varepsilon}+\frac{1}{\sqrt{\beta_2^2-\sigma_1^2}}e^{-\sqrt{\beta_2^2-\sigma_1^2}\frac{(b-y)}{\varepsilon}}-\frac{1}{\sqrt{\beta_2^2-\sigma_1^2}}\right\} \qquad (3.18f)$$

Expanded for small ε:

As $\varepsilon\rightarrow 0$, $e^{-\sqrt{\beta_1^2-\sigma_1^2}\frac{(b-y)}{\varepsilon}}$ becomes $e^{-\infty}=0$

$$\bar{\bar{r}}_o(x)\frac{b-y}{\varepsilon}+\frac{\bar{\bar{r}}_o(x)}{\sqrt{\beta_2^2-\sigma_1^2}}e^{-\sqrt{\beta_2^2-\sigma_1^2}\frac{(b-y)}{\varepsilon}}-\frac{\bar{\bar{r}}_o(x)}{\sqrt{\beta_2^2-\sigma_1^2}}$$

$$+\bar{\bar{r}}_1(x)(b-y)+\frac{\varepsilon\bar{\bar{\sigma}}(x)}{\sqrt{\beta_2^2-\sigma_1^2}}e^{-\sqrt{\beta_2^2-\sigma_1^2}\frac{(b-x)}{\varepsilon}}-\frac{\varepsilon\bar{\bar{\sigma}}_1(x)}{\sqrt{\beta_2^2-\sigma_1^2}} \qquad (3.18g)$$

One term outer expansion:

$$\bar{\bar{r}}_1(x)(b-y) \qquad (3.18h)$$

Equating (3.18d) and (3.18h) according to matching principle, we obtain

$$[2C_2\sin n\pi x-2\Gamma_s\sin n\pi x\sin\alpha y_0]\cos\alpha b+[2D_2\sin n\pi x+2\Gamma_s\sin n\pi x\cos\alpha y_0]\sin\alpha b$$

$$+\varepsilon Y\{(2C_2 \sin n\pi x - 2\Gamma_1 \sin n\pi x \sin \alpha y_0)\alpha \sin \alpha b - (2D_2 \sin n\pi x + 2\Gamma_1 \sin n\pi x \cos \alpha y_0)\alpha \cos \alpha b$$

$$-(-1)^n \frac{m\pi}{b}\left[\frac{2}{b}E_3 \cos \alpha x + \frac{2}{b}F_3 \sin \alpha x - \frac{2}{b}\Gamma_2 e^{-\frac{1}{\varepsilon}x}\right]\} = \bar{\bar{r}}_1(x)(b-y)$$

which implies that

$$[2C_2 \sin n\pi x - 2\Gamma_1 \sin n\pi x \sin \alpha y_0]\cos \alpha b + [2D_2 \sin n\pi x + 2\Gamma_1 \sin n\pi x \cos \alpha y_0]\sin \alpha b = 0$$

and

$$(2C_2 \sin n\pi x - 2\Gamma_1 \sin n\pi x \sin \alpha y_0)\alpha \sin \alpha b - (2D_2 \sin n\pi x + 2\Gamma_1 \sin n\pi x \cos \alpha y_0)\alpha \cos \alpha b$$

$$-(-1)^n \frac{m\pi}{b}\left[\frac{2}{b}E_3 \cos \alpha x + \frac{2}{b}F_3 \sin \alpha x - \frac{2}{b}\Gamma_2 e^{-\frac{1}{\varepsilon}x}\right] = \bar{\bar{r}}_1(x)$$

so that

$$\bar{\bar{r}}_1(x) = \frac{2(-1)^n m\pi P_0 u \sin \frac{m\pi}{b} y_0}{b^2\left(\beta_1^2 - \sigma_1^2\right)\left(s^2 + \sigma^2 u^2\right)}\left[e^{-\frac{1}{\varepsilon}x} - \cos \alpha x - e^{-\frac{1}{\varepsilon}} \frac{\sin \alpha x}{\sin \sigma} + \cot \sigma \sin \alpha x\right] \tag{3.19}$$

Matching (2-2 matching) procedure for the boundary condition x = 0.

We seek an asymptotic outer solution of the form

$$V^o = V_0^o + \varepsilon V_1^o \tag{3.20}$$

which implies

$$V^o = A_4\left[\cos \alpha x - e^{-\frac{1}{\varepsilon}x} + e^{-\frac{1}{\varepsilon}} \frac{\sin \alpha x}{\sin \sigma} - \frac{\cos \sigma \sin \alpha x}{\sin \sigma}\right]\sin \frac{m\pi}{b} y$$

$$+ 2\varepsilon[C_3 \cos \alpha y + D_3 \sin \alpha y]\sin n\pi x + \frac{2}{b}\varepsilon[E_4 \cos \alpha x + F_4 \sin \alpha x]\sin \frac{m\pi}{b} y \tag{3.21}$$

where

$$A_4 = \frac{2P_o u \sin\frac{m\pi}{b} y_o}{b(\beta_1^2 - \sigma_1^2)(s^2 + \sigma^2 u^2)} \tag{3.22}$$

We now match two terms outer expansion written in inner variable with two terms inner expansion written in outer variable (2 – 2 matching) for the boundary condition x = 0.

We proceed systematically as follows:

Two terms outer expansion:

$$V^0 = V_0^0 + \varepsilon V_1^0$$

$$= A_4\left[\cos\sigma x - e^{-\frac{1}{\varepsilon}x} + e^{-\frac{1}{\varepsilon}x}\frac{\sin\sigma x}{\sin\sigma} - \frac{\cos\sigma\sin\sigma x}{\sin\sigma}\right]\sin\frac{m\pi}{b}y$$

$$+ 2\varepsilon C_3\cos\alpha y\sin n\pi x + 2\varepsilon D_3\sin\alpha y\sin n\pi x$$

$$+ \frac{2}{b}\varepsilon E_4\cos\sigma x\sin\frac{m\pi}{b}y + \frac{2}{b}\varepsilon F_4\sin\sigma x\sin\frac{m\pi}{b}y \tag{3.23a}$$

Rewritten in inner variable:

For inner variable $X = \frac{x}{\varepsilon}$ which implies that $x = b - \varepsilon X$ is given by

$$= A_4\sin\frac{m\pi}{b}y\cos\sigma\varepsilon X - A_4\sin\frac{m\pi}{b}ye^{-\frac{1}{\varepsilon}\varepsilon X} + A_4\frac{e^{-\frac{1}{\varepsilon}}}{\sin\sigma}\sin\frac{m\pi}{b}y\sin\sigma\varepsilon X$$

$$- A_4\frac{\sin\frac{m\pi}{b}y}{\sin\sigma}\cos\sigma\sin\sigma\varepsilon X + 2\varepsilon C_3\cos\alpha y\sin n\pi\varepsilon X$$

$$+ 2\varepsilon D_3\sin\alpha y\sin n\pi\varepsilon X + \frac{2}{b}\varepsilon E_4\sin\tfrac{m\pi}{b}y\cos\sigma\varepsilon X + \frac{2}{b}\varepsilon F_4\sin\tfrac{m\pi}{b}y\sin\sigma\varepsilon X \tag{3.23b}$$

Expanded for small ε:

$$A_4 \sin\frac{m\pi}{b} y\left[1-\frac{\sigma^2\varepsilon^2X^2}{2!}+\frac{\sigma^4\varepsilon^4X^4}{4!}-.............\right]$$

$$-A_4 \sin\frac{m\pi}{b} y\left[1-\frac{s\varepsilon X}{u}+\frac{s^2\varepsilon^2X^2}{2!u^2}.............\right]$$

$$+A_4\frac{e^{-\frac{1}{\varepsilon}}}{\sin\sigma}\sin\frac{m\pi}{b} y\left[\sigma\varepsilon X-\frac{\sigma^3\varepsilon^3X^3}{3!}+\frac{\sigma^5\varepsilon^5X^5}{5!}-.............\right]$$

$$-A_4\frac{\sin\frac{m\pi}{b}y}{\sin\sigma}\cos\sigma\left[\sigma\varepsilon X-\frac{\sigma^3\varepsilon^3X^3}{3!}+\frac{\sigma^5\varepsilon^5X^5}{5!}-.............\right]$$

$$+2\varepsilon C_3\cos\alpha y\left[n\pi\varepsilon X-\frac{n^3\pi^3\varepsilon^3X^3}{3!}+\frac{n^5\pi^5\varepsilon^5X^5}{5!}-............\right]$$

$$+2\varepsilon D_3\sin\alpha y\left[n\pi\varepsilon X-\frac{n^3\pi^3\varepsilon^3X^3}{3!}+\frac{n^5\pi^5\varepsilon^5X^5}{5!}-............\right]$$

$$+\frac{2}{b}\varepsilon E_4\sin\frac{m\pi}{b} y\left[1-\frac{\sigma^2\varepsilon^2X^2}{2!}+\frac{\sigma^4\varepsilon^4X^4}{4!}-.............\right]$$

$$+\frac{2}{b}\varepsilon F_4\sin\frac{m\pi}{b} y\left[\sigma\varepsilon X-\frac{\sigma^3\varepsilon^3X^3}{3!}+\frac{\sigma^5\varepsilon^5X^5}{5!}-.............\right] \qquad (3.23c)$$

Two terms inner expansion:

$$A_4\sin\frac{m\pi}{b}y-A_4\sin\frac{m\pi}{b}y+\varepsilon X\left\{A_4\frac{s}{u}\sin\frac{m\pi}{b}y+A_4\frac{\sigma e^{-\frac{1}{\varepsilon}}}{\sin\sigma}\sin\frac{m\pi}{b}y\right.$$

$$\left.-A_4\frac{\sigma\cos\sigma}{\sin\sigma}\sin\frac{m\pi}{b}y\right\}+\frac{2}{b}\varepsilon E_4\sin\frac{m\pi}{b}y \qquad (3.23d)$$

Two terms inner expansion:

$$\psi = \psi_0 + \varepsilon\psi_1$$

$$= \bar{q}_o(y)\left\{X + \frac{1}{\sqrt{\beta_1^2-\sigma_1^2}}e^{-\sqrt{\beta_1^2-\sigma_1^2}X} - \frac{1}{\sqrt{\beta_1^2-\sigma_1^2}}\right\}$$

$$+\varepsilon\bar{q}_1(y)\left\{X + \frac{1}{\sqrt{\beta_1^2-\sigma_1^2}}e^{-\sqrt{\beta_1^2-\sigma_1^2}X} - \frac{1}{\sqrt{\beta_1^2-\sigma_1^2}}\right\} \tag{3.23e}$$

Rewritten in outer variable:

$$\bar{q}_o(y)\left\{\frac{x}{\varepsilon} + \frac{1}{\sqrt{\beta_1^2-\sigma_1^2}}e^{-\sqrt{\beta_1^2-\sigma_1^2}\frac{x}{\varepsilon}} - \frac{1}{\sqrt{\beta_1^2-\sigma_1^2}}\right\}$$

$$+\varepsilon\bar{q}_1(y)\left\{\frac{x}{\varepsilon} + \frac{1}{\sqrt{\beta_1^2-\sigma_1^2}}e^{-\sqrt{\beta_1^2-\sigma_1^2}\frac{x}{\varepsilon}} - \frac{1}{\sqrt{\beta_1^2-\sigma_1^2}}\right\} \tag{3.23f}$$

Expanded for small ε:

As $\varepsilon \to 0$, $e^{-\sqrt{\beta_1^2-\sigma_1^2}\frac{x}{\varepsilon}}$ becomes $e^{-\infty} = 0$

$$\bar{q}_o(y)\frac{x}{\varepsilon} + \frac{\bar{q}_o(y)}{\sqrt{\beta_1^2-\sigma_1^2}}e^{-\sqrt{\beta_1^2-\sigma_1^2}\frac{x}{\varepsilon}} - \frac{\bar{q}_o(y)}{\sqrt{\beta_1^2-\sigma_1^2}}$$

$$+\bar{q}_1(y)x + \frac{\varepsilon\bar{q}_1(y)}{\sqrt{\beta_1^2-\sigma_1^2}}e^{-\sqrt{\beta_1^2-\sigma_1^2}\frac{x}{\varepsilon}} - \frac{\varepsilon\bar{q}_1(y)}{\sqrt{\beta_1^2-\sigma_1^2}} \tag{3.23g}$$

Two terms outer expansion:

$$\bar{q}_1(y)x = \frac{\bar{\varepsilon} q_1(y)}{\sqrt{\beta_1^2 - \sigma_1^2}} \tag{3.23h}$$

Equating (3.23d) and (3.23h) according to matching principle, we obtain

$$\varepsilon X\left\{A_4 \frac{s}{u}\sin\frac{m\pi}{b}y + A_4\frac{\sigma e^{-\frac{s}{u}}}{\sin\sigma}\sin\frac{m\pi}{b}y - A_4\frac{\sigma\cos\sigma}{\sin\sigma}\sin\frac{m\pi}{b}y\right\}$$

$$+\frac{2}{b}\varepsilon E_4 \sin\frac{m\pi}{b}y = \bar{q}_1(y)x - \frac{\varepsilon\bar{q}_1(y)x}{\sqrt{\beta_1^2 - \sigma_1^2}}$$

which implies that

$$\bar{q}_1(y) = A_4\left[\frac{\sigma e^{-\frac{s}{u}}}{\sin\sigma} - \frac{\sigma\cos\sigma}{\sin\sigma} + \frac{s}{u}\right]\sin\frac{m\pi}{b}y$$

and

$$\frac{2}{b}E_4\sin\frac{m\pi}{b}y = -\frac{\bar{q}_1(y)}{\sqrt{\beta_1^2 - \sigma_1^2}}$$

$$= \frac{A_4}{\sqrt{\left(\beta_1^2 - \sigma_1^2\right)}}\left[\frac{\sigma e^{-\frac{s}{u}}}{\sin\sigma} - \frac{\sigma\cos\sigma}{\sin\sigma} + \frac{s}{u}\right]\sin\frac{m\pi}{b}y$$

Hence

$$E_4 = \frac{-P_0\mu\sin\frac{m\pi}{b}y_0}{\left(\beta_1^2 - \sigma_1^2\right)^{\frac{3}{2}}\left(s^2 + \sigma^2 u^2\right)}\left[\frac{\sigma e^{-\frac{s}{u}}}{\sin\sigma} - \frac{\sigma\cos\sigma}{\sin\sigma} + \frac{s}{u}\right] \tag{3.24}$$

Matching (2-2 matching) procedure for the boundary condition x =1

Two terms outer expansion:

$$V^{o} = A_4\left[\cos\sigma x - e^{-\frac{s}{\varepsilon}} + e^{-\frac{s}{\varepsilon}}\frac{\sin\sigma x}{\sin\sigma} - \frac{\cos\sigma\sin\sigma x}{\sin\sigma}\right]\sin\frac{m\pi}{b}y$$

$$+ 2\varepsilon\left[C_3\cos\alpha y + D_3\sin\alpha y\right]\sin n\pi x + \frac{2}{b}\varepsilon\left[E_4\cos\alpha x + F_4\sin\alpha x\right]\sin\frac{m\pi}{b}y \qquad (3.25a)$$

Rewritten in inner variable:

For inner variable $X = \frac{1-x}{\varepsilon}$ which implies that $x = 1-\varepsilon X$ is given by

$$= A_4\sin\frac{m\pi}{b}y\cos\sigma(1-\varepsilon X) - A_4\sin\frac{m\pi}{b}ye^{-\frac{s}{\varepsilon}(1-\varepsilon X)} + A_4\frac{e^{-\frac{s}{\varepsilon}}}{\sin\sigma}\sin\frac{m\pi}{b}y\sin\sigma(1-\varepsilon X)$$

$$- A_4\frac{\sin\frac{m\pi}{b}y}{\sin\sigma}\cos\sigma\sin\sigma(1-\varepsilon X) + 2\varepsilon C_3\cos\alpha y\sin n\pi(1-\varepsilon X)$$

$$+ 2\varepsilon D_3\sin\alpha y\sin n\pi(1-\varepsilon X) + \frac{2}{b}\varepsilon E_4\sin\tfrac{m\pi}{b}y\cos\sigma(1-\varepsilon X) + \frac{2}{b}\varepsilon F_4\sin\tfrac{m\pi}{b}y\sin\sigma(1-\varepsilon X)$$

$$(3.25b)$$

Expanded for small ε:

$$A_4\sin\frac{m\pi}{b}y\cos\sigma\left[1 - \frac{\sigma^2\varepsilon^2X^2}{2!} + \frac{\sigma^4\varepsilon^4X^4}{4!} - \ldots\ldots\ldots\ldots\right]$$

$$+ A_4\sin\frac{m\pi}{b}y\sin\sigma\left[\sigma\varepsilon X - \frac{\sigma^3\varepsilon^3X^3}{3!} + \frac{\sigma^5\varepsilon^5X^5}{5!} - \ldots\ldots\ldots\ldots\right]$$

$$- A_4\sin\frac{m\pi}{b}ye^{-\frac{s}{\varepsilon}}\left[1 + \frac{s\varepsilon X}{u} + \frac{s^2\varepsilon^2X^2}{2!u^2}\ldots\ldots\ldots\ldots\right]$$

$$+ A_4\sin\frac{m\pi}{b}ye^{-\frac{s}{\varepsilon}}\left[1 - \frac{\sigma^2\varepsilon^2X^2}{2!} + \frac{\sigma^4\varepsilon^4X^4}{4!} - \ldots\ldots\ldots\ldots\right]$$

$$-A_4\frac{e^{-\frac{1}{\varepsilon}}}{\sin\sigma}\sin\frac{m\pi}{b}y\cos\sigma\left[\sigma\varepsilon X-\frac{\sigma^3\varepsilon^3X^3}{3!}+\frac{\sigma^5\varepsilon^5X^5}{5!}-\ldots\ldots\ldots\ldots\right]$$

$$-A_4\sin\frac{m\pi}{b}y\cos\sigma\left[1-\frac{\sigma^2\varepsilon^2X^2}{2!}+\frac{\sigma^4\varepsilon^4X^4}{4!}-\ldots\ldots\ldots\ldots\right]$$

$$-A_4\frac{\cos^2\sigma}{\sin\sigma}\sin\frac{m\pi}{b}y\left[\sigma\varepsilon X-\frac{\sigma^3\varepsilon^3X^3}{3!}+\frac{\sigma^5\varepsilon^5X^5}{5!}-\ldots\ldots\ldots\ldots\right]$$

$$-2(-1)^n\varepsilon C_3\cos\alpha y\left[n\pi\varepsilon X-\frac{n^3\pi^3\varepsilon^3X^3}{3!}+\frac{n^5\pi^5\varepsilon^5X^5}{5!}-\ldots\ldots\ldots\ldots\right]$$

$$-2(-1)^n\varepsilon D_3\sin\alpha y\left[n\pi\varepsilon X-\frac{n^3\pi^3\varepsilon^3X^3}{3!}+\frac{n^5\pi^5\varepsilon^5X^5}{5!}-\ldots\ldots\ldots\ldots\right]$$

$$+\frac{2}{b}\varepsilon E_4\sin\frac{m\pi}{b}y\cos\sigma\left[1-\frac{\sigma^2\varepsilon^2X^2}{2!}+\frac{\sigma^4\varepsilon^4X^4}{4!}-\ldots\ldots\ldots\ldots\right]$$

$$+\frac{2}{b}\varepsilon E_4\sin\frac{m\pi}{b}y\sin\sigma\left[\sigma\varepsilon X-\frac{\sigma^3\varepsilon^3X^3}{3!}+\frac{\sigma^5\varepsilon^5X^5}{5!}-\ldots\ldots\ldots\ldots\right]$$

$$+\frac{2}{b}\varepsilon F_4\sin\frac{m\pi}{b}y\sin\sigma\left[1-\frac{\sigma^2\varepsilon^2X^2}{2!}+\frac{\sigma^4\varepsilon^4X^4}{4!}-\ldots\ldots\ldots\ldots\right]$$

$$+\frac{2}{b}\varepsilon F_4\sin\frac{m\pi}{b}y\cos\sigma\left[\sigma\varepsilon X-\frac{\sigma^3\varepsilon^3X^3}{3!}+\frac{\sigma^5\varepsilon^5X^5}{5!}-\ldots\ldots\ldots\ldots\right]\qquad(3.25c)$$

Two terms inner expansion:

$$A_4\sin\frac{m\pi}{b}y\cos\sigma-A_4e^{-\frac{1}{\varepsilon}}\sin\frac{m\pi}{b}y+A_4e^{-\frac{1}{\varepsilon}}\sin\frac{m\pi}{b}y-A_4\sin\frac{m\pi}{b}y\cos\sigma$$

$$+\varepsilon\left\{\frac{2}{b}E_4\cos\sigma\sin\frac{m\pi}{b}y+\frac{2}{b}F_4\sin\sigma\sin\frac{m\pi}{b}y\right\}+\varepsilon X\left\{A_4\sigma\sin\sigma\sin\frac{m\pi}{b}y-A_4e^{-\frac{x}{\varepsilon}}\frac{s}{u}\sin\frac{m\pi}{b}y\right.$$

$$\left.-A_4e^{-\frac{1}{\varepsilon}}\sigma\frac{\cos\sigma}{\sin\sigma}\sin\frac{m\pi}{b}y+A_4\sigma\frac{\cos^2\sigma}{\sin\sigma}\sin\frac{m\pi}{b}y\right\} \tag{3.25d}$$

Two terms inner expansion:

$$\psi=\psi_0+\varepsilon\psi_1$$

$$=\bar{\bar{q}}_o(y)\left\{X+\frac{1}{\sqrt{\beta_1^2-\sigma_1^2}}e^{-\sqrt{\beta_1^2-\sigma_1^2}\,x}-\frac{1}{\sqrt{\beta_1^2-\sigma_1^2}}\right\}$$

$$+\varepsilon\bar{\bar{q}}_1(y)\left\{X+\frac{1}{\sqrt{\beta_1^2-\sigma_1^2}}e^{-\sqrt{\beta_1^2-\sigma_1^2}\,x}-\frac{1}{\sqrt{\beta_1^2-\sigma_1^2}}\right\} \tag{3.25e}$$

Rewritten in outer variable:

$$\bar{\bar{q}}_o(y)\left\{\frac{1-x}{\varepsilon}+\frac{1}{\sqrt{\beta_1^2-\sigma_1^2}}e^{-\sqrt{\beta_1^2-\sigma_1^2}\frac{(1-x)}{\varepsilon}}-\frac{1}{\sqrt{\beta_1^2-\sigma_1^2}}\right\}$$

$$+\varepsilon\bar{\bar{q}}_1(y)\left\{\frac{1-x}{\varepsilon}+\frac{1}{\sqrt{\beta_1^2-\sigma_1^2}}e^{-\sqrt{\beta_1^2-\sigma_1^2}\frac{(1-x)}{\varepsilon}}-\frac{1}{\sqrt{\beta_1^2-\sigma_1^2}}\right\} \tag{3.25f}$$

Expanded for small ε:

As $\varepsilon\to 0$, $e^{-\sqrt{\beta_1^2-\sigma_1^2}\frac{(1-x)}{\varepsilon}}$ becomes $e^{-\infty}=0$

$$\bar{\bar{q}}_o(y)\frac{(1-x)}{\varepsilon}+\frac{\bar{\bar{q}}_o(y)}{\sqrt{\beta_1^2-\sigma_1^2}}e^{-\sqrt{\beta_1^2-\sigma_1^2}\frac{(1-x)}{\varepsilon}}-\frac{\bar{\bar{q}}_o(y)}{\sqrt{\beta_1^2-\sigma_1^2}}$$

$$+\bar{\bar{q}}_1(y)(1-x)+\frac{\varepsilon\bar{\bar{q}}_1(y)}{\sqrt{\beta_1^2-\sigma_1^2}}e^{-\sqrt{\beta_1^2-\sigma_1^2}\frac{(1-x)}{\varepsilon}}-\frac{\varepsilon\bar{\bar{q}}_1(y)}{\sqrt{\beta_1^2-\sigma_1^2}} \qquad (3.25g)$$

Two terms outer expansion:

$$\bar{\bar{q}}_1(y)(1-x)-\frac{\varepsilon\bar{\bar{q}}_1(y)}{\sqrt{\beta_1^2-\sigma_1^2}} \qquad (3.25h)$$

Equating (3.25d) and (3.25h) according to matching principle, we obtain

$$\left\{\frac{2}{b}E_4\cos\sigma\sin\frac{m\pi}{b}y+\frac{2}{b}F_4\sin\sigma\sin\frac{m\pi}{b}y\right\}+\varepsilon X\left\{A_4\sigma\sin\sigma\sin\frac{m\pi}{b}y-A_4e^{-\frac{s}{u}}\frac{s}{u}\sin\frac{m\pi}{b}y\right.$$

$$\left.-A_4e^{-\frac{s}{u}}\sigma\frac{\cos\sigma}{\sin\sigma}\sin\frac{m\pi}{b}y+A_4\sigma\frac{\cos^2\sigma}{\sin\sigma}\sin\frac{m\pi}{b}y\right\}=\bar{\bar{q}}_1(y)(1-x)-\frac{\varepsilon\bar{\bar{q}}_1(y)}{\sqrt{\beta_1^2-\sigma_1^2}}$$

which implies that

$$\bar{\bar{q}}_1(y)=\frac{2P_ou\sin\frac{m\pi}{b}y_o}{b\left(\beta_1^2-\sigma_1^2\right)\left(s^2+\sigma^2u^2\right)}\left[\sigma\sin\sigma-\sigma e^{-\frac{s}{u}}\frac{\cos\sigma}{\sin\sigma}+\sigma\cos\sigma\cot\sigma-\frac{s}{u}e^{-\frac{s}{u}}\right]\sin\frac{m\pi}{b}y$$

and

$$\frac{2}{b}E_4\cos\sigma\sin\frac{m\pi}{b}y=-\frac{\bar{\bar{q}}_1(y)}{\sqrt{\beta_1^2-\sigma_1^2}}-\frac{2}{b}F_4\sin\sigma\sin\frac{m\pi}{b}y$$

Hence

$$F_4=\frac{-P_ou\sin\frac{m\pi}{b}y_o}{\left(\beta_1^2-\sigma_1^2\right)^{\frac{3}{2}}\left(s^2+\sigma^2u^2\right)}\left[\sigma-2e^{-\frac{s}{u}}\frac{\cot\sigma}{\sin\sigma}+2\sigma\cot^2\sigma-\frac{se^{-\frac{s}{u}}}{u\sin\sigma}-\frac{s}{u}\cot\sigma\right] \qquad (3.26)$$

Matching (2-2 matching) procedure for the boundary condition y = 0.

Two term outer expansion:

$V^o = V_0^o + \varepsilon V_1^o$

which implies

$$A_4\left[\cos\sigma x - e^{-\frac{1}{\varepsilon}x} + e^{-\frac{1}{\varepsilon}x}\frac{\sin\sigma x}{\sin\sigma} - \frac{\cos\sigma\sin\sigma x}{\sin\sigma}\right]\sin\frac{m\pi}{b}y$$

$$+2\varepsilon C_3\cos\alpha y\sin n\pi x + 2\varepsilon D_3\sin\alpha y\sin n\pi x$$

$$+\frac{2}{b}\varepsilon E_4\cos\sigma x\sin\frac{m\pi}{b}y + \frac{2}{b}\varepsilon F_4\sin\sigma x\sin\frac{m\pi}{b}y \qquad (3.27a)$$

Rewritten in inner variable:

For inner variable $Y = \frac{y}{\varepsilon}$ which implies that $y = \varepsilon Y$ is given by

$$= A_4\left[\cos\sigma x - e^{-\frac{1}{\varepsilon}x} + \frac{e^{-\frac{1}{\varepsilon}}\sin\sigma x}{\sin\sigma} - \frac{\cos\sigma\sin\sigma x}{\sin\sigma}\right]\sin\frac{m\pi}{b}\varepsilon Y$$

$$+2\varepsilon C_3 y\sin n\pi x\cos\alpha\varepsilon Y + 2\varepsilon D_3\sin n\pi x\sin\alpha\varepsilon Y$$

$$+\frac{2}{b}\varepsilon\left[E_4\cos\sigma x + F_4\sin\sigma x\right]\sin\frac{m\pi}{b}\varepsilon Y \qquad (3.27b)$$

Expanded for small ε:

$$A_4\left[\cos\sigma x - e^{-\frac{1}{\varepsilon}x} + \frac{e^{-\frac{1}{\varepsilon}}\sin\sigma x}{\sin\sigma} - \frac{\cos\sigma\sin\sigma x}{\sin\sigma}\right]\left[\frac{m\pi\varepsilon Y}{b} - \frac{m^3\pi^3\varepsilon^3Y^3}{3!b^3} + \dots\dots\dots\right]$$

$$+2\varepsilon C_3 y \sin n\pi x\left[1-\frac{\alpha^2\varepsilon^2Y^2}{2!}+\frac{\alpha^4\varepsilon^4Y^4}{4!}-\ldots\ldots\ldots\ldots\right]$$

$$+2\varepsilon D_3 \sin n\pi x\left[\alpha\varepsilon Y-\frac{\alpha^3\varepsilon^3Y^3}{3!}+\frac{\alpha^5\varepsilon^5Y^5}{5!}-\ldots\ldots\ldots\ldots\right]$$

$$+\frac{2}{b}\varepsilon\left[E_4\cos\sigma x+F_4\sin\sigma x\right]\left[\frac{m\pi\varepsilon Y}{b}-\frac{m^3\pi^3\varepsilon^3Y^3}{3!b^3}+\frac{m^5\pi^5\varepsilon^5Y^5}{5!b^5}\ldots\ldots\ldots\ldots\right] \tag{3.27c}$$

Two terms inner expansion:

$$\varepsilon[2C_3\sin n\pi x]+\varepsilon Y\left\{A_4\frac{m\pi}{b}\left[\cos\sigma x-e^{-\frac{1}{\varepsilon}}+\frac{e^{-\frac{1}{\varepsilon}}\sin\sigma x}{\sin\sigma}-\frac{\cos\sigma\sin\sigma x}{\sin\sigma}\right]\right\}$$

$$\varepsilon^2Y\left\{2\alpha D_3\sin n\pi x+\frac{2m\pi}{b^2}\varepsilon\left[E_4\cos\sigma x+F_4\sin\sigma x\right]\right\} \tag{3.27d}$$

Two terms inner expansion:

$$\psi=\psi_0+\varepsilon\psi_1$$

$$=\bar{F}_0(x)\left\{Y+\frac{1}{\sqrt{\beta_2^2-\sigma_1^2}}e^{-\sqrt{\beta_2^2-\sigma_1^2}Y}-\frac{1}{\sqrt{\beta_2^2-\sigma_1^2}}\right\}$$

$$+\varepsilon\bar{F}_1(x)\left\{Y+\frac{1}{\sqrt{\beta_2^2-\sigma_1^2}}e^{-\sqrt{\beta_2^2-\sigma_1^2}Y}-\frac{1}{\sqrt{\beta_2^2-\sigma_1^2}}\right\} \tag{3.27e}$$

Rewritten in outer variable:

$$\bar{F}_0(x)\left\{\frac{y}{\varepsilon}+\frac{1}{\sqrt{\beta_2^2-\sigma_1^2}}e^{-\sqrt{\beta_2^2-\sigma_1^2}\frac{y}{\varepsilon}}-\frac{1}{\sqrt{\beta_2^2-\sigma_1^2}}\right\},$$

$$+\varepsilon\bar{r}_1(x)\left\{\frac{y}{\varepsilon}+\frac{1}{\sqrt{\beta_2^2-\sigma_1^2}}e^{-\sqrt{\beta_2^2-\sigma_1^2}\frac{y}{\varepsilon}}-\frac{1}{\sqrt{\beta_2^2-\sigma_1^2}}\right\} \tag{3.27f}$$

Expanded for small ε:

As $\varepsilon \to 0$, $e^{-\sqrt{\beta_2^2-\sigma_1^2}\frac{y}{\varepsilon}}$ becomes $e^{-\infty}=0$

$$\bar{r}_o(x)\frac{y}{\varepsilon}+\frac{\bar{r}_o(x)}{\sqrt{\beta_2^2-\sigma_1^2}}e^{-\sqrt{\beta_2^2-\sigma_1^2}\frac{y}{\varepsilon}}-\frac{\bar{r}_o(x)}{\sqrt{\beta_2^2-\sigma_1^2}}$$

$$+\bar{r}_1(x)y+\frac{\varepsilon\bar{r}_1(x)}{\sqrt{\beta_2^2-\sigma_1^2}}e^{-\sqrt{\beta_2^2-\sigma_1^2}\frac{y}{\varepsilon}}-\frac{\varepsilon\bar{r}_1(x)}{\sqrt{\beta_2^2-\sigma_1^2}} \tag{3.27g}$$

Two terms outer expansion:

$$\bar{r}_1(x)y-\frac{\varepsilon\bar{r}_1(x)}{\sqrt{\beta_2^2-\sigma_1^2}} \tag{3.27h}$$

Equating (3.27d) and (3.27h) according to matching principle, we obtain

$$\varepsilon[2C_3\sin n\pi x]+\varepsilon Y\left\{A_4\frac{m\pi}{b}\left[\cos\sigma x-e^{-\frac{1}{\varepsilon}x}+\frac{e^{-\frac{1}{\varepsilon}}\sin\sigma x}{\sin\sigma}-\frac{\cos\sigma\sin\sigma x}{\sin\sigma}\right]\right\}$$

$$\varepsilon^2 Y\left\{2\alpha D_3\sin n\pi x+\frac{2m\pi}{b^2}\varepsilon\left[E_4\cos\sigma x+F_4\sin\sigma x\right]\right\}=\bar{r}_1(x)y-\frac{\varepsilon\bar{r}_1(x)}{\sqrt{\beta_2^2-\sigma_1^2}}$$

which implies that

$$2C_3\sin n\pi x=-\frac{\bar{r}_1(x)}{\sqrt{\beta_2^2-\sigma_1^2}}$$

and

$$\bar{r}_1(x)=\frac{2m\pi P_0 u\sin\frac{m\pi}{b}y_0}{b^3\left(\beta_1^2-\sigma_1^2\right)\left(s^2+\sigma^2u^2\right)}\left[\cos\sigma x+e^{-\frac{s}{u}}\frac{\sin\sigma x}{\sin\sigma}-\cot\sigma\sin\sigma x-e^{-\frac{s}{u}x}\right]$$

Hence

$$C_3=\frac{P_0 u m\pi\sin\frac{m\pi}{b}y_0}{b^2(\beta_1^2-\sigma_1^2)(\beta_2^2-\sigma_1^2)^{\frac{1}{2}}(s^2+\sigma^2u^2)\sin n\pi x}\Big\{\cos\sigma x-e^{\frac{-s}{u}}$$

$$+e^{-\frac{s}{u}}\frac{\sin\sigma x}{\sin\sigma}-\frac{\cos\sigma\sin\sigma x}{\sin\sigma}\Big\}\tag{3.28}$$

Matching (2-2 matching) procedure for the boundary condition y = b.

Two term outer expansion:

$V^o=V_0^o+\varepsilon V_1^o$

which implies

$$A_4\left[\cos\sigma x-e^{-\frac{s}{u}x}+e^{-\frac{s}{u}x}\frac{\sin\sigma x}{\sin\sigma}-\frac{\cos\sigma\sin\sigma x}{\sin\sigma}\right]\sin\frac{m\pi}{b}y$$

$$+2\varepsilon C_3\cos\sigma y\sin n\pi x+2\varepsilon D_3\sin\sigma y\sin n\pi$$

$$+\frac{2}{b}\varepsilon E_4\cos\sigma x\sin\frac{m\pi}{b}y+\frac{2}{b}\varepsilon F_4\sin\sigma x\sin\frac{m\pi}{b}y\tag{3.29a}$$

Rewritten in inner variable:

For inner variable $Y=\frac{b-y}{\varepsilon}$ which implies that $y=b-\varepsilon Y$ is given by

$$= A_4\left[\cos\sigma x - e^{-\frac{1}{\varepsilon}x} + \frac{e^{-\frac{1}{\varepsilon}}\sin\sigma x}{\sin\sigma} - \frac{\cos\sigma\sin\sigma x}{\sin\sigma}\right]\sin\frac{m\pi}{b}(b-\varepsilon Y)$$

$$+ 2\varepsilon C_3 y\sin n\pi x\cos\alpha(b-\varepsilon Y) + 2\varepsilon D_3\sin n\pi x\sin\alpha(b-\varepsilon Y)$$

$$+ \frac{2}{b}\varepsilon\left[E_4\cos\sigma x + F_4\sin\sigma x\right]\sin\frac{m\pi}{b}(b-\varepsilon Y) \qquad (3.29b)$$

Expanded for small ε:

$$-(-1)^m A_4\left[\cos\sigma x - e^{-\frac{1}{\varepsilon}x} + \frac{e^{-\frac{1}{\varepsilon}}\sin\sigma x}{\sin\sigma} - \frac{\cos\sigma\sin\sigma x}{\sin\sigma}\right]\left[\frac{m\pi\varepsilon Y}{b} - \frac{m^3\pi^3\varepsilon^3 Y^3}{3!b^3} + \ldots\ldots\ldots\ldots\right]$$

$$+ 2\varepsilon C_3 y\sin n\pi x\cos\alpha b\left[1 - \frac{\alpha^2\varepsilon^2 Y^2}{2!} + \frac{\alpha^4\varepsilon^4 Y^4}{4!} - \ldots\ldots\ldots\ldots\right]$$

$$+ 2\varepsilon C_3\sin n\pi x\sin\alpha b\left[\alpha\varepsilon Y - \frac{\alpha^3\varepsilon^3 Y^3}{3!} + \frac{\alpha^5\varepsilon^5 Y^5}{5!} - \ldots\ldots\ldots\ldots\right]$$

$$+ 2\varepsilon D_3 y\sin n\pi x\sin\alpha b\left[\alpha\varepsilon Y - \frac{\alpha^3\varepsilon^3 Y^3}{3!} + \frac{\alpha^5\varepsilon^5 Y^5}{5!} - \ldots\ldots\ldots\ldots\right]$$

$$- 2\varepsilon D_3\sin n\pi x\cos\alpha b\left[\alpha\varepsilon Y - \frac{\alpha^3\varepsilon^3 Y^3}{3!} + \frac{\alpha^5\varepsilon^5 Y^5}{5!} - \ldots\right.$$

$$-\frac{2}{b}(-1)^m\varepsilon\left[E_4\cos\sigma x + F_4\sin\sigma x\right]\left[\frac{m\pi\varepsilon Y}{b} - \frac{m^3\pi^3\varepsilon^3 Y^3}{3!b^3} + \ldots\ldots\ldots\ldots\right] \qquad (3.29c)$$

Two terms inner expansion:

$$\varepsilon[2C_3\sin n\pi x\cos\alpha b + 2C_3\sin n\pi x\sin\alpha b]$$

$$-\varepsilon Y\left\{A_4(-1)^m\frac{m\pi}{b}\left[\cos\sigma x-e^{-\frac{1}{\varepsilon}x}+\frac{e^{-\frac{1}{\varepsilon}}\sin\sigma x}{\sin\sigma}-\frac{\cos\sigma\sin\sigma x}{\sin\sigma}\right]\right\}$$

$$+\varepsilon^2 Y\left\{2\alpha C_3\sin n\pi x\sin\alpha b-2\alpha D_3\sin n\pi x\cos\alpha b+\frac{2m\pi}{b^2}(-1)^n\varepsilon\left[E_4\cos\sigma x+F_4\sin\sigma x\right]\right\}$$

(3.29d)

Two terms inner expansion:

$$\psi=\psi_0+\varepsilon\psi_1$$

$$=\bar{\bar{r}}_a(x)\left\{Y+\frac{1}{\sqrt{\beta_2^2-\sigma_1^2}}e^{-\sqrt{\beta_2^2-\sigma_1^2}Y}-\frac{1}{\sqrt{\beta_2^2-\sigma_1^2}}\right\}$$

$$+\varepsilon\bar{\bar{r}}_1(x)\left\{Y+\frac{1}{\sqrt{\beta_2^2-\sigma_1^2}}e^{-\sqrt{\beta_2^2-\sigma_1^2}Y}-\frac{1}{\sqrt{\beta_2^2-\sigma_1^2}}\right\} \tag{3.29e}$$

Rewritten in outer variable:

$$\bar{\bar{r}}_o(x)\left\{\frac{b-y}{\varepsilon}+\frac{1}{\sqrt{\beta_2^2-\sigma_1^2}}e^{-\sqrt{\beta_2^2-\sigma_1^2}\frac{(b-y)}{\varepsilon}}-\frac{1}{\sqrt{\beta_2^2-\sigma_1^2}}\right\}$$

$$+\varepsilon\bar{\bar{r}}_1(x)\left\{\frac{b-y}{\varepsilon}+\frac{1}{\sqrt{\beta_2^2-\sigma_1^2}}e^{-\sqrt{\beta_2^2-\sigma_1^2}\frac{(b-y)}{\varepsilon}}-\frac{1}{\sqrt{\beta_2^2-\sigma_1^2}}\right\} \tag{3.29f}$$

Expanded for small ε:

As $\varepsilon\to 0$, $e^{-\sqrt{\beta_2^2-\sigma_1^2}\frac{(b-y)}{\varepsilon}}$ becomes $e^{-\infty}=0$

$$\bar{\bar{r}}_o(x)\frac{b-y}{\varepsilon}+\frac{\bar{\bar{r}}_o(x)}{\sqrt{\beta_2^2-\sigma_1^2}}e^{-\sqrt{\beta_2^2-\sigma_1^2}\frac{(b-y)}{\varepsilon}}-\frac{\bar{\bar{r}}_o(x)}{\sqrt{\beta_2^2-\sigma_1^2}}$$

$$+\bar{\bar{r}}_1(x)y+\frac{\varepsilon\bar{\bar{r}}_1(x)}{\sqrt{\beta_2^2-\sigma_1^2}}e^{-\sqrt{\beta_2^2-\sigma_1^2}\frac{(b-y)}{\varepsilon}}-\frac{\varepsilon\bar{\bar{r}}_1(x)}{\sqrt{\beta_2^2-\sigma_1^2}} \tag{3.29g}$$

Two terms outer expansion:

$$\bar{\bar{r}}_1(x)y-\frac{\varepsilon\bar{\bar{r}}_1(x)}{\sqrt{\beta_2^2-\sigma_1^2}} \tag{3.29h}$$

Equating (3.29d) and (3.29h) according to matching principle, we obtain

$$\varepsilon[2C_3\sin n\pi x\cos\alpha b+2C_3\sin n\pi x\sin\alpha b]$$

$$-\varepsilon Y\left\{A_4(-1)^m\frac{m\pi}{b}\left[\cos\sigma x-e^{-\frac{1}{\varepsilon}x}+\frac{e^{-\frac{1}{\varepsilon}}\sin\sigma x}{\sin\sigma}-\frac{\cos\sigma\sin\sigma x}{\sin\sigma}\right]\right\}$$

$$+\varepsilon^2Y\left\{2\alpha C_3\sin n\pi x\sin\alpha b-2\alpha D_3\sin n\pi x\cos\alpha b+\frac{2m\pi}{b^2}(-1)^m\varepsilon\left[E_4\cos\sigma x+F_4\sin\sigma x\right]\right\}$$

$$=\bar{\bar{r}}_1(x)y-\frac{\varepsilon\bar{\bar{r}}_o(x)}{\sqrt{\beta_2^2-\sigma_1^2}}$$

which implies that

$$\bar{\bar{r}}_1(x)=\frac{2(-1)^m m\pi P_o u\sin\frac{m\pi}{b}y_o}{b^2\left(\beta_1^2-\sigma_1^2\right)\left(s^2+\sigma^2u^2\right)}\left[e^{-\frac{1}{\varepsilon}x}-\cos\sigma x-e^{-\frac{1}{\varepsilon}}\frac{\sin\sigma x}{\sin\sigma}+\cot\sigma\sin\sigma x\right]$$

and

$$2C_3\sin n\pi x\cos\alpha b+2C_3\sin n\pi x\sin\alpha b=-\frac{\bar{\bar{r}}_1(x)}{\sqrt{\beta_2^2-\sigma_1^2}}$$

Hence

$$D_3=\frac{2P_o u m\pi\sin\frac{m\pi}{b}y_o}{b^2\left(\beta_1^2-\sigma_1^2\right)\left(\beta_2^2-\sigma_1^2\right)^{\frac{1}{2}}\left(s^2+\sigma^2u^2\right)\sin n\pi x}\left\{(-1)^m\frac{\cos\sigma x}{\sin\alpha b}-\frac{(-1)^n e^{-\frac{s}{u}x}}{\sin\alpha b}\right.$$

$$+\frac{(-1)^m e^{-\frac{s}{u}}\sin\sigma x}{\sin\sigma\sin\alpha b}-\frac{(-1)^m\cos\sigma\sin\sigma x}{\sin\sigma\sin\alpha b}+\cos\sigma x\cot\alpha b$$

$$\left.-e^{-\frac{s}{u}}\cos\alpha b+\frac{e^{-\frac{s}{u}}\sin\sigma x\cot\alpha b}{\sin\sigma b}-\cot\sigma\sin\sigma x\cot\alpha b\right\} \tag{3.30}$$

Substituting (3.24), (3.26), (3.28), and(3.30) into (3.8), it is straight forward to show that

$$V_1(x,y)=\frac{2P_o u m\pi\sin\frac{m\pi}{b}y_o}{b^2\left(\beta_1^2-\sigma_1^2\right)\left(\beta_2^2-\sigma_1^2\right)^{\frac{1}{2}}\left(s^2+\sigma^2u^2\right)}\left\{-\cos\sigma x\cos\alpha y+e^{-\frac{s}{u}x}\cos\alpha y\right.$$

$$-e^{-\frac{s}{u}x}\frac{\sin\sigma x\cos\alpha y}{\sin\sigma}+\cot\sigma\sin\sigma x\cos\alpha y+\frac{(-1)^m\cos\sigma x\sin\alpha y}{\sin\alpha b}$$

$$-(-1)^n e^{-\frac{s}{u}x}\frac{\sin\alpha y}{\sin\alpha b}+(-1)^m e^{-\frac{s}{u}}\frac{\sin\sigma x\sin\alpha y}{\sin\sigma\sin\alpha b}-(-1)^m\frac{\cos\sigma\sin\sigma x\sin\alpha y}{\sin\sigma\sin\alpha b}$$

$$+\cos\sigma x\cot\alpha b\sin\alpha y-e^{-\frac{s}{u}x}\cot\alpha b\sin\alpha y+e^{-\frac{s}{u}}\frac{\sin\sigma x\cot\alpha b\sin\alpha y}{\sin\sigma}$$

$$\left.-\cot\sigma\sin\sigma x\cot\alpha b\sin\alpha y\right\}$$

$$+\frac{2P_o u\sin\frac{m\pi}{b}y_o\sin\frac{m\pi}{b}y}{\left(\beta_1^2-\sigma_1^2\right)^{\frac{3}{2}}\left(s^2+\sigma^2u^2\right)}\left\{\frac{\sigma\cos\sigma\cos\sigma x}{\sin\sigma}-\sigma e^{-\frac{s}{u}}\frac{\cos\sigma x}{\sin\sigma}-\frac{s}{u}\cos\sigma x\right.$$

$$\left.-\sigma\sin\sigma x+2e^{-\frac{s}{u}}\frac{\cot\sigma\sin\sigma x}{\sin\sigma}-2\sigma\cot^2\sigma\sin\sigma x+\frac{s}{u}e^{-\frac{s}{u}}\frac{\sin\sigma x}{\sin\sigma}+\frac{s}{u}\cot\sigma\sin\sigma x\right\}$$

$$\tag{3.31}$$

It remains to obtain the Laplace inversion of equation (3.31).

3.3 LAPLACE INVERSION OF FIRST ORDER SOLUTION

It is observed that α and σ are complex expressions, as such the first order equation (3.31) can be rewritten as

$$V_1(x,y) = P_{a_1}\left\{\frac{-\cosh\sigma^{\epsilon}x\cosh\alpha^{\epsilon}y}{s^2+\sigma^2u^2}+\frac{e^{\frac{s}{u}x}\cosh\alpha^{\epsilon}y}{s^2+\sigma^2u^2}-\frac{(-1)^m e^{-\frac{s}{u}}\sinh\sigma^{\epsilon}x\cosh\alpha^{\epsilon}y}{\left(s^2+\sigma^2u^2\right)\sinh\sigma^{\epsilon}}\right.$$

$$+\frac{\coth\sigma^{\epsilon}\sinh\sigma^{\epsilon}x\cosh\alpha^{\epsilon}y}{s^2+\sigma^2u^2}+\frac{(-1)^m\cosh\sigma^{\epsilon}x\sinh\alpha^{\epsilon}y}{\left(s^2+\sigma^2u^2\right)\sinh\alpha^{\epsilon}y}-\frac{(-1)^m e^{\frac{s}{u}y}\sinh\alpha^{\epsilon}y}{\left(s^2+\sigma^2u^2\right)\sinh\alpha^{\epsilon}b}$$

$$+\frac{(-1)^m e^{-\frac{s}{u}}\sinh\sigma^{\epsilon}x\sinh\alpha^{\epsilon}y}{\left(s^2+\sigma^2u^2\right)\sinh\sigma^{\epsilon}\sinh\alpha^{\epsilon}y}-\frac{(-1)^m\cosh\sigma\sinh\sigma^{\epsilon}x\sinh\alpha^{\epsilon}y}{\left(s^2+\sigma^2u^2\right)\sinh\sigma^{\epsilon}\sinh\alpha^{\epsilon}y}$$

$$+\frac{\cosh\sigma^{\epsilon}x\coth\alpha^{\epsilon}b\sinh\alpha^{\epsilon}y}{s^2+\sigma^2u^2}-\frac{e^{-\frac{s}{u}x}\coth\alpha^{\epsilon}b\sinh\alpha^{\epsilon}y}{s^2+\sigma^2u^2}$$

$$\left.+\frac{e^{-\frac{s}{u}}\sinh\sigma^{\epsilon}x\coth\alpha^{\epsilon}b\sinh\alpha^{\epsilon}y}{\left(s^2+\sigma^2u^2\right)\sinh\sigma^{\epsilon}}-\frac{\coth\sigma^{\epsilon}\sinh\sigma^{\epsilon}x\coth\alpha^{\epsilon}b\sinh\alpha^{\epsilon}y}{s^2+\sigma^2u^2}\right\}$$

$$+P_{a_2}\left\{\frac{\sigma\cosh\sigma^{\epsilon}\cosh\sigma^{\epsilon}x}{\left(s^2+\sigma^2u^2\right)\sinh\sigma^{\epsilon}}-\frac{\sigma e^{-\frac{s}{u}}\cosh\sigma^{\epsilon}x}{\left(s^2+\sigma^2u^2\right)\sinh\sigma^{\epsilon}}\right.$$

$$-\frac{\frac{s}{u}\cosh\sigma^{\epsilon}x+\sigma\sinh\sigma^{\epsilon}x-\frac{s}{u}\coth\sigma^{\epsilon}\sinh\sigma^{\epsilon}x+2\sigma\coth^2\sigma^{\epsilon}\sinh\sigma^{\epsilon}x}{s^2+\sigma^2u^2}$$

$$+\frac{2e^{-\frac{s}{u}}\coth\sigma^{\varepsilon}\sinh\sigma^{\varepsilon}x}{\left(s^{2}+\sigma^{2}u^{2}\right)\sinh\sigma^{\varepsilon}}+\frac{se^{-\frac{s}{u}}\sinh\sigma^{\varepsilon}x}{\left(s^{2}+\sigma^{2}u^{2}\right)u\sinh\sigma^{\varepsilon}}\Bigg\} \tag{3.32}$$

where

$$P_{a_1}=\frac{2P_o u m\pi\sin\frac{m\pi}{b}y_o}{b^{2}\left(\beta_1^{2}-\sigma_1^{2}\right)\left(\beta_2^{2}-\sigma_1^{2}\right)^{\frac{1}{2}}} \tag{3.33}$$

$$P_{a_2}=\frac{2P_o u\sin\frac{m\pi}{b}y_o\sin\frac{m\pi}{b}y}{\left(\beta_1^{2}-\sigma_1^{2}\right)^{\frac{1}{2}}} \tag{3.34}$$

The Laplace inversion of (3.32) is defined as

$$\begin{aligned}V_1(x,y)=P_{a_1}\{&-G_1(x,y;t)+G_2(x,y;t)-G_3(x,y;t)+G_4(x,y;t)+G_5(x,y;t)\\&-G_6(x,y;t)+G_7(x,y;t)-G_8(x,y;t)+G_9(x,y;t)-G_{10}(x,y;t)\\&+G_{11}(x,y;t)-G_{12}(x,y;t)\}\\&+P_{a_2}\{-G_{13}(x,y;t)-G_{14}(x,y;t)-G_{15}(x,y;t)+G_{16}(x,y;t)+G_{17}(x,y;t)\}\end{aligned} \tag{3.35}$$

where

$$G_1(x,y;t)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{e^{st}\cosh\sigma^{\varepsilon}x\cosh\alpha^{\varepsilon}y}{s^{2}+\sigma^{2}u^{2}}ds$$

$$G_2(x,y;t)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{e^{s\left(t-\frac{x}{u}\right)}\cosh\alpha^{\varepsilon}y}{s^{2}+\sigma^{2}u^{2}}ds$$

$$G_3(x,y;t)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{e^{s\left(t-\frac{1}{u}\right)}\sinh\sigma^{\varepsilon}x\cosh\alpha^{\varepsilon}y}{\left(s^{2}+\sigma^{2}u^{2}\right)\sinh\sigma^{\varepsilon}}ds$$

$$G_4(x,y;t)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{e^{st}\coth\sigma^{\varepsilon}\sinh\sigma^{\varepsilon}x\cosh\alpha^{\varepsilon}y}{s^{2}+\sigma^{2}u^{2}}ds$$

$$G_5(x,y;t)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{(-1)^m e^{st}\cosh\sigma^{\epsilon}x\sinh\alpha^{\epsilon}y}{\left(s^2+\sigma^2u^2\right)\sinh\sigma^{\epsilon}b}ds$$

$$G_6(x,y;t)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{(-1)^m e^{s\left(t-\frac{x}{u}\right)}\sinh\alpha^{\epsilon}y}{\left(s^2+\sigma^2u^2\right)\sinh\alpha^{\epsilon}b}ds$$

$$G_7(x,y;t)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{(-1)^m e^{s\left(t-\frac{1}{u}\right)}\sinh\sigma^{\epsilon}x\sinh\alpha^{\epsilon}y}{\left(s^2+\sigma^2u^2\right)\sinh\sigma^{\epsilon}\sinh\alpha^{\epsilon}b}ds$$

$$G_8(x,y;t)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{(-1)e^{st}\cosh\sigma^{\epsilon}\sinh\sigma^{\epsilon}x\sinh\alpha^{\epsilon}y}{\left(s^2+\sigma^2u^2\right)\sinh\sigma^{\epsilon}\sinh\alpha^{\epsilon}b}ds$$

$$G_9(x,y;t)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{e^{st}\cosh\sigma^{\epsilon}x\coth\alpha^{\epsilon}b\sinh\alpha^{\epsilon}y}{s^2+\sigma^2u^2}ds$$

$$G_{10}(x,y;t)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{e^{s\left(t-\frac{x}{u}\right)}\coth\alpha^{\epsilon}b\sinh\alpha^{\epsilon}y}{\left(s^2+\sigma^2u^2\right)}ds$$

$$G_{11}(x,y;t)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{e^{s\left(t-\frac{1}{u}\right)}\sinh\sigma^{\epsilon}x\coth\alpha^{\epsilon}b\sinh\alpha^{\epsilon}y}{\left(s^2+\sigma^2u^2\right)\sinh\sigma^{\epsilon}}ds$$

$$G_{12}(x,y;t)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{e^{st}\coth\sigma^{\epsilon}\sinh\sigma^{\epsilon}x\coth\alpha^{\epsilon}b\sinh\alpha^{\epsilon}y}{s^2+\sigma^2u^2}ds$$

$$G_{13}(x,y;t)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{e^{st}\sigma\cosh\sigma^{\epsilon}\cosh\sigma^{\epsilon}x}{\left(s^2+\sigma^2u^2\right)\sinh\sigma^{\epsilon}}ds$$

$$G_{14}(x,y;t)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{e^{s\left(t-\frac{1}{u}\right)}\sigma\cosh\sigma^{\epsilon}x}{\left(s^2+\sigma^2u^2\right)\sinh\sigma^{\epsilon}}ds$$

$$G_{15}(x,y;t)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}e^{st}H(x;s)ds$$

where

$$H(x;s)=\frac{\frac{s}{u}\cosh\sigma^{\epsilon}x+\sigma\sinh\sigma^{\epsilon}x-\frac{s}{u}\coth\sigma^{\epsilon}\sinh\sigma^{\epsilon}x+2\sigma\coth^{2}\sigma^{\epsilon}\sinh\sigma^{\epsilon}x}{s^{2}+\sigma^{2}u^{2}}$$

$$G_{16}(x,y;t)=\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{2e^{s\left(t-\frac{1}{u}\right)}\coth\sigma^{\epsilon}\sinh\sigma^{\epsilon}x}{\left(s^{2}+\sigma^{2}u^{2}\right)\sinh\sigma^{\epsilon}}ds$$

$$G_{17}(x,y;t)=\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{se^{s\left(t-\frac{1}{u}\right)}\sinh\sigma^{\epsilon}x}{\left(s^{2}+\sigma^{2}u^{2}\right)u\sinh\sigma^{\epsilon}}ds \tag{3.36}$$

$$\alpha^{\epsilon}=\frac{1}{\left(\beta_{2}^{2}-\sigma_{1}^{2}\right)^{\frac{1}{2}}}\left[\left(\beta_{1}^{2}-\sigma_{1}^{2}\right)n^{2}\pi^{2}+\eta+s^{2}\right]^{\frac{1}{2}} \tag{3.37}$$

To evaluate the integrals (3.36), we shall follow the procedure earlier outlined for $F_1(x,y;t)-F_4(x,y;t)$. Thus, we notice that integrals $G_1(x,y;t)$, $G_2(x,y;t)$, $G_4(x,y;t)$, $G_9(x,y;t)$, $G_{10}(x,y;t)$, $G_{12}(x,y;t)$ and $G_{15}(x,y;t)$ have simple poles at $s=\pm\Omega_1$.

Thus, their results are listed as follows

$$G_1(x,y;t)=\frac{1}{2A_1\Omega_1}\cosh\Omega_3 x\cosh\Omega_6 y\left(e^{\Omega_1 t}-e^{-\Omega_1 t}\right)$$

Where $\Omega_5=\left(\beta_1^2-\sigma_1^2\right)n^2\pi^2+\eta$ (3.38)

$$\Omega_6=\frac{1}{\left(\beta_1^2-\sigma_1^2\right)^{\frac{1}{2}}}\left(\Omega_1^2+\Omega_5\right)^{\frac{1}{2}} \tag{3.39}$$

$$G_2(x,y;t)=\frac{1}{2A_1\Omega_1}\cosh\Omega_6 y\left[e^{\Omega_1\left(t-\frac{x}{u}\right)}-e^{-\Omega_1\left(t-\frac{x}{u}\right)}\right]$$

$$G_4(x,y;t)=\frac{1}{2A_1\Omega_1}\coth\Omega_3\sinh\Omega_3 x\cosh\Omega_6 y\left(e^{\Omega_1 t}-e^{-\Omega_1 t}\right)$$

$$G_9(x,y;t)=\frac{1}{2A_1\Omega_1}\cosh\Omega_3 x\sinh\Omega_6 b\sinh\Omega_6 y\left(e^{\Omega_1 t}-e^{-\Omega_1 t}\right)$$

$$G_{10}(x,y;t)=\frac{1}{2A_1\Omega_1}\coth\Omega_6 b\sinh\Omega_6 y\left[e^{\Omega_1\left(t-\frac{x}{u}\right)}-e^{-\Omega_1\left(t-\frac{x}{u}\right)}\right]$$

$$G_{12}(x,y;t)=\frac{1}{2A_1\Omega_1}\coth\Omega_3\sinh\Omega_3 x\coth\Omega_6\sinh\Omega_6 y\left(e^{\Omega_1 t}-e^{-\Omega_1 t}\right)$$

$$G_{15}(x,y;t)=\frac{1}{2A_1\Omega_1}\left(G_{15a}(x,y;t)+G_{15b}(x,y;t)+G_{15c}(x,y;t)\right)\left(e^{\Omega_1 t}-e^{-\Omega_1 t}\right)$$

$$G_{15a}(x,y;t)=\frac{\Omega_1}{u}\cosh\Omega_3 x+i\Omega_3\sinh\Omega_3 x \tag{3.40}$$

$$G_{15b}(x,y;t)=-\frac{\Omega_1}{u}\coth\Omega_3\sinh\Omega_3 x \tag{3.41}$$

$$G_{15c}(x,y;t)=2i\Omega_3\coth^2\Omega_3\sinh\Omega_3 x \tag{3.42}$$

Furthermore, integrals

$G_3(x,y;t),\ G_{11}(x,y;t),\ G_{13}(x,y;t),\ G_{14}(x,y;t),\quad G_{16}(x,y;t)\ and\ G_{17}(x,y;t)$

have simple poles at $s=\pm\Omega_1$ and $\sigma^c=\pm i\nu\pi$, a similar case to those of $F_3(x,y;t)$ and $F_4(x,y;t)$ in the leading order solution. Thus, the solutions are

$$G_3(x,y;t)=\frac{\sinh\Omega_3 x\cosh\Omega_6 y}{2A_1\Omega_1\sinh\Omega_3}\left[e^{\Omega_1\left(t-\frac{x}{u}\right)}-e^{-\Omega_1\left(t-\frac{x}{u}\right)}\right]$$

$$+\frac{(-1)^{\nu+1}\left(\beta_1^2-\sigma_1^2\right)\nu\pi\sin\nu\pi x\cosh\alpha^{c*}y}{A_1\Omega_7\Omega_8}\left[e^{i\Omega_7\left(t-\frac{1}{u}\right)}-e^{-i\Omega_7\left(t-\frac{1}{u}\right)}\right]$$

where

$$\alpha^{c*} = \frac{1}{\left(\beta_1^2 - \sigma_1^2\right)^{1/2}}\left[-\nu^2\pi^2\left(\beta_1^2 - \sigma_1^2\right) - \left(\beta_2^2 - \sigma_1^2\right)\frac{m^2\pi^2}{b^2} + \left(\beta_1^2 - \sigma_1^2\right)n^2\pi^2\right]^{\frac{1}{2}} \tag{3.43}$$

$$\Omega_7 = \sqrt{\Omega_4} = \left(\nu^2\pi^2\left(\beta_1^2 - \sigma_1^2\right) + \Omega_2\right)^{\frac{1}{2}} \tag{3.44}$$

$$\Omega_8 = \left(\nu^2\pi^2\left(\beta_1^2 - \sigma_1^2\right) + \Omega_2 + \Omega_1^2\right)^{\frac{1}{2}} \tag{3.45}$$

$$G_{11}(x, y; t) = \frac{\sinh\Omega_3 x \coth\Omega_6 b \cosh\Omega_6 y}{2A_1\Omega_1 \sinh\Omega_6}\left[e^{\Omega_1\left(t-\frac{1}{u}\right)} - e^{-\Omega_1\left(t-\frac{1}{u}\right)}\right]$$

$$+ \frac{(-1)^{\nu+1}\left(\beta_1^2 - \sigma_1^2\right)\nu\pi \sin\nu\pi x \coth\alpha^{c*} \sinh\alpha^{c*} y}{A_1\Omega_7\Omega_8}\left[e^{i\Omega_7\left(t-\frac{1}{u}\right)} - e^{-i\Omega_7\left(t-\frac{1}{u}\right)}\right]$$

$$G_{13}(x, y; t) = \frac{i\Omega_3 \cosh\Omega_3 \cos\Omega_3 x}{2A_1\Omega_1 \sinh\Omega_3}\left(e^{\Omega_1 t} - e^{-\Omega_1 t}\right)$$

$$+ \frac{(-1)^{2\nu+1}\left(\beta_1^2 - \sigma_1^2\right)\nu^2\pi^2 \cos\nu\pi x}{2A_1\Omega_7\Omega_8}\left(e^{i\Omega_7 t} - e^{-i\Omega_7 t}\right)$$

$$G_{14}(x, y; t) = \frac{i\Omega_3 \cosh\Omega_3 x}{2A_1\Omega_1 \sinh\Omega_3}\left[e^{\Omega_1\left(t-\frac{1}{u}\right)} - e^{-\Omega_1\left(t-\frac{1}{u}\right)}\right]$$

$$+ \frac{(-1)^{\nu}\left(\beta_1^2 - \sigma_1^2\right)\nu^2\pi^2 \cos\nu\pi x}{A_1\Omega_7\Omega_8}\left[e^{i\Omega_7\left(t-\frac{1}{u}\right)} - e^{-i\Omega_7\left(t-\frac{1}{u}\right)}\right]$$

$$G_{16}(x, y; t) = \frac{\coth\Omega_3 \sinh\Omega_3 x}{A_1\Omega_1 \sinh\Omega_3}\left[e^{\Omega_1\left(t-\frac{1}{u}\right)} - e^{-\Omega_1\left(t-\frac{1}{u}\right)}\right]$$

$$+ \frac{(-1)^{2\nu}\left(\beta_1^2 - \sigma_1^2\right)\sin\nu\pi x}{A_1\Omega_7\Omega_8}\left[e^{i\Omega_7\left(t-\frac{1}{u}\right)} - e^{-i\Omega_7\left(t-\frac{1}{u}\right)}\right]$$

$$G_{17}(x, y; t) = \frac{\frac{\Omega_1}{u}\sinh\Omega_3 x}{2A_1\Omega_1 \sinh\Omega_3}\left[e^{\Omega_1\left(t-\frac{1}{u}\right)} - e^{-\Omega_1\left(t-\frac{1}{u}\right)}\right]$$

$$+\frac{(-1)^{v}\left(\beta_1^2-\sigma_1^2\right)\frac{v\pi}{u}\Omega_7\sin v\pi x}{A_1\Omega_7\Omega_8}\left[e^{i\Omega_7\left(t-\frac{1}{u}\right)}-e^{-i\Omega_7\left(t-\frac{1}{u}\right)}\right]$$

The integral $G_5(x,y;t)$ and $G_6(x,y;t)$ have simple poles at $s=\pm\Omega_1$ and $\alpha^{c}=\pm\frac{ik\pi}{b}$.

The poles emanating from $\sinh\alpha^{c}$ are obtained by setting it to zero. i.e

$$\sinh\alpha^{c}=0 \qquad (3.46)$$

which implies

$$\begin{aligned}\alpha^{c}&=+\frac{ik\pi}{b}\\ \alpha^{c}&=-\frac{ik\pi}{b}\end{aligned}\qquad \begin{cases}\alpha^{c^2}=-\frac{k^2\pi^2}{b^2}\\ \alpha^{c^2}=-\frac{k^2\pi^2}{b^2}\end{cases} \qquad (3.47)$$

Explicitly, equation (3.47) with (3.37) implies

$$s^2=-\left(\beta_1^2-\sigma_1^2\right)n^2\pi^2-\eta-\frac{k^2\pi^2}{b^2}\left(\beta_2^2-\sigma_1^2\right) \qquad (3.48)$$

so that

$$s=\pm i\left[\left(\beta_1^2-\sigma_1^2\right)n^2\pi^2+\eta+\frac{k^2\pi^2}{b^2}\left(\beta_2^2-\sigma_1^2\right)\right]^{\frac{1}{2}} \qquad (3.49)$$

Therefore the four poles are

$$s=\Omega_1 \qquad (3.50a)$$

$$s=-\Omega_1 \qquad (3.50b)$$

$$s=i\Omega_9 \qquad (3.50c)$$

$$s=-i\Omega_9 \qquad (3.50d)$$

Contribution due to the poles $s=\Omega_1$ and $s=-\Omega_1$

$$G_{51}(x,y;t)=\frac{(-1)^m e^{\Omega_1 t}\cosh\Omega_3 x\sinh\Omega_6 y}{2A_1\Omega_1\sinh\Omega_6 b}$$

$$G_{52}(x,y;t)=-\frac{(-1)^m e^{-\Omega_1 t}\cosh\Omega_3 x\sinh\Omega_6 y}{2A_1\Omega_1\sinh\Omega_6 b}$$

Contribution due to the poles $s=i\Omega_9$ and $s=-i\Omega_9$ are

$$G_{53}(x,y;t)=\frac{(-1)^{m+k}\left(\beta_2^2-\sigma_1^2\right)k\pi\cosh\sigma^{c*}x\sin\frac{k\pi}{b}y}{bA_1\Omega_9\Omega_{10}}e^{i\Omega_9 t}$$

$$G_{54}(x,y;t)=-\frac{(-1)^{m+k}\left(\beta_2^2-\sigma_1^2\right)k\pi\cosh\sigma^{c*}x\sin\frac{k\pi}{b}y}{bA_1\Omega_9\Omega_{10}}e^{-i\Omega_9 t}$$

Thus, summing $G_{53}(x,y;t)$ - $G_{54}(x,y;t)$ yields

$$G_5(x,y;t)=\frac{(-1)^m\cosh\Omega_3 x\sin\Omega_6 y}{2A_1\Omega_1\sinh\Omega_6 b}\left(e^{\Omega_1 t}-e^{-\Omega_1 t}\right)$$

$$\frac{(-1)^{m+k}\left(\beta_2^2-\sigma_1^2\right)k\pi\cosh\sigma^{c*}x\sin\frac{k\pi}{b}y}{bA_1\Omega_9\Omega_{10}}\left(e^{i\Omega_9 t}-e^{-i\Omega_9 t}\right)$$

Also

$$G_6(x,y;t)=\frac{(-1)^m\sinh\Omega_6 y}{2A_1\Omega_1\sinh\Omega_6 y}\left[e^{\Omega_1\left(t-\frac{x}{u}\right)}-e^{-\Omega_1\left(t-\frac{x}{u}\right)}\right]$$

$$+\frac{(-1)^{m+k}\left(\beta_2^2-\sigma_1^2\right)k\pi\sin\frac{k\pi}{b}y}{bA_1\Omega_9\Omega_{10}}\left[e^{i\Omega_9\left(t-\frac{x}{u}\right)}-e^{-i\Omega_9\left(t-\frac{x}{u}\right)}\right]$$

where

$$\Omega_9=\left[\left(\beta_1^2-\sigma_1^2\right)n^2\pi^2+\eta+\frac{k^2\pi^2}{b^2}\left(\beta_2^2-\sigma_1^2\right)\right]^{\frac{1}{2}},\qquad(3.51)$$

$$\Omega_{10} = \left[\Omega_1^2 + \Omega_9^2\right] \tag{3.52}$$

$$\sigma^{\varepsilon *} = \frac{1}{\left(\beta_1^2 - \sigma_1^2\right)^{\frac{1}{2}}}\left[-\left(\beta_1^2 - \sigma_1^2\right)n^2\pi^2 - \frac{k^2\pi^2}{b^2}\left(\beta_2^2 - \sigma_1^2\right) + \left(\beta_2^2 - \sigma_1^2\right)\frac{m^2\pi^2}{b^2}\right]^{\frac{1}{2}} \tag{3.53}$$

Furthermore, one observes that the integrals $G_7(x, y;t)$ and $G_8(x, y;t)$ have simple poles at $s = \pm\Omega_1$, $\sigma^\varepsilon = \pm i\nu\pi$ and $\alpha^\varepsilon = \pm\frac{ik\pi}{b}$

Using equation (2.83) with (2.62) and (3.47) with (3.37), one obtains

$$s = \pm i\left[\left(\beta_1^2 - \sigma_1^2\right)n^2\pi^2 + \eta + \frac{k^2\pi^2}{b^2}\left(\beta_2^2 - \sigma_1^2\right)\right]^{\frac{1}{2}} \tag{3.54}$$

and

$$s = \pm i\left[\left(\beta_1^2 - \sigma_1^2\right)\nu^2\pi^2 + \left(\beta_2^2 - \sigma_1^2\right)\frac{m^2\pi^2}{b^2} + \eta\right]^{\frac{1}{2}} \tag{3.55}$$

Clearly, there are six simple poles. These are

$$s = \Omega_1 \quad ; \quad s = -\Omega_1 \tag{3.56a}$$

$$s = i\Omega_7 \quad ; \quad s = -i\Omega_7 \tag{3.56b}$$

$$s = i\Omega_9 \quad ; \quad s = -i\Omega_9 \tag{3.56c}$$

The contribution due to the poles $s = \Omega_1$ and $s = -\Omega_1$ are

$$G_{71}(x, y;t) = \frac{(-1)^n e^{-\Omega_1\left(t - \frac{1}{u}\right)} \sinh\Omega_3 x \sinh\Omega_6 y}{2A_1\Omega_1 \sinh\Omega_3 \sinh\Omega_6 b}$$

and

$$G_{72}(x, y;t) = \frac{(-1)^n e^{-\Omega_1\left(t - \frac{1}{u}\right)} \sinh\Omega_3 x \sinh\Omega_6 y}{2A_1\Omega_1 \sinh\Omega_3 \sinh\Omega_6 b}$$

The contribution due to the poles $s = i\Omega_7$ and $s = -i\Omega_7$ are

$$G_{73}(x,y;t) = \frac{(-1)^{\nu+m}\left(\beta_1^2 - \sigma_1^2\right)\nu\pi \sin \nu\pi x \sinh \alpha^{c^*} y}{A_1\Omega_7\Omega_8 \sinh \alpha^{c^*} b} e^{i\Omega_7\left(t-\frac{1}{u}\right)}$$

and

$$G_{74}(x,y;t) = \frac{(-1)^{\nu+m}\left(\beta_1^2 - \sigma_1^2\right)\nu\pi \sin \nu\pi x \sinh \alpha^{c^*} y}{A_1\Omega_7\Omega_8 \sinh \alpha^{c^*} b} e^{-i\Omega_7\left(t-\frac{1}{u}\right)}$$

The contribution due to the poles $s = i\Omega_9$ and $s = -i\Omega_9$ are

$$G_{75}(x,y;t) = \frac{(-1)^{k+m}\left(\beta_2^2 - \sigma_1^2\right)k\pi \sin \frac{k\pi}{b} y \sinh \sigma^{c^*} x}{bA_1\Omega_9\Omega_{10} \sinh \sigma^{c^*}} e^{i\Omega_9\left(t-\frac{1}{u}\right)}$$

and

$$G_{76}(x,y;t) = \frac{(-1)^{k+m}\left(\beta_2^2 - \sigma_1^2\right)k\pi \sin \frac{k\pi}{b} y \sinh \sigma^{c^*}}{bA_1\Omega_9\Omega_{10} \sinh \sigma^{c^*}} e^{-i\Omega_9\left(t-\frac{1}{u}\right)}$$

Thus, summing up $G_{71}(x,y;t)$ - $G_{76}(x,y;t)$ yields

$$G_7(x,y;t) = \frac{(-1)^m \sinh \Omega_3 x \sinh \Omega_6 y}{2A_1\Omega_1 \sinh \Omega_3 \sinh \Omega_6 b}\left[e^{\Omega_1\left(t-\frac{1}{u}\right)} - e^{-\Omega_1\left(t-\frac{1}{u}\right)}\right]$$

$$+ \frac{(-1)^{\nu+m}\left(\beta_1^2 - \sigma_1^2\right)\nu\pi \sin \nu\pi x \sinh \alpha^{c^*} y}{A_1\Omega_7\Omega_8 \sinh \alpha^{c^*} b}\left[e^{i\Omega_7\left(t-\frac{1}{u}\right)} - e^{-i\Omega_7\left(t-\frac{1}{u}\right)}\right]$$

$$+ \frac{(-1)^{k+m}\left(\beta_2^2 - \sigma_1^2\right)k\pi \sin \frac{k\pi}{b} y \sinh \sigma^{c^*} x}{bA_1\Omega_9\Omega_{10} \sinh \sigma^{c^*}}\left[e^{i\Omega_9\left(t-\frac{1}{u}\right)} - e^{-i\Omega_9\left(t-\frac{1}{u}\right)}\right]$$

The solution $G_8(x,y;t)$ is obtained through the same procedure as

$$G_4(x,y;t)=\frac{(-1)^m \cosh\Omega_3 \sinh\Omega_3 x \sinh\Omega_6 y}{2A_1\Omega_1 \sinh\Omega_3 \sinh\Omega_6 b}\left(e^{\Omega_1 t}-e^{-\Omega_1 t}\right)$$

$$+\frac{(-1)^{2\nu+m}\left(\beta_1^2-\sigma_1^2\right)\nu\pi \sin\nu\pi x \sinh\alpha^{c^*} y}{bA_1\Omega_7\Omega_8 \sinh\alpha^{c^*} b}\left(e^{i\Omega_7 t}-e^{-i\Omega_7 t}\right)$$

$$+\frac{(-1)^{k+m}\left(\beta_2^2-\sigma_1^2\right)k\pi \cosh\sigma^{c^*} \sinh\sigma^{c^*} x \sin\frac{k\pi}{b} y}{bA_1\Omega_9\Omega_{10} \sinh\sigma^{c^*}}\left(e^{i\Omega_9 t}-e^{-i\Omega_9 t}\right)$$

Substitution of integrals $G_1(x,y;t)$ - $G_{17}(x,y;t)$ into equation (3.35) gives the complete inversion of $V_1(x,y;t)$.

$$V_1(x,y)=\frac{2P_a u m\pi \sin\frac{m\pi}{b} y_o}{2A_1 b^2(\beta_1^2-\sigma_1^2)(\beta_2^2-\sigma_1^2)^{\frac{1}{2}}}\left\{\frac{\cosh\Omega_3 \cosh\Omega_6 y\left[e^{-\Omega_1 t}-e^{\Omega_1 t}\right]}{\Omega_1}+\frac{\cosh\Omega_6 y}{\Omega_1}\left[e^{-\Omega_1(t-\frac{1}{u})}-e^{\Omega_1(t-\frac{1}{u})}\right]\right.$$

$$-\frac{\sinh\Omega_3 x \cosh\Omega_6 y}{\Omega_1 \sinh\Omega_3}\left[e^{\Omega_1(t-\frac{1}{u})}-e^{-\Omega_1(t-\frac{1}{u})}\right]-\frac{2(-1)^{\nu+1}(\beta_1^2-\sigma_1^2)\nu\pi \sin\nu\pi x \cosh\alpha^{c^*} y}{\Omega_7\Omega_8}\left[e^{i\Omega_7(t-\frac{1}{u})}-e^{-i\Omega_7(t-\frac{1}{u})}\right]$$

$$+\frac{\coth\Omega_3 \sinh\Omega_3 x \cosh\Omega_6 y}{\Omega_1}\left[e^{\Omega_1 t}-e^{-\Omega_1 t}\right]+\frac{(-1)^m \cosh\Omega_3 x \sinh\Omega_6 y}{\Omega_1 \sinh\Omega_6 b}\left[e^{\Omega_1 t}-e^{-\Omega_1 t}\right]$$

$$+\frac{2(-1)^{m+k}(\beta_2^2-\sigma_1^2)k\pi \cosh\sigma^{c^*} x \sin\frac{k\pi}{b} y}{b\Omega_9\Omega_{10}}\left[e^{i\Omega_9 t}-e^{-i\Omega_9 t}\right]-(-1)^m\frac{\sinh\Omega_6 y}{\Omega_1 \sinh\Omega_6 b}\left[e^{\Omega_1(t-\frac{1}{u})}-e^{-\Omega_1(t-\frac{1}{u})}\right]$$

$$-\frac{2(-1)^{m+k}(\beta_2^2-\sigma_1^2)k\pi \sinh\frac{k\pi}{b} y}{b\Omega_9\Omega_{10}}\left[e^{i\Omega_9(t-\frac{1}{u})}-e^{-i\Omega_9(t-\frac{1}{u})}\right]+(-1)^m\frac{\sinh\Omega_3 x \sinh\Omega_6 y}{\Omega_1 \sinh\Omega_3 \sinh\Omega_6 b}\left[e^{\Omega_1(t-\frac{1}{u})}-e^{-\Omega_1(t-\frac{1}{u})}\right]$$

$$+\frac{2(-1)^{\nu+m}(\beta_1^2-\sigma_1^2)\nu\pi \sin\nu\pi x \sinh\alpha^{c^*} y}{\Omega_7\Omega_8 \sinh\alpha^{c^*} b}\left[e^{i\Omega_7(t-\frac{1}{u})}-e^{-i\Omega_7(t-\frac{1}{u})}\right]+\frac{2(-1)^{m+k}(\beta_2^2-\sigma_1^2)k\pi \sin\frac{k\pi}{b} y \sinh\sigma^{c^*} x}{b\Omega_9\Omega_{10} \sinh\sigma^{c^*}}\left[e^{i\Omega_9(t-\frac{1}{u})}-e^{-i\Omega_9(t-\frac{1}{u})}\right]$$

$$-\frac{(-1)^m \cosh\Omega_3 \sinh\Omega_3 x \sinh\Omega_6 y}{\Omega_1 \sinh\Omega_3 \sinh\Omega_6 b}\left[e^{\Omega_1 t}-e^{-\Omega_1 t}\right]-\frac{2(-1)^{2\nu+m}(\beta_1^2-\sigma_1^2)\nu\pi \sin\nu\pi x \sinh\alpha^{c^*} y}{\Omega_7\Omega_8 \sinh\alpha^{c^*} b}\left[e^{i\Omega_7 t}-e^{-i\Omega_7 t}\right]$$

$$-\frac{2(-1)^{k+m}(\beta_2^2-\sigma_1^2)k\pi\cosh\sigma^{c*}\sinh\sigma^{c*}x\sin\frac{k\pi}{b}y}{b\Omega_9\Omega_{10}\sinh\sigma^{c*}}\left[e^{i\Omega_9 t}-e^{-i\Omega_9 t}\right]+\frac{\cosh\Omega_3 x\coth\Omega_6 b\sinh\Omega_6 y}{\Omega_1}\left[e^{\Omega_1 t}-e^{-\Omega_1 t}\right]$$

$$-\frac{\coth\Omega_6 b\sinh\Omega_6 y}{\Omega_1}\left[e^{\Omega_1(t-\frac{1}{u})}-e^{-\Omega_1(t-\frac{1}{u})}\right]+\frac{\sinh\Omega_3 x\coth\Omega_6 b\sinh\Omega_6 y}{\Omega_1\sinh\Omega_3}\left[e^{\Omega_1(t-\frac{1}{u})}-e^{-\Omega_1(t-\frac{1}{u})}\right]$$

$$+\frac{2(-1)^{\nu+1}(\beta_1^2-\sigma_1^2)\nu\pi\sin\nu\pi x\coth\alpha^{c*}\sinh\alpha^{c*}y}{\Omega_7\Omega_8}\left[e^{i\Omega_9(t-\frac{1}{u})}-e^{-i\Omega_9(t-\frac{1}{u})}\right]-\frac{\coth\Omega_3\sinh\Omega_3 x\coth\Omega_6\sinh\Omega_6 y}{\Omega_1}\left[e^{\Omega_1 t}-e^{-\Omega_1 t}\right]\Bigg\}$$

$$+\frac{2P_a u\sin\frac{m\pi}{b}y_0\sin\frac{m\pi}{b}y}{2A_1(\beta_1^2-\sigma_1^2)^{\frac{3}{2}}}\left\{\frac{i\Omega_3\cosh\Omega_3\cosh\Omega_3 x}{\Omega_1\sinh\Omega_3}\left[e^{\Omega_1 t}-e^{-\Omega_1 t}\right]+\frac{2(-1)^{2\nu+1}(\beta_1^2-\sigma_1^2)\nu^2\pi^2\cos\nu\pi x}{\Omega_7\Omega_8}\left[e^{i\Omega_7 t}-e^{-i\Omega_7 t}\right]\right.$$

$$-\frac{i\Omega_3\cosh\Omega_3 x}{\Omega_1\sinh\Omega_3}\left[e^{\Omega_1(t-\frac{1}{u})}-e^{-\Omega_1(t-\frac{1}{u})}\right]-\frac{2(-1)^{\nu}(\beta_1^2-\sigma_1^2)\nu^2\pi^2\cos\nu\pi x}{\Omega_7\Omega_8}\left[e^{i\Omega_7(t-\frac{1}{u})}-e^{-i\Omega_7(t-\frac{1}{u})}\right]$$

$$\frac{1}{u}\cosh\Omega_3 x-\frac{i}{\Omega_1}\Omega_3\sinh\Omega_3 x+\frac{1}{u}\coth\Omega_3\sinh\Omega_3 x-\frac{2i}{\Omega_1}\Omega_3\coth^2\Omega_3\sinh\Omega_3 x$$

$$+\frac{2\coth\Omega_3\sinh\Omega_3 x}{\Omega_1\sinh\Omega_3}\left[e^{\Omega_1(t-\frac{1}{u})}-e^{-\Omega_1(t-\frac{1}{u})}\right]+\frac{2(-1)^{2\nu}(\beta_1^2-\sigma_1^2)\sin\nu\pi x}{\Omega_7\Omega_8}\left[e^{i\Omega_7(t-\frac{1}{u})}-e^{-i\Omega_7(t-\frac{1}{u})}\right]$$

$$\left.+\frac{\frac{\Omega_1}{u}\sinh\Omega_3 x}{\sinh\Omega_3}\left[e^{\Omega_1(t-\frac{1}{u})}-e^{-\Omega_1(t-\frac{1}{u})}\right]+\frac{2(-1)^{\nu}(\beta_1^2-\sigma_1^2)\Omega_7\frac{\nu\pi}{u}\sin\nu\pi x}{\Omega_7\Omega_8}\left[e^{i\Omega_7(t-\frac{1}{u})}-e^{-i\Omega_7(t-\frac{1}{u})}\right]\right\}$$

(3.57)

From equation (2.15), the perturbation scheme of a uniformly valid solution in the entire domain of definition of the plate problem is given by

$$V(x,y;t)=V_o(x,y;t)+\varepsilon V_1(x,y;t) \qquad (3.58)$$

Where $V_o(x, y, t)$ is the leading order solution and $V_1(x, y, t)$ is the first order correction.

Thus substituting $V_0(x,y,t)$ and $V_1(x, y, t)$ into equation (3.58) gives the required solution.

3.4 REMARKS ON THEORY

Equations (2.106) and (3.57) are the leading order and first order (transformed) solutions of the problem. The leading order and the first order solutions are combined in equation (3.58) to form the composite solution which is uniformly valid in the entire domain of the highly prestressed plate.

Form equation (2.106), it is found that the anisotropic prestress, shear modulus and the foundation stiffness affect the response to $o(\varepsilon)$ of the rectangular plate. In an undamped system such as this, it is pertinent to examine the phenomenon of resonance.

It is observed from the leading order and the first order correction results that fully clamped prestressed isotropic plate resting on a Pasternak-type foundation and transversed by a moving force reaches the state of resonance whenever at

$$\beta_1^2 = \sigma_1^2 \tag{3.59}$$

other conditions when the system reaches a state of resonance are

$$\beta_2^2 = \sigma_1^2 \tag{3.60}$$

$$(\beta_1^2 - \sigma_1^2)^{\frac{1}{2}} = u \tag{3.61}$$

$$u^2\left[\frac{\left(\beta_2^2 - \sigma_1^2\right)m^2\pi^2 + \eta b^2}{b^2\left(\beta_1^2 - \sigma_1^2 - u^2\right)}\right]^{\frac{1}{2}} = -\left[v^2\pi^2\left(\beta_1^2 - \sigma_1^2\right) + \left(\beta_2^2 - \sigma_1^2\right)\frac{m^2\pi^2}{b^2} + \eta\right] \tag{3.62}$$

$$u^2\left[\frac{\left(\beta_2^2-\sigma_1^2\right)m^2\pi^2+\eta b^2}{b^2\left(\beta_1^2-\sigma_1^2-u^2\right)}\right]^{\frac{1}{2}}=-\left[n^2\pi^2\left(\beta_1^2-\sigma_1^2\right)+\left(\beta_2^2-\sigma_1^2\right)\frac{k^2\pi^2}{b^2}+\eta\right] \tag{3.63}$$

From (3.59) to (3.63), it is observed that the resonance conditions of the plate are dependent on the anisotropic prestress and the elastic foundation. It is also evident that to any order of calculation, resonance conditions are affected by both the shear modulus G and foundation stiffness K.

At this juncture, the critical velocities for the system of a highly prestressed isotropic rectangular plate on an elastic foundation traversed by a moving load are sought. The three distinct critical velocities that exist in the dynamical system are given as

$$U_1(v,m,\pi)=-\frac{1}{v\pi b}\left[v^2\pi^2b^2\left(\beta_1^2-\sigma_1^2\right)+\left(\beta_2^2-\sigma_1^2\right)m^2\pi^2+\eta b^2\right]^{\frac{1}{2}}$$

$$U_2(k,n,\pi)=-\frac{1}{\pi}\left[\frac{\left(\beta_1^2-\sigma_1^2\right)\left[n^2\pi^2b^2\left(\beta_1^2-\sigma_1^2\right)+\left(\beta_2^2-\sigma_1^2\right)k^2\pi^2+\eta b^2\right]}{\left(k^2-m^2\right)\left(\beta_2^2-\sigma_1^2\right)+n^2b^2\left(\beta_1^2-\sigma_1^2\right)}\right]^{\frac{1}{2}}$$

and

$$U_3=\left(\beta_1^2-\sigma_1^2\right)^{\frac{1}{2}}$$

where

U_n = velocity u at which resonance occurs.

3.5 NUMERICAL CALCULATIONS AND DISCUSSIONS

In order to illustrate the analytical results, for example, the isotropic rectangular plate is taken to be of length L_X = 1.0m and width 0.5m. Other values used for the analysis in this section are b = 0.5 m, v = 1, $\pi = \frac{22}{7}$. The values of the prestress in x - direction β_1^2 range between 0 and 100000. The critical velocities are plotted against prestress and foundation stiffness for various values of shear modulus G and subgrade K. Values of shear modulus G between 0 and 100000 were used while the values of foundation stiffness K were varied between 0N/m^3 and 2000000 N/m^3.

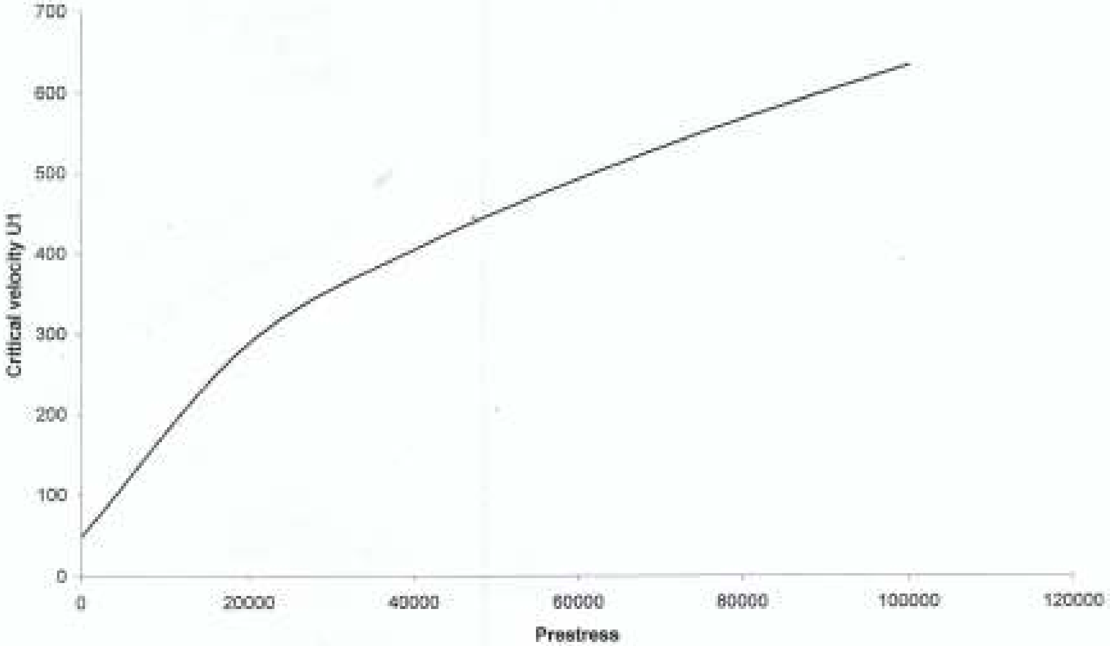

Fig. 3.1: The graph of Critical Velocity U_1against Prestress for fixed values of Shear modulus G (100000) and Foundation stiffness K (2000000).

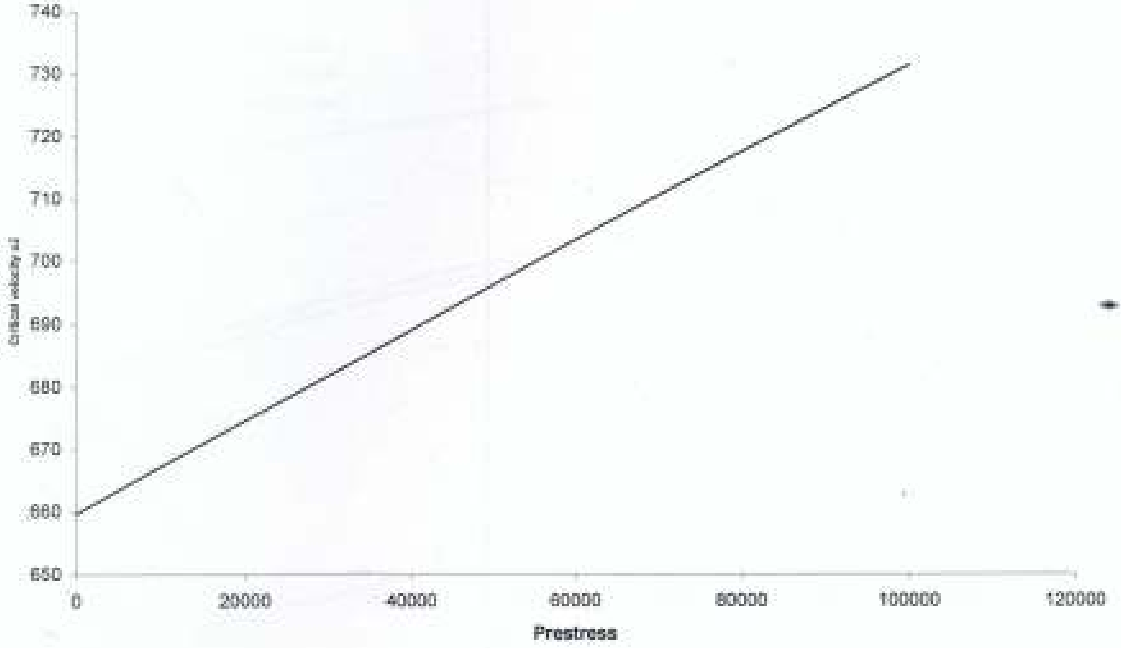

Fig.3.2: The graph of Critical Velocity U_2 against Prestress for fixed values of Shear modulus G (50000) and Foundation stiffness K (2000000).

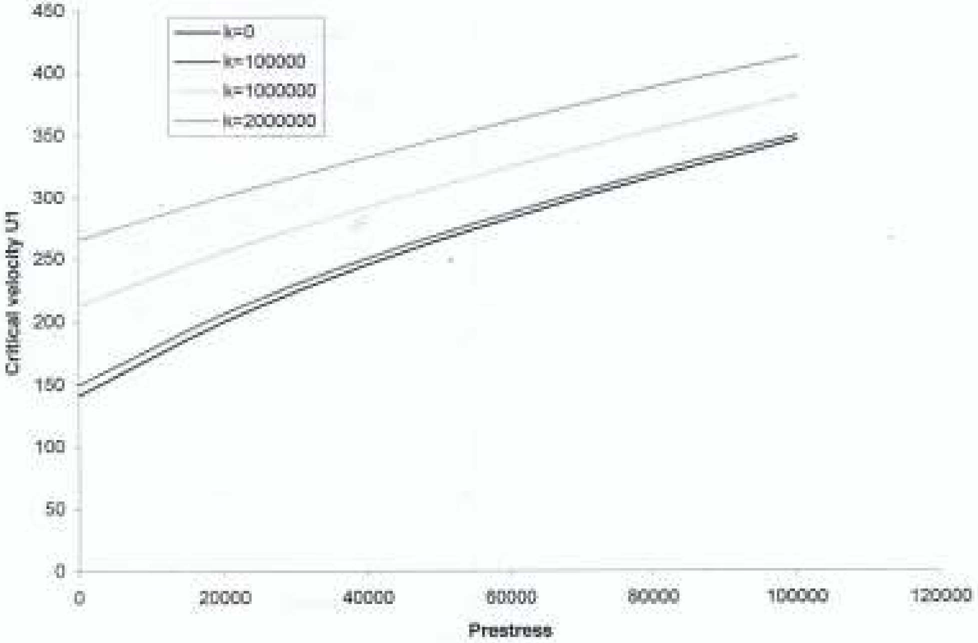

Fig. 3.3: The graph of Critical velocity U_1 against Prestress for various values of Foundation stiffness K and fixed value of Shear modulus G (100000)

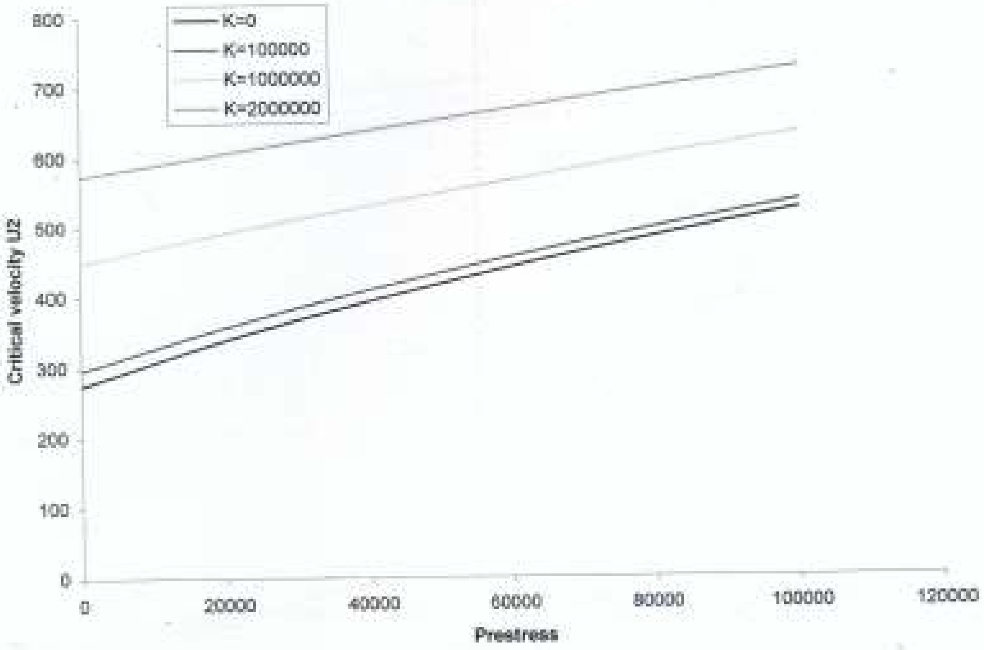

Fig. 3.4: The graph of Critical velocity U_1 against Prestress for various values of Foundation stiffness K and fixed value of Shear modulus G (100000).

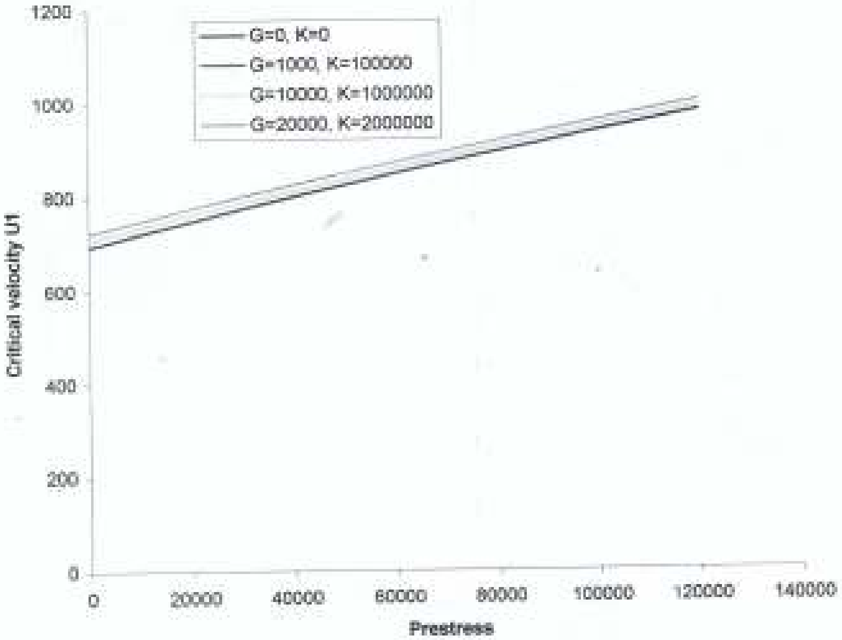

Fig.3.5: The graph of Critical velocity U_1 against prestress for various values of Shear modulus G and Foundation stiffness K.

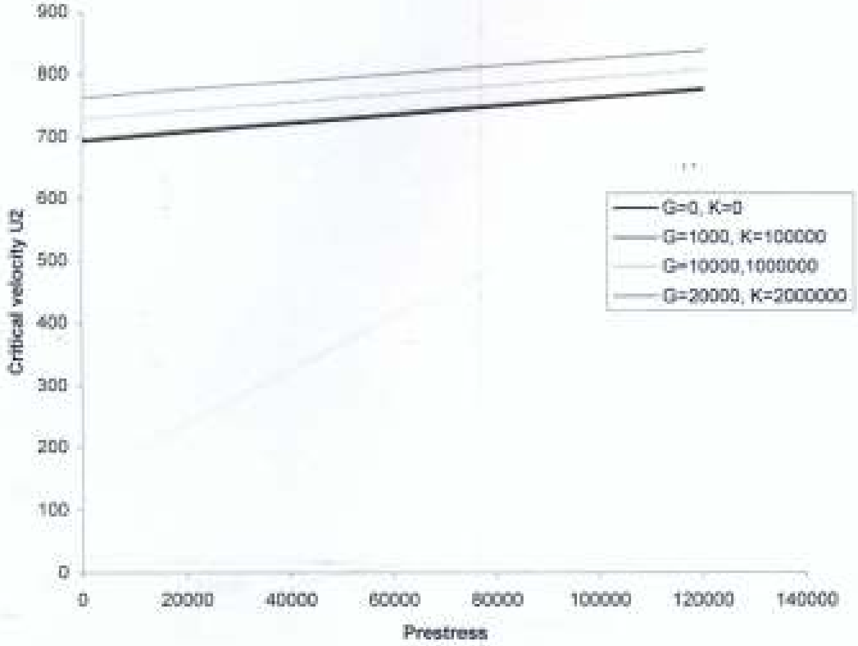

Fig. 3.6: The graph of Critical velocity U_2 against Prestress for various values of Shear modulus and Foundation stiffness.

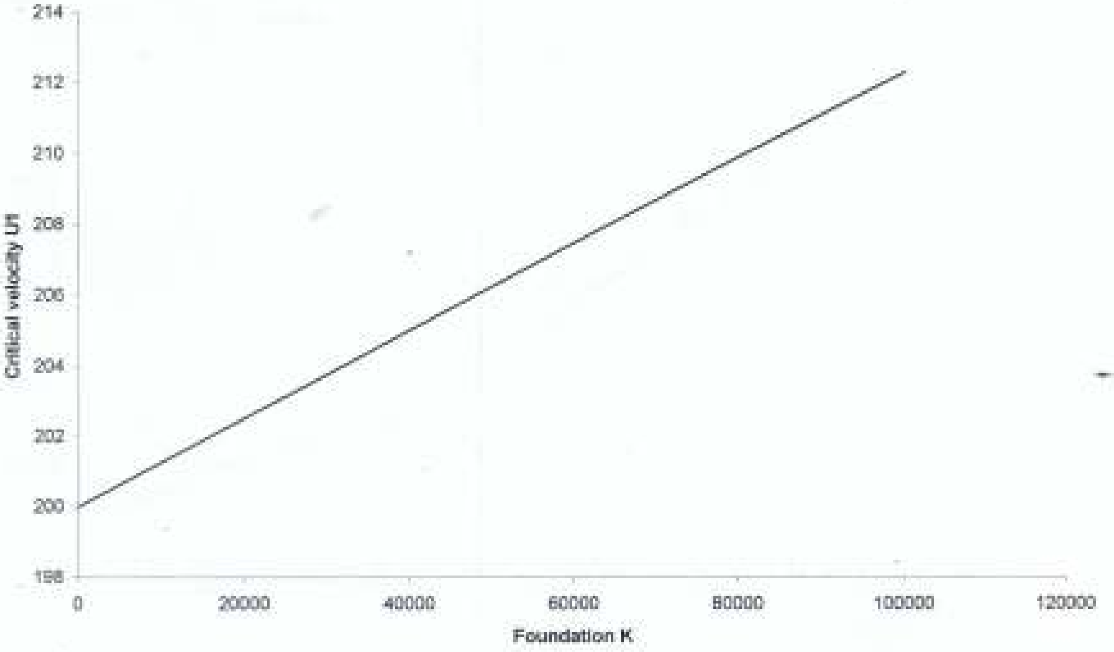

Fig. 3.7: The graph of Critical velocity U_1 against Foundation stiffness K for fixed value of Shear modulus G (100000).

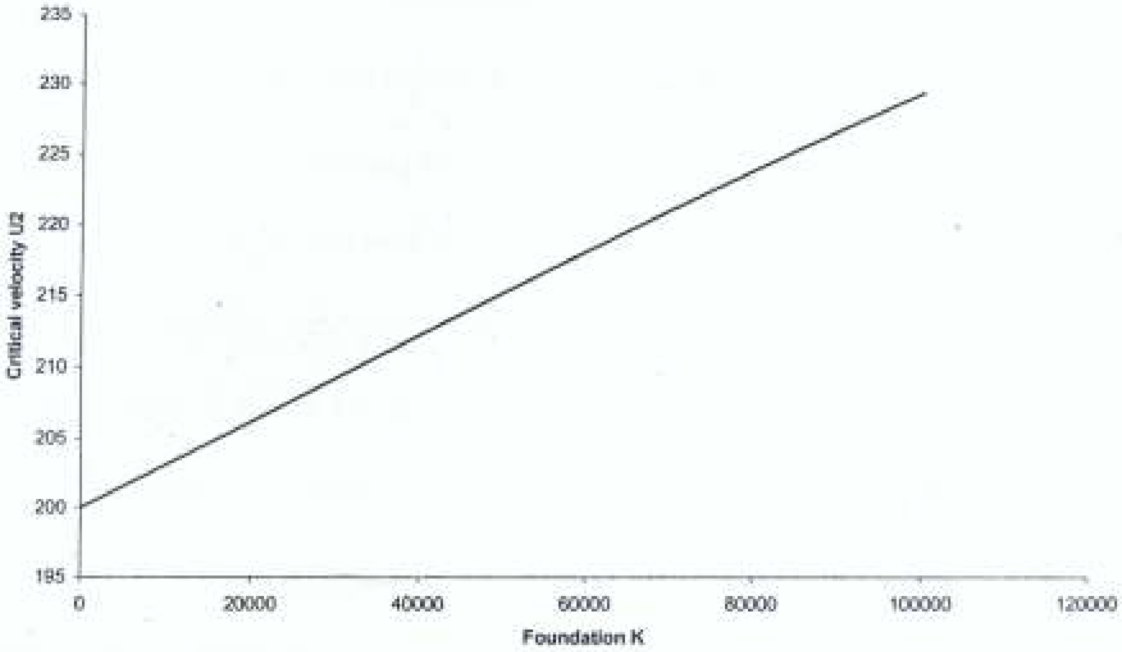

Fig. 3.8: The graph of Critical velocity U_2 against Foundation stiffness K for fixed value of Shear modulus G (100000).

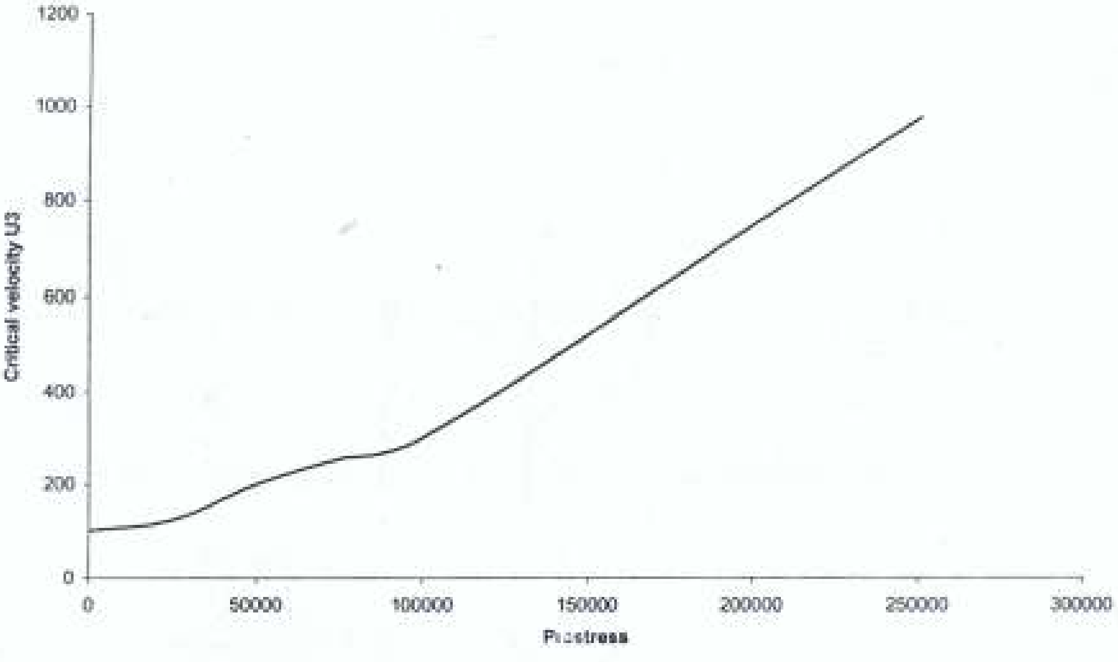

Fig. 3.9: The graph of Critical velocity U_3 against Prestress for fixed value of Shear modulus G (100000).

Figure 3.1 displays the graph of critical velocity U_1 against prestress. From the graph, it is observed that the critical velocity U_1 increases with prestress for fixed values of shear modulus G and foundation stiffness K. Thus, for high value of prestress, our design is more stable and reliable. In a similar manner, in figure 3.2, the critical velocity U_2 behaves exactly the same way as U_1. Results and analysis similar to those of figure 3.1 are obtained.

The graph of U_1 against the prestress for various values of foundation stiffness is shown in figure 3.3. Evidently, the critical velocity increases with prestress for all values of foundation stiffness used. Thus resonance is reached earlier for lower values of prestress than for high values of prestress. Thus the design is more stable and the risk of resonance is remote for high values of prestress. Also, the critical velocity U_2 in figure 3.4 against the foundation stiffness behaves the same way as U_1.

Figure 3.5 shows the plotted curves of U_1 against prestress for various values of shear modulus and foundation stiffness. The graph shows that as prestress increases, the critical velocity increases as well. Thus, with these, the likelihood of collapsed structure is very remote. The critical velocity U_2 in figure 3.6 behaves exactly the same way as U_1. Results and analysis similar to those of figure 3.5 are obtained.

The graph of U_1 against foundation is shown in figure 3.7. Evidently, the critical velocity increases as the foundation stiffness increases. Thus resonance is reached earlier for lower values of foundation stiffness. Also, the stability and reliability of the structural design is enhanced. The critical velocity U_2 in figure 3.8 behaves exactly the same way as U_1. Results and analysis similar to those of figure 3.7 are obtained.

Fig 3.9 displays the graph of critical velocity U_3 against prestress. From the graph, the critical velocity increases with prestress for fixed value of shear modulus. Thus, resonance is reached earlier for lower values of prestress.

CHAPTER FOUR

4.0 GENERAL CONCLUSION

4.1 SUMMARY OF RESEARCH WORK

The problem of the dynamic response of a highly prestressed isotropic rectangular plate under a travelling load is considered in this thesis. The problem is governed by a fourth order non–homogenous differential equation.

For the purpose of solution, the equation is presented in a non–dimensionalized form. It is observed that a small parameter multiplies the highest derivatives in the governing differential equation. Thus, this type of dynamical problem is usually amenable to singular perturbation technique. In particular the Method of Matched Asymptotic Expansion (MMAE) is used. This technique constructs outer (core) and inner (boundary layer) solutions that are valid in partly disjoint domains. These solutions are then matched in an intermediate domain where both asymptotic expansions are valid. Consequently, an approximate uniformly valid solution in the entire domain of definition of the rectangular plate is obtained with the rigorous use of Laplace transformation and the Cauchy residue theorem. This solution is analysed and three distinct resonance conditions are obtained in the dynamical system.

Numerical analysis is carried out and the study exhibits the following results:

1. The leading order solutions and the first order correction are affected by the bi-parametric subgrade moduli and anisotropic prestress.

2. As the foundation stiffness increases, the critical velocities of the isotropic rectangular plate transversed by moving load increased.
3. The critical velocities of the dynamical system increases with increase in prestress for fixed values of shear modulus and foundation stiffness.
4. The critical velocity of the dynamical system also increases with increase in shear modulus for fixed foundation stiffness and prestress.
5. There may be more than one resonance condition in a dynamical system such as this which involves plate flexure under moving loads.

Finally, this work has showcased the use of a valuable method for the solution of this class of dynamical problems.

4.2 CONTRIBUTIONS TO KNOWLEDGE

Closed form uniformly valid analytical solution has been obtained for the problem of the dynamic analysis of a highly prestressed isotropic rectangular plate resting on a bi-parametric subgrade under a travelling load and analysis have indicated

(i) the resonance conditions for the dynamical system

(ii) the influence of prestress, shear modulus and foundation stiffness on the response of a highly prestressed plate under moving load.

4.3 LIMITATION TO STUDY AND RECOMMENDATIONS FOR FURTHER RESEARCH

The response of an isotropic rectangular plate on a bi-parametric subgrade under the action of moving loads is the main objective of this study. The study assumes that the load moves at a constant velocity and the inertia effects of the moving load are not considered. The theory could be extended to the case when the load moves at a variable velocity and the inertia effects of the moving load considered.

Plate models resting on variable elastic Winkler foundation or visco-elastic foundation and under the action of moving masses are not considered in this study and hence left for further study.

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