DYNAMIC RESPONSE TO MOVING CONCENTRATED MASSES OF RAYLEIGH BEAMS ON VARIABLE WINKLER ELASTIC FOUNDATIONS.

BY

AWODOLA, THOMAS OLUBUNMI

(IMC/91/3302)

A THESIS IN THE DEPARTMENT OF INDUSTRIAL MATHEMATICS AND COMPUTER SCIENCE SUBMITTED TO THE SCHOOL OF POST GRADUATE STUDIES IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE AWARD OF THE DEGREE OF MASTERS OF TECHNOLOGY (M.TECH.) IN INDUSTRIAL MATHEMATICS OF THE FEDERAL UNIVERSITY OF TECHNOLOGY, AKURE, NIGERIA.

JULY, 2003.

CERTIFICATION

This is to certify that this project was carried out by AWODOLA Thomas Olubunmi of Industrial Mathematics and Computer Science Department, Federal University of Technology, Akure, Nigeria.

31-7-03

DR. S.T. ONI
SUPERVISOR.

DATE

14-08-2003

PROF. O.C. AKINYOKUN
CHIEF EXAMINER

DATE

31st July, 2003

DR. S.A. OLORUNŠOLA
EXTERNAL EXAMINER

DATE

DEDICATION

This work is dedicated to the ALMIGHTY GOD who has never failed to renew my strength and also to all friends and relatives who have contributed in one way or the other to my academic success.

ACKNOWLEDGEMENT

I wish to appreciate the efforts of my supervisor –Dr. S. T. Oni, for his fatherly love, care and guidance and for making constructive suggestions, corrections and contributions towards the successful completion of this project.

My gratitude goes to my lecturers in the department –Dr. R.A. Ademiluyi, Dr. O. K. Koriko, Dr. E.O. Olowofeso, Mr. O. Fashoranbaku, Dr. F.A. Alao, Mr. O. Olotu and a host of others, for their useful contributions toward the success of this project. My gratitude also goes to my Head of Department –Prof. O.C. Akinyokun for his fatherly care and administrative assistance throughout the course of this work. Once more, I appreciate the moral and inspirational support and suggestions from my project supervisor. These have indeed contributed to the improvement of the work.

I must express my appreciation to all the efforts of Mr. B.S. Ige and Mrs. Wumi Aladeyelu of the computing laboratory towards the success of this research work. They have contributed immensely to the development of the computer programmes for generating the values used in the work and also the type setting and printing of the project write-up.

Moreover, I express my profound gratitude to the department of staff-development and the authority of the Federal University of Technology, Akure for being my sponsors throughout the period of this programme and for making this project a reality.

Again, my thanks go to my wife -Mrs. Awodola Olusola Stella for her cooperation at all times. I thank my parents, brothers and sisters and all the people that have contributed, morally, financially, academically and spiritually toward the success of this work, may the Lord bless them all.

TABLE OF CONTENTS

PAGES

PAGES

ABSTRACT

The response of Rayleigh beams carrying moving masses, resting on variable Winkler elastic foundations is investigated in this thesis. The problem is investigated for both cases of uniform and non-uniform Rayleigh beams. In each case, the governing equation is a fourth order partial differential equation. In order to solve this problem, the versatile Galerkin's method is used to reduce the governing fourth order partial differential equations with variable coefficients to a sequence of second order ordinary differential equations. For the solutions of these equations, a modification of the Struble's technique is employed. Numerical results in plotted curves are then presented.

The results show that response amplitudes of the uniform Rayleigh beam decrease as the rotatory inertia correction factor R^o increases for all variants of classical boundary conditions considered. These same results obtain for the non-uniform Rayleigh beams. Furthermore, for fixed value of R^o, the dispiacements of both uniform and non-uniform Rayleigh beams resting on variable elastic foundations decrease as the foundation modulli K increases.

The results further show that, for fixed R^o and K, the transverse deflections of both uniform and non-uniform Rayleigh beams under the actions of moving masses are higher than those when only the force effects of the moving load are considered. Therefore, the moving force solution is not a safe approximation to the moving mass problem. Hence safety is not guaranteed for a design based on the moving force solution. Also the analyses show that the response amplitudes of both moving force and moving mass problems decrease both with increasing Foundation constant and with increasing Rotatory inertia.

Finally, the critical speed for the moving mass problem is reached prior to that of the moving force for both uniform and non-uniform Rayleigh beam problems in all variants of illustrative examples considered.

CHAPTER ONE

1.0 INTRODUCTION

Various structures, ranging from bridges and roads to space vehicles and submarines, are constantly acted upon by moving masses and, hence, the problem of analyzing the dynamic response of elastic structures under the action of moving masses continues to motivate a variety of investigations, Fryba (1972). In particular, the study of beam and plate flexure under moving loads forms a very important structural element in Engineering design and Construction. It has also become the objective of various investigators in the field of Applied Mathematics. These structures may be elastic, viscoelastic or inelastic and the moving loads may either be of constant or variable magnitude. For moving loads, load effects are variable functions of the position of the load. Below are the definitions of key terms relevant in this study:

(i) A **beam** may be considered as a slender bar acted upon by forces and moments producing primarily bending. A beam is also considered as a one-dimensional body whose physical properties (Stiffness, mass, length etc) are described with reference to a single dimension, the position along the elastic axis. Thus, a typical beam problem is governed by a partial differential equation of two independent variables, distance along the axis and time.

(ii) **Plates** are initially flat structural elements for which the thickness is much smaller than the other dimensions. Thus they are considered as a two-dimensional structures. Practical examples are tabletop, bridges etc.

(iii) **Moving loads** may be defined as forces acting on a structure and continually changing position. Common examples of these include trains, vehicles, cranes etc.

(iv) **Inertia** is defined as the tendency of the body to remain in its state of rest or uniform motion in a straight line.

Pertinent to investigators in the field of structural dynamics is the quest for an effective and reliable method in accurately determining the response of an elastic structure under the actions of heavy masses traversing it at various speeds. Such important fundamental dynamical problems as the analysis and design of railway bridges, bridge girders, road tracks etc in our era of heavy motor-vehicles of various masses and speeds continuously traversing these structures have intensified the need for the study of the dynamic response to moving masses of elastic structures.

Several authors have extensively studied the flexural vibration of prismatic rods and beams. In most of these previous works, it has tacitly been assumed that the Cross-sectional dimensions of the rods and beams are small in comparison with its length. The account of the effects of the Cross-sectional dimensions on the frequencies has been neglected. Nonetheless, it is easy to see that a typical element of an elastic system performs not only a translatory motion but also rotates, Timoshenko (1974). These corrections are of considerable importance in studying the modes of vibration of higher frequencies when an elastic system is subdivided into comparatively short portions. It is remarked at this juncture that in most of the previous studies on moving load problems involving rods, beams or plates, the structures have been idealized by one whose cross section is uniform. However, in the recent years, such important engineering problems as the vibration of turbines, hulls of ships and bridge girders of variable depth etc involving the theory of vibration of structures of variable cross-section have intensified the need for the study of the response of non-uniform elastic systems under the action of moving loads, Oni (1996).

The effects of moving loads on solid bodies are dual, Fryba (1972). On one hand is the gravitational effect of the moving load while on the other hand are the inertial effects of the mass of the load on the vibrating solid bodies. Dynamical problems involving moving loads can be generally grouped into the following three classes:

(i) In the first case, the mass of the moving load is considered much smaller than the mass of the structure it is traversing

(ii) The second class comprises of the system for which the mass of the structure is assumed to be much smaller than that of the moving load, and thirdly, we have

(iii) The case in which the mass of the structure and that of the moving load are of comparable magnitudes.

The first case is much simpler than the second and the third. Infact, the first is the commonest problem treated in literature. In this problem, the inertia effects of the moving load are assumed negligible and only the force effects of the moving load are taken into consideration. Thus, this type of problem is termed the "moving force" problem. Though, the problem on this assumption has greatly been simplified, the following question arises: how safe is a design based on this assumption? The justification of this assumption would have been established had the solution of this approximate model been proved to be an upper bound for the actual deflection of the elastic system.

The most difficult of all the three types of the problem is the third, while both the second and the third problems involve not only the consideration of the force effects of the moving load but also its inertia effects, the moving load in the former does not have mass commensurable with the mass of the structure. The third type of the problem may be termed "moving mass" problem.

In general, the moving load problems are mathematically complex when the inertia effect of the moving load is taken into consideration. Thus, most of the research works available in the literature are those in which this effect has been neglected. This is due, at least in part, to the great amount of computational labour, which is required both to set up and to solve the necessary equations. One important problem that arises when the inertia effects of the masses are considered is the singularity which occurs in the inertia terms of the governing differential equation of motion.

Aside the problem arising from the inclusion of the inertia terms in moving mass problems, difficulties often arise from the type of specified end-conditions. These end-conditions can be classified into two, Bishop (1979), viz:

(a) Geometric boundary conditions

(b) Dynamic/force boundary conditions

The geometric boundary conditions relate to the deflection, say $U(x,t)$ and slope $\frac{\partial U}{\partial x}(x,t)$, (where x is the spatial coordinate and t is the time) while the dynamic or force boundary conditions relate to the bending moments ($\frac{\partial^2 U(x,t)}{\partial x^2}$) and shear force ($\frac{\partial^3 U(x,t)}{\partial x^3}$) .

In practice, in structural dynamics, boundary conditions may be classified into two, namely (i) classical and (ii) non-classical boundary conditions.

There are four classical boundary conditions. They are described as follows:

1. Pinned end conditions: In this case the lateral displacement U(x,t) and the bending moment $\frac{\partial^2 U(x,t)}{\partial x^2}$ must vanish at the end considered, giving two conditons

$$U(x,t) = 0 \quad , \quad \frac{\partial^2 U(x,t)}{\partial x^2} = 0$$

2. Fixed/clamped end conditions: Here the displacement U(x,t) and the slope $\frac{\partial U(x,t)}{\partial x}$ must vanish

3. Free end conditions: In this case the moment and the shearing force must vanish at the ends. These imply

$$\frac{\partial^2 U(x,t)}{\partial x^2} = 0 \text{ and } \frac{\partial^3 U(x,t)}{\partial x^3} = 0$$

respectively.

4. Sliding end conditions: If one end is constrained to retain zero slope but is completely free to move vertically, the slope $\frac{\partial U(x,t)}{\partial x}$ and the shear force $\frac{\partial^3 U(x,t)}{\partial x^3}$ must vanish

and it follows that

$$\frac{\partial U(x,t)}{\partial x} = 0 \text{ and } \frac{\partial^3 U(x,t)}{\partial x^3} = 0$$

at such ends for sliding end conditions.

Other class of boundary conditions termed non-classical boundary conditions includes, Masri (1976):

(a) Elastically supported end conditions

(b) Time dependent end conditions

(a) Elastically supported end conditions: Suppose a beam is hinged or pinned at one of its ends and supported by an elastic spring, with modulus k at the other end, the magnitude of the shearing force must be k times the displacement, i.e.

$$\frac{\partial^2 U(x,t)}{\partial x^2} = \pm kU(x,t)$$

Thus for such elastically supported end, the boundary conditions are

$$\frac{\partial^2 U(x,t)}{\partial x^2} = 0 \ , \ \frac{\partial^2 U(x,t)}{\partial x^2} - k_1 U(x,t) = 0$$

where k_1 is an arbitrary spring modulus.

(b) Time dependent end conditions: In this case, at least one of the geometric or dynamic boundary conditions does not vanish but is time dependent. Problems involving this type of boundary conditions are, in general, resistant to the classical methods of solving dynamical problems.

In this thesis we shall be concerned with the problem of assessing the dynamic response to moving concentrated masses of Rayleigh beams on variable Winkler elastic foundations. This work incorporates the inertia effect of the moving load, the effect of cross sectional dimensions of the beam and the effect of variable Winkler elastic foundation in the governing differential equations of the dynamical problems and sets at solving them. The analyses of the effects of these parameters on the response of the beams when they are being traversed by moving loads shall be carried out. The work covers both uniform and non-uniform Rayleigh beams.

1.1 REVIEW OF PREVIOUS WORK

The study of the behaviour of elastic solid bodies (beams, plates or shell) subjected to moving loads has been the concern of several researchers in applied Mathematics and Engineering. More specifically, several dynamical problems involving the response of beams on a foundation and without foundation have variously been tackled ,Gbadeyan and Aiyesimi (1990). Among the earliest work in this area of study was the work of Stokes reported in Fryba (1972), who obtained an approximate solution for the response of a beam by neglecting the mass of the beam. This is because the introduction of the inertia effect of the moving mass would make the governing equation cumbersome to solve as reported in Stanisic et al (1974). Recognizing this difficulty Pestel (1951) applied Rayleigh-Ritz techniques to reduce the moving mass problem defined by a continuous differential equation to an approximate system of discrete differential equations with analytic coefficients. The system was reduced by a finite difference scheme for solution, but no numerical results were presented. After this, several researchers have approached this problem by assuming that the inertia of the moving load was negligible. Infact, Arye et al (1951) pointed out that the fundamental mathematical difficulties encountered in the problem lie in the fact that one of the coefficients of the linear operator describing the motion is a function of space and time. They added that it is caused by the presence of the Dirac-delta function as a coefficient necessary for a proper description of the motion. It is remarked at this juncture that, physically, this term represents the interplay of the inertial forces due to the discrete masses distributed over the structure during the motion, Fryba (1972). Some of the studies reported by Arye et al (1951) include the work of Willis et al, reported in Fryba (1972), who considered the problem of elastic beam under the action of moving loads. He assumed the mass of the beam to be smaller than the mass of the moving load and obtained an approximate solution to the problem. This is followed by the other extreme case when the mass of the load was smaller than the mass of the beam. In particular, the dynamic response of a simply supported beam traversed by a constant force moving at a uniform speed was first studied by Krylov (1905). He used method of expansion of eigen function to obtain his results. Lowan (1935) also considered the problem of transverse oscillations of beams under the action of moving loads for the general case of any arbitrary prescribed law of motion. He obtained his solution using Green's functions.

The problem of a load moving along elastic beam on an elastic foundation is of great theoretical and practical significance, Oni (2000). Extensive theoretical and experimental investigations have been carried out, particularly, when the foundation modulus is constant along the span of the beam. This class of problems is generally mathematically complex. Infact, to overcome the complexity, the mechanical behaviour of the subgrade has often been idealized. The simplest mechanical foundation model was proposed by Winkler in 1867. The model expresses the relation between the pressure and the deflection of the foundation surface. The Winkler foundation model consists of an infinite number of closely spaced springs uniformly distributed along the structure, Arnold (1964). When the spring constant also called the foundation modulus, is constant along the length of the structure, the differential equation of motion of the structure has constant coefficients. If the foundation stiffness varies along the structure, the differential equation is of variable coefficients and becomes more cumbersome to solve. The analysis of beam on Winkler foundation when the foundation modulus is constant is very common in literature. The work of Timoshenko (1921) gave impetus to research work in this area of study. He used energy methods to obtain solutions in series form for simply supported finite beams on elastic foundations subjected to time dependent point loads moving with uniform velocity across the beam. Kenny (1954) also investigated the dynamic response of infinite beams on elastic foundation under the action of moving load of constant speed. He included in the governing equation the effect of viscous damping. The response of a finite, simply supported Bernoulli-Euler beam to a unit force at a constant speed is investigated by steele (1967). The effects of this moving force on beams with or without an elastic foundation are analyzed. Furthermore Clastonik et al (1986) studied the problem of vibrations of Bernoulli-Euler beams on variable Winkler elastic foundation. The load acting on the beam in this problem was static.

In all the aforementioned investigations, problems have been largely restricted to the case when the inertia effects of the moving load have been neglected. The more complicated case in which consideration is given to the inertial effects of the moving load has been neglected. However, recently, the advent of long highway bridges, together with increased velocity and mass of automobiles, has forced a renewal of consideration of this problem.

In general, problems of this type are mathematically complex. A major break-through in this field of research is the work of stanisic et al (1968) who solved the problem of simply supported non-mindling plate under a multi-masses moving system by making use of an approximation of the Dirac-delta function. Only the inertia term which measures the effect of local acceleration in the direction of the deflection was considered. The method of solution was based on the Fourier sine transform technique suitable only for simply supported boundary conditions. A one dimensional analogue of this problem was taken up later by Milomir et al (1969) who developed a theory describing the response of a beam under an arbitrary number of moving masses. The theory is based on the Fourier technique and shows that, for a simply supported beam, the resonance frequency is lower with no corresponding decrease in maximum amplitude when the inertia is considered. This work was later extended by Stanisic et al (1974) to include all the components of the inertia term. They considered only Bernoulli-Euler Beams with only simply supported end conditions. It is remarked at this juncture that in all these aforementioned investigations, methods of solution have been suitable only for simply supported end conditions. To address this problem, an interesting attempt was made by Sadiku and Leipholz (1981) who developed an elegant technique capable of solving Bernoulli-Euler moving load problems for all variants of classical boundary conditions. The technique involves transforming the differential equation governing the moving mass problem into an integro-differential equation by using the Green function of the associated moving force problem. Although this work is impressive, its application is limited only to the case of beams executing flexural vibrations according to the simple

Bernoulli-Euler theory of flexure. The extension of the method to flexural vibrations of structures with height-span ratios larger than about 1/10 has not been effected. Nonetheless, it is known that during vibration, a typical element of a beam performs not only a translatory motion but also rotates, Timoshenko et al (1974). Thus, there is the need to consider beams whose motion is not governed by Bernoulli-Euler theory (thin beams). More recently, Oni (1991) and Gbadeyan and Oni (1995) presented a theory for determining the response of a finite Rayleigh beam (a thick beam) under an arbitrary number of moving concentrated masses. The theory advanced involves the development of an analytical versatile technique which is based on the modified generalized finite integral transform and the modified Struble's method. An important feature of this technique is that it is applicable to all classical end conditons, as well as both thin and thick beam moving load problems. This technique was further extended by Oni (2000) and Gbadeyan and Oni (1995) to solve the problem of dynamic response of an elastic plate traversed by several moving concentrated masses. In particular, a two-dimensional analogue of the analytical method developed in Oni (2000) was developed. The extension was carried out in such a way that the technique is also suitable for all variants of the corresponding two-dimensional dynamical problems for all variants of classical boundary conditions.

As impressive, though, these methods are, they are incapable of handling dynamical problems involving beams and plates:

(i) of non-uniform cross section

(ii) resting on a variable elastic foundation

1.2 OBJECTIVES OF THE RESEARCH

This study considers beams, resting on variable Winkler elastic foundations and traversed by moving concentrated masses, the motion of which is not governed by Bernoulli-Euler (thin beam) theory of flexure, with the following specific objectives:

(a) To obtain the analytical solution of the fourth order partial differential equation with variable and singular coefficients, of (i) Uniform Rayleigh beam and (ii) Non-uniform Rayleigh beam for all variants of boundary conditions.

(b) To investigate the influence of Rotatory inertia on the response to moving masses of both uniform and non-uniform Rayleigh beams resting on a variable elastic foundation.

(c) To analyze the effect of the variable elastic foundation on the transverse displacement response of both types of Rayleigh beam.

(d) To examine the reliability of the moving force solution as a safe approximation to the moving mass problem.

(e) To establish the resonance conditions for both moving force and moving mass problems and the effect of rotatory inertia and foundation moduli on the resonance conditions.

GOVERNING DIFFERENTIAL EQUATION

The displacement response to a moving concentrated load P(x,t) of a Rayleigh beam resting on a variable elastic Winkler foundation f(x) is governed by the partial differential equation [1]

$$\frac{\partial^2}{\partial x^2}\left[EI \frac{\partial^2 U(x,t)}{\partial x^2} \right] + \mu\frac{\partial^2 U(x,t)}{\partial t^2} - \frac{\partial}{\partial x}\mu R^o \left[\frac{\partial^3 U(x,t)}{\partial x \partial t^2} \right] + f(x)\, U(x,t) = P(x,t) \tag{1.1}$$

where x is the spatial coordinate, t is the time, U(x,t) is the transverve displacement, E is the Young's modulus, I is the moment of inertia, μ is the mass per unit length of the beam and R^o is the measure of rotatory inertia effect.

For a uniform Rayleigh beam, equation (1.1) becomes

$$EI \frac{\partial^4 U(x,t)}{\partial x^4}] + \mu\frac{\partial^2 U(x,t)}{\partial t^2} - \mu R^o \left[\frac{\partial^4 U(x,t)}{\partial x^2 \partial t^2} \right] + f(x)\, U(x,t) = P(x,t) \tag{1.2}$$

On the other hand for the non-unifrom Rayleigh beam, equation (1.1) takes the form

$$\frac{\partial^2}{\partial x^2}\left[EI(x) \frac{\partial^2 U(x,t)}{\partial x^2} \right] + \mu(x)\frac{\partial^2 U(x,t)}{\partial t^2} - \frac{\partial}{\partial x} R^o \mu(x) \left[\frac{\partial^3 U(x,t)}{\partial x \partial t^2} \right] + f(x)\, U(x,t) = P(x,t) \tag{1.3}$$

Where I and μ are now functions of the spatial coordinate x.

FEATURES OF THE THESIS

The procedure adopted in the remaining part of this dissertation is as follows:

In chapter two, the initial-boundary value problem of the dynamic response to a moving mass of a uniform Rayleigh beam resting on a variable elastic foundation is solved in general form. Illustrative examples involving the various classical boundary conditions, numerical calculations and discussions of results are presented in chapter three. Chapter four considers the initial-boundary value problem of the dynamic response to a moving mass of a non-uniform Rayleigh beam resting on a variable elastic foundation. The analytical solution is obtained in general form. This is followed immediately by illustrative examples involving the various classical boundary conditions, numerical calculations and discussions of results in chapter five.

Finally, conclusions and suggestions for future work are given in chapter six of the thesis.

CHAPTER TWO

UNIFORM RAYLEIGH BEAM ON A VARIABLE WINKLER ELASTIC FOUNDATION

2.1 GOVERNING EQUATION

The problem of the dynamic response to a moving mass of beam resting on a variable winkler elastic fcundation is considered in this chapter. This problem is governed by the fourth order partial differential equation given by

$$\frac{\partial^2}{\partial x^2}\left[EI\frac{\partial^2 U(x,t)}{\partial x^2}\right] + \mu\frac{\partial^2 U(x,t)}{\partial t^2} - \frac{\partial \mu R^0}{\partial x}\left(\frac{\partial^3 U(x,t)}{\partial x \partial t^2}\right) + f(x)\,U(x,t) = P(x,t). \tag{2.1}$$

Where x is the Spacial Coordinate, t is Time, U(x,t) is the Transverse Displacement, E is the Young's Modulus, I is the Moment of Inertia, μ is the Mass per unit length of the beam, R^0 is the Rotatory Inertia of the cross section and f(x) is the variable elastic Foundation.

The moving load on the beam under consideration has mass commeasurable with the mass of the beam. Thus, the load P(x,t) takes the form Timoshenko (1922)

$$P(x,t) = P_f(x,t)\left[1 - \frac{1}{g}\frac{d(U(x,t))}{dx}\right] \tag{2.2}$$

where $P_f(x,t)$ is the continuos moving force, $\frac{d}{dx}$ is the convective acceleration operator and g is the acceleration due to gravity.

The load on the beam is assumed to be of mass M moving with constant velocity c. The operator $\frac{d}{dx}$ used in equation (2.2) is defined as

$$\frac{d}{dx} = \frac{\partial^2}{\partial t^2} + 2c\frac{\partial^2}{\partial x \partial t} + c^2\frac{\partial^2}{\partial x^2} \tag{2.3}$$

Furthermore, the moving force acting on the beam model here is defined as

$$P_f(x,t) = \sum_{i=1}^{N} \mu\, g\, \delta(x - ct) \tag{2.4}$$

where the time t is assumed to be limited to that interval of time within which the mass μ are on the beam, that is

$$0 \le ct \le L \tag{2.5}$$

and δ (x – ct) is the Dirac delta function defined as

$$\delta(x - ct) = \begin{cases} 0, & x \neq ct \\ \infty, & x = ct \end{cases} \tag{2.6}$$

with the properties

$$\text{(i)} \qquad \delta(-x) = \delta(x) \tag{2.7}$$

$$\text{(ii)} \qquad \int_a^b \delta(x - k) f(x)\, dx = \begin{cases} 0, & k< <a <b \\ f(k), & a< k< <b \\ 0, & a< b< k \end{cases} \tag{2.8}$$

In mechanics, the Dirac delta function δ(x) may be thought of as a unit concentrated force acting at point x = 0, Timoshenko (1922).

In this chapter, the Rayleigh beam under consideration is assumed to be uniform, that is, the beam's properties such as the young's modulus E, the moment of inertia I and the mass per unit length of the beam μ do not vary along the span L.

As an example in this problem, a variable elastic foundation of the form

$$f(x) = K(4x - 3x^2 + x^3) \tag{2.9}$$

where k is the Foundation constant, is considered.

Thus substituting (2.2), (2.3), (2.4) and (2.9) into (2.1), one obtains

$$EI \frac{\partial^4 U(x,t)}{\partial x^4} + \mu \frac{\partial^2 U(x,t))}{\partial t^2} - \mu R^o \frac{\partial^4 U(x,t)}{\partial x^2 \partial t^2} + M\delta(x - ct) \left(\frac{\partial^2}{\partial t^2} + \frac{2c\partial^2}{\partial x \partial t} + \frac{c^2\partial^2}{\partial x^2} \right) U(x,t)$$

$$+ K(4x - 3x^2 + x^3)\, U(x, t) = Mg\delta(x - ct)\,. \qquad (2.10)$$

The boundary conditions of the above problem are assumed to be arbitrary, that is, it can take any form of the classical boundary conditions. The initial conditions, without any loss of generality is taken as

$$U(x,0) = 0 = \frac{\partial U\,(x,\,0)}{\partial t} \qquad (2.11)$$

2.2 METHOD OF SOLUTION

Evidently, the method of separation of variables is unapplicable because it becomes difficult to get separate equations where functions are functions of a single variable. In fact, a closed form solution of the above singular differential equation does not exist. As a result of the above difficulty, an approximate solution is sought. One of the approximate methods best suited for solving diverse problems in dynamics of structures is the Galerkin's method.

2.3 GALERKIN'S METHOD

This is one of the best methods used in solving problems involving mechanical vibrations. The Galerkin's method is used to solve equations of the form

$$\Gamma[V] - P = 0 \qquad (2.12)$$

where

Γ= the differential operator, (linear or non-linear),

V= the structural displacement,

P = the transverse load acting on the structure.

A solution of the form

$$V_j\,(x,t) = q_j(t)\phi_j(x) \qquad (2.13)$$

is sought, where j = 1, 2, 3, ..., n.

The functions $\phi_j(x)$ are chosen to satisfy the appropriate boundary conditions. The Galerkin's method requires that the expression (2.13) be orthogonal to functions ϕ_1, ϕ_2, ϕ_3, ..., ϕ_n, that is,

$$\int_0^L [\Gamma \sum_{j=1}^{n} q_j(t)\phi_j(x) - P]\,\phi_k(x)dx = 0, \tag{2.14}$$

$$k = 1, 2, 3, \ldots, n$$

This gives us a set of ordinary differential equations in $q_j(t)$ to be solved. These differential equations are called Galerkin's equations.

2.4 ANALYTICAL APPROXIMATE SOLUTION

The Galerkin's method requires that the solution of equation (2.10) takes the form

$$U_n(x,t) = \sum_{m=1}^{n} W_m(t)V_m(x) \tag{2.15}$$

where $V_m(x)$ is chosen such that the desired boundary conditions are satisfied.

Equation (2.15) when substituted into equation (2.10) yields

$$\sum_{m=1}^{n}\{EI\, V^{Iv}{}_m(x)W_m(t) + \mu V_m(x)\ddot{W}_m(t) - \mu R^o\, V^{11}{}_m(x)\ddot{W}_m(t)$$

$$+ M\delta(x-ct)\,[V_m(x)\ddot{W}_m(t) + 2cV^{1}{}_m(x)\,\dot{W}_m(t) + c^2\, V^{11}{}_m(x)\, W_m(t)]$$

$$+ k(4x - 3x^2 + x^3)V_m(x)W_m(t) - Mg\,\delta(x-ct)\} = 0 \tag{2.16}$$

In order to determine $W_m(t)$, it is required that the expression on the left-hand side of equation (2.16) be orthogonal to the functions $V_k(x)$.

Hence

$$\int_0^L \{\sum_{m=1}^{n}[\,EI\, V^{Iv}{}_m(x)W_m(t) + \mu V_m(x)\ddot{W}_m(t)$$

$$-\ \mu R^o V^{11}{}_m(x)\ddot{W}_m(t) + M\,\delta(x-ct)\,(V_m(x)\ddot{W}_m(t)$$

$$+ 2cV^{1}{}_m(x)\dot{W}_m(t) + c^2V^{11}{}_m(x)W_m(t) + k(4x - 3x^2 + x^3)V_m(x)W_m(t)$$

$$- Mg\,\delta(x-ct)]\,\}V_k(x)dx = 0 . \quad (2.17)$$

Further simplification and rearrangement of (2.17) yields

$$\sum_{m=1}^{n}\{(\Omega_{oA} - R^{o}\,\Omega_{oB})\ddot{W}_m(t) + [\,\frac{EI}{\mu}\,\Omega_{1A} + \frac{k}{\mu}\,(\,4\,\Omega_{1B} - 3\Omega_{1C} + \Omega_{1D})]W_m(t)$$

$$+ \frac{M}{\mu}\,[\Omega_2(t)\ddot{W}_m(t) + 2c\Omega_3(t)\dot{W}_m(t) + c^2\Omega_4(t)W_m(t)]\,\} = \frac{M_gV_k(ct)}{\mu} \quad (2.18)$$

where

$$\Omega_{oA} = \int_0^L V_m(x)V_k(x)dx \quad (2.19)$$

$$\Omega_{oB} = \int_0^L V^{11}{}_m(x)V_k(x)\,dx \quad (2.20)$$

$$\Omega_{1A} = \int_0^L V^{1v}{}_m(x)\,V_k(x)\,dx \quad (2.21)$$

$$\Omega_{1B} = \int_0^L xV_m(x)\,V_k(x)\,dx \quad (2.22)$$

$$\Omega_{1C} = \int_0^L x^2\,V_m(x)\,V_k\,(x)\,dx \quad (2.23)$$

$$\Omega_{1D} = \int_0^L x^3\,V_m(x)\,V_k\,(x)\,dx \quad (2.24)$$

$$\Omega_2(t) = \int_0^L \delta(x-ct)\,V_m(x)\,V_k\,(x)\,dx \quad (2.25)$$

$$\Omega_3(t) = \int_0^L \delta(x-ct)\,V^1{}_m(x)\,V_k(x)\,dx \quad (2.26)$$

$$\Omega_4\,(t) = \int_0^L \delta(x-ct)\,V^{11}{}_m(x)\,V_k(x)\,dx \quad (2.27)$$

Using the property of the Dirac – delta function as an even function, it can easily be shown that

$$\delta(x-ct) = \frac{1}{L} + \frac{2}{L}\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}\cos\frac{n\pi x}{L} \quad (2.28)$$

When use is made of (2.28) in (2.18), one obtains

$$\sum_{m=1}^{n}\{\Omega_0(m,k)\ddot{W}_m(t) + \Omega_1(m,k)W_m(t) + \frac{M}{L\mu}[(\Omega_{2A}(m,k)$$

$$+ 2\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}\ \Omega_{2B}(n,m,k))\ddot{W}_m(t) + 2c(\Omega_{3A}(m,k) + 2\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}\Omega_{3B}(n,m,k))\dot{W}_m(t)$$

$$+ c^2(\Omega_{4A}(m,k) + 2\sum_{n=1}^{\infty}\cos n\pi ct\ \Omega_{4B}(n,m,k)W_m(t)]\} = \frac{Mg}{\mu}V_k(ct) \tag{2.29}$$

where

$$\Omega_0(m,k) = \Omega_{0A} - R^0\Omega_{0B} \tag{2.30}$$

$$\Omega_1(m,k) = \frac{EI}{\mu}\Omega_{1A} + \frac{K}{\mu}\ [4\Omega_{1B} - 3\Omega_{1C} + \Omega_{1D}] \tag{2.31}$$

$$\Omega_{2A}(m,k) = \int_0^L V_m(x)\, V_k(x)\, dx \tag{2.32}$$

$$\Omega_{2B}(n,m,k) = \int_0^L \cos\frac{n\pi x}{L} V_m(x)\, V_k(x)\, dx \tag{2.33}$$

$$\Omega_{3A}(m,k) = \int_0^L V^1{}_m(x)\, V_k(x)\, dx \tag{2.34}$$

$$\Omega_{3B}(n,m,k) = \int_0^L \cos\frac{n\pi x}{L} V^1{}_m(x)\, V_k(x)\, dx \tag{2.35}$$

$$\Omega_{4A}(m,k) = \int_0^L V^{11}{}_m(x)\, V_k(x)\, dx \tag{2.36}$$

$$\Omega_{4B}(n,m,k) = \int_0^L \cos\frac{n\pi x}{L} V^{11}{}_m(x)\, V_k(x)\, dx \tag{2.37}$$

In this section, a solution valid for all cases of classical boundary conditions is sought. Consequently, $V_m(x)$ is chosen as the beam function given as

$$V_m(x) = \sin\frac{\lambda_m x}{L} + A_m\cos\frac{\lambda_m x}{L} + B_m\sinh\frac{\lambda_m x}{L} + C_m\cosh\frac{\lambda_m x}{L} \tag{2.38}$$

Thus

$$V_k(ct) = \sin\frac{\lambda_k ct}{L} + A_k\cos\frac{\lambda_k ct}{L} + B_k\sinh\frac{\lambda_k ct}{L} + C_k\cosh\frac{\lambda_k ct}{L} \tag{2.39}$$

Using equations (2.38) and (2.39) and rearranging, one obtains

$$\Omega_{OA} = I_1 + A_mI_2 + B_mI_3 + C_mI_4 + A_kI_5 + A_kA_mI_6 + A_kB_mI_7 + A_kC_mI_8 + B_kI_9$$

$$+ B_k A_m I_{10} + B_k B_m I_{11} + B_k C_m I_{12} + C_k I_{13} + C_k A_m I_{14} + C_k B_m I_{15} + C_k C_m I_{16} \quad (2.40)$$

$$\Omega_{oB} = \frac{\lambda_m^2}{L^2} [- I_1 - A_k I_2 - B_k I_3 - C_k I_4 - A_m I_5 - A_m A_k I_6 - A_m B_k I_7 - A_m C_k I_8 + B_m I_9 + B_m A_k I_{10} + B_m B_k I_{11} + B_m C_k I_{12} + C_m I_{13} + C_m A_k I_{14} + C_m B_k I_{15} + C_m C_k I_{16}] \quad (2.41)$$

$$\Omega_{1A} = \frac{\lambda_m^4}{L^4} \Omega_{oA} \quad (2.42)$$

$$\Omega_{1B} = I_{17C} + A_m I_{18C} + B_m I_{19C} + C_m I_{20C} + A_k I_{21C} + A_m A_k I_{22C} + B_m A_k I_{23C} + C_m A_k I_{24C} + B_k I_{25C} + A_m B_k I_{26C} + B_m B_k I_{27C} + C_m B_k I_{28C} + C_k I_{29C} + A_m C_k I_{30C} + B_m C_k I_{31C} + C_m C_k I_{32C} \quad (2.43)$$

$$\Omega_{1C} = I_{17A} + A_m I_{18A} + B_m I_{19A} + C_m I_{20A} + A_k I_{21A} + A_m A_k I_{22A} + B_m A_k I_{23A} + C_m A_k I_{24A} + B_k I_{25A} + A_m B_k I_{26A} + B_m B_k I_{27A} + C_m B_k I_{28A} + C_k I_{29A} + A_m C_k I_{30A} + B_m C_k I_{31A} + C_m C_k I_{32A} \quad (2.44)$$

$$\Omega_{1D} = I_{17B} + A_m I_{18B} + B_m I_{19B} + C_m I_{20B} + A_k I_{21B} + A_m A_k I_{22B} + B_m A_k I_{23B} + C_m A_k I_{24B} + B_k I_{25B} + A_m B_k I_{26B} + B_m B_k I_{27B} + C_m B_k I_{28B} + C_k I_{29B} + A_m C_k I_{30B} + B_m C_k I_{31B} + C_m C_k I_{32B} \quad (2.45)$$

$$\Omega_{2A}(m,k) = I_1 + A_m I_2 + B_m I_3 + C_m I_4 + A_k I_5 + A_m A_k I_6 + B_m A_k I_7 + C_m A_k I_8 + B_k I_9 + A_m B_k I_{10} + B_m B_k I_{11} + C_m B_k I_{12} + C_k I_{13} + A_m C_k I_{14} + B_m C_k I_{15} + C_m C_k I_{16} \quad (2.46)$$

$$\Omega_{2B}(n,m,k) = I_{17} + A_m I_{18} + B_m I_{19} + C_m I_{20} + A_k I_{21} + A_m A_k I_{22} + B_m A_k I_{23} + C_m A_k I_{24} + B_k I_{25} + A_m B_k I_{26} + B_m B_k I_{27} + C_m B_k I_{28} + C_k I_{29} + A_m C_k I_{30} + B_m C_k I_{31} + C_m C_k I_{32} \quad (2.47)$$

$$\Omega_{3A}(m,k) = \frac{\lambda_m}{L} [-A_m I_1 + I_2 + C_m I_3 + B_m I_4 - A_m A_k I_5 + A_k I_6 + C_m A_k I_7 + B_m A_k I_8$$

$$- A_mB_kI_9 + B_kI_{10} + C_mB_kI_{11} + B_mB_kI_{12} - A_mC_kI_{13} + C_kI_{14}$$

$$+ \; C_mC_kI_{15} + B_mC_kI_{16}] \tag{2.48}$$

$$\Omega_{3B}(n,m,k) = \frac{\lambda_m}{L} [-A_mI_{17} + I_{18} + C_mI_{19} + B_mI_{20} - A_mA_kI_{21} + A_kI_{22} + C_mA_kI_{23}$$

$$+ B_mA_kI_{24} - A_mB_kI_{25} + B_kI_{26} + C_mB_kI_{27} + B_mB_kI_{28}$$

$$- A_mC_kI_{29} + C_kI_{30} + C_mC_kI_{31} + B_mC_kI_{32}] \tag{2.49}$$

$$\Omega_{4A}(m,k) = \frac{\lambda_m^2}{L^2} [-I_1 - A_mI_2 + B_mI_3 + C_mI_4 - A_kI_5 - A_mA_kI_6 + B_mA_kI_7$$

$$+ C_mA_kI_8 - B_kI_9 - A_mB_kI_{10} + B_mB_kI_{11} + C_mB_kI_{12} - C_kI_{13}$$

$$- A_mC_kI_{14} + B_mC_kI_{15} + C_mC_kI_{16}] \tag{2.50}$$

$$\Omega_{4B}(n,m,k) = \frac{\lambda_m^2}{L^2} [- I_{17} - A_mI_{18} + B_mI_{19} + C_mI_{20} - A_kI_{21} - A_mA_kI_{22} + B_mA_kI_{23}$$

$$+ C_mA_kI_{24} - B_kI_{25} - A_mB_kI_{26} + B_mB_kI_{27} + C_mB_kI_{28} - C_kI_{29}$$

$$- A_mC_kI_{30} + B_mC_kI_{31} + C_mC_kI_{32} \tag{2.51}$$

where

$$I_1 = \int_0^L \sin \frac{\lambda_k x}{L} \sin \frac{\lambda_m x}{L} dx , \qquad I_2 = \int_0^L \sin \frac{\lambda_k x}{L} \cos \frac{\lambda_m x}{L} dx ,$$

$$I_3 = \int_0^L \sin \frac{\lambda_k x}{L} \sinh \frac{\lambda_m x}{L} dx , \qquad I_4 = \int_0^L \sin \frac{\lambda_k x}{L} \cosh \frac{\lambda_m x}{L} dx ,$$

$$I_5 = \int_0^L \cos \frac{\lambda_k x}{L} \sin \frac{\lambda_m x}{L} dx , \qquad I_6 = \int_0^L \cos \frac{\lambda_k x}{L} \cos \frac{\lambda_m x}{L} dx ,$$

$$I_7 = \int_0^L \cos \frac{\lambda_k x}{L} \sinh \frac{\lambda_m x}{L} dx , \qquad I_8 = \int_0^L \cos \frac{\lambda_k x}{L} \cosh \frac{\lambda_m x}{L} dx ,$$

$$I_9 = \int_0^L \sinh \frac{\lambda_k x}{L} \sin \frac{\lambda_m x}{L} dx , \qquad I_{10} = \int_0^L \sinh \frac{\lambda_k x}{L} \cos \frac{\lambda_m x}{L} dx ,$$

$$I_{11} = \int_0^L \sinh \frac{\lambda_k x}{L} \sinh \frac{\lambda_m x}{L} dx , \qquad I_{12} = \int_0^L \sinh \frac{\lambda_k x}{L} \cosh \frac{\lambda_m x}{L} dx,$$

$$I_{13} = \int_0^L \cosh \frac{\lambda_k x}{L} \sin \frac{\lambda_m x}{L} dx , \qquad I_{14} = \int_0^L \cosh \frac{\lambda_k x}{L} \cos \frac{\lambda_m x}{L} dx,$$

$$I_{15} = \int_0^L \cosh \frac{\lambda_k x}{L} \sinh \frac{\lambda_m x}{L} \, dx , \quad I_{16} = \int_0^L \cosh \frac{\lambda_k x}{L} \cosh \frac{\lambda_m x}{L} \, dx,$$

$$I_{17} = \int_0^L \cos \frac{n\pi x}{L} \sin \frac{\lambda_k x}{L} \sin \frac{\lambda_m x}{L} \, dx , \quad I_{18} = \int_0^L \cos \frac{n\pi x}{L} \sin \frac{\lambda_k x}{L} \cos \frac{\lambda_m x}{L} \, dx,$$

$$I_{19} = \int_0^L \cos \frac{n\pi x}{L} \sin \frac{\lambda_k x}{L} \sinh \frac{\lambda_m x}{L} \, dx , \quad I_{20} = \int_0^L \cos \frac{n\pi x}{L} \sin \frac{\lambda_k x}{L} \cosh \frac{\lambda_m x}{L} \, dx,$$

$$I_{21} = \int_0^L \cos \frac{n\pi x}{L} \cos \frac{\lambda_k x}{L} \sin \frac{\lambda_m x}{L} \, dx , \quad I_{22} = \int_0^L \cos \frac{n\pi x}{L} \cos \frac{\lambda_k x}{L} \cos \frac{\lambda_m x}{L} \, dx,$$

$$I_{23} = \int_0^L \cos \frac{n\pi x}{L} \cos \frac{\lambda_k x}{L} \sinh \frac{\lambda_m x}{L} \, dx , \quad I_{24} = \int_0^L \cos \frac{n\pi x}{L} \cos \frac{\lambda_k x}{L} \cosh \frac{\lambda_m x}{L} dx,$$

$$I_{25} = \int_0^L \cos \frac{n\pi x}{L} \sinh \frac{\lambda_k x}{L} \sin \frac{\lambda_m x}{L} \, dx , \quad I_{26} = \int_0^L \cos \frac{n\pi x}{L} \sinh \frac{\lambda_k x}{L} \cos \frac{\lambda_m x}{L} \, dx,$$

$$I_{27} = \int_0^L \cos \frac{n\pi x}{L} \sinh \frac{\lambda_k x}{L} \sinh \frac{\lambda_m x}{L} dx , \quad I_{28} = \int_0^L \cos \frac{n\pi x}{L} \sinh \frac{\lambda_k x}{L} \cosh \frac{\lambda_m x}{L} \, dx,$$

$$I_{29} = \int_0^L \cos \frac{n\pi x}{L} \cosh \frac{\lambda_k x}{L} \sin \frac{\lambda_m x}{L} \, dx , \quad I_{30} = \int_0^L \cos \frac{n\pi x}{L} \cosh \frac{\lambda_k x}{L} \cos \frac{\lambda_m x}{L} \, dx,$$

$$I_{31} = \int_0^L \cos \frac{n\pi x}{L} \cosh \frac{\lambda_k x}{L} \sinh \frac{\lambda_m x}{L} \, dx , \quad I_{32} = \int_0^L \cos \frac{n\pi x}{L} \cosh \frac{\lambda_k x}{L} \cosh \frac{\lambda_m x}{L} \, dx,$$

$$I_{17A} = \int_0^L x^2 \sin \frac{\lambda_k x}{L} \sin \frac{\lambda_m x}{L} \, dx , \quad I_{18A} = \int_0^L x^2 \sin \frac{\lambda_k x}{L} \cos \frac{\lambda_m x}{L} \, dx,$$

$$I_{19A} = \int_0^L x^2 \sin \frac{\lambda_k x}{L} \sinh \frac{\lambda_m x}{L} \, dx , \quad I_{20A} = \int_0^L x^2 \sin \frac{\lambda_k x}{L} \cosh \frac{\lambda_m x}{L} \, dx,$$

$$I_{21A} = \int_0^L x^2 \cos \frac{\lambda_k x}{L} \sin \frac{\lambda_m x}{L} \, dx , \quad I_{22A} = \int_0^L x^2 \cos \frac{\lambda_k x}{L} \cos \frac{\lambda_m x}{L} \, dx,$$

$$I_{23A} = \int_0^L x^2 \cos \frac{\lambda_k x}{L} \sinh \frac{\lambda_m x}{L} \, dx , \quad I_{24A} = \int_0^L x^2 \cos \frac{\lambda_k x}{L} \cosh \frac{\lambda_m x}{L} \, dx,$$

$$I_{25A} = \int_0^L x^2 \sinh \frac{\lambda_k x}{L} \sin \frac{\lambda_m x}{L} \, dx , \quad I_{26A} = \int_0^L x^2 \sinh \frac{\lambda_k x}{L} \cos \frac{\lambda_m x}{L} \, dx,$$

$$I_{27A} = \int_0^L x^2 \sinh \frac{\lambda_k x}{L} \sinh \frac{\lambda_m x}{L} \, dx , \quad I_{28A} = \int_0^L x^2 \sinh \frac{\lambda_k x}{L} \cosh \frac{\lambda_m x}{L} \, dx,$$

$$I_{29A} = \int_0^L x^2 \cosh\frac{\lambda_k x}{L} \sin\frac{\lambda_m x}{L}\, dx\,, \quad I_{30A} = \int_0^L x^2 \cosh\frac{\lambda_k x}{L} \cos\frac{\lambda_m x}{L}\, dx,$$

$$I_{31A} = \int_0^L x^2 \cosh\frac{\lambda_k x}{L} \sinh\frac{\lambda_m x}{L}\, dx\,, \quad I_{32A} = \int_0^L x^2 \cosh\frac{\lambda_k x}{L} \cosh\frac{\lambda_m x}{L}\, dx,$$

$$I_{17B} = \int_0^L x^3 \sin\frac{\lambda_k x}{L} \sin\frac{\lambda_m x}{L}\, dx\,, \quad I_{18B} = \int_0^L x^3 \sin\frac{\lambda_k x}{L} \cos\frac{\lambda_m x}{L}\, dx,$$

$$I_{19B} = \int_0^L x^3 \sin\frac{\lambda_k x}{L} \sinh\frac{\lambda_m x}{L}\, dx\,, \quad I_{20B} = \int_0^L x^3 \sin\frac{\lambda_k x}{L} \cosh\frac{\lambda_m x}{L}\, dx,$$

$$I_{21B} = \int_0^L x^3 \cos\frac{\lambda_k x}{L} \sin\frac{\lambda_m x}{L}\, dx\,, \quad I_{22B} = \int_0^L x^3 \cos\frac{\lambda_k x}{L} \cos\frac{\lambda_m x}{L}\, dx,$$

$$I_{23B} = \int_0^L x^3 \cos\frac{\lambda_k x}{L} \sinh\frac{\lambda_m x}{L}\, dx\,, \quad I_{24B} = \int_0^L x^3 \cos\frac{\lambda_k x}{L} \cosh\frac{\lambda_m x}{L}\, dx,$$

$$I_{25B} = \int_0^L x^3 \sinh\frac{\lambda_k x}{L} \sin\frac{\lambda_m x}{L}\, dx\,, \quad I_{26B} = \int_0^L x^3 \sinh\frac{\lambda_k x}{L} \cos\frac{\lambda_m x}{L}\, dx,$$

$$I_{27B} = \int_0^L x^3 \sinh\frac{\lambda_k x}{L} \sinh\frac{\lambda_m x}{L}\, dx\,, \quad I_{28B} = \int_0^L x^3 \sinh\frac{\lambda_k x}{L} \cosh\frac{\lambda_m x}{L}\, dx,$$

$$I_{29B} = \int_0^L x^3 \cosh\frac{\lambda_k x}{L} \sin\frac{\lambda_m x}{L}\, dx\,, \quad I_{30B} = \int_0^L x^3 \cosh\frac{\lambda_k x}{L} \cos\frac{\lambda_m x}{L}\, dx,$$

$$I_{31B} = \int_0^L x^3 \cosh\frac{\lambda_k x}{L} \sinh\frac{\lambda_m x}{L}\, dx\,, \quad I_{32B} = \int_0^L x^3 \cosh\frac{\lambda_k x}{L} \cosh\frac{\lambda_m x}{L}\, dx,$$

$$I_{17c} = \int_0^L x \sin\frac{\lambda_k x}{L} \sin\frac{\lambda_m x}{L}\, dx\,, \quad I_{18c} = \int_0^L x \sin\frac{\lambda_k x}{L} \cos\frac{\lambda_m x}{L}\, dx,$$

$$I_{19c} = \int_0^L x \sin\frac{\lambda_k x}{L} \sinh\frac{\lambda_m x}{L}\, dx\,, \quad I_{20c} = \int_0^L x \sin\frac{\lambda_k x}{L} \cosh\frac{\lambda_m x}{L}\, dx,$$

$$I_{21c} = \int_0^L x \cos\frac{\lambda_k x}{L} \sin\frac{\lambda_m x}{L}\, dx\,, \quad I_{22c} = \int_0^L x \cos\frac{\lambda_k x}{L} \cos\frac{\lambda_m x}{L}\, dx,$$

$$I_{23c} = \int_0^L x \cos\frac{\lambda_k x}{L} \sinh\frac{\lambda_m x}{L}\, dx\,, \quad I_{24c} = \int_0^L x \cos\frac{\lambda_k x}{L} \cosh\frac{\lambda_m x}{L}\, dx,$$

$$I_{25c} = \int_0^L x \sinh\frac{\lambda_k x}{L} \sin\frac{\lambda_m x}{L}\, dx\,, \quad I_{26c} = \int_0^L x \sinh\frac{\lambda_k x}{L} \cos\frac{\lambda_m x}{L}\, dx,$$

$$I_{27c} = \int_0^L x \sinh \frac{\lambda_k x}{L} \sinh \frac{\lambda_m x}{L} dx, \quad I_{28c} = \int_0^L x \sinh \frac{\lambda_k x}{L} \cosh \frac{\lambda_m x}{L} dx,$$

$$I_{29c} = \int_0^L x \cosh \frac{\lambda_k x}{L} \sin \frac{\lambda_m x}{L} dx, \quad I_{30c} = \int_0^L x \cosh \frac{\lambda_k x}{L} \cos \frac{\lambda_m x}{L} dx,$$

$$I_{31c} = \int_0^L x \cosh \frac{\lambda_k x}{L} \sinh \frac{\lambda_m x}{L} dx, \quad I_{32c} = \int_0^L x \cosh \frac{\lambda_k x}{L} \cosh \frac{\lambda_m x}{L} dx,$$

(2.52)

The solutions to these integrals are contained in the Appendix.

Equation (2.29) can be rewritten as

$$\sum_{m=1}^{n} \{\Omega_0(m,k)\ddot{W}_m(t) + \Omega_1(m,k)W_m(t) + \Gamma\,[\,(\,\Omega_{2A}(m,k)$$

$$+ 2\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}\;\Omega_{2B}(n,m,k))\ddot{W}_m(t) + 2c(\,\Omega_{3A}(m,k)$$

$$+ 2\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}\;\Omega_{3B}(n,m,k))\dot{W}_m(t) + c^2(\Omega_{4A}(m,k) + 2\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}\;\Omega_{4B}(n,m,k))W_m(t)]\}$$

$$= \frac{Mg}{\mu}\,[\,\sin\frac{\lambda_k ct}{L} + A_k\cos\frac{\lambda_k ct}{L} + B_k\sinh\frac{\lambda_k ct}{L} + C_k\cosh\frac{\lambda_k ct}{L}]$$

(2.53)

where

$$\Gamma = \frac{M}{L\mu} \tag{2.54}$$

Equation (2.53) is the transformed equation governing the problem of a uniform Rayleigh beam on a variable Winkler elastic foundation. This second order differential equation holds for all variants of the classical boundary conditions.

2.5 SOLUTION OF THE TRANSFORMED EQUATION

In this section, two cases of equation (2.53) are considered.

2.5.1 CASE I

Setting $\Gamma = 0$ in the transformed equation (2.53), one obtains

$$\Omega_o(m,k)\ddot{W}_m(t) + \Omega_1(m,k)W_m(t)$$

$$= \frac{Mg}{\mu}\left[\sin \frac{\lambda_k ct}{L} + A_k\cos \frac{\lambda_k ct}{L} + B_k\sinh \frac{\lambda_k ct}{L} + C_k\cosh \frac{\lambda_k ct}{L} \right] \tag{2.55}$$

This represents the classical case of a moving force problem associated with our system. It is an approximate model, which assumes the inertia effect of the moving mass as negligible.

Further rearrangement of (2.55) yields

$$\ddot{W}_m(t) + \theta^2_m W_m(t) = P_m\left[\sin \frac{\lambda_k ct}{L} + A_k\cos \frac{\lambda_k ct}{L} + B_k\sinh \frac{\lambda_k ct}{L} + C_k\cosh \frac{\lambda_k ct}{L} \right] \tag{2.56}$$

where

$$\theta^2_m = \frac{\Omega_1(m,k)}{\Omega_o(m,k)} \quad \text{and} \quad P_m = \frac{Mg}{\mu\, \Omega_o(m,k)}$$

To obtain the solution to equation (2.56), it is subjected to a Laplace transform defined as

$$(\tilde{\cdot}) = \int_0^\infty (\,.\,)\, e^{-st}\, dt \tag{2.57}$$

Where s is the Laplace parameter. Applying the initial conditions (2.11), one obtains the simple algebraic equation given by

$$W_m(s) = \frac{p_m}{S^2 + \theta^2_m}\left[\frac{\lambda_k c / L}{S^2 + (\lambda_k c / L)^2} + A_k \frac{S}{S^2 + (\lambda_k c / L)^2} + B_k \frac{\lambda_k c / L}{S^2 - (\lambda_k c / L)^2} + C_k \frac{S}{S^2 - (\lambda_k c / L)^2}\right] \tag{2.58}$$

Thus, the problem reduces to that of finding the Laplace inversion of (2.58). To do this we adopt the following representations:

$$f(s) = \frac{P_m}{S^2 + \theta_m^2} \tag{2.59}$$

$$g(s) = \frac{\lambda_k c / L}{S^2 + (\lambda_k c / L)^2} + A_k \frac{S}{S^2 + (\lambda_k c / L)^2} + B_k \frac{\lambda_k c / L}{S^2 - (\lambda_k c / L)^2} + C_k \frac{S}{S^2 - (\lambda_k c / L)^2} \tag{2.60}$$

So that the Laplace inversion of $W_m(s)$ is the convolution of f(s) and g(s) defined as

$$f(s) * g(s) = \int_0^t f(t-u)\, g(u)\, du \tag{2.61}$$

Using (2.61), $W_m(t)$ is easily expressed as a sum of eight integrals namely:

$$W_m(t) = P_m[X_a - X_b + X_c - X_d + X_e - X_f + X_g - X_h] \tag{2.62}$$

where

$$X_a = \frac{\sin\theta_m t}{\theta m} \int_0^t \sin \frac{\lambda_k cu}{L} \cos\theta_m u du\ , \quad X_b = \frac{\cos\theta_m t}{\theta m} \int_0^t \sin \frac{\lambda_k cu}{L} \sin\theta_m u du$$

$$X_c = A_k \frac{\sin\theta_m t}{\theta m} \int_0^t \cos \frac{\lambda_k cu}{L} \cos\theta_m u du, \quad X_d = A_k \frac{\cos\theta_m t}{\theta m} \int_0^t \cos \frac{\lambda_k cu}{L} \sin\theta_m u du$$

$$X_e = B_k \frac{\sin\theta_m t}{\theta m} \int_0^t \sinh \frac{\lambda_k cu}{L} \cos\theta_m u du, \quad X_f = B_k \frac{\cos\theta_m t}{\theta m} \int_0^t \sinh \frac{\lambda_k cu}{L} \sin\theta_m u du$$

$$X_g = C_k \frac{\sin\theta_m t}{\theta m} \int_0^t \cosh \frac{\lambda_k cu}{L} \cos\theta_m u du, \quad X_h = C_k \frac{\cos\theta_m t}{\theta m} \int_0^t \cosh \frac{\lambda_k cu}{L} \sin\theta_m u du$$

It is readily shown that

$$X_a = - \frac{\sin\theta_m t}{2\theta_m} \left[\frac{\cos(\frac{\lambda_k c}{L} + \theta_m)t}{\frac{\lambda_k c}{L} + \theta_m} + \frac{\cos(\frac{\lambda_k c}{L} - \theta_m)t}{\frac{\lambda_k c}{L} - \theta_m} - \frac{2\frac{\lambda_k c}{L}}{\left[\frac{\lambda_k c}{L}\right]^2 - \theta^2_m} \right]$$

$$X_b = \frac{\cos\theta_m t}{2\theta_m}\left[\frac{\sin(\frac{\lambda_k c}{L} - \theta_m)t}{\frac{\lambda_k c}{L} - \theta_m} - \frac{\sin(\frac{\lambda_k c}{L} + \theta_m)t}{\frac{\lambda_k c}{L} + \theta_m}\right]$$

$$X_c = A_k \frac{\sin\theta_m t}{2\theta_m}\left[\frac{\sin(\theta_m + \frac{\lambda_k c}{L})t}{\theta_m + \frac{\lambda_k c}{L}} + \frac{\sin(\theta_m - \frac{\lambda_k c}{L})t}{\theta_m - \frac{\lambda_k c}{L}}\right]$$

$$X_d = -A_k \frac{\cos\theta_m t}{2\theta_m}\left[\frac{\cos(\theta_m + \frac{\lambda_k c}{L})t}{\theta_m + \frac{\lambda_k c}{L}} + \frac{\cos(\theta_m - \frac{\lambda_k c}{L})t}{\theta_m - \frac{\lambda_k c}{L}} - \frac{2\theta_m}{\theta_m^2 - \left[\frac{\lambda_k c}{L}\right]^2}\right]$$

$$X_e = B_k \frac{\frac{\lambda_k c}{L}\sin\theta_m t}{\theta_m[(\lambda_k c/_L)^2 + \theta_m^2]}\left[\cos\theta_m t \cosh\frac{\lambda_k c t}{L} - 1 + \frac{\theta_m}{\lambda_k c/_L}\sin\theta_m t \sinh\frac{\lambda_k c t}{L}\right]$$

$$X_f = B_k \frac{\frac{\lambda_k c}{L}\cos\theta_m t}{\theta_m[(\lambda_k c/_L)^2 + \theta_m^2]}\left[\sin\theta_m t \cosh\frac{\lambda_k c t}{L} - \frac{\theta_m}{\lambda_k c/_L}\cos\theta_m t \sinh\frac{\lambda_k c t}{L}\right]$$

$$X_g = C_k \frac{\frac{\lambda_k c}{L}\sin\theta_m t}{\theta_m[(\lambda_k c/_L)^2 + \theta_m^2]}\left[\cos\theta_m t \sinh\frac{\lambda_k c t}{L} + \frac{\theta_m}{\lambda_k c/_L}\sin\theta_m t \cosh\frac{\lambda_k c t}{L}\right]$$

$$X_h = C_k \frac{\frac{\lambda_k c}{L}\cos\theta_m t}{\theta_m[(\lambda_k c/_L)^2 + \theta_m^2]}\left[\sin\theta_m t \sinh\frac{\lambda_k c t}{L} - \frac{\theta_m}{\lambda_k c/_L}\cos\theta_m t \cosh\frac{\lambda_k c t}{L} + \frac{\theta_m}{\lambda_k c/_L}\right]$$

When $X_a - X_h$ are substituted into equation (2.62), after some rearrangements, one obtains

$$W_m(t) = \frac{P_m}{\theta_m[\theta_m^4 - \omega_k^4]}\Bigg\{[\theta_m^2 - \omega_k^2]\Big[C_k\theta_m[\cosh\omega_k t - \cos\theta_m t]$$

$$+ B_k[\theta_m \sinh\omega_k t - \omega_k \sin\theta_m t]\Big]$$

$$+ [\theta_m^2 + \omega_k^2]\Big[A_k\theta_m[\cos\omega_k t - \cos\theta_m t]$$

$$- [\omega_k \sin\theta_m t - \theta_m \sin \omega_k t] \Big] \Big\} \qquad (2.63)$$

where $\omega_k = \frac{\lambda kc}{L}$

Hence in view of equations (2.15) and (2.38)

$$U_n(x,t) = \sum_{m=1}^{n} \frac{P_m}{\theta_m(\theta^4_m - \omega^4_k)} \Big\{ [\theta^2_m - \omega^2_k] \Big[C_k\theta_m [\cosh \omega_k t - \cos\theta_m t]$$

$$+ B_k [\theta_m \sinh \omega_k t - \omega_k \sin\theta_m t] \Big] + [\theta^2_m + \omega^2_k] \Big[A_k\theta_m [\cos\omega_k t - \cos\theta_m t]$$

$$- [\omega_k \sin\theta_m t - \theta_m \sin \omega_k t] \Big] \Big\} [\sin\frac{\lambda_m x}{L} + A_m \cos \frac{\lambda_m x}{L}$$

$$+ B_m \sinh\frac{\lambda_m x}{L} + C_m \cosh \frac{\lambda_m x}{L}]$$

(2.64)

Equation (2.64) represents the transverse – displacement response to a moving force of a uniform Rayleigh beam on a variable Winkler elastic foundation.

2.5.2 CASE II

If the moving load has mass commensurable with that of the structure, the inertial effect of the moving mass is not negligible. Thus $\Gamma \neq 0$ and we are required to solve the entire equation (2.53). This, we term, the moving mass problem.

Unlike case I, it is obvious that an exact analytical solution to this equation is not possible. Thus, we resort to an approximate analytical method due to struble, Sadiku et al (1981).

To this end, equation (2.53) is rewritten to take the form

$$\Omega_0(m,k)\ddot{W}_m(t) + \Omega_1(m,k)W_m t + \Gamma \Big[[\Omega_{2A}(m,k) + 2\Omega_{2B}(m,k)\cos\frac{\pi ct}{L}]\ddot{W}_m(t)$$

$$+ 2c [\Omega_{3A}(m,k) + 2\Omega_{3B}(m,k)\cos\frac{\pi ct}{L}]\dot{W}_m(t)$$

$$+ c^2 [\Omega_{4A}(m,k) + 2\Omega_{4B}(m,k) \cos \frac{\pi ct}{L}] W_m(t) \Big]$$

$$= \Gamma g L \, [\sin \frac{\lambda_k ct}{L} + A_k \cos \frac{\lambda_k ct}{L} + B_k \sinh \frac{\lambda_k ct}{L} + C_k \cosh \frac{\lambda_k ct}{L}] \qquad (2.65)$$

and a further rearrangement yields

$$\ddot{W}_m(t) + \frac{2\Gamma c[\Omega_{3A}(m,k) + 2\Omega_{3B}(m,k) \cos \frac{\pi ct}{L}] \dot{W}_m(t)}{\Omega_o(m,k) + \Gamma [\Omega_{2A}(m,k) + 2\Omega_{2B}(m,k) \cos \frac{\pi ct}{L}]}$$

$$+ \frac{[\Omega_1(m,k) + \Gamma c^2[\Omega_{4A}(m,k) + 2\Omega_{4B}(m,k) \cos \frac{\pi ct}{L}] W_m(t)}{\Omega_o(m,k) + \Gamma [\Omega_{2A}(m,k) + 2\Omega_{2B}(m,k) \cos \frac{\pi ct}{L}]}$$

$$= \frac{\Gamma g L \, [\sin \frac{\lambda_k ct}{L} + A_k \cos \frac{\lambda_k ct}{L} + B_k \sinh \frac{\lambda_k ct}{L} + C_k \cosh \frac{\lambda_k ct}{L}]}{\Omega_o(m,k) + \Gamma [\Omega_{2A}(m,k) + 2\Omega_{2B}(m,k) \cos \frac{\pi ct}{L}]} \qquad (2.66)$$

First, we shall consider the homogeneous part of (2.66) and obtain a modified frequency corresponding to the frequency of the free system due to the presence of the moving mass. An equivalent free system operator defined by the modified frequency then replaces equation (2.66). We shall consider a parameter $\varepsilon < 1$ for any arbitrary mass ratio Γ defined as

$$\varepsilon = \frac{\Gamma}{1 + \Gamma} \quad .$$

It is then clear that

$$\Gamma = \varepsilon [1 + o(\varepsilon) + o(\varepsilon^2) + \ldots\ldots\ldots\ldots\ldots] \qquad (2.67)$$

All the various time dependent coefficients of the differential operator which acts on $W_m(t)$ in equation (2.66) can be written in terms of ε when we notice that to $o(\varepsilon)$

$$\Gamma = \varepsilon$$

and

$$\frac{1}{\Omega_o(m,k) + \varepsilon[\,\Omega_{2A}(m,k) + 2(\,\Omega_{2B}(m,k)\cos\frac{\pi ct}{L}\,]} =$$

$$\frac{1}{\Omega_o(m,k)}\left[1 - \frac{1}{\Omega_o(m,k)}\varepsilon[\,\Omega_{2A}(m,k) + 2\Omega_{2B}(m,k)\cos\frac{\pi ct}{L}\,] + o(\varepsilon^2)\,\right] \tag{2.68}$$

where

$$\left|\frac{\varepsilon}{\Omega_o(m,k)}[\Omega_{2A}(m,k) + 2\Omega_{2B}(m,k)\cos\frac{\pi ct}{L}]\right| < 1 \tag{2.69}$$

Now, using (2.67) and (2.68), equation (2.66) takes the form

$$\ddot{W}_m(t) + \frac{2c\varepsilon}{\Omega o(m,k)}[\,\Omega_{3A}(m,k) + 2\Omega_{3B}(m,k)\cos\frac{\pi ct}{L})]\,\dot{W}_m(t)$$
$$+ \left\{\frac{\Omega_1(m,k)}{\Omega_o(m,k)}\left[1 - \frac{\varepsilon\,[\Omega_{2A}(m,k) + 2\Omega_{2B}(m,k)\cos\frac{\pi ct}{L}]}{\Omega_o(m,k)}\right]\right.$$
$$\left. + \frac{C^2\varepsilon}{\Omega_o(m,k)}[\,\Omega_{4A}(m,k) + 2\Omega_{4B}(m,k)\cos\frac{\pi ct}{L}\,]\right\}W_m(t)$$
$$= \frac{\varepsilon g L}{\Omega_o(m,k)}[\sin\frac{\lambda_k ct}{L} + A_k\cos\frac{\lambda_k ct}{L} + B_k\sinh\frac{\lambda_k ct}{L} + C_k\cosh\frac{\lambda_k ct}{L}\,] \tag{2.70}$$

to $o(\varepsilon)$ only.

When $\varepsilon = 0$, a case corresponding to the case when the inertial effect of the mass of the system is neglected, the solution of equation (2.70) can be written as

$$W_{mA}(t) = c_1\cos(\theta_m t - \Phi_m) \tag{2.71}$$

where $\theta^2_m = \frac{\Omega_1(m,k)}{\Omega_o(m,k)}$ and c_1 and Φ_m are constants.

Since ε < 1 for any arbitrary mass ratio Γ, Struble's technique requires that the asymptotic solution of the homogeneous part of (2.70) be of the form

$$W_m(t) = \Phi(m,t)\cos[\theta_m t - \Omega(m,t)] + \varepsilon W_1(t) + 0(\varepsilon^2) \quad (2.72)$$

where $\Phi(m,t)$ and $\Omega(m,t)$ are slowly time varying functions or equivalently

$$\frac{d\Phi(m,t)}{dt} \longrightarrow o(\varepsilon)\ ;\quad \frac{d^2\Phi(m,t)}{dt^2} \longrightarrow o(\varepsilon^2)$$

$$\frac{d\Omega(m,t)}{dt} \longrightarrow o(\varepsilon)\ ;\quad \frac{d^2\Omega(m,t)}{dt^2} \longrightarrow o(\varepsilon^2) \quad (2.73)$$

where $\longrightarrow$ implies "is of"

In view of equation (2.72), it can be shown that

$$\dot{W}_m(t) = \dot{\Phi}(m,t)\cos[\theta_m t - \Omega(m,t)] - \Phi(m,t)\theta_m\sin[\theta_m t - \Omega(m,t)]$$
$$+ \dot{\Omega}(m,t)\,\Phi(m,t)\sin[\theta_m t - \Omega(m,t)] + \varepsilon\dot{W}_1(t) + o(\varepsilon^2) \quad (2.74)$$

and

$$\ddot{W}_m(t) = \ddot{\Phi}(m,t)\cos[\theta_m t - \Omega(m,t)] - \dot{\Phi}(m,t)\,\theta_m\sin[\theta_m t - \Omega(m,t)]$$
$$+ \dot{\Phi}(m,t)\,\dot{\Omega}(m,t)\sin[\theta_m t - \Omega(m,t)] - \dot{\Phi}(m,t)\,\theta_m\sin[\theta_m t - \Omega(m,t)]$$
$$- \Phi(m,t)\,\theta_m^2\cos[\theta_m t - \Omega(m,t)] + \dot{\Omega}(m,t)\,\theta_m\Phi(m,t)\cos[\theta_m t - \Omega(m,t)]$$
$$+ \ddot{\Omega}(m,t)\Phi(m,t)\sin[\theta_m t - \Omega(m,t)] + \dot{\Omega}(m,t)\dot{\Phi}(m,t)\sin[\theta_m t - \Omega(m,t)]$$
$$+ \dot{\Omega}(m,t)\Phi(m,t)\,\theta_m\cos[\theta_m t - \Omega(m,t)] - \dot{\Omega}^2(m,t)\Phi(m,t)\cos[\theta_m t - \Omega(m,t)]$$
$$+ \varepsilon\ddot{W}_1(t) + o(\varepsilon^2) \quad (2.75)$$

To obtain the modified frequency, equations (2.72), (2.74) and (2.75) are substituted into the homogeneous part of equation (2.70). Subsequently, only the variational part of the equation describing the behaviours of $\Phi(m,t)$ and $\Omega(m,t)$ during the motion of the mass is extracted.

Thus, substituting equations (2.72), (2.74) and (2.75) into the homogeneous part of equation (2.70) and taking into account (2.67), (2.68) and (2.69), we then have

$$- 2\theta_m \dot{\Phi}(m,t) \sin [\theta_m t - \Omega(m,t)] + 2\theta_m \Phi(m,t) \dot{\Omega}(m,t) \cos[\theta_m t - \Omega(m,t)]$$

$$- \Phi(m,t)\, \theta^2_m \cos [\theta_m t - \Omega(m,t)]$$

$$+ \frac{2c\varepsilon}{\Omega_o(m,t)} [\Omega_{3A}(m,k) + 2\Omega_{3B}(m,k) \cos \frac{\pi ct}{L}] \{ - \Phi(m,t)\, \theta_m \sin[\theta_m t - \Omega(m,t)] \}$$

$$+ \{ \frac{\Omega_1(m,t)}{\Omega_o(m,k)} - \frac{\varepsilon\Omega_1(m,k)}{\Omega^2_o(m,k)} [\Omega_{2A}(m,k) + 2\Omega_{2B}(m,k) \cos \frac{\pi ct}{L}]$$

$$+ \frac{c^2\varepsilon}{\Omega_o(m,k)} [\Omega_{4A}(m,k) + 2\Omega_{4B}(m,k)\cos \frac{\pi ct}{L}] \} [\Phi(m,t)\cos[\theta_m t - \Omega(m,t)] = 0 \qquad (2.76)$$

retaining terms to o(ε) only.

The variational equations are obtained by equating the coefficients of Sin[θ_mt - Ω(m,t)] and cos[θ_mt - Ω(m,t)] terms on both sides of the equation. Thus, noting that

$$\cos \frac{\pi ct}{L} \sin [\theta_m t - \Omega(m,t)] = \tfrac{1}{2} \sin [\frac{\pi ct}{L} + \theta_m t - \Omega(m,t)] + \tfrac{1}{2} \sin [\theta_m t - \Omega(m,t) - \frac{\pi ct}{L}]$$

and

$$\cos \frac{\pi ct}{L} \cos [\theta_m t - \Omega(m,t)] = \tfrac{1}{2} \cos [\frac{\pi ct}{L} + \theta_m t - \Omega(m,t)] + \tfrac{1}{2} \cos [\frac{\pi ct}{L} - \theta_m t + \Omega(m,t)]$$

and neglecting those terms that do not contribute to the variational equations, equation (2.76) reduces to

$$[-\theta^2_m\Phi(m,t) + 2\theta_m\Phi(m,t)\dot{\Omega}(m,t) + \Phi(m,t)\theta^2_m - \varepsilon\Phi(m,t)\theta^2_m \frac{\Omega_{2A}(m,k)}{\Omega_o(m,k)}$$

$$+\ \varepsilon\Phi(m,t)c^2\,\frac{\Omega_{4A}(m,k)}{\Omega_o(m,k)}\,\Big]\ \cos\,[\theta_m t - \Omega(m,t)]$$

$$+\Big[\,-2\theta_m\dot{\Phi}(m,t) - 2\varepsilon c\,\frac{\Omega_{3A}(m,k)}{\Omega_o(m,k)}\theta_m\Phi(m,t)\,\Big]\sin[\theta_m t - \Omega(m,t)] \quad = \quad 0$$

that is,

$$\Big[2\theta_m\Phi(m,t)\dot{\Omega}(m,t) - \varepsilon\Phi(m,t)\theta^2_m\,\frac{\Omega_{2A}(m,k)}{\Omega_o(m,k)} + \varepsilon\Phi(m,t)c^2\,\frac{\Omega_{4A}(m,k)}{\Omega_o(m,k)}\Big]\cos[\theta_m t - \Omega(m,t)]$$

$$-\Big[\,2\theta_m\,\dot{\Phi}(m,t) + 2\varepsilon c\,\frac{\Omega_{3A}(m,k)}{\Omega_o(m,k)}\theta_m\Phi(m,t)\,\Big]\sin\,[\theta_m t - \Omega(m,t)] \ = \ 0 \qquad (2.77)$$

The variational equations of the problem are obtained by setting the coefficients of $\cos\,[\theta_m t - \Omega(m,t)]$ and $\sin\,[\theta_m t - \Omega(m,t)]$ in (2.77) to zero. Thus, we have

$$2\theta_m\Phi(m,t)\,\dot{\Omega}(m,k) - \varepsilon\Phi(m,t)\theta^2_m\frac{\Omega_{2A}(m,k)}{\Omega_o(m,k)} + \varepsilon\Phi(m,t)c^2\,\frac{\Omega_{4A}(m,k)}{\Omega_o(m,k)} \ = \ 0 \qquad (2.78)$$

and

$$\theta_m\dot{\Phi}(m,t) + \varepsilon c\,\frac{\Omega_{3A}(m,k)}{\Omega_o(m,k)}\,\theta_m\Phi(m,t) \quad = \quad 0 \qquad (2.79)$$

Rearranging (2.78) and (2.79), we have

$$\dot{\Omega}(m,t) = \ \varepsilon\,\frac{[\theta^2_m\Omega_{2A}(m,k) - c^2\,\Omega_{4A}(m,t)]}{2\Omega_o(m,k)\theta_m} \qquad (2.80)$$

and

$$\dot{\Phi}(m,t) \ = \ -\,\varepsilon c\,\frac{\Phi(m,t)\Omega_{3A}(m,k)}{\Omega_o(m,k)} \qquad (2.81)$$

Solving equations (2.80) and (2.81) respectively, we have

$$\Omega(m,t) = \ \varepsilon\,\frac{[\theta^2_m\,\Omega_{2A}(m,k) - c^2\,\Omega_{4A}(m,k)]}{2\theta_m\,\Omega_o(m,k)}\,t + \Omega_m \qquad (2.82)$$

where Ω_m is a constant and $\Phi(m,t) = c^o e^{-\gamma t}$ (2.83)

where $\gamma \ = \ \varepsilon c\,\dfrac{\Omega_{3A}(m,k)}{\Omega_o(m,k)}$ and c^o is a constant.

Therefore, when the effect of the mass of the particle is considered, the first approximation to the homogeneous system is

$$W_m(t) = \Phi(m,t)\cos[\beta_m t - \Omega_m] \tag{2.84}$$

where

$$\beta_m = \theta_m - \frac{\varepsilon\,[\theta^2_m \Omega_{2A}(m,k) - c^2 \Omega_{4A}(m,k)]}{2\theta_m \Omega_o(m,k)}$$

is called the modified natural frequency representing the frequency of the free system due to the presence of the moving mass.

Using equation (2.84), the homogeneous part of the equation (2.66) can be written as

$$\frac{d^2 W_m(t)}{dt^2} + \beta^2_m W_m(t) = 0 \tag{2.85}$$

Hence the entire equation (2.66), taking into account (2.68), takes the form:

$$\frac{d^2 W_m(t)}{dt^2} + \beta^2_m W_m(t) = R_m[\sin \frac{\lambda_k ct}{L} + A_k \cos\frac{\lambda_k ct}{L} + B_k \sinh\frac{\lambda_k ct}{L} + C_k \cosh\frac{\lambda_k ct}{L}] \tag{2.86}$$

where

$$R_m = \frac{\varepsilon L.g}{\Omega_o(m,t)}$$

It is observed that equation (2.56) and (2.86) are similar. Thus, going through the same argument as in the previous section, equation (2.86) when solved yields

$$W_m(t) = \frac{R_m}{\beta_m(\beta^4_m - \theta^4_k)}\{(\beta^2_m - \theta^2_k)[C_k \beta_m(\cosh\theta_k t - \cos\beta_m t) + B_k(\beta_m \sinh\theta_k t - \theta_k \sin\beta_m t)]$$
$$+ (\beta^2_m + \theta^2_k)[A_k\beta_m(\cos\theta_k t - \cos\beta_m t) - (\theta_k \sin\beta_m t - \beta_m \sin\theta_k t)]\} \tag{2.87}$$

where $\theta_k = \frac{\lambda_k c}{L}$

Hence, in view of equations (2.15) and (2.38)

$$U_n(x,t) = \sum_{m=1}^{n} \frac{R_m}{\beta_m(\beta^4{}_m - \theta^4{}_k)} \{(\beta^2{}_m - \theta^2{}_k)[C_k\, \beta_m(\cosh\theta_k t - \cos\beta_m t)$$

$$+ B_k\,(\beta_m \sinh\theta_k t - \theta_k \sin\beta_m t)] + (\beta^2{}_m + \theta^2{}_k)\,[A_k\beta_m(\cos\theta_k t - \cos\beta_m t)$$

$$- (\theta_k \sin\beta_m t - \beta_m \sin\theta_k t)]\}\,[\sin\frac{\lambda_m x}{L} + A_m \cos\frac{\lambda_m x}{L}$$

$$+ B_m \sinh\frac{\lambda_m x}{L} + C_m \cosh\frac{\lambda_m x}{L}] \qquad (2.88)$$

This represents the transverse-displacement response to a moving mass of a uniform Rayleigh beam on a variable Winkler elastic foundation.

CHAPTER THREE

ILLUSTRATIVE EXAMPLES, NUMERICAL CALCULATIONS AND DISCUSSION OF RESULTS (UNIFORM RAYLEIGH BEAM)

3.1.0 ILLUSTRATIVE EXAMPLES.

In this section, the foregoing analysis is illustrated by various practical examples. In particular, classical boundary conditions such as simply supported boundary conditions, free ends condition, clamped ends condition and clamped-free ends condition are considered.

3.1.1 SIMPLY SUPPORTED BOUNDARY CONDITIONS

In this case, the uniform Rayleigh beam has simple supports at ends x = 0 and x=L. thus the conditions are expressed as

$$U(0,t) = 0 = U(L,t), \quad \frac{\partial^2 U(0,t)}{\partial x^2} = 0 = \frac{\partial^2 U(L,t)}{\partial x^2} \tag{3.1}$$

and hence for normal modes

$$V_m(0) = 0 = V_m(L), \quad \frac{d^2 V_m(0)}{dx^2} = 0 = \frac{d^2 V_m(L)}{dx^2} \tag{3.2}$$

which implies that

$$V_k(0) = 0 = V_k(L), \quad \frac{d^2 V_k(0)}{dx^2} = 0 = \frac{d^2 V_k(L)}{dx^2} \tag{3.3}$$

Thus, it can be shown that

$$A_m = A_k = 0 \ ; \quad B_m = B_k = 0 \ ; C_m = C_k = 0 \tag{3.4}$$

and the frequency equation becomes

$$\sin \lambda_m = \sin \lambda_k = 0$$

which implies

$$\lambda_m = m\pi \text{ and } \lambda_k = k\pi \tag{3.5}$$

respectively.

Thus the moving force problem is reduced to a non-homogeneous ordinary differential equation

$$\frac{d^2W_m(t)}{dt^2} + \theta^2{}_m W_m(t) = R_0 \sin \frac{k\pi ct}{L} \tag{3.6}$$

where

$$R_0 = \frac{2gM}{LP_o\mu}, \quad P_0 = 1 + \frac{R^0 m^2\pi^2}{L^2} \text{ and } \theta^2{}_m = \frac{2R(k,m)}{LP_0} \tag{3.7}$$

and

$$R(k,m) = \left\{ \frac{EIm^4\pi^4 L}{2\mu L^4} + \frac{kL^2}{\mu\pi^4 (k-m)^4 (k+m)^4} \left[16\pi^2 km(k-m)^2(k+m)^2 + 48L^2 km(k^2+m^2) + 6Lkm\pi^2(L-2)(k-m)^2(k+m)^2(-1)^{k+m} \right] \right\} \tag{3.8}$$

Equation (3.6) when solved in conjunction with the initial conditions one obtains expression for $W_m(t)$. Thus in view of (2.15)

$$U_n(x,t) = \sum_{m=1}^{n} \frac{R_0}{\theta_m[\theta^2{}_m - (k\pi c/L)^2]} \left[\theta_m \sin \frac{k\pi ct}{L} - \frac{k\pi c}{L} \sin \theta_m t\right] \sin \frac{m\pi x}{L} \tag{3.9}$$

Equation (3.9) represents the transverse-displacement response to a moving force of a simply supported Uniform Rayleigh beam on a variable Winkler elastic foundation.

Next, the moving mass problem, that is when $\Gamma \neq 0$ is considered. Following arguments in the previous sections, the modified frequency corresponding to the frequency of the free system due to the presence of the moving mass of this model is obtained as

$$\gamma_m = \theta_m - \varepsilon \left[\frac{4L^2 \theta^2_m + c^2 m^2 \pi^2}{4L^2 \theta_m P_0} \right] \qquad (3.10)$$

neglecting higher order terms of ε. Thus, the moving mass problem reduces to

$$\frac{d^2 W_m(t)}{dt^2} + \gamma^2_m W_m(t) = \frac{2\varepsilon g}{P_0} \sin \frac{k\pi ct}{L} \qquad (3.11)$$

which when solved in conjunction with the initial conditions yields expression for $W_m(t)$. Thus, using (2.15), one obtains

$$U_n(x,t) = \sum_{m=1}^{n} \frac{2\varepsilon g}{P_0 \gamma_m [\gamma^2_m - (k\pi c/L)^2]} \left[\gamma_m \sin \frac{k\pi ct}{L} - \frac{k\pi c}{L} \sin \gamma_m t \right] \sin \frac{m\pi x}{L}$$

(3.12)

This represents the transverse-displacement response to a moving mass of a simply supported Uniform Rayleigh beam on a variable Winkler elastic fundation.

3.1.2 FREE ENDS CONDITION

For free ends condition at x = 0 and x = L, the pertinent boundary conditions are

$$\frac{\partial^2 U(0,t)}{\partial x^2} = 0 = \frac{\partial^2 U(L,t)}{\partial x^2} \text{ and } \frac{\partial^3 U(0,t)}{\partial x^3} = 0 = \frac{\partial^3 U(L,t)}{\partial x^3} \qquad (3.13)$$

and for the normal modes one has

$$\frac{d^2 V_m(0)}{dx^2} = 0 = \frac{d^2 V_m(L)}{dx^2} \text{ and } \frac{d^3 V_m(0)}{dx^3} = 0 = \frac{d^3 V_m(L)}{dx^3} \qquad (3.14)$$

which implies that

$$\frac{d^2V_k(0)}{dx^2} = 0 = \frac{d^2V_k(L)}{dx^2} \text{ and } \frac{d^3V_k(0)}{dx^3} = 0 = \frac{d^3V_k(L)}{dx^3} \tag{3.15}$$

Thus, it can be shown that

$$A_m = \frac{\sin\lambda_m - \sinh\lambda_m}{\cosh\lambda_m - \cos\lambda_m} = \frac{\cos\lambda_m - \cosh\lambda_m}{\sin\lambda_m + \sinh\lambda_m} = C_m \text{ and } B_m = 1 \tag{3.16}$$

and from (3.16), one obtains

$$\cos\lambda_m \cosh\lambda_m = 1 \tag{3.17}$$

which is termed the frequency equation for the dynamical problem, such that, Gbadeyan et al (1990),

$$\lambda_1 = 4.73004,\ \lambda_2 = 7.85320,\ \lambda_3 = 10.99561 \tag{3.18}$$

Using (3.16) and (3.18) in equations (2.64) and (2.88) one obtains the displacement response respectively to a moving force and a moving mass of free-ends uniform Rayleigh beam on a variable Winkler elastic foundation.

3.1.3 CLAMPED ENDS CONDITION.

At a clamped end, both deflection and slope vanish. Thus when the Rayleigh beam is clamped at x = 0 and x = L, the conditions are expressed as

$$U(0,t) = 0 = U(L,t) \text{ and } \frac{\partial U(0,t)}{\partial x} = 0 = \frac{\partial U(L,t)}{\partial x} \tag{3.19}$$

Thus, for normal modes

$$V_m(0) = 0 = V_m(L) \text{ and } \frac{dV_m(0)}{dx} = 0 = \frac{dV_m(L)}{dx} \tag{3.20}$$

which implies that

$$V_k(0) = 0 = V_k(L) \text{ and } \frac{dV_k(0)}{dx} = 0 = \frac{dV_k(L)}{dx} \tag{3.21}$$

Thus, it can be shown that

$$A_m = \frac{\sinh\lambda_m - \sin\lambda_m}{\cos\lambda_m - \cosh\lambda_m} = \frac{\cos\lambda_m - \cosh\lambda_m}{\sin\lambda_m + \sinh\lambda_m} = -C_m \quad \text{and} \quad B_m = -1 \tag{3.22}$$

The frequency equation, from (3.22) becomes

$$\cos\lambda_m \cosh\lambda_m = 1 \tag{3.23}$$

This is similar to equation (3.17) and one has

$$\lambda_1 = 4.73004,\ \lambda_2 = 7.85320,\ \lambda_3 = 10.99561 \tag{3.24}$$

Substituting (3.22) and (3.24) into equations (2.64) and (2.88) one obtains the displacement response respectively to a moving force and a moving mass of a clamped uniform Rayleigh beam on a variable Winkler elastic foundation.

3.1.4 ONE END CLAMPED AND ONE END FREE CONDITION.

Next at $x = 0$, the Rayleigh beam is taken to be clamped and at $x = L$, the beam model is free. Thus, the boundary conditions of the Rayleigh beam can be written as

$$\frac{\partial^2 U(L,t)}{\partial x^2} = 0 = \frac{\partial^3 U(L,t)}{\partial x^3} \quad \text{and} \quad U(0,t) = 0 = \frac{\partial U(0,t)}{\partial x} \tag{3.25}$$

Similarly, for normal modes

$$\frac{d^2 V_m(L)}{dx^2} = 0 = \frac{d^3 V_m(L)}{dx^3} \quad \text{and} \quad V_m(0) = 0 = \frac{dV_m(0)}{dx} \tag{3.26}$$

which implies that

$$\frac{d^2 V_k(L)}{dx^2} = 0 = \frac{d^3 V_k(L)}{dx^3} \quad \text{and} \quad V_k(0) = 0 = \frac{dV_k(0)}{dx} \tag{3.27}$$

Using (3.26), it is straight forward to show that at $x = 0$,

$$A_m = -C_m \quad \text{and} \quad B_m = -1 \tag{3.28}$$

and at x = L, using (3.28)

$$A_m = -\frac{\sin\lambda_m - \sinh\lambda_m}{\cos\lambda_m + \cosh\lambda_m} = -\frac{\cos\lambda_m - \cosh\lambda_m}{\sinh\lambda_m - \sin\lambda_m} = -C_m \tag{3.29}$$

and the frequency equation for both end conditions is

$$\cos\lambda_m \cosh\lambda_m = -1 \tag{3.30}$$

such that

$$\lambda_1 = 1.875,\ \lambda_2 = 4.694,\ \lambda_3 = 7.855 \text{ and so on.} \tag{3.31}$$

Substituting (3.28), (3.29) and (3.31) into equations (2.64) and (2.88), one obtains the displacement response respectively to a moving force and a moving mass of a uniform clamped-free ends Rayleigh beam on a variable elastic foundation.

3.2.0 DISCUSSION OF THE ANALYTICAL SOLUTIONS

In studying undamped system such as this, it is desirable to examine the phenomenon of resonance. Equation (3.9) clearly shows that the simply supported Rayleigh beam on a variable Winkler elastic foundation and traversed by a moving force reaches a state of resonance whenever

$$\theta_m = \frac{k\pi c}{L} \tag{3.32}$$

while equation (3.12) shows that the same beam under the action of a moving mass experiences resonance when

$$\gamma_m = \frac{k\pi c}{L} \tag{3.33}$$

where

$$\gamma_m = \theta_m \{ 1 - \varepsilon [\frac{1}{2(1+\frac{R^0 m^2\pi^2}{L^2})} + \frac{c^2 m^2 \pi^2}{2L^2\theta^2_m(1 + \frac{R^0 m^2\pi^2}{L^2})}] \} \tag{3.34}$$

From equations (3.33) and (3.34), it can be shown that

$$\frac{\theta_m [1 + \frac{R^0 m^2\pi^2}{L^2} - \frac{\varepsilon}{2} (1 + \frac{c^2 m^2\pi^2}{L^2\theta^2_m})]}{1 + \frac{R^0 m^2\pi^2}{L^2}} = \frac{k\pi c}{L} \tag{3.35}$$

Since $1 + \frac{R^0 m^2\pi^2}{L^2} > 1 + \frac{R^0 m^2\pi^2}{L^2} - \frac{\varepsilon}{2} (1 + \frac{c^2 m^2\pi^2}{L^2\theta^2_m})$ for all m.

It can be deduced from equation (3.35) that, for the same natural frequency, the critical speed (and the natural frequency) for the system of a simply supported Rayleigh beam traversed by a moving mass is smaller than that of the same system traversed by a moving force. Thus, for the same natural frequency of the Rayleigh beam, the resonance is reached earlier when we consider the moving mass system than when we consider the moving force system.

Next, the phenomenon of resonance for other classical boundary conditions is examined. Equation (2.64) clearly shows that the uniform Rayleigh beam on a variable Winkler elastic foundation and traversed by a moving force reaches a state of resonance whenever

$$\theta_m = \frac{\lambda_k c}{L} \tag{3.36}$$

while equation (2.88) shows that the same beam under the action of a moving mass experiences resonance effect whenever

$$\beta_m = \frac{\lambda_k c}{L} \qquad (3.37)$$

where

$$\beta_m = \theta_m - \varepsilon \left[\frac{\theta^2_m \Omega_{2A}(m,k) - c^2 \Omega_{4A}(m,k)}{2\theta_m \Omega_0(m,k)} \right]$$

This implies

$$\beta_m = \theta_m \left[\frac{\Omega_0(m,k) - \frac{\varepsilon}{2}\left(\Omega_{2A}(m,k) - \frac{c^2 \Omega_{4A}(m,k)}{\theta^2_m} \right)}{\Omega_0(m,k)} \right] = \frac{\lambda_k c}{L} \qquad (3.38)$$

Consequently from equation (3.37) and (3.38), the same results and analysis obtained in the case of a Rayleigh beam simply supported at both ends are obtained for the other examples of end support conditions.

3.3.0. NUMERICAL CALCULATIONS AND DISCUSSION OF RESULTS.

In order to present the calculations of practical interests in dynamics of structures and Engineering design for all the illustrative examples, an elastic uniform Rayleigh beam of length 12.192m has been considered. It is assumed that the mass travels at the constant velocity 8.123m/s. Also EI and ε are chosen to be $6.068 \times 10^6 m^3/s^2$ and 0.25 respectively. The results are as shown on the various graphs below for the various classes of boundary conditions considered.

3.3.1 SIMPLY SUPPORTED ENDS

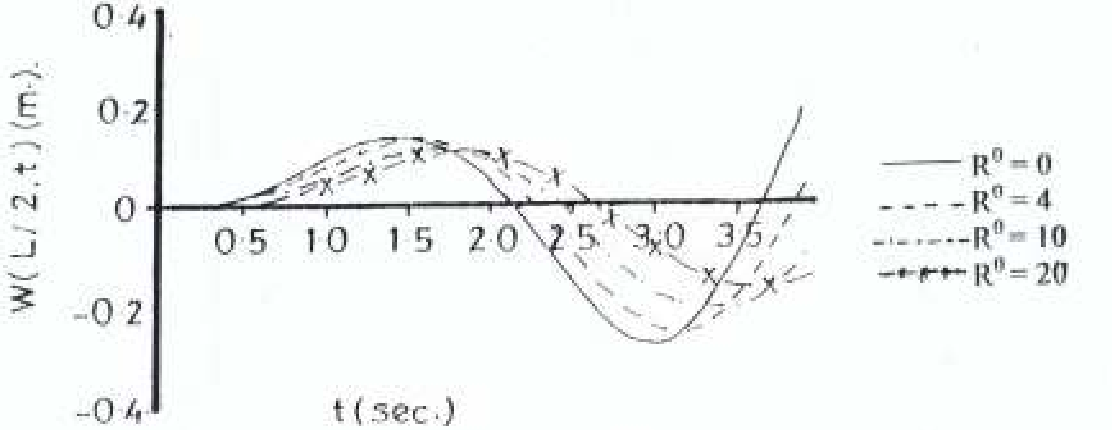

Fig 3·01 : Deflection profile of moving force at various values of R^0 for simply supported uniform Rayleigh beam.

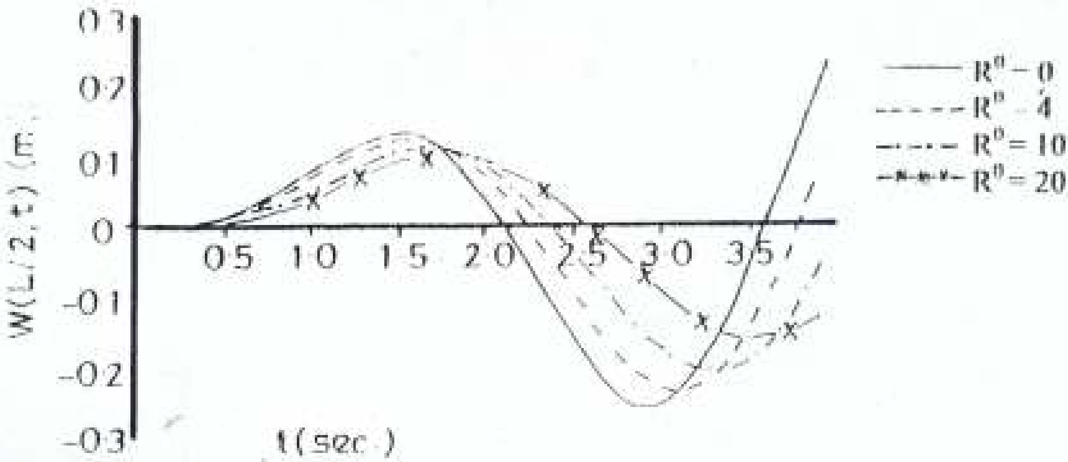

Fig 3·02 : Deflection profile of moving mass at various values of R^0 for simply supported uniform Rayleigh beam

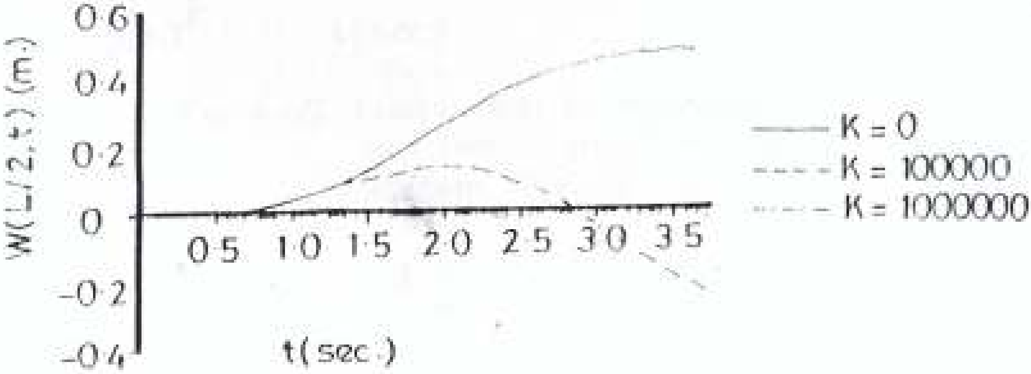

Fig 3·03 : Displacement response of moving force for simply supported uniform Rayleigh beam for various values of foundation moduli K.

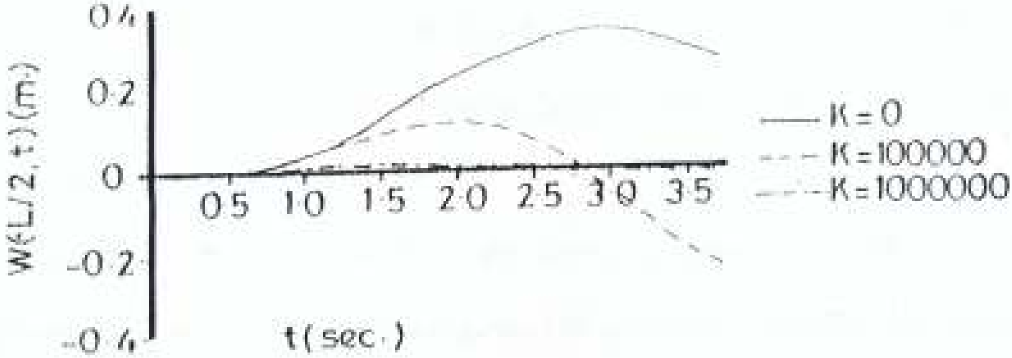

Fig 3.04 Displacement response of moving mass for simply supported uniform Rayleigh beam for various values of foundation moduli K

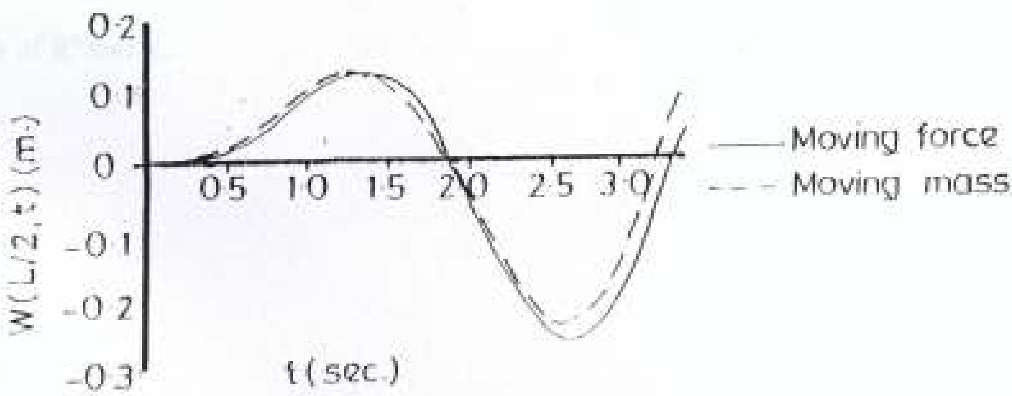

Fig 3.05: Comparison of the deflection of moving force and moving mass for simply supported uniform Rayleigh beam

Figures 3.01 and 3.02 display the effect of Rotatory inertia (R^o) on the transverse deflection of the simply supported uniform beam in both cases of moving force and moving mass respectively. The graphs show that the response amplitude of the uniform beam decreases as the value of the rotatory inertia correction factor increases. Values of R^o between 0 m and 20m are used.

The effect of foundation constant K on the transverse deflection in both cases of moving force and moving mass displayed in figures 3.03 and 3.04 respectively show that an increase in the value of the foundation constant K decreases the deflection of the simply supported uniform beam. Here, values of K between $0N/m^3$ and $1m\ N/m^3$ are used.

For the purpose of comparison, the displacement curves of the moving force and moving mass for a simply supported uniform Rayleigh beam with $R^o = 4$ and $K = 100000N/m$ are illustrated in figure 3.05. It can be noted that the response amplitude of a moving mass is greater than that of a moving force problem. This result also holds for other choice of R^o and K.

3.3.2 FREE ENDS

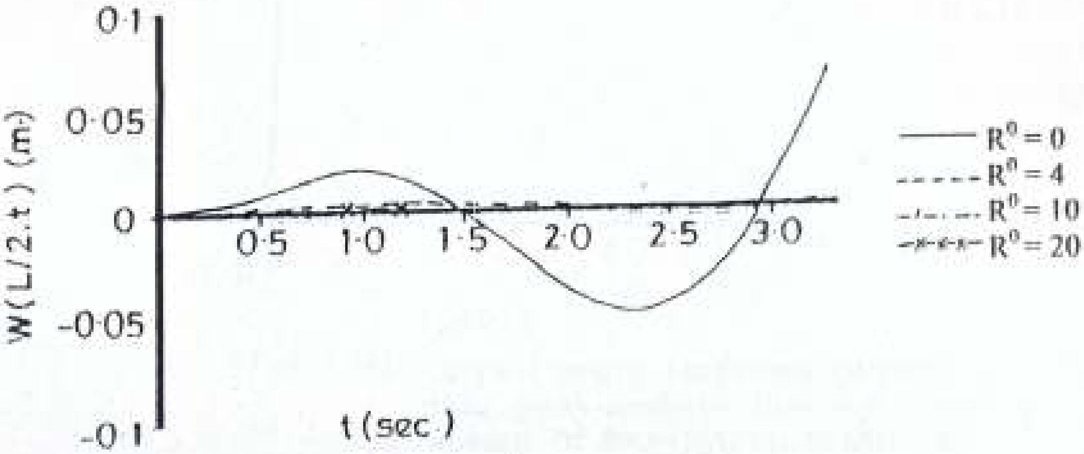

Fig 3·06 : Deflection profile of moving force for free ends uniform Rayleigh beam for various values of R^0

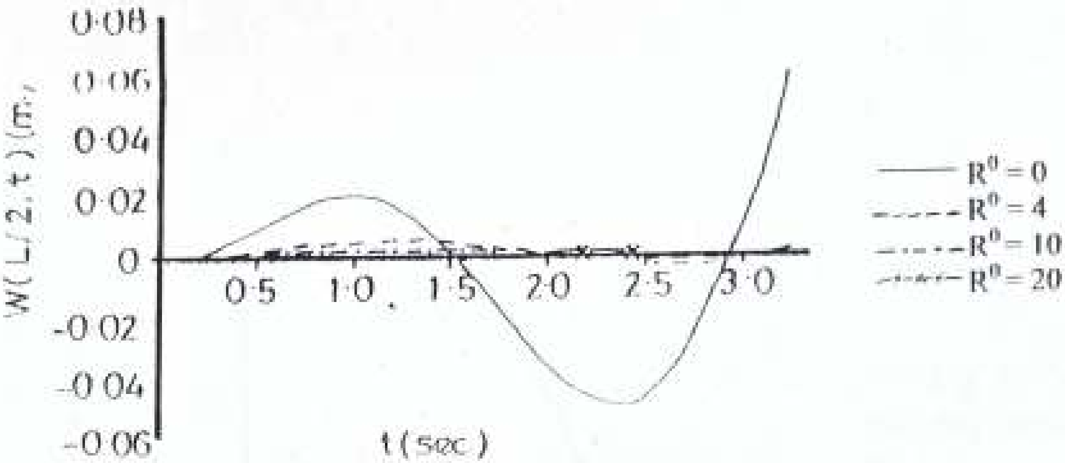

Fig 3·07 : Deflection profile of moving mass for free ends uniform Rayleigh beam for various values of R^0

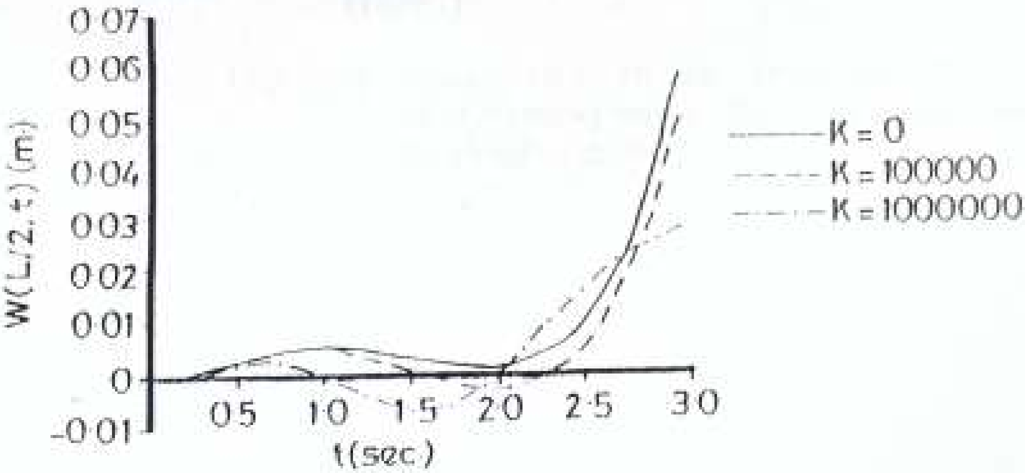

Fig 3·08 : Displacement response of moving force for free ends uniform Rayleigh beam for various values of foundation moduli K

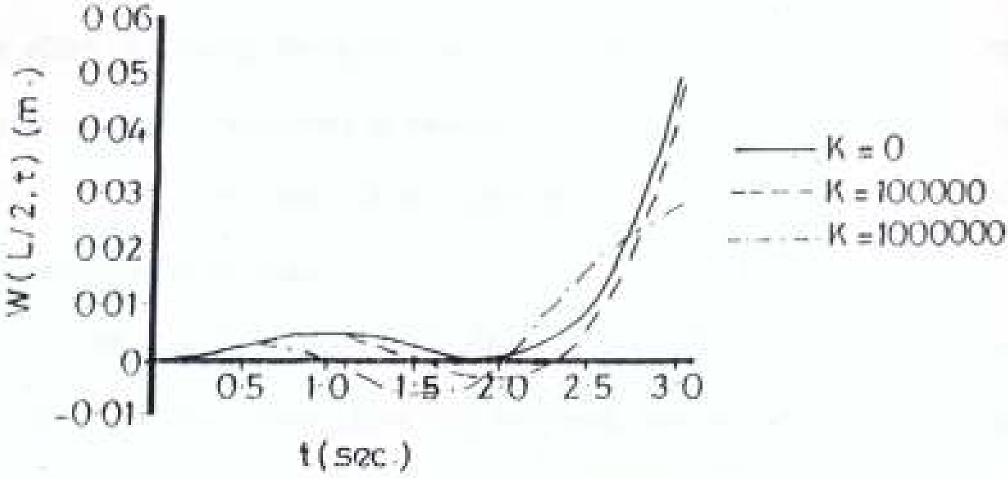

Fig 3.09 : Displacement response of moving mass for free ends uniform Rayleigh beam for various values of foundation moduli K.

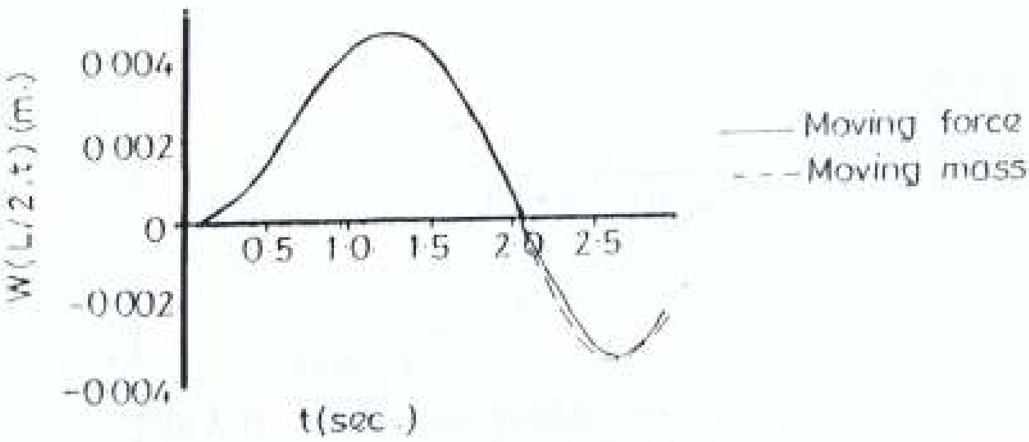

Fig 3.10 : Comparison of the deflection of moving force and moving mass for free ends uniform Rayleigh beam

The effect of Rotatory inertia (R^0) on the transverse deflection of the free-free uniform Rayleigh beam in both cases of moving force and moving mass is displayed in figures 3.06 and 3.07 respectively. It is shown that an increase in the value of R^0 decreases the deflection of the beam.

Figures 3.08 and 3.09 display the effect of foundation constant K on the transverse deflection of the uniform beam, with both ends free, in both cases of moving force and moving mass respectively. As K increases, the transverse deflection decreases.

Figure 3.10 compares the displacement curves of the moving force and moving mass for a free-free uniform Rayleigh beam for fixed values of R^0 and K. It is evident that the displacement response of the moving mass problem is greater than that of the moving force problem.

3.3.3 CLAMPED ENDS

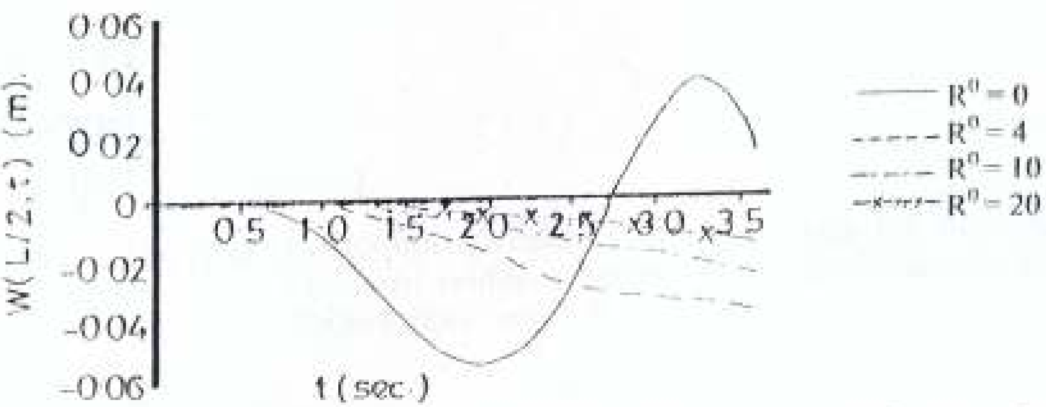

Fig. 3.11 : Deflection profile of moving force for clamped-clamped uniform Rayleigh beam for various values of R^0

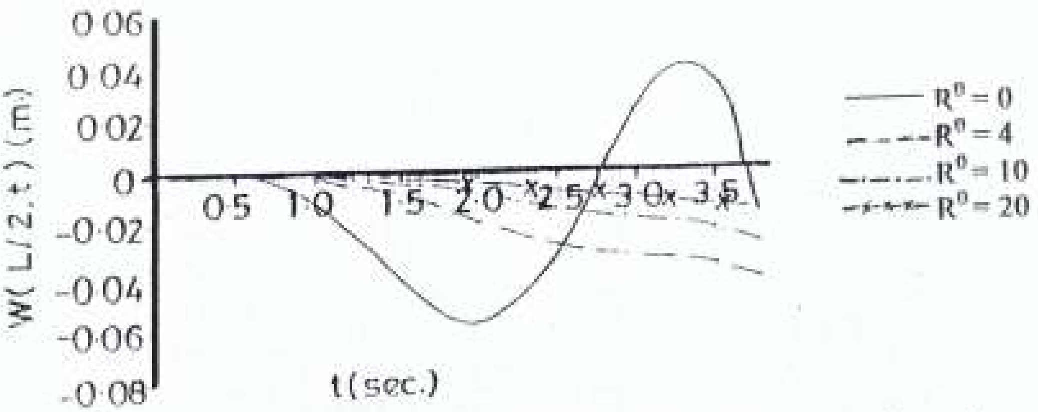

Fig 3·12 : Deflection profile of moving mass for clamped-clamped uniform beam for various values of R^0

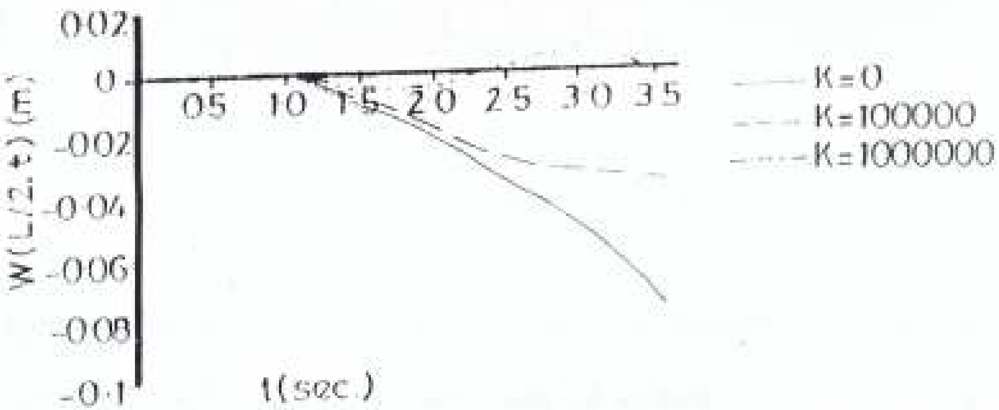

Fig 3·13 : Deflection profile of moving force for clamped-clamped uniform beam for various values of foundation moduli K.

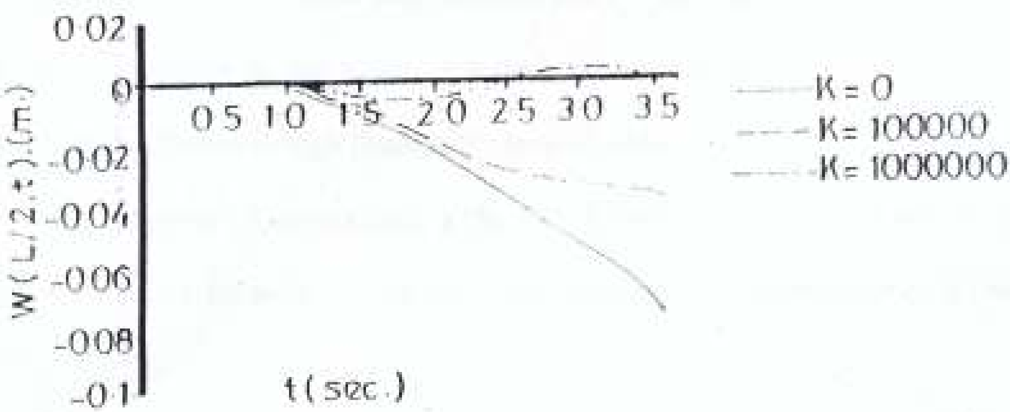

Fig 3·14 : Deflection profile of moving mass for clamped-clamped uniform Rayleigh beam for various values of foundation moduli K.

Time, t (sec.)	Displacement, W(L/2,t) (m.) (Moving force)	Displacement, W(L/2,t) (m) (Moving mass)
0	0	0
0.1	-3.240335E-06	-3.2404E-06
0.2	-4.833567E-05	-4.833896E-05
0.3	-2.267223E-04	-2.267578E-04
0.4	-6.594771E-04	-6.596653E-04
0.5	-1.471216E-03	-1.471893E-03
0.6	-2.766456E-03	-2.768348E-03
0.7	-4.61019E-03	-4.614635E-03
0.8	-7.01424E-03	-7.023435E-03
0.9	-9.930572E-03	-9.947784E-03
1.0	-.0132521	-1.328186E-02
1.1	-1.682112E-02	-1.686933E-02
1.2	-2.044461E-02	-2.051855E-02
1.3	-2.391537E-02	-2.402355E-02
1.4	-2.703732E-02	-2.718923E-02
1.5	-2.965324E-02	-2.985905E-02
1.6	-3.167319E-02	-3.194321E-02
1.7	-3.310218E-02	-3.344645E-02
1.8	-3.406624E-02	-3.449393E-02
1.9	-3.483677E-02	-.035356

Table 3.01: Comparison of the displacement of moving force and moving mass for clamped-clamped uniform Rayleigh beam.

Figures 3.11 and 3.12 display the effect of R^0 on the transverse deflection of the clamped-clamped uniform beam in both cases of moving force and moving mass problems respectively. It is evident that as the value of R^0 increases, the deflection of the beam decreases.

Figures 3.13 and 3.14 show that, for both cases of moving force and moving mass respectively, an increase in the value of foundation moduli K reduces the transverse deflection of the uniform Rayleigh beam with clamped ends.

For the purpose of comparison, table 3.01 presents the generated values of the displacement response for both moving force and moving mass problems when R^0 and K are fixed for a clamped ends uniform Reyleigh beam. It can be noted that the displacement response of a moving mass is greater than that of a moving force problem.

3.3.4 ONE END CLAMPED AND ONE END FREE.

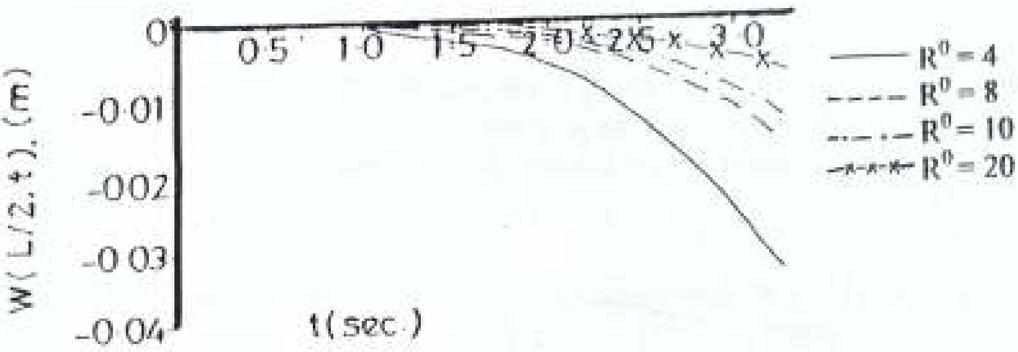

Fig 3.15: Deflection profile of moving force for clamped - free uniform Rayleigh beam for various values of R^0

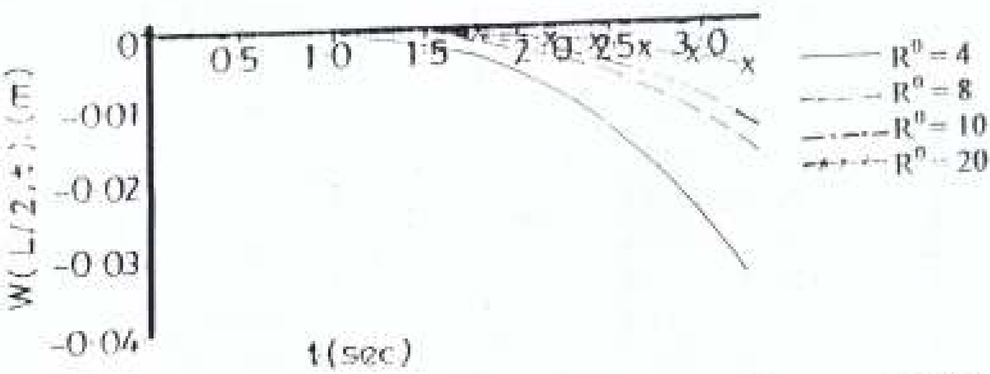

Fig 3.16: Deflection profile of moving mass for clamped - free uniform beam for various values of R^0

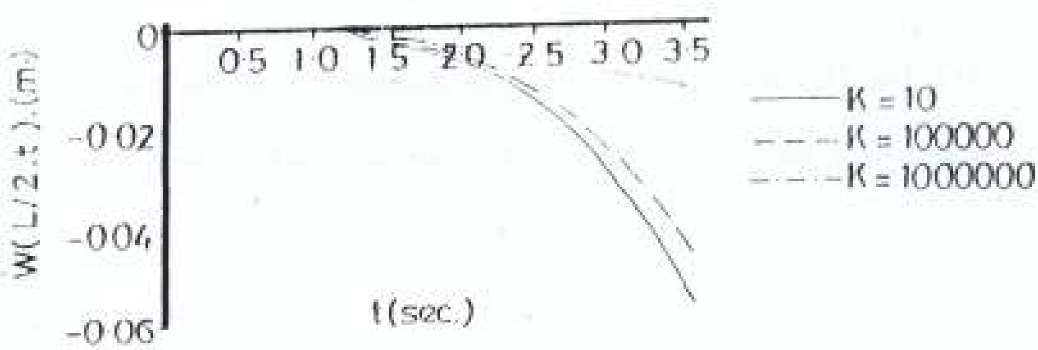

Fig 3.17: Displacement response of moving force for clamped - free uniform Rayleigh beam for various values of foundation moduli K.

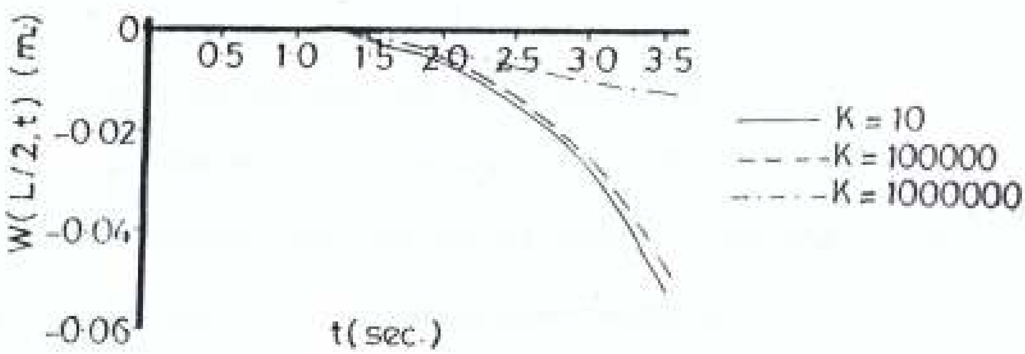

Fig. 3.18: Displacement response of moving mass for clamped-free uniform beam for various values of foundation moduli K.

Time, t (sec.)	Displacement, W(L/2,t) (m.) (Moving force)	Displacement, W(L/2,t) (m) (Moving mass)
0	0	0
0.1	-3.679376E-07	-3.67973E-07
0.2	-5.820625E-06	-5.820652E-06
0.3	-2.909273E-05	-2.909324E-05
0.4	-9.069086E-05	-9.069208E-05
0.5	-2.182052E-04	-2.182098E-04
0.6	-4.455388E-04	-4.455505E-04
0.7	-8.120924E-04	-8.12121E-04
0.8	-1.361884E-03	-1.361949E-03
0.9	-2.14267E-03	-2.142797E-03
1.0	-3.205016E-03	-3.20525E-03
1.1	-4.601392E-03	-4.601797E-03
1.2	-6.385264E-03	-6.385935E-03
1.3	-8.610255E-03	-8.611321E-03
1.4	-1.132931E-02	-1.133094E-02
1.5	-1.459396E-02	-1.459637E-02
1.6	-1.845366E-02	-1.845715E-02
1.7	-2.295525E-02	-2.296016E-02
1.8	-2.814249E-02	-2.814925E-02
1.9	-3.405576E-02	-.0340649

Table 3.02: Comparison of the displacement of moving force and moving mass for clamped-free uniform Rayleigh beam.

As in the previous boundary conditions, it is observed in figures 3.15 and 3.16 that as the value of the Rotatory inertia increases, the deflection amplitude of the beam

decreases for both cases of moving force and moving mass respectively. Also, figures 3.17 and 3.18 show that as the value of the foundation moduli K increases, the deflection amplitude of the uniform Rayleigh beam, with one end clamped and the other end free, decreases for both cases of moving force and moving mass respectively.

Table 3.02 compares the displacement response of the moving force and moving mass for a clamped–free uniform Rayleigh beam for fixed values of R^0 and K. It is evident that the displacement response of the moving mass problem is greater than that of the moving force problem.

CHAPTER FOUR

NON-UNIFORM RAYLEIGH BEAM ON A VARIABLE WINKLER ELASTIC FOUNDATION.

4.1 INTRODUCTION

In the previous chapter, the problem of the dynamic response of Rayleigh beam resting on a variable elastic foundation to moving concentrated masses is restricted to the case of uniform beam, that is, when the beam properties do not vary along the span L of the beam. However, in many practical problems involving dynamic of structures (beams or plates) under moving loads, the structures have variable cross-sections, Gbadeyan et al (1990). Thus, the problem of a non-uniform Rayleigh beam under the action of a moving load is considered in this chapter. In particular, the Garlerkin's method already alluded to is employed to simplify the governing fourth order singular variable coefficient partial differential equation. The resulting Garlerkin's equations are solved via the modified struble's asymptotic technique

4.2 GOVERNING EQUATION

Consider a non-uniform Rayleigh beam resting on a variable elastic foundation where the beam's properties such as the moment of inertia I and the mass per unit length of the beam μ vary along the span L of the beam.

The transverse displacement of the beam when it is under the action of a moving load of mass M which is moving with velocity c is governed by the fourth order partial differential equation given by

$$\frac{\partial^2}{\partial x^2}\left[EI(x) \frac{\partial^2 U(x,t)}{\partial x^2}\right] + \mu(x) \frac{\partial^2 U(x,t)}{\partial t^2} - \frac{\partial}{\partial x}\left[\mu(x) R^{o}\left(\frac{\partial^3 U(x,t)}{\partial x \partial t^2} \right)\right]$$

$$+ M\,\delta(x-ct)\left(\frac{\partial^2}{\partial t^2} + 2c\frac{\partial^2}{\partial x\partial t} + c^2\frac{\partial^2}{\partial x^2}\right)U(x,t) + f(x)U(x,t) = Mg\delta(x-ct)$$

$$(4.1)$$

where all parameters are as defined in the previous chapter.

As in the last chapter, we shall take the elastic foundation f(x) to be of the form (2.9). Since R^o and E are constants equation (4.1) can be rewritten as

$$E\frac{\partial^2}{\partial x^2}\left[I(x)\frac{\partial^2 U(x,t)}{\partial x^2}\right] + \mu(x)\frac{\partial^2 U(x,t)}{\partial t^2} - R^o\frac{\partial}{\partial x}\left[\mu(x)\frac{\partial^3 U(x,t)}{\partial x\partial t^2}\right]$$

$$+ M\delta(x-ct)\left(\frac{\partial^2}{\partial t^2} + 2c\frac{\partial^2}{\partial x\partial t} + c^2\frac{\partial^2}{\partial x^2}\right)U(x,t) + K(4x - 3x^2 + x^3)U(x,t) = Mg\delta(x-ct)$$

$$(4.2)$$

Next, the example in Sadiku et al (1981) shall be adopted and I(x) and $\mu(x)$ take the forms

$$I(x) = I_o\left(1 + \sin\frac{\pi x}{L}\right)^3 \qquad (4.3)$$

and

$$\mu(x) = \mu_o\left(1 + \sin\frac{\pi x}{L}\right) \qquad (4.4)$$

where I_o and μ_o are constants.

Substituting equations (4.3) and (4.4) into (4.2), one obtains.

$$EI_o\frac{\partial^2}{\partial x^2}\left[\left(1 + \sin\frac{\pi x}{L}\right)^3\frac{\partial^2 U(x,t)}{\partial x^2}\right] + \mu_o\left[1 + \sin\frac{\pi x}{L}\right]\frac{\partial^2 U(x,t)}{\partial t^2}$$

$$- R^o\mu_o\frac{\partial}{\partial x}\left[\left(1 + \sin\frac{\pi x}{L}\right)\left(\frac{\partial^3 U(x,t)}{\partial x\partial t^2}\right)\right] + M\delta(x-ct)\left(\frac{\partial^2}{\partial t^2} + 2c\frac{\partial^2}{\partial x\partial t} + c^2\frac{\partial^2}{\partial x^2}\right)U(x,t)$$

$$+ K(4x - 3x^2 + x^3)U(x,t) = Mg\delta(x-ct) \qquad (4.5)$$

which, on further simplification, yields

$$\frac{EI_o}{4}\frac{\partial^2}{\partial x^2}\left[\left(10 + 15\sin\frac{\pi x}{L} - 6\cos\frac{2\pi x}{L} - \sin\frac{3\pi x}{L}\right)\frac{\partial^2 U(x,t)}{\partial x^2}\right]$$

$$+ \mu_o\left[1 + \sin\frac{\pi x}{L}\right]\frac{\partial^2 U(x,t)}{\partial t^2} - R^o\mu_o\frac{\partial}{\partial x}\left[\left(1 + \sin\frac{\pi x}{L}\right)\left(\frac{\partial^3 U(x,t)}{\partial x\partial t^2}\right)\right]$$

$$+ M\delta(x-ct)\left(\frac{\partial^2}{\partial t^2} + 2c\frac{\partial^2}{\partial x\partial t} + c^2\frac{\partial^2}{\partial x^2}\right)U(x,t) + k(4x - 3x^2 + x^3)U(x,t) = Mg\delta(x-ct) \tag{4.6}$$

4.3. ANALYTICAL APPROXIMATE SOLUTION

Evidently, a closed form solution of the partial differential equation (4.6) does not exist. Thus, the Garlerkin's method described in chapter two is employed to reduce the equation to a sequence of ordinary differential equations. Thus a solution of the form

$$U_n(x,t) = \sum_{m=1}^{n} W_m(t)\,V_m(x), \tag{4.7}$$

where $V_m(x)$ is chosen such that the desired boundary conditions are satisfied, is sought.

As in the previous chapter, equation (4.7) when substituted into equation (4.6) yields

$$\sum_{m=1}^{n}\Big\{\frac{EI_o}{4}\frac{\partial^2}{\partial x^2}\Big[10V^{11}_m(x) + 15\sin\frac{\pi x}{L}V^{11}_m(x) - 6\cos\frac{2\pi x}{L}V^{11}_m(x)$$

$$- \sin\frac{3\pi x}{L}V^{11}_m(x)\Big]W_m(t) + \mu_o\Big[V_m(x) + V_m(x)\sin\frac{\pi x}{L}\Big]\ddot{W}_m(t)$$

$$- R^o\frac{\partial\mu_o}{\partial x}\Big[\big(V^1_m(x) + V^1_m(x)\sin\frac{\pi x}{L}\big)\ddot{W}_m(t)$$

$$+ M\delta(x-ct)\big[V_m(x)\ddot{W}_m(t) + 2cV^1_m(x)\dot{W}_m(t) + c^2V^{11}_m(x)W_m(t)\big]$$

$$+ K(4x - 3x^2 + x^3)V_m(x)W_m(t) - Mg\delta(x-ct)\Big\} = 0 \tag{4.8}$$

Simplifying equation (4.8) further, we have

$$\sum_{m=1}^{n}\Big\{\frac{EI_o}{4}\Big[10V^{iv}_m(x) + 15V^{iv}_m(x)\sin\frac{\pi x}{L} + \frac{30\pi}{L}V^{111}_m(x)\cos\frac{\pi x}{L}$$

$$- \frac{15\pi^2}{L^2}V^{11}_m(x)\sin\frac{\pi x}{L} - 6V^{iv}_m(x)\cos\frac{2\pi x}{L} + \frac{24\pi}{L}V^{111}_m(x)\sin\frac{2\pi x}{L}$$

$$+ \frac{24\pi^2}{L^2}V^{11}_m(x)\cos\frac{2\pi x}{L} - V^{iv}_m(x)\sin\frac{3\pi x}{L} - \frac{6\pi}{L}V^{111}_m(x)\cos\frac{3\pi x}{L}$$

$$+ \frac{9\pi^2}{L^2} V^{11}{}_m(x) \sin \frac{3\pi x}{L}] W_m(t) + \mu_o [V_m(x) + V_m(x) \sin \frac{\pi x}{L}] \ddot{W}_m(t)$$

$$- R^o\mu_o [V^{11}{}_m(x) + V^{11}{}_m(x)\sin \frac{\pi x}{L} + \frac{\pi}{L} V^1{}_m(x) \cos \frac{\pi x}{L}] \ddot{W}_m(t)$$

$$+ M\delta(x - ct) [V_m(x)\ddot{W}_m(t) + 2cV^1{}_m(x)\dot{W}_m(t) + c^2V^{11}{}_m(x)W_m(t)]$$

$$+ K(4x - 3x^2 + x^3)V_m(x)W_m(t) - Mg\delta(x - ct) \Big\} = 0 \qquad (4.9)$$

In order to determine $W_m(t)$, it is required that the expression on the left hand side of equation (4.9) be orthogonal to the function $V_k(x)$. Thus,

$$\int_o^L \sum_{m=1}^{n} \Big\{ \frac{EI_o}{4} [(10 + 15\sin \frac{\pi x}{L} - 6\cos\frac{2\pi x}{L} - \sin \frac{3\pi x}{L})V^{iv}{}_m(x)$$

$$+ \frac{6\pi}{L} (5\cos \frac{\pi x}{L} + 4\sin \frac{2\pi x}{L} - \cos \frac{3\pi x}{L})V^{111}{}_m(x)$$

$$+ \frac{3\pi^2}{L^2} (3\sin \frac{3\pi x}{L} + 8\cos\frac{2\pi x}{L} - 5\sin \frac{\pi x}{L})V^{11}{}_m(x)] W_m(t)$$

$$+ \mu_o [1 + \sin \frac{\pi x}{L}] V_m(x) \ddot{W}_m(t)$$

$$- \mu_o R^0 [(1 + \sin \frac{\pi x}{L})V^{11}{}_m(x) + \frac{\pi}{L} V^1{}_m(x) \cos \frac{\pi x}{L}] \ddot{W}_m(t)$$

$$+ M\delta(x - ct) [V_m(x) \ddot{W}_m(t) + 2cV^1{}_m(x)\dot{W}_m(t) + c^2V^{11}{}_m(x)W_m(t)]$$

$$+ K(4x - 3x^2 + x^3)V_m(x)W_m(t) - Mg\delta(x - ct) \Big\} V_k(x)\, dx = 0 \qquad (4.10)$$

When equation (4.10) is further simplified and rearranged, one obtains

$$\sum_{m=1}^{n} \Big\{ [(\theta_1 + \theta_2) - R^o (\theta_3 + \theta_4 + \frac{\pi}{L} \theta_5)] \ddot{W}_m(t)$$

$$+ \{ \frac{EI_0}{4\mu_o} [10\theta_6 + 15\theta_7 - 6\theta_8 - \theta_9 + \frac{6\pi}{L} (5\theta_{10} + 4\theta_{11} - \theta_{12})$$

$$+ \frac{3\pi^2}{L^2} (3\theta_{13} + 8\theta_{14} - 5\theta_{15})] + \frac{K}{\mu_o} (4\theta_{16} - 3\theta_{17} + \theta_{18}) \} W_m(t)$$

$$+ \frac{M}{\mu_o}\left[\theta_{19}(t)\ddot{W}_m(t) + 2c\theta_{20}(t)\dot{W}_m(t) + c^2\theta_{21}(t)W_m(t) \right] \Big\} = \frac{Mg}{\mu_o} V_k(ct) \qquad (4.11)$$

where

$$\theta_1 = \int_0^L V_m(x)V_k(x)\,dx \;;\quad \theta_2 = \int_0^L \sin\frac{\pi x}{L} V_m(x)V_k(x)\,dx \;;$$

$$\theta_3 = \int_0^L V''_m(x)\,V_k(x)\,dx \;;\quad \theta_4 = \int_0^L \sin\frac{\pi x}{L} V''_m(x)\,V_k(x)\,dx \;;$$

$$\theta_5 = \int_0^L \cos\frac{\pi x}{L} V'_m(x)V_k(x)\,dx \;;\quad \theta_6 = \int_0^L V^{iv}_m(x)\,V_k(x)\,dx \;;$$

$$\theta_7 = \int_0^L \sin\frac{\pi x}{L} V^{iv}_m(x)V_k(x)\,dx \;;\quad \theta_8 = \int_0^L \cos\frac{2\pi x}{L} V^{iv}_m(x)V_k(x)\,dx \;;$$

$$\theta_9 = \int_0^L \sin\frac{3\pi x}{L} V^{iv}_m(x)V_k(x)\,dx \;;\quad \theta_{10} = \int_0^L \cos\frac{\pi x}{L} V'''_m(x)V_k(x)\,dx \;;$$

$$\theta_{11} = \int_0^L \sin\frac{2\pi x}{L} V'''_m(x)V_k(x)\,dx \;;\quad \theta_{12} = \int_0^L \cos\frac{3\pi x}{L} V'''_m(x)V_k(x)\,dx \;;$$

$$\theta_{13} = \int_0^L \sin\frac{3\pi x}{L} V''_m(x)V_k(x)\,dx \;;\quad \theta_{14} = \int_0^L \cos\frac{2\pi x}{L} V''_m(x)\,V_k(x)\,dx \;;$$

$$\theta_{15} = \int_0^L \sin\frac{\pi x}{L} V''_m(x)V_k(x)\,dx \;;\quad \theta_{16} = \int_0^L xV_m(x)V_k(x)\,dx \;;$$

$$\theta_{17} = \int_0^L x^2V_m(x)V_k(x)\,dx \;;\quad \theta_{18} = \int_0^L x^3V_m(x)V_k(x)\,dx \;;$$

$$\theta_{19}(t) = \int_0^L \delta(x - ct)V_m(x)V_k(x)\,dx \;;\quad \theta_{20}(t) = \int_0^L \delta(x - ct)V'_m(x)V_k(x)\,dx \;;$$

and

$$\theta_{21}(t) = \int_0^L \delta(x - ct)V''_m(x)V_k(x)\,dx\,.$$

When use is made of the property of the Dirac-delta function as an even function, and substitutes it into equation (4.11), equation (4.11), after some rearrangements takes the form:

$$\sum_{m=1}^{n} \left\{ \alpha_o(m,k)\ddot{W}_m(t) + \alpha_1(m,k)W_m(t) \right.$$

$$+ \frac{M}{L\mu_O} \left[\left(\alpha_{2A}(m,k) + 2\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}\,\alpha_{2B}(n,m,k) \right)\ddot{W}_m(t) \right.$$

$$+ 2c \left(\alpha_{3A}(m,k) + 2\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}\,\alpha_{3B}(n,m,k) \right)\dot{W}_m(t)$$

$$\left. \left. + c^2 \left(\alpha_{4A}(m,k) + 2\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}\,\alpha_{4B}(n,m,k) \right) W_m(t) \right] \right\} = \frac{mg}{\mu_O} V_k(ct)$$

(4.12)

where

$$\alpha_o(m,k) = \theta_1 + \theta_2 - R^o\left(\theta_3 + \theta_4 + \frac{\pi}{L}\theta_5\right);$$

$$\alpha_1(m,k) = \frac{EI_0}{4\mu_o}\left[10\theta_6 + 15\theta_7 - 6\theta_8 - \theta_9 + \frac{6\pi}{L}(5\theta_{10} + 4\theta_{11} - \theta_{12}) + \frac{3\pi^2}{L^2}(3\theta_{13} + 8\theta_{14} - 5\theta_{15})\right] + \frac{K}{\mu_O}\left[4\theta_{16} - 3\theta_{17} + \theta_{18}\right];$$

$$\alpha_{2A}(m,k) = \int_0^L V_m(x)V_k(x)\,dx \ ; \ \alpha_{2B}(n,m,k) = \int_0^L \cos\frac{n\pi x}{L} V_m(x)V_k(x)\,dx \ ;$$

$$\alpha_{3A}(m,k) = \int_0^L V^1{}_m(x)V_k(x)\,dx \ ; \ \alpha_{3B}(n,m,k) = \int_0^L \cos\frac{n\pi x}{L} V^1{}_m(x)V_k(x)\,dx \ ;$$

$$\alpha_{4A}(m,k) = \int_0^L V^{11}{}_m(x)V_k(x)\,dx \ ; \ \alpha_{4B}(n,m,k) = \int_0^L \cos\frac{n\pi x}{L} V^{11}{}_m(x)V_k(x)\,dx \ ;$$

Using (2.38), it is straight forward to show that

$$\theta_1 = I_1 + A_mI_2 + B_mI_3 + C_mI_4 + A_kI_5 + A_kA_mI_6 + A_kB_mI_7 + A_kC_mI_8 + B_kI_9 + B_kA_mI_{10} + B_kB_mI_{11} + B_kC_mI_{12} + C_kI_{13} + C_kA_mI_{14} + C_kB_mI_{15} + C_kC_mI_{16}$$

$$\theta_2 = [\, I_{33} + A_mI_{34} + B_mI_{35} + C_mI_{36} + A_kI_{37} + A_kA_mI_{38} + A_kB_mI_{39} + A_kC_mI_{40} + B_kI_{41} + B_kA_mI_{42} + B_kB_mI_{43} + B_kC_mI_{44} + C_kI_{45} + C_kA_mI_{46} + C_kB_mI_{47} + C_kC_mI_{48}\,]_{at\ n=1}$$

$$\theta_3 = \frac{\lambda^2_m}{L^2}\,[\, -I_1 - A_mI_2 + B_mI_3 + C_mI_4 - A_kI_5 - A_mA_kI_6 + B_mA_kI_7 + C_mA_kI_8 - B_kI_9$$

$$- A_mB_kI_{10} + B_mB_kI_{11} + C_mB_kI_{12} - C_kI_{13} - A_mC_kI_{14} + B_mC_kI_{15} + C_mC_kI_{16}]$$

$$\theta_4 = \frac{\lambda^2_m}{L^2} [-I_{33} - A_mI_{34} + B_mI_{35} + C_mI_{36} - A_kI_{37} - A_mA_kI_{38} + B_mA_kI_{39} + C_mA_kI_{40} - B_kI_{41} - A_mB_kI_{42} + B_mB_kI_{43} + C_mB_kI_{44} - C_kI_{45} - A_mC_kI_{46} + B_mC_kI_{47} + C_mC_kI_{48}]_{at\ n=1}$$

$$\theta_5 = \frac{\lambda_m}{L} [-A_mI_{17} + I_{18} + C_mI_{19} + B_mI_{20} - A_mA_kI_{21} + A_kI_{22} + C_mA_kI_{23} + B_mA_kI_{24} - A_mB_kI_{25} + B_kI_{26} + C_mB_kI_{27} + B_mB_kI_{28} - A_mC_kI_{29} + C_kI_{30} + C_mC_k I_{31} + B_m C_kI_{32}]_{at\ n=1}$$

$$\theta_6 = \frac{\lambda^4_m}{L^4} [I_1 + A_mI_2 + B_mI_3 + C_mI_4 + A_kI_5 + A_kA_mI_6 + A_kB_mI_7 + A_kC_mI_8 + B_kI_9 + B_kA_mI_{10} + B_kB_mI_{11} + B_kC_mI_{12} + C_kI_{13} + C_kA_mI_{14} + C_kB_mI_{15} + C_kC_mI_{16}]$$

$$\theta_7 = \frac{\lambda^4_m}{L^4} [I_{33} + A_mI_{34} + B_mI_{35} + C_mI_{36} + A_kI_{37} + A_kA_mI_{38} + A_kB_mI_{39} + A_kC_mI_{40} + B_kI_{41} + B_kA_mI_{42} + B_kB_mI_{43} + B_kC_mI_{44} + C_kI_{45} + C_kA_mI_{46} + C_kB_mI_{47} + C_kC_mI_{48}]_{at\ n=1}$$

$$\theta_8 = \frac{\lambda^4_m}{L^4} [I_{17} + A_mI_{18} + B_mI_{19} + C_mI_{20} + A_kI_{21} + A_kA_mI_{22} + A_kB_mI_{23} + A_kC_mI_{24} + B_kI_{25} + B_kA_mI_{26} + B_kB_mI_{27} + B_kC_mI_{28} + C_kI_{29} + C_kA_mI_{30} + C_kB_mI_{31} + C_kC_mI_{32}]_{at\ n=2}$$

$$\theta_9 = \frac{\lambda^4_m}{L^4} [I_{33} + A_mI_{34} + B_mI_{35} + C_mI_{36} + A_kI_{37} + A_kA_mI_{38} + A_kB_mI_{39} + A_kC_mI_{40} + B_kI_{41} + B_kA_mI_{42} + B_kB_mI_{43} + B_kC_mI_{44} + C_kI_{45} + C_kA_mI_{46} + C_kB_mI_{47} + C_kC_mI_{48}]_{at\ n=3}$$

$$\theta_{10} = \frac{\lambda^3_m}{L^3} [A_mI_{17} - I_{18} + C_mI_{19} + B_mI_{20} + A_kA_mI_{21} - A_kI_{22} + A_kC_mI_{23} + A_kB_mI_{24} + B_kA_mI_{25} - B_kI_{26} + B_kC_mI_{27} + B_kB_mI_{28} + C_kA_mI_{29} - C_kI_{30} + C_kC_mI_{31} + C_kB_mI_{32}]_{at\ n=1}$$

$$\theta_{11} = \frac{\lambda^3_m}{L^3}[\, A_mI_{33} - I_{34} + C_mI_{35} + B_mI_{36} + A_kA_mI_{37} - A_kI_{38} + A_kC_mI_{39} + A_kB_mI_{40}$$

$$+ B_kA_mI_{41} - B_kI_{42} + B_kC_mI_{43} + B_kB_mI_{44} + C_kA_mI_{45} - C_kI_{46} + C_kC_mI_{47}$$

$$+ C_kB_mI_{48}\,]_{at\ n=2}$$

$$\theta_{12} = \frac{\lambda^3_m}{L^3}[\, A_mI_{17} - I_{18} + C_mI_{19} + B_mI_{20} + A_kA_mI_{21} - A_kI_{22} + A_kC_mI_{23} + A_kB_mI_{24}$$

$$+ B_kA_mI_{25} - B_kI_{26} + B_kC_mI_{27} + B_kB_mI_{28} + C_kA_mI_{29} - C_kI_{30} + C_kC_mI_{31}$$

$$+ C_kB_mI_{32}\,]_{at\ n=3}$$

$$\theta_{13} = \frac{\lambda^2_m}{L^2}[\, -I_{33} - A_mI_{34} + B_mI_{35} + C_mI_{36} - A_kI_{37} - A_mA_kI_{38} + B_mA_kI_{39} + C_mA_kI_{40}$$

$$- B_kI_{41} - A_mB_kI_{42} + B_mB_kI_{43} + C_mB_kI_{44} - C_kI_{45} - A_mC_kI_{46} + B_mC_kI_{47} +$$

$$C_mC_kI_{48}\,]_{at\ n=3}$$

$$\theta_{14} = \frac{\lambda^2_m}{L^2}[\, -I_{17} - A_mI_{18} + B_mI_{19} + C_mI_{20} - A_kI_{21} - A_mA_kI_{22} + B_mA_kI_{23} + C_mA_kI_{24}$$

$$- B_kI_{25} - A_mB_kI_{26} + B_mB_kI_{27} + C_mB_kI_{28} - C_kI_{29} - A_mC_kI_{30} + B_mC_kI_{31} +$$

$$C_mC_kI_{32}\,]_{at\ n=2}$$

$$\theta_{15} = \frac{\lambda^2_m}{L^2}[\, -I_{33} - A_mI_{34} + B_mI_{35} + C_mI_{36} - A_kI_{37} - A_mA_kI_{38} + B_mA_kI_{39} + C_mA_kI_{40}$$

$$- B_kI_{41} - A_mB_kI_{42} + B_mB_kI_{43} + C_mB_kI_{44} - C_kI_{45} - A_mC_kI_{46} + B_mC_kI_{47}$$

$$+ C_mC_kI_{48}\,]_{at\ n=1}$$

$$\theta_{16} = I_{17}c + A_mI_{18}c + B_mI_{19}c + C_mI_{20}c + A_kI_{21}c + A_mA_kI_{22}c + B_mA_kI_{23}c$$

$$+ C_mA_kI_{24}c + B_kI_{25}c + A_mB_kI_{26}c + B_mB_kI_{27}c + C_mB_kI_{28}c + C_kI_{29}c$$

$$+ A_mC_kI_{30}c + B_mC_kI_{31}c + C_mC_kI_{32}c$$

$$\theta_{17} = I_{17A} + A_mI_{18A} + B_mI_{19A} + C_mI_{20A} + A_kI_{21A} + A_mA_kI_{22A} + B_mA_kI_{23A}$$

$$+ C_mA_kI_{24A} + B_kI_{25A} + A_mB_kI_{26A} + B_mB_kI_{27A} + C_mB_kI_{28A} + C_kI_{29A}$$

$$+ A_mC_kI_{30A} + B_mC_kI_{31A} + C_mC_kI_{32A}$$

$$\theta_{18} = I_{17B} + A_mI_{18B} + B_mI_{19B} + C_mI_{20B} + A_kI_{21B} + A_mA_kI_{22B} + B_mA_kI_{23B}$$

$$+ C_mA_kI_{24B} + B_kI_{25B} + A_mB_kI_{26B} + B_mB_kI_{27B} + C_mB_kI_{28B} + C_kI_{29B}$$

$$+ A_mC_kI_{30B} + B_mC_kI_{31B} + C_mC_kI_{32B}$$

$$\alpha_{2A}(m,k) = 0_i$$

$$\alpha_{2B}(m,k) = I_{17} + A_mI_{18} + B_mI_{19} + C_mI_{20} + A_kI_{21} + A_mA_kI_{22} + B_mA_kI_{23} + C_mA_kI_{24} + B_kI_{25} + A_mB_kI_{26} + B_mB_kI_{27} + C_mB_kI_{28} + C_kI_{29} + A_mC_kI_{30} + B_mC_kI_{31} + C_mC_kI_{32}$$

$$\alpha_{3A}(m,k) = \frac{\lambda_m}{L}[-A_mI_1 + I_2 + C_mI_3 + B_mI_4 - A_mA_kI_5 + A_kI_6 + C_mA_kI_7 + B_mA_kI_8 - A_mB_kI_9 + B_kI_{10} + C_mB_kI_{11} + B_mB_kI_{12} - A_mC_kI_{13} + C_kI_{14} + C_mC_kI_{15} + B_mC_kI_{16}]$$

$$\alpha_{3B}(m,k) = \frac{\lambda_m}{L}[-A_mI_{17} + I_{18} + C_mI_{19} + B_mI_{20} - A_mA_kI_{21} + A_kI_{22} + C_mA_kI_{23} + B_mA_kI_{24} - A_mB_kI_{25} + B_kI_{26} + C_mB_kI_{27} + B_mB_kI_{28} - A_mC_kI_{29} + C_kI_{30} + C_mC_kI_{31} + B_mC_kI_{32}]$$

$$\alpha_{4A}(m,k) = \frac{\lambda^2_m}{L^2}[-I_1 - A_mI_2 + B_mI_3 + C_mI_4 - A_kI_5 - A_mA_kI_6 + B_mA_kI_7 + C_mA_kI_8 - B_kI_9 - A_mB_kI_{10} + B_mB_kI_{11} + C_mB_kI_{12} - C_kI_{13} - A_mC_kI_{14} + B_mC_kI_{15} + C_mC_kI_{16}]$$

$$\alpha_{4B}(m,k) = \frac{\lambda^2_m}{L^2}[-I_{17} - A_mI_{18} + B_mI_{19} + C_mI_{20} - A_kI_{21} - A_mA_kI_{22} + B_mA_kI_{23} + C_mA_kI_{24} - B_kI_{25} - A_mB_kI_{26} + B_mB_kI_{27} + C_mB_kI_{28} \;.\; C_kI_{29} - A_mC_kI_{30} + B_mC_kI_{31} + C_mC_kI_{32}]$$

where

I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_8, I_9, I_{10}, I_{11}, I_{12}, I_{13}, I_{14}, I_{15}, I_{16}, I_{17}, I_{18}, I_{19}, I_{20}, I_{21}, I_{22}, I_{23}, I_{24}, I_{25}, I_{26}, I_{27}, I_{28}, I_{29}, I_{30}, I_{31}, I_{32}, $I_{17}c$, $I_{18}c$, $I_{19}c$, $I_{20}c$, $I_{21}c$, $I_{22}c$, $I_{23}c$, $I_{24}c$, $I_{25}c$, $I_{26}c$, $I_{27}c$, $I_{28}c$, $I_{29}c$, $I_{30}c$, $I_{31}c$, $I_{32}c$, I_{17A}, I_{18A}, I_{19A}, I_{20A}, I_{21A}, I_{22A}, I_{23A}, I_{24A}, I_{25A}, I_{26A}, I_{27A}, I_{28A}, I_{29A}, I_{30A}, I_{31A}, I_{32A}, I_{17B}, I_{18B}, I_{19B}, I_{20B}, I_{21B}, I_{22B}, I_{23B}, I_{24B}, I_{25B}, I_{26B}, I_{27B}, I_{28B}, I_{29B}, I_{30B}, I_{31B} and I_{32B} have been defined in the previous chapter and

$$I_{33} = \int_0^L \sin\frac{n\pi x}{L} \sin\frac{\lambda_k x}{L} \sin\frac{\lambda_m x}{L}\, dx \ ; I_{34} = \int_0^L \sin\frac{n\pi x}{L} \sin\frac{\lambda_k x}{L} \cos\frac{\lambda_m x}{L}\, dx \ ;$$

$$I_{35} = \int_0^L \sin\frac{n\pi x}{L} \sin\frac{\lambda_k x}{L} \sinh\frac{\lambda_m x}{L}\, dx \ ; I_{36} = \int_0^L \sin\frac{n\pi x}{L} \sin\frac{\lambda_k x}{L} \cosh\frac{\lambda_m x}{L}\, dx \ ;$$

$$I_{37} = \int_0^L \sin\frac{n\pi x}{L} \cos\frac{\lambda_k x}{L} \sin\frac{\lambda_m x}{L}\, dx \ ; I_{38} = \int_0^L \sin\frac{n\pi x}{L} \cos\frac{\lambda_k x}{L} \cos\frac{\lambda_m x}{L}\, dx \ ;$$

$$I_{39} = \int_0^L \sin\frac{n\pi x}{L} \cos\frac{\lambda_k x}{L} \sinh\frac{\lambda_m x}{L}\, dx \ ; I_{40} = \int_0^L \sin\frac{n\pi x}{L} \cos\frac{\lambda_k x}{L} \cosh\frac{\lambda_m x}{L}\, dx \ ;$$

$$I_{41} = \int_0^L \sin\frac{n\pi x}{L} \sinh\frac{\lambda_k x}{L} \sin\frac{\lambda_m x}{L}\, dx \ ; I_{42} = \int_0^L \sin\frac{n\pi x}{L} \sinh\frac{\lambda_k x}{L} \cos\frac{\lambda_m x}{L}\, dx \ ;$$

$$I_{43} = \int_0^L \sin\frac{n\pi x}{L} \sinh\frac{\lambda_k x}{L} \sinh\frac{\lambda_m x}{L}\, dx \ ; I_{44} = \int_0^L \sin\frac{n\pi x}{L} \sinh\frac{\lambda_k x}{L} \cosh\frac{\lambda_m x}{L}\, dx \ ;$$

$$I_{45} = \int_0^L \sin\frac{n\pi x}{L} \cosh\frac{\lambda_k x}{L} \sin\frac{\lambda_m x}{L}\, dx \ ; I_{46} = \int_0^L \sin\frac{n\pi x}{L} \cosh\frac{\lambda_k x}{L} \cos\frac{\lambda_m x}{L}\, dx \ ;$$

$$I_{47} = \int_0^L \sin\frac{n\pi x}{L} \cosh\frac{\lambda_k x}{L} \sinh\frac{\lambda_m x}{L}\, dx \ ; I_{48} = \int_0^L \sin\frac{n\pi x}{L} \cosh\frac{\lambda_k x}{L} \cosh\frac{\lambda_m x}{L}\, dx \ .$$

Next, equation (4.12) is simplified and rearranged to take the form

$$\sum_{m=1}^{n} \{ \alpha_0(m,k)\ddot{W}_m(t) + \alpha_1(m,k)W_m(t)$$

$$+ \Gamma_a [(\alpha_{2A}(m,k) + \sum_{n=1}^{\infty} \cos\frac{n\pi ct}{L} \alpha_{2B}(n,m,k))\ddot{W}_m(t)$$

$$+ 2c (\alpha_{3A}(m,k) + 2 \sum_{n=1}^{\infty} \cos\frac{n\pi ct}{L} \alpha_{3B}(n,m,k))\dot{W}_m(t)$$

$$+ c^2 (\alpha_{4A}(m,k) + 2 \sum_{n=1}^{\infty} \cos\frac{n\pi ct}{L} \alpha_{4B}(n,m,k))W_m(t)] \}$$

$$= \frac{Mg}{\mu_0} [\sin\frac{\lambda_k ct}{L} + A_k \cos\frac{\lambda_k ct}{L} + B_k \sinh\frac{\lambda_k ct}{L} + C_k \cosh\frac{\lambda_k ct}{L}] \quad (4.13)$$

where

$$\Gamma_a = \frac{M}{L\mu_o}$$

Equation (4.13) is the transformed equation governing the problem of a non-uniform Rayleigh beam resting on a variable Winkler elastic foundation and traversed by a moving load.

These second order differential equations are valid for all variants of the classical boundary conditions.

4.4. SOLUTION OF THE TRANSFORMED EQUATION

As in the previous chapter, we shall discuss two cases of the transformed equation.

4.4.1 CASE I

Setting $\Gamma_a = 0$ in the transformed equation (4.13), one obtains

$$\alpha_o(m,k)\ddot{W}_m(t) + \alpha_1(m,k)W_m(t) = \frac{Mg}{\mu_o}\left[\sin\frac{\lambda_k ct}{L} + A_k\cos\frac{\lambda_k ct}{L} + B_k\sinh\frac{\lambda_k ct}{L} + C_k\cosh\frac{\lambda_k ct}{L}\right] \tag{4.14}$$

This is the classical case of a moving force problem associated with the system. It is an approximate model which assumes the inertial effect of the moving mass as negligible.

A rearrangement of equation (4.14) yields

$$\ddot{W}_m(t) + \gamma^2_m W_m(t) = F_m\left[\sin\frac{\lambda_k ct}{L} + A_k\cos\frac{\lambda_k ct}{L} + B_k\sinh\frac{\lambda_k ct}{L} + C_k\cosh\frac{\lambda_k ct}{L}\right] \tag{4.15}$$

where $$\gamma^2_m = \frac{\alpha_1(m,k)}{\alpha_o(m,k)} \tag{4.16}$$

Consequently, it can be shown that

$$W_m(t) = \frac{F_m}{\gamma_m[\gamma^4_m - \theta^4_k]}\Big\{[\gamma^2_m - \theta^2_k][C_k\gamma_m(\cosh\theta_k t - \cos\gamma_m t) + B_k(\gamma_m \sinh\theta_k t - \theta_k \sin\gamma_m t)] + [\gamma^2_m + \theta^2_k][A_k\gamma_m(\cos\theta_k t - \cos\gamma_m t) - (\theta_k \sin\gamma_m t - \gamma_m \sin\theta_k t)]\Big\} \qquad (4.17)$$

where $\theta_k = \frac{\lambda_k c}{L}$

Hence in view of equations (2.15) and (2.38), one obtains

$$U_n(x,t) = \sum_{m=1}^{n} \frac{F_m}{\gamma_m(\gamma^4_m - \theta^4_k)}\Big\{[\gamma^2_m - \theta^2_k][C_k\gamma_m(\cosh\theta_k t - \cos\gamma_m t) + B_k(\gamma_m \sinh\theta_k t - \theta_k \sin\gamma_m t)] + [\gamma^2_m + \theta^2_k][A_k\gamma_m(\cos\theta_k t - \cos\gamma_m t) - (\theta_k \sin\gamma_m t - \gamma_m \sin\theta_k t)]\Big\}\Big[\sin\frac{\lambda_m x}{L} + A_m\cos\frac{\lambda_m x}{L} + B_m\sinh\frac{\lambda_m x}{L} + C_m\cosh\frac{\lambda_m x}{L}\Big] \qquad (4.18)$$

as the transverse - displacement response to a moving force of a non-uniform Rayleigh beam resting on a variable Winkler elastic foundation.

4.4.2 CASE II

As discussed in the previous chapter, if the mass of the moving load is Commensurable with that of the structure, the inertial effect of the moving mass is not negligible. Thus $\Gamma_a \neq 0$ and one is required to solve the entire equation (4.13). This is termed the moving mass problem.

Unlike case I, it is obvious that an exact analytical solution to this equation is not possible. Thus, one resorts to the approximate analytical method due to Struble enumerated in chapter two.

To this end, equation (4.13) is rearranged to take the form

$$\alpha_0(m,k)\ddot{W}_m(t) + \alpha_1(m,k)W_m(t)$$
$$+ \Gamma_a [(\alpha_{2A}(m,k) + 2\alpha_{2B}(m,k)\cos\frac{\pi ct}{L})\ddot{W}_m(t)$$
$$+ 2c (\alpha_{3A}(m,k) + 2\alpha_{3B}(m,k)\cos\frac{\pi ct}{L})\dot{W}_m(t)$$
$$+ c^2 (\alpha_{4A}(m,k) + 2\alpha_{4B}(m,k)\cos\frac{\pi ct}{L}) W_m(t)$$
$$= \Gamma_a gL[\sin\frac{\lambda_k ct}{L} + A_k\cos\frac{\lambda_k ct}{L} + B_k\sinh\frac{\lambda_k ct}{L} + C_k\cosh\frac{\lambda_k ct}{L}] \qquad (4.19)$$

which implies

$$\ddot{W}_m(t) + \frac{2\Gamma_a c (\alpha_{3A}(m,k) + 2\alpha_{3B}(m,k)\cos\frac{\pi ct}{L})\dot{W}_m(t)}{\alpha_0(m,k) + \Gamma_a(\alpha_{2A}(m,k) + 2\alpha_{2B}(m,k)\cos\frac{\pi ct}{L})}$$
$$+ \frac{[\alpha_1(m,k) + \Gamma_a c^2 (\alpha_{4A}(m,k) + 2\alpha_{4B}(m,k)\cos\frac{\pi ct}{L})] W_m(t)}{\alpha_0(m,k) + \Gamma_a(\alpha_{2A}(m,k) + 2\alpha_{2B}(m,k)\cos\frac{\pi ct}{L})}$$
$$= \frac{\Gamma_a gL[\sin\frac{\lambda_k ct}{L} + A_k\cos\frac{\lambda_k ct}{L} + B_k\sinh\frac{\lambda_k ct}{L} + C_k\cosh\frac{\lambda_k ct}{L}]}{\alpha_0(m,k) + \Gamma_a(\alpha_{2A}(m,k) + 2\alpha_{2B}(m,k)\cos\frac{\pi ct}{L})} \qquad (4.20)$$

As in chapter two, the homogeneous part of (4.20) is first considered and a modified frequency corresponding to the frequency of the free system due to the presence of the moving mass is sought. An equivalent free system operator defined by the modified frequency then replaces equation (4.20).

Thus, consider a parameter $\lambda < 1$ for any arbitrary mass ratio Γ_a defined as

$$\lambda = \frac{\Gamma_a}{1+\Gamma_a}\ .$$

Obviously

$$\Gamma_a = \lambda\,[\,1 + o(\lambda) + o(\lambda^2) + \ldots\ldots\ldots\ldots] \tag{4.21}$$

All the various time dependent coefficients of the differential operator which acts on $W_m(t)$ in equation (4.20) can be written in terms of λ when one considers that to $o(\lambda)$.

$$\Gamma_a = \lambda \tag{4.22}$$

and

$$\frac{1}{\alpha_0(m,k) + \lambda(\,\alpha_{2A}(m,k) + 2\alpha_{2B}(m,k)\cos\frac{\pi ct}{L}\,)}$$

$$= \frac{1}{\alpha_0(m,k)}\left[\,1 - \frac{1}{\alpha_0(m,k)}\,\lambda(\,\alpha_{2A}(m,k) + 2\alpha_{2B}(m,k)\cos\frac{\pi ct}{L}\,) + o(\lambda^2) + \ldots\ldots\,\right] \tag{4.23}$$

where

$$\left|\,\frac{\lambda}{\alpha_0(m,k)}\,(\,\alpha_{2A}(m,k) + 2\alpha_{2B}(m,k)\cos\frac{\pi ct}{L}\,)\,\right| < 1 \tag{4.24}$$

Now, using (4.22) and (4.23), equation (4.20) takes the form

$$\ddot{W}_m(t) + \frac{2c\lambda}{\alpha_0(m,k)}\left[\,(\,\alpha_{3A}(m,k) + 2\alpha_{3B}(m,k)\cos\frac{\pi ct}{L}\,\right]\dot{W}_m(t)$$

$$+ \left\{\,\frac{\alpha_1(m,k)}{\alpha_0(m,k)}\left[\,1 - \frac{\lambda}{\alpha_0(m,k)}\,(\,\alpha_{2A}(m,k) + 2\alpha_{2B}(m,k)\cos\frac{\pi ct}{L}\,\right]\right.$$

$$\left. + \frac{c^2\lambda}{\alpha_0(m,k)}\left[\,\alpha_{4A}(m,k) + 2\alpha_{4B}(m,k)\cos\frac{\pi ct}{L}\,\right]\,\right\} W_m(t)$$

$$= \frac{\lambda gL}{\alpha_0(m,k)}\left[\,\sin\frac{\lambda_k ct}{L} + A_k\cos\frac{\lambda_k ct}{L} + B_k\sinh\frac{\lambda_k ct}{L} + C_k\cosh\frac{\lambda_k ct}{L}\,\right] \tag{4.25}$$

to $o(\lambda)$ only.

When $\lambda = 0$, a case corresponding to the case when the inertial effect of the mass of the system is neglected, the solution of equation (4.25) takes the form

$$W_{mc}(t) = C_0\cos(\gamma_m t - \phi_m)$$

where $\gamma^2_m = \dfrac{\alpha_1(m,k)}{\alpha_o(m,k)}$ and C_0 and ϕ_m are constants

Since $\lambda < 1$ for any arbitrary mass ratio Γ_a, Struble's technique requires that the asymptotic solution of the homogeneous part of equation (4.25) be of the form

$$W_m(t) = \Phi(m,t) \cos [\gamma_m t - \Omega(m,t)] + \lambda W_1(t) + o(\lambda^2) \tag{4.26}$$

where the conditions (2.73) hold.

To obtain the modified frequency, equation (4.26) and its derivatives are substituted into the homogeneous part of equation (4.25). The resulting variational equations describing the behaviour of $\Phi(m,t)$ and $\Omega(m,t)$ during the motion of the mass determine the modified frequency. To this end, substituting (4.26) and its derivatives into the homogeneous part of equation (4.25) and taking into account (4.22), (4.23) and (4.24) one obtains

$$\begin{aligned}
& - 2\gamma_m \dot{\Phi}(m,t) \sin [\gamma_m t - \Omega(m,t)] + 2\gamma_m \Phi(m,t) \dot{\Omega}(m,t) \cos [\gamma_m t - \Omega(m,t)] \\
& - \Phi(m,t)\gamma^2_m \cos [\gamma_m t - \Omega(m,t)] \\
& + \frac{2c\lambda}{\alpha_o(m,k)} [\alpha_{3A}(m,k) + 2\alpha_{3B}(m,k) \cos \frac{\pi c t}{L}] \{ -\Phi(m,t)\gamma_m \sin[\gamma_m t - \Omega(m,t)] \} \\
& + \{ \frac{\alpha_1(m,k)}{\alpha_o(m,k)} - \lambda \frac{\alpha_1(m,k)}{\alpha_o(m,k)} [\alpha_{2A}(m,k) + 2\alpha_{2B}(m,k)\cos \frac{\pi c t}{L}] \\
& + \frac{c^2\lambda}{\alpha_o(m,k)} [\alpha_{4A}(m,k) + 2\alpha_{4B}(m,k) \cos \frac{\pi c t}{L}] \} \{ \Phi(m,k) \cos [\gamma_m t - \Omega(m,t)] \} \\
& = 0
\end{aligned} \tag{4.27}$$

where terms higher than $o(\lambda)$ have been neglected.

The variational equations are obtained by equating the coefficients of

$\sin[\gamma_m t - \Omega(m,t)]$ and $\cos[\gamma_m t - \Omega(m,t)]$ terms on both sides of the equation. Thus, noting that

$$\cos\frac{\pi ct}{L}\sin[\gamma_m t - \Omega(m,t)] = \tfrac{1}{2}\sin[\frac{\pi ct}{L} + \gamma_m t - \Omega(m,t)] + \tfrac{1}{2}\sin[\gamma_m t - \Omega(m,t) - \frac{\pi ct}{L}]$$

and

$$\cos\frac{\pi ct}{L}\cos[\gamma_m t - \Omega(m,t)] = \tfrac{1}{2}\cos[\frac{\pi ct}{L} + \gamma_m t - \Omega(m,t)] + \tfrac{1}{2}\cos[\frac{\pi ct}{L} - \gamma_m t + \Omega(m,t)]$$

and neglecting those terms that do not contribute to the variational equations and taking into account (4.16), equation (4.27) reduces to

$$\begin{aligned} &- 2\gamma_m\dot{\Phi}(m,t)\sin[\gamma_m t - \Omega(m,t)] + 2\gamma_m\Phi(m,t)\dot{\Omega}(m,t)\cos[\gamma_m t - \Omega(m,t)] \\ &- \Phi(m,t)\gamma^2_m\cos[\gamma_m t - \Omega(m,t)] - \frac{2c\lambda}{\alpha_o(m,k)}\Phi(m,t)\gamma_m\,\alpha_{3A}(m,k)\sin[\gamma_m t - \Omega(m,t)] \\ &+ \gamma^2_m\,\Phi(m,t)\cos[\gamma_m t - \Omega(m,t)] - \frac{\lambda\gamma^2_m}{\alpha_o(m,k)}\alpha_{2A}(m,k)\Phi(m,t)\cos[\gamma_m t - \Omega(m,t)] \\ &+ \frac{c^2\lambda}{\alpha_o(m,k)}\alpha_{4A}(m,k)\Phi(m,t)\cos[\gamma_m t - \Omega(m,t)] = 0 \end{aligned} \tag{4.28}$$

The variational equations of the problem are obtained by setting the coefficients of $\cos[\gamma_m t - \Omega(m,t)]$ and $\sin[\gamma_m t - \Omega(m,t)]$ to zero in (4.28), thus one obtains

$$2\gamma_m\Phi(m,t)\dot{\Omega}(m,t) - \frac{\lambda\Phi(m,t)\gamma^2_m\alpha_{2A}(m,k)}{\alpha_o(m,k)} + \frac{\lambda\Phi(m,t)c^2\alpha_{4A}(m,k)}{\alpha_o(m,k)} = 0 \tag{4.29}$$

and

$$\gamma_m\dot{\Phi}(m,t) + \frac{\lambda c\,\alpha_{3A}(m,k)\,\gamma_m\Phi(m,t)}{\alpha_o(m,k)} = 0 \tag{4.30}$$

Rearranging (4.29) and (4.30), we have

$$\dot{\Omega}(m,t) = \frac{\lambda\,[\gamma^2_m\,\alpha_{2A}(m,k) - c^2\alpha_{4A}(m,k)]}{2\alpha_o(m,k)\,\gamma_m} \tag{4.31}$$

and

$$\dot{\Phi}(m,t) = -\frac{\lambda c\, \Phi(m,t)\, \alpha_{3A}(m,k)}{\alpha_o(m,k)} \tag{4.32}$$

Solving equations (4.31) and (4.32) respectively yields

$$\Omega(m,t) = \lambda \left[\frac{\gamma^2_m \alpha_{2A}(m,k) - c^2\alpha_{4A}(m,k)}{2\gamma_m \alpha_o(m,k)} \right] t + \Omega_m \tag{4.33}$$

where Ω_m is a constant and

$$\Phi(m,t) = \psi\, e^{(-\gamma_o t)} \tag{4.34}$$

where $\gamma_o = \frac{\lambda\, c\, \alpha_{3A}(m,k)}{\alpha_o(m,k)}$ and ψ is a constant.

Therefore, when the effect of the mass of the particle is considered, the first approximation to the homogeneous system is

$$W_m(t) = \Phi(m,t) \cos[\, \alpha_m t - \Omega_m\,] \tag{4.35}$$

where

$$\alpha_m = \gamma_m - \lambda \left[\frac{\gamma^2_m \alpha_{2A}(m,k) - c^2\alpha_{4A}(m,k)}{2\gamma_m \alpha_o(m,k)} \right] \tag{4.36}$$

is called the modified natural frequency representing the frequency of the free system due to the presence of the moving mass.

In view of (4.35), the homogeneous part of the equation (4.20) can be written as

$$\frac{d^2W_m(t)}{dt^2} + \alpha^2_m W_m(t) = 0 \tag{4.37}$$

while the entire equation (4.20) takes the form, taking into account (4.23),

$$\frac{d^2W_m(t)}{dt^2} + \alpha^2_m W_m(t) = A^o \left[\sin \frac{\lambda_k ct}{L} + A_k \cos \frac{\lambda_k ct}{L} + B_k \sinh \frac{\lambda_k ct}{L} + C_k \cosh \frac{\lambda_k ct}{L} \right] \tag{4.38}$$

where $A^o = \frac{\lambda L g}{\alpha_o(m,k)}$

Clearly, equations (4.15) and (4.38) are similar. Thus going through the same argument as in CASE I, equation (4.38) when solved yields

$$W_m(t) = \frac{A^o}{\alpha_m(\alpha^4_m - \theta^4_0)} \{ (\alpha^2_m - \theta^2_0) [C_k \alpha_m (\cosh \theta_0 t - \cos \alpha_m t)$$

$$+ B_k(\alpha_m \sinh\theta_0 t - \theta_0 \sin\alpha_m t)] + (\alpha^2_m + \theta^2_0) [A_k \alpha_m (\cos\theta_0 t - \cos \alpha_m t)$$

$$- (\theta_0 \sin\alpha_m t - \alpha_m \sin\theta_0 t)] \} \quad (4.39)$$

where

$$\theta_0 = \frac{\lambda_k c}{L}$$

Hence, in view of equations (2.15) and (2.38)

$$U_n(x,t) = \sum_{m=1}^{n} \frac{A^o}{\alpha_m(\alpha^4_m - \theta^4_0)} \{ (\alpha^2_m - \theta^2_0) [C_k \alpha_m (\cosh \theta_0 t - \cos \alpha_m t)$$

$$+ B_k(\alpha_m \sinh\theta_0 t - \theta_0 \sin\alpha_m t)] + (\alpha^2_m + \theta^2_0) [A_k \alpha_m (\cos\theta_0 t - \cos \alpha_m t)$$

$$- (\theta_0 \sin\alpha_m t - \alpha_m \sin\theta_0 t)] \} [\sin\frac{\lambda_m x}{L} + A_m \cos \frac{\lambda_m x}{L} + B_m \sinh\frac{\lambda_m x}{L} + C_m \cosh\frac{\lambda_m x}{L}] \quad (4.40)$$

This represents the transverse - displacement response to a moving mass of a non – uniform Rayleigh beam on a variable Winkler elastic foundation.

CHAPTER FIVE

ILLUSTRATIVE EXAMPLES, NUMERICAL CALCULATIONS AND DISCUSSION OF RESULTS (NON-UNIFORM RAYLEIGH BEAM).

5.1.0 ILLUSTRATIVE EXAMPLES.

As in the previous chapter, we shall illustrate the foregoing analysis by various practical examples. Particularly we shall consider classical boundary conditions such as simply supported boundary conditions, free ends condition, clamped ends condition and clamped-free end conditions.

5.1.1 SIMPLY SUPPORTED BOUNDARY CONDITIONS.

For the non-uniform Rayleigh beam having simple supports at ends x = 0 and x = L, the deflection and bending moment at both ends vanish. Thus

$$U(0,t) = 0 = U(L,t) \text{ and } \frac{\partial^2 U(0,t)}{\partial x^2} = 0 = \frac{\partial^2 U(L,t)}{\partial x^2} \tag{5.1}$$

and hence for the normal modes

$$V_m(0) = 0 = V_m(L) \text{ and } \frac{d^2V_m(0)}{dx^2} = 0 = \frac{d^2V_m(L)}{dx^2} \tag{5.2}$$

which implies

$$V_k(0) = 0 = V_k(L) \text{ and } \frac{d^2V_k(0)}{dx^2} = 0 = \frac{d^2V_k(L)}{dx^2} \tag{5.3}$$

It can then be shown that

$$A_m = A_k = 0;\ B_m = B_k = 0;\ C_m = C_k = 0$$

with the frequency equation

$$\sin\lambda_m = \sin\lambda_k = 0$$

which implies

$\lambda_m = m\pi$ and $\lambda_k = k\pi$ respectively.

Thus the moving force problem for the non-uniform Rayleigh beam is reduced to the non-homogeneous ordinary differential equation given as

$$\frac{d^2W_m(t)}{dt^2} + \omega^2_m W_m(t) = A_0 \sin \frac{k\pi ct}{L} \tag{5.4}$$

where

$$A_0 = \frac{gM}{\mu_0 L[A_1 + \frac{R^0\pi^2m^2B_1}{L^2}]}. \tag{5.5}$$

and

$$\omega^2_m = \frac{1}{2\mu_0[A_1 + \frac{R^0\pi^2m^2B_1}{L^2}]}\left\{\frac{EI_0m^2\pi^3}{2L^3}\left[m\pi(5m-6)\right.\right.$$

$$\left. + \frac{30(1-k^2-m^2)(1+2km-m^2)}{[(1+k)^2-m^2][(1-k)^2-m^2]} + \frac{3(9-k^2-m^2)(9-4km+m^2)}{[(3+k)^2-m^2][(3-k)^2-m^2]}\right]$$

$$+ \frac{KL}{\pi^4(k-m)^4(k+m)^4}\left[3L^2[\pi^2(2-L)(k+m)^2+2L](k-m)^4(-1)^{k+m}\right.$$

$$- 3L^2[\pi^2(2-L)(k-m)^2+2L](k+m)^4(-1)^{k-m}$$

$$\left.\left. - 32k\pi^2m(k^2-m^2)^2 + 12L^3(k^4+6k^2m^2+m^4)\right]\right\} \tag{5.6}$$

where

$$A_1 = \frac{1}{2L}[L + (L - m\pi^2)(2B_1 - 1)],$$

$$B_1 = ½ + \frac{2k(1+m^2-k^2)}{\pi[(1+k)^2-m^2][(1-k)^2-m^2]}.$$

When equation (5.4) is solved in conjunction with the initial conditions, one obtains expression for $W_m(t)$. Thus in view of (2.15) and (2.38), one obtains

$$U_n(x,t) = \sum_{m=1}^{n} \frac{A_0}{\omega_m[\omega^2_m - (k\pi c/_L)^2]} \left[\omega_m \sin \frac{k\pi ct}{L} - \frac{k\pi c}{L}\sin \omega_m t\right] \sin \frac{m\pi x}{L} \tag{5.7}$$

as the transverse-displacement response to a moving force of a simply supported non-uniform Rayleigh beam on a variable Winkler elastic foundation.

Next, we consider the moving mass problem, that is, when $\Gamma_a \neq 0$. Following arguments in the previous section, the modified frequency corresponding to the frequency of the free system due to the presence of the moving mass of the model is obtained as

$$\theta_m = \omega_m - \lambda \left[\frac{4L^2\omega^2_m + c^2m^2\pi^2}{8L\omega_m[A_I + \frac{R^0\pi^2m^2B_I}{L^2}]} \right] \tag{5.8}$$

neglecting higher order terms of λ. Thus, the moving mass problem becomes

$$\frac{d^2W_m(t)}{dt^2} + \theta^2_m W_m(t) = \frac{Lg\lambda}{[A_I + \frac{R^0\pi^2m^2B_I}{L^2}]} \left[\sin \frac{k\pi ct}{L} \right] \tag{5.9}$$

which when solved in conjunction with the initial conditions yields expression for $W_m(t)$ and in view of (2.15), one has

$$U_n(x,t) = \sum_{m=1}^{n} \frac{Lg\lambda}{\theta_m[A_I + \frac{R^0\pi^2m^2B_I}{L^2}][\theta^2_m - (k\pi c/_L)^2]} \left[\theta_m \sin \frac{k\pi ct}{L} - \frac{k\pi c}{L} \sin \theta_m t \right] \sin \frac{m\pi x}{L} \tag{5.10}$$

This represents the transverse-displacement response to a moving mass of a simply supported non-uniform Rayleigh beam on a variable Winkler elastic foundation.

5.1.2 FREE ENDS CONDITION.

For free end conditions at x = 0 and x = L, the pertinent boundary conditions are

$$\frac{\partial^2 U(0,t)}{\partial x^2} = 0 = \frac{\partial^2 U(L,t)}{\partial x^2} \quad \text{and} \quad \frac{\partial^3 U(0,t)}{\partial x^3} = 0 = \frac{\partial^3 U(L,t)}{\partial x^3} \tag{5.11}$$

and for normal modes

$$\frac{d^2V_m(0)}{dx^2} = 0 = \frac{d^2V_m(L)}{dx^2} \text{ and } \frac{d^3V_m(0)}{dx^3} = 0 = \frac{d^3V_m(L)}{dx^3} \tag{5.12}$$

which implies that

$$\frac{d^2V_k(0)}{dx^2} = 0 = \frac{d^2V_k(L)}{dx^2} \text{ and } \frac{d^3V_k(0)}{dx^3} = 0 = \frac{d^3V_k(L)}{dx^3} \tag{5.13}$$

Going through the same procedure as in the previous chapter, we have

$$A_m = \frac{\sin\lambda_m - \sinh\lambda_m}{\cosh\lambda_m - \cos\lambda_m} = \frac{\cos\lambda_m - \cosh\lambda_m}{\sin\lambda_m + \sinh\lambda_m} = C_m \text{ and } B_m = 1 \tag{5.14}$$

and the frequency equation for the dynamical problem is

$$\cos\lambda_m \cosh\lambda_m = 1 \tag{5.15}$$

Such that

$$\lambda_1 = 4.73004,\ \lambda_2 = 7.85320,\ \lambda_3 = 10.99561,\ \ldots.. \tag{5.16}$$

Substituting (5.14) and (5.16) into equation (4.18) and (4.40) one obtains the displacement response respectively to a moving force and a moving mass of a free-ends non-uniform Rayleigh beam on a variable Winkler elastic foundation.

5.1.3 CLAMPED ENDS CONDITION.

Both the deflection and slope vanish at a clamped end. Thus, considering the Rayleigh beam when it is clamped at x = 0 and x = L, the conditions are expressed as

$$U(0,t) = 0 = U(L,t) \text{ and } \frac{\partial U(0,t)}{\partial x} = 0 = \frac{\partial U(L,t)}{\partial x} \tag{5.17}$$

and for normal modes

$$V_m(0) = 0 = V_m(L) \text{ and } \frac{dV_m(0)}{dx} = 0 = \frac{dV_m(L)}{dx} \tag{5.18}$$

which implies that

$$V_k(0) = 0 = V_k(L) \text{ and } \frac{dV_k(0)}{dx} = 0 = \frac{dV_k(L)}{dx} \, . \tag{5.19}$$

Following similar argument as in the previous section, it is straight forward to show that

$$A_m = \frac{\sinh\lambda_m - \sin\lambda_m}{\cos\lambda_m - \cosh\lambda_m} = \frac{\cos\lambda_m - \cosh\lambda_m}{\sin\lambda_m + \sinh\lambda_m} = -C_m \text{ and } B_m = 1 \tag{5.20}$$

and

$$\cos\lambda_m \cosh\lambda_m = 1 \tag{5.21}$$

as the corresponding frequency equation.

Thus,

$$\lambda_1 = 4.73004, \lambda_2 = 7.85320, \lambda_3 = 10.99566, \ldots\ldots \tag{5.22}$$

When (5.20) and (5.22) are substituted into equations (4.18) and (4.40) the displacement response respectively to moving force and moving mass of a non-uniform clamped Rayleigh beam on a variable Winkler elastic foundation is obtained.

5.1.4 ONE END CLAMPED AND ONE END FREE CONDITION.

At $x = 0$, the Rayleigh beam is taken to be clamped and at $x = L$, the beam model is free as obtains in the last chapter. The boundary conditions of the beam are

$$\frac{\partial^2 U(L,t)}{\partial x^2} = 0 = \frac{\partial^3 U(L,t)}{\partial x^3} \text{ and } U(0,t) = 0 = \frac{\partial U(0,t)}{\partial x} \tag{5.23}$$

and for normal modes

$$\frac{d^2 V_m(L)}{dx^2} = 0 = \frac{d^3 V_m(L)}{dx^3} \text{ and } V_m(0) = 0 = \frac{dV_m(0)}{dx} \tag{5.24}$$

which implies that

$$\frac{d^2 V_k(L)}{dx^2} = 0 = \frac{d^3 V_k(L)}{dx^3} \text{ and } V_k(0) = 0 = \frac{dV_k(0)}{dx} \tag{5.25}$$

Thus, following the same procedures, we have

$$A_m = -\frac{\sin\lambda_m - \sinh\lambda_m}{\cos\lambda_m + \cosh\lambda_m} = -\frac{\cos\lambda_m - \cosh\lambda_m}{\sinh\lambda_m - \sin\lambda_m} = -C_m \text{ and } B_m = -1 \tag{5.26}$$

and

$$\cos\lambda_m \cosh\lambda_m = -1 \tag{5.27}$$

as the frequency equation for the system such that

$$\lambda_1 = 1.875,\ \lambda_2 = 4.694,\ \lambda_3 = 7.855,\ \ldots\ldots\ldots \tag{5.28}$$

Using (5.26) and (5.28) in equations (4.18) and (4.40) one obtains the displacement response to a moving force and a moving mass respectively of a non-uniform clamped-free-ends Rayleigh beam resting on a variable elastic foundation.

5.2.0 DISCUSSIONS OF THE ANALYTICAL SOLUTIONS

As in the previous chapter, we shall examine the phenomenon of resonance in this section.

Equation (5.7) reveals clearly that the simply supported non-uniform Rayleigh beam resting on a variable Winkler elastic foundation and traversed by a moving force reaches a state of resonance whenever

$$\omega_m = \frac{k\pi c}{L} \tag{5.29}$$

While equation (5.10) shows that the same Rayleigh beam under the action of a moving mass experiences resonance when

$$\theta_m = \frac{k\pi c}{L} \tag{5.30}$$

where

$$\theta_m = \omega_m \left[1 - \lambda\left(\frac{1}{4\left(A_I + \frac{R^0\pi^2 m^2 B_I}{L^2}\right)} + \frac{c^2 m^2 \pi^2}{4L^2\omega_m^2\left(A_I + \frac{R^0\pi^2 m^2 B_I}{L^2}\right)} \right) \right] \tag{5.31}$$

From (5.30) and (5.31), it is easily shown that

$$\frac{\omega_m \left[A_I + \frac{R^0\pi^2m^2B_I}{L^2} - \frac{\lambda}{4}\left(1 + \frac{c^2m^2\pi^2}{L^2\omega^2_m}\right)\right]}{A_I + \frac{R^0\pi^2m^2B_I}{L^2}} = \frac{k\pi c}{L} \tag{5.32}$$

Clearly

$$A_I + \frac{R^0\pi^2m^2B_I}{L^2} > A_I + \frac{R^0\pi^2m^2B_I}{L^2} - \frac{\lambda}{4}\left(1 + \frac{c^2m^2\pi^2}{L^2\omega^2_m}\right) \quad \text{for all m.}$$

Consequently, for the same natural frequency, the critical speed (and the natural frequency) for the moving mass problem is smaller than that of the moving force problem. Thus, the resonance is reached earlier in the moving mass system than in the moving force system.

Furthermore we examine the phenomenon of resonance for other classical boundary conditions. From equation (4.18), it is evident that the non-uniform Rayleigh beam on a variable Winkler elastic foundation and traversed by a moving force encounters a resonance effect when

$$\eta_m = \frac{\lambda_k c}{L} \tag{5.33}$$

While equation (4.40) reveals that the same beam under the action of a moving mass reaches the state of resonance whenever

$$\alpha_m = \frac{\lambda_k c}{L} \tag{5.34}$$

where

$$\alpha_m = \eta_m - \lambda\left[\frac{\eta^2_m\alpha_{2A}(m,k) - c^2\alpha_{4A}(m,k)}{2\eta_m\alpha_0(m,k)}\right]$$

Consequently,

$$\alpha_m = \frac{\eta_m\left[\alpha_0(m,k) - \frac{\lambda}{2}\left(\alpha_{2A}(m,k) - \frac{c^2\alpha_{4A}(m,k)}{\eta^2_m}\right)\right]}{\alpha_0(m,k)} = \frac{\lambda_k c}{L} \tag{5.35}$$

Thus, from equations (5.33) and (5.35), results and analysis similar to those of the simply supported Rayleigh beam are obtained for other examples of classical end support conditions.

5.3.0 NUMERICAL CALCULATIONS AND DISCUSSION OF RESULTS.

In this section, calculations of practical interests in dynamics of structures are presented for the illustrative examples.

An elastic non-uniform Rayleigh beam of length 121.92m has been considered. The mass is assumed to travel at the constant velocity 8.123m/s. Furthermore, EI and λ are chosen to be $6.068 \times 10^6 m^3/s^2$ and 0.025 respectively. The results are as presented on the various graphs below for the various classes of boundary conditions considered.

5.3.1 SIMPLY SUPPORTED ENDS.

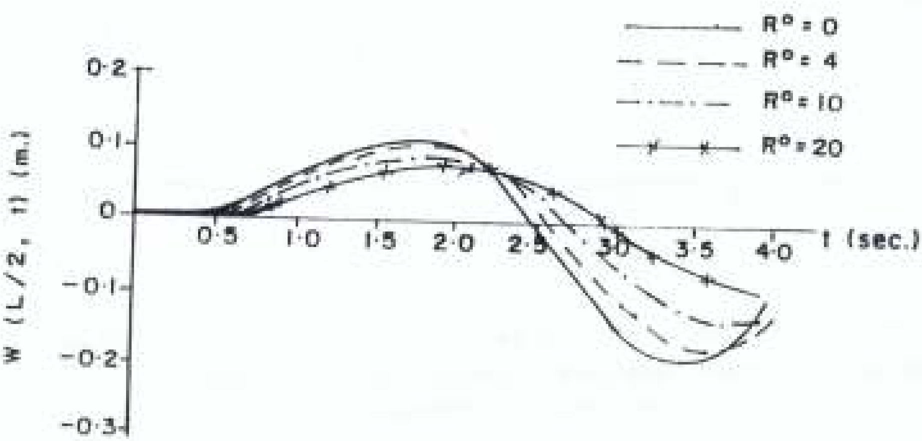

Fig. 5·01: Deflection profile of moving force at various values of R° for simply supported non-uniform Rayleigh beam.

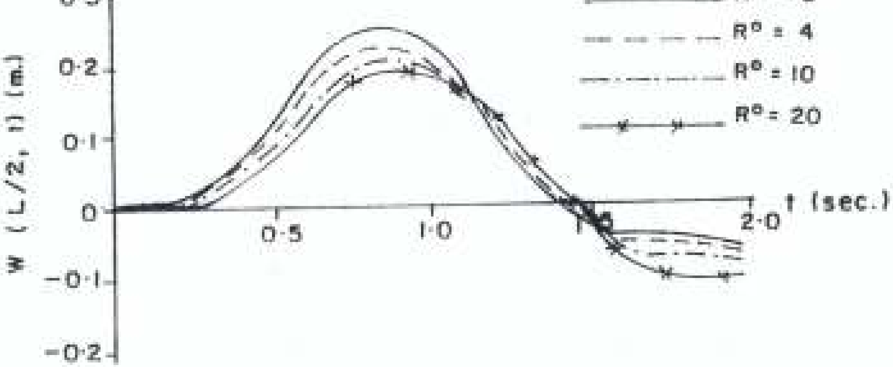

Fig. 5·02: Deflection profile of moving mass at various values of R° for simply supported non-uniform Rayleigh beam.

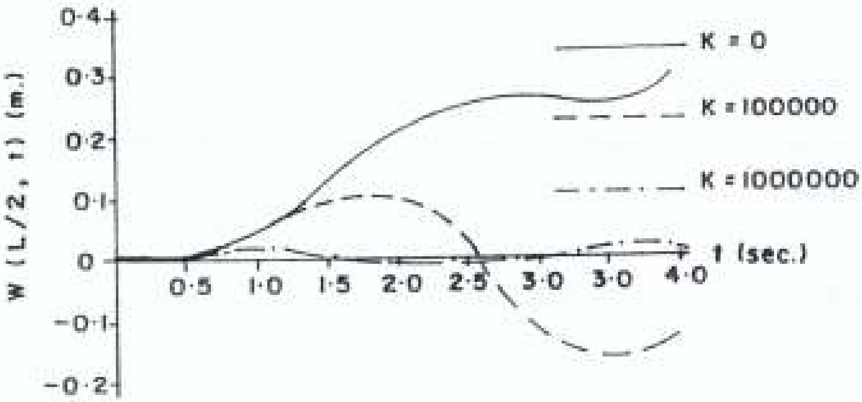

Fig. 5·03: Displacement response of moving force for simply supported non-uniform beam for various values of foundation moduli K.

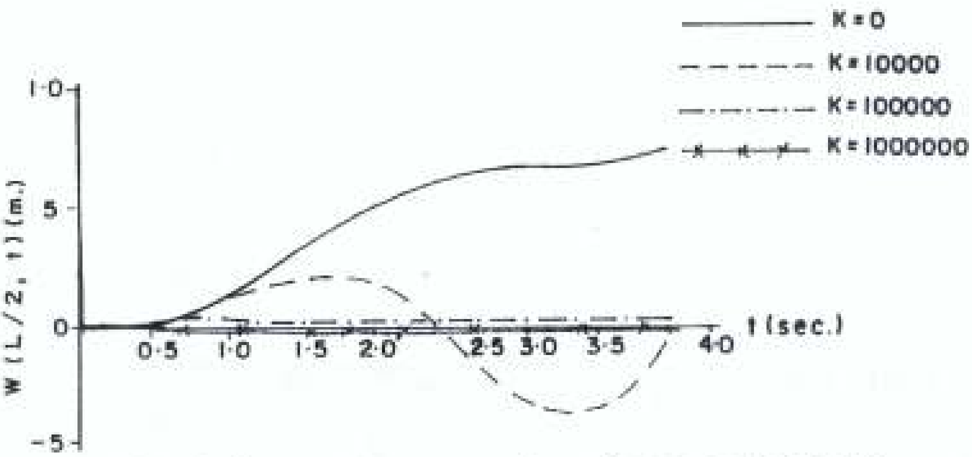

Fig. 5·04: Displacement response of moving mass for simply supported non-uniform Rayleigh beam for various values of foundation moduli. K

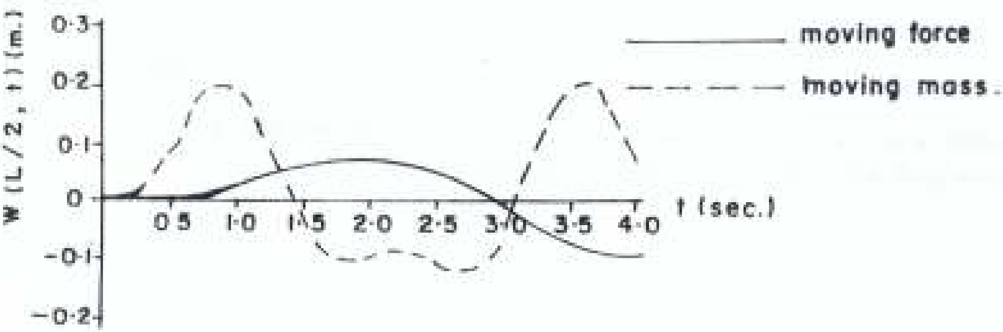

Fig. 5·05: Comparison of the deflection of moving force and moving mass for simply supported non-uniform beam.

The effect of Rotatory inertia R^0 on the transverse deflection of the simply supported non-uniform Rayleigh beam in both cases of moving force and moving mass is shown in figures 5.01 and 5.02 respectively. It is shown that the response amplitude of the non-uniform Rayleigh beam decreases as the value of the Rotatory inertia correction factor increases.

Figures 5.03 and 5.04 present the effect of foundation moduli K on the transverse deflection in both cases of moving force and moving mass respectively. It is observed that as K increases, the deflection of the simply supported non-uniform beam decreases. Here, values of K between 0 N/m^3 and 1m N/m^3 are used.

For the purpose of comparison, the displacement curves of the moving force and moving mass for a simply supported non- uniform beam with fixed R^0 and K are presented in figure 5.05. It can be noted that the response amplitude of a moving mass is greater than that of a moving force problem. The result holds for every choice of R^0 and K.

5.3.2 CLAMPED ENDS.

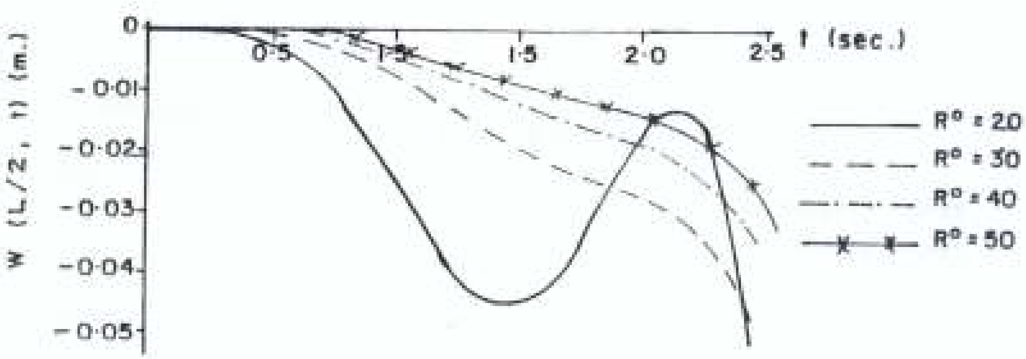

Fig. 5·06: Deflection profile of moving force at various values of R° for clamped - clamped non-uniform Rayleigh beam.

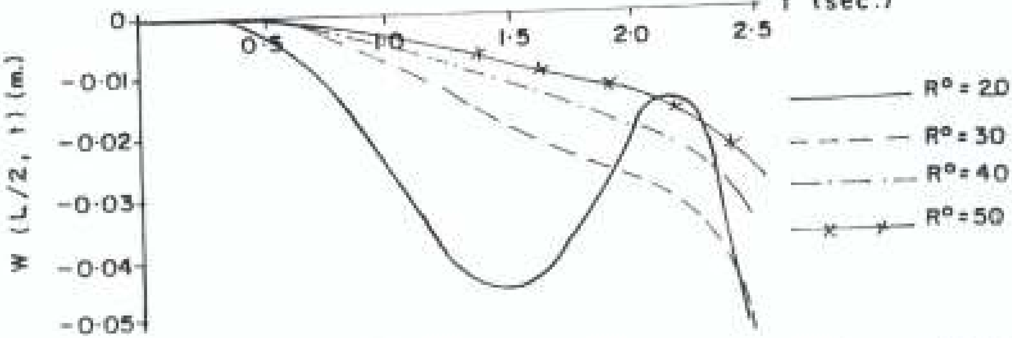

Fig. 5·07: Deflection profile of moving mass at various values of R° for clamped-clamped non-uniform Rayleigh beam.

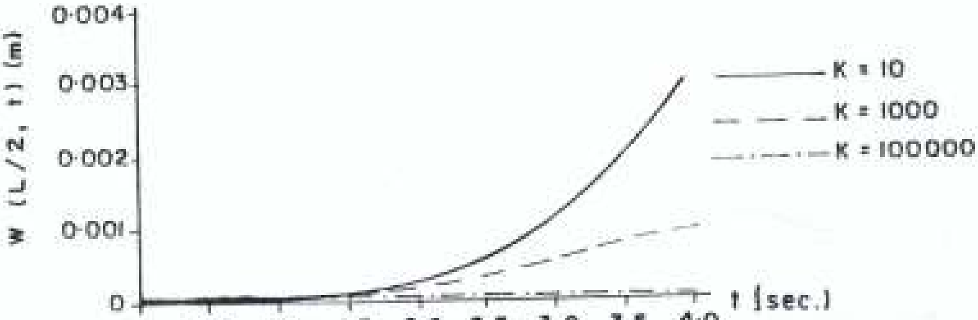

Fig. 5·08: Displacement response of moving force for clampled-clamped non-uniform beam for various values of foundation moduli K.

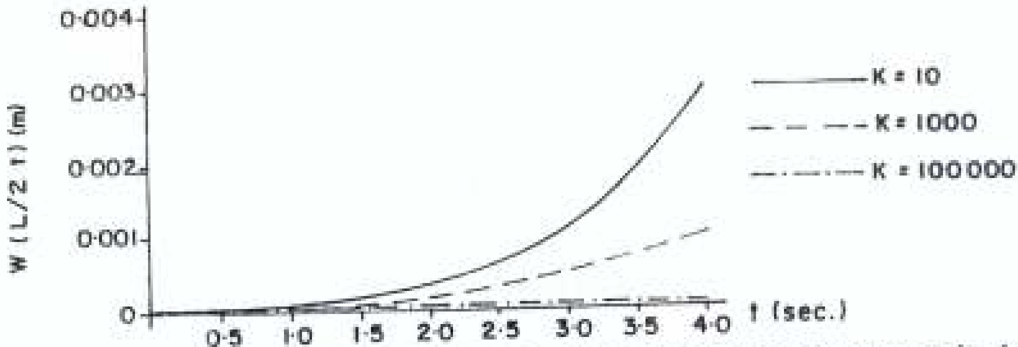

Fig. 5·09: Displacement response of moving mass for clamped-clamped non-uniform Rayleigh beam for various values of foundation moduli K.

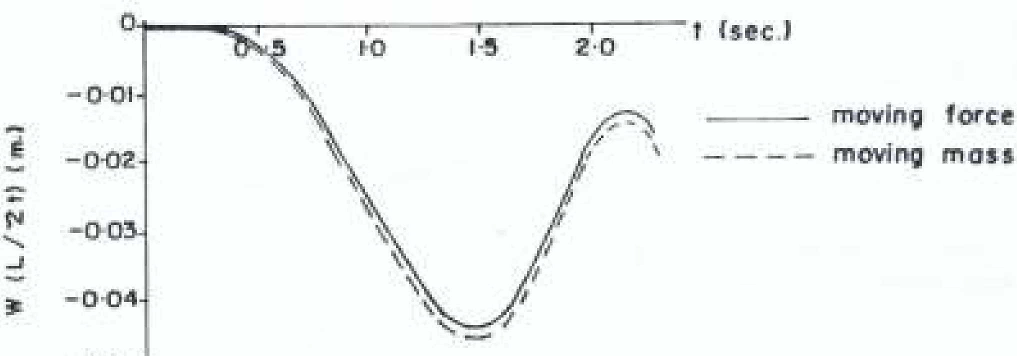

Fig. 5·10: Comparison of the deflection of moving force and moving mass for clamped-clamped non-uniform beam.

Figure 5.06 and 5.07 display the effect of Rotatory inertia R^0 on the transverse deflection of the clamped-clamped non-uniform beam in both cases of moving force and moving mass respectively. It is observed that as the value of R^0 increases the deflection of the beam decreases.

The effect of foundation constant K on the deflection of the non-uniform clamped – clamped Rayleigh beam in both cases of moving force and moving mass is displayed in figures 5.08 and 5.09 respectively. The graphs show that an increase in the value of K gives a decrease in the transverse deflection of the clamped-clamped non- uniform beam.

Figure 5.10 compares the displacement curves of the moving force and moving mass for a clamped-clamped non-uniform Rayleigh beam for fixed values of R^0 and K. it is evident that the displacement response of the moving mass problem is greater than that of the moving force problem.

5.3.3 ONE END CLAMPED AND ONE END FREE.

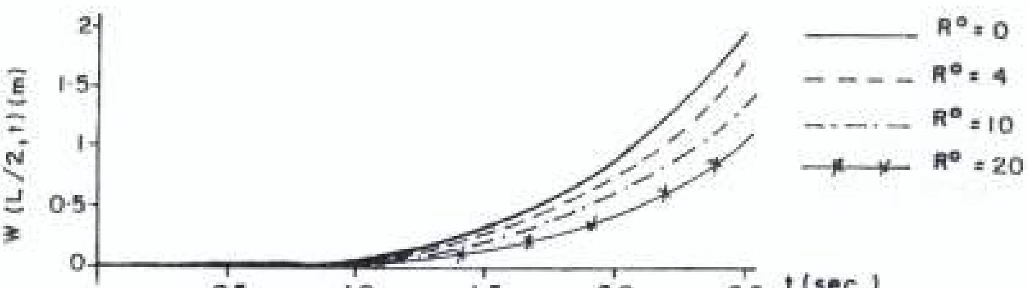

Fig. 5·11 : Deflection profile of moving force at various values of R° for clamped-free non-uniform Rayleigh beam.

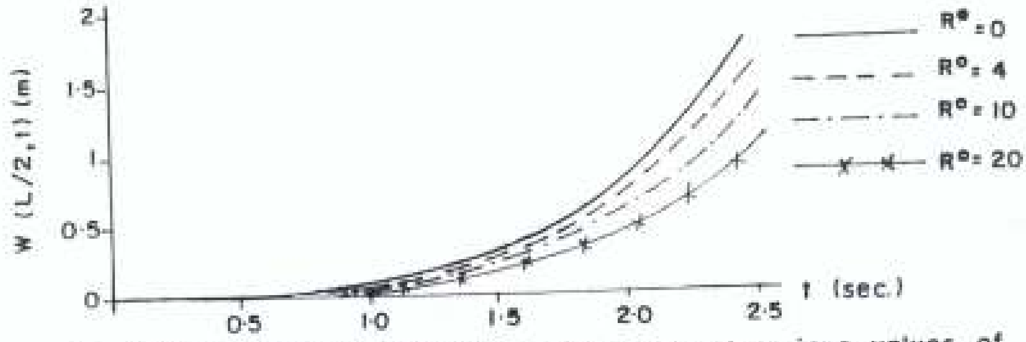

Fig. 5·12: Deflection profile of moving mass at various values of R° for clamped-free non-uniform Rayleigh beam.

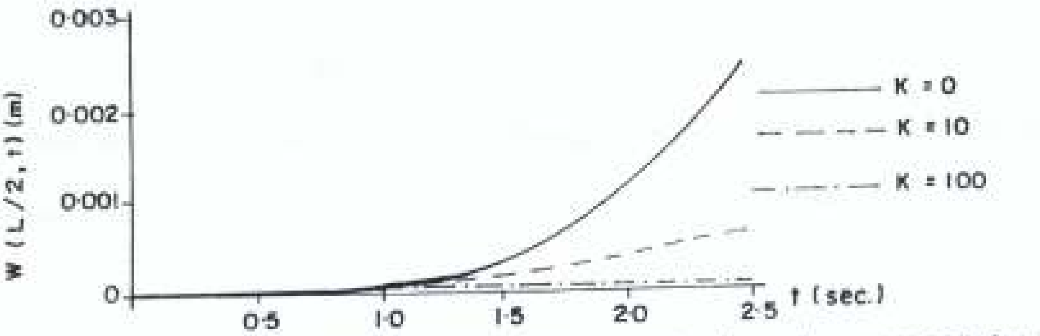

Fig. 5·13: Displacement response of moving force for clamped-free non-uniform beam for various values of foundation moduli K.

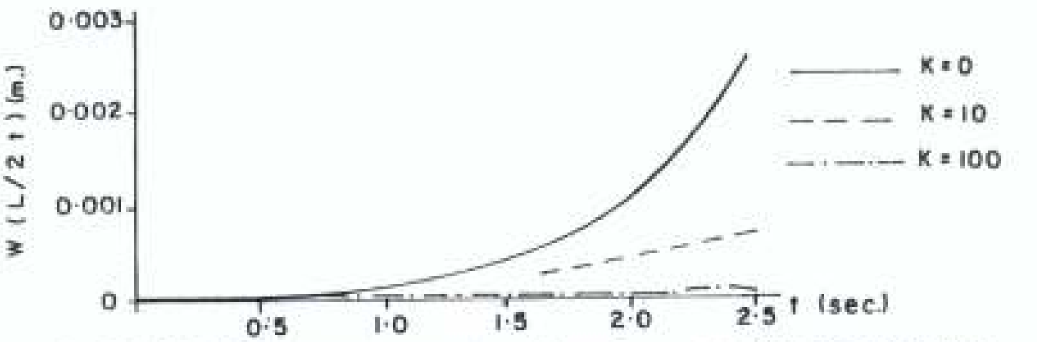

Fig. 5·14: Displacement response of moving mass for clamped-free non-uniform Rayleigh beam for various values of foundation moduli K.

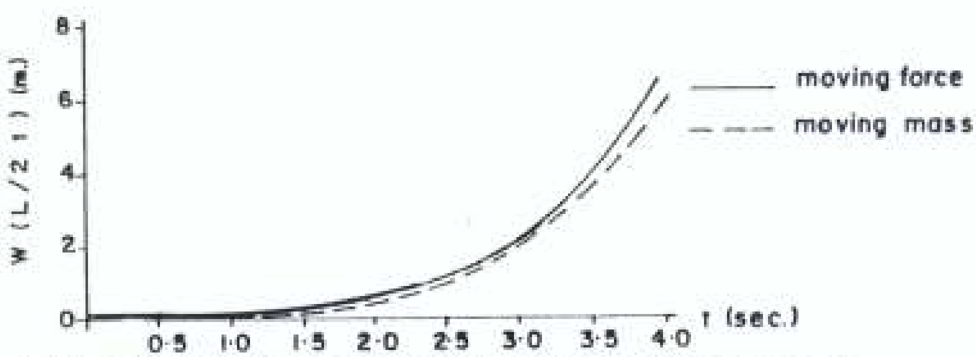

Fig. 5·15: Comparison of the deflection of moving force and moving mass for clamped-free non-uniform beam.

As in the previous sections, figures 5.11 and 5.12 display the effect of R^0 on the transverse deflection of the clamped-free non-uniform beam in both cases of moving force and moving mass problems respectively. It is shown that as the value of R^0 increases, the deflection of the beam decreases.

Figures 5.13 and 5.14 show that, for both cases of moving force and moving mass problems respectively, an increase in the value of foundation moduli K reduces the transverse deflection of the non-uniform Rayleigh beam with one end clamped and the other end free.

Figure 5.15 compares the displacement curves of the moving force and moving mass for a clamped – free non-uniform beam for fixed values of R^0 and K. It is observed that the displacement response of the moving force problem is greater than that of the moving mass problem.

CHAPTER SIX

GENERAL CONCLUSION

6.1 SUMMARY OF RESEARCH WORK.

The objective of this work has been to study the problem of the dynamic response to moving concentrated masses of Rayleigh beams on variable Winkler elastic foundations. In particular, the closed form solutions of the fourth order partial differential equations with variable and singular coefficients of

(i) Uniform Rayleigh beam and

(ii) Non-uniform Rayleigh beam.

moving mass problems are obtained. The method of solution is based on (i) The Galerkin method, (ii) The Modified Struble's Technique and (iii) The method of integral transformations.

Important features of this technique include the following:

(i) It can be used to solve moving mass problem involving Bernoulli-Euler beams theory of flexure.

(ii) It can also handle moving mass Rayleigh beams problems having uniform cross-section.

(iii) It also readily yields solutions to moving mass non-uniform Rayleigh beam problems having arbitrary classical end conditions.

In both problems, illustrative examples involving (i) Simply supported end conditions, (ii) Free end conditions, (iii) Clamped end conditions and (iv) One end fixed and the other free are given.

These solutions are analyzed and resonance conditions for the various problems are obtained. Numerical analyses for both moving force and moving mass problems are carried out and results in plotted curves are presented.

The major findings from the various analyses are summarized as follows:

(i) For all the four illustrative examples considered, the moving force solution is not an upper bound for the accurate solution of the moving mass solution in both uniform and non-uniform Rayleigh beam moving mass problems.

(ii) As the rotatory inertia correction factor increases, the response amplitudes of both uniform and non-uniform Rayleigh beams decrease.

(iii) When the rotatory inertia correction factor is fixed, the displacements of a uniform Rayleigh beam resting on a variable elastic foundation decrease as the foundation moduli increases for all variants of the boundary conditions. The same results obtain for non-uniform Rayleigh beams.

(iv) Higher values of rotatory inertia factor are required for a more noticeable effect in the case of clamped-clamped end conditions than those of simply supported end conditions for both moving force and moving mass problems of both uniform and non-uniform beams.

(v) For fixed rotatory inertia factor and foundation moduli, the response amplitude for the moving mass problem is greater than that of the moving force problem for all illustrative end conditions considered.

(vi) In all illustrative examples considered, for the same natural frequency, the critical speed for moving mass problem is smaller than that of the moving force problem.

(vii) As rotatory inertia correction factor increases, the critical speeds of both uniform and non-uniform Rayleigh beams increase.

(viii) In general, higher values of rotatory inertia correction factor are required for a more noticeable effect on the response amplitude of non-uniform Rayleigh beams than would be required for similar uniform Rayleigh beam moving mass problems.

Finally, this work has suggested valuable method of analytical solution for this category of problems for all variants of classical boundary conditions.

6.2 CONTRIBUTION TO KNOWLEDGE

Analytical solutions have been provided for both problems of uniform and non-uniform Rayleigh beams resting on variable Winkler elastic foundations for all variants of classical boundary conditions and analyses have indicated

(a) the influence of Rotatory inertia on the transverse-displacement response of both uniform and non-uniform Rayleigh beams under the action of a moving load.

(b) the effects of the variable Winkler elastic foundation modulli on the response of both uniform and non-uniform Rayleigh beam problems.

(c) the resonance conditions for both moving force and moving mass of uniform and non-uniform Rayleigh beam problems.

(d) the reliability of the moving force solution as a safe approximation to the moving mass problem for all variants of classical boundary conditions.

These findings are useful tools in the hands of practicing Engineers in structural design and analysis.

6.3 LIMITATIONS TO STUDY AND RECOMMENDATIONS FOR FURTHER RESEARCH.

The dynamic response of Rayleigh beams resting on variable Winkler elastic foundations to moving concentrated masses is the main objective of this study. Illustrative examples have been limited to classical boundary conditions only. Non-classical boundary conditions such as (i) elastically supported end conditions and (ii) time dependent boundary conditions are not considered and as such are suggested for future research. The two-dimensional analogue of the theory developed in this thesis could be extended to moving load rectangular plate problems. Structures (beams or plates) on other foundation models are left for further research. Other beam models under the action of moving loads such as Shear beams and Timoshenko beams described by simple equations resting on Winkler or non-Winkler elastic foundation and Visco-elastic foundation are not considered in this study.

REFERENCES

1. **Ayre, R.S, Jacobsen L. S. and Hsu C. S. (1951):** Transverse vibration of one- and two-span beams under the action of a moving mass load. Proc. First U.S. National Congress Applied Mechanic, pp. 81 - 90.

2. **Arnold D. Kerr (1964):** Elastic and Viscoelastic foundation Models. Journal of applied Mechanics. pp. 491 - 498.

3. **Bishop, R.E.D. and Johnson, D.C. (1960):** The mechanics of Vibration. Cambridge University Press. Reissued with minor revisions 1979.

4. **Clastornic, J., Eisenberger, M., Yankelevsky, D. Z. and Adin, M. A. (1986):** Beams on variable elastic foundation. Journal of Applied Mechanics. Vol. 53 pp 925 - 928.

5. **Fryba, L. (1972):** Vibration of solids and structures under moving loads. Groningen: Noordhoff .

6. **Gbadeyan, J.A. and Aiyesimi, Y.M. (1990):** Response of an elastic beam resting on Viscoelastic foundation to a load moving at non-uniform speed. Nigerian Journal of Mathematics and Applications Vol. 3, pp.73 - 90.

7. **Gbadeyan, J.A. and Oni S.T. (1995):** Dynamic behaviour of beams and rectangular plates under moving loads. Journal of sound and Vibration 182(5), pp.677 - 695.

8. **Jaeger, J.C. and Strarfield, A. M. (1974):** An introduction to applied Mathematics. Oxford University Press. Second Edition and reprinted in 1979.

pp.295 -296.

9. **Kenny, J. (1954):** Steady state Vibrations of a beam on an elastic foundation for a moving load. Journal of Applied Mechanics (ASME) Vol 76.

10. **Krylov, A.N. (1905):** Mathematical Collection of papers of the Academy of sciences. Vol. 61, Peterburg.

11. **Lowan, A.N. (1935):** On transverse Oscillations of beams under the action of moving variable loads. Phil. Mag. Ser. 7, Vol. 19, No 127, pp. 708 – 715.

12. **Milornir, M., Stanisic, M.M. and Hardin, J. C. (1969):** On the response of beams to an arbitrary Number of concentrated moving masses. Journal of the franklin institute, Vol 287, No 2.

13. **Masri, S. F. (1976):** Response of beams to propagating boundary Excitation. Earthquake Engineering and structural Dynamics Vol. 4, pp. 497 - 509.

14. **Oni S.T. (1991):** On the dynamic response of elastic structures to moving multi-mass system. Ph.D. thesis. University of Ilorin.

15. **Oni, S.T. (1996):** Response of a non-uniform beam resting on an elastic foundation to several moving masses. Abacus, Journal of Mathematical Association of Nigeria. Vol. 24, No 2.

16. **Oni, S.T. (2000):** Flexural Vibrations under moving loads of Isotropic rectangular plates on a non-Winkler elastic foundation. Journal of the Nigerian society of Engineers Vol. 35, No 1, pp. 18 - 27, 40 – 41.

17. **Oni, S.T. (2001):** Dynamic response to a moving concentrated mass of prestressed bernoulli-Euler beam on variable Winkler's foundation. Nigerian Journal of Mathematics and applications. In press.

18. **Pestel, E. (1951):** Tragwerk sanslenkung unter bewegter last. Ing. Arch. 19 pp 378 -383.

19. **Steele C.R. (1967):** The finite beam with a moving load. Journal of Applied Mechanics. Vol 34, No 1 pp 111 – 118.

20. **Stanisic, M.M., Hardin, J. C. and Lou, Y. C. (1968):** On the response of the plate to a Multi-masses moving system - Acta mechanical 5, pp.37-53.

21. **Stanisic, M.M, Euler J. A. and Montgomery S.T. (1974):** On a theory concerning the dynamical behaviour of structures carrying moving masses. Ing. Archiv. 43 pp 295 - 305.

22. **Sadiku, S. and Leipholz, H.H.E. (1981):** On the dynamics of elastic systems with moving concentrated masses. Ing. Archiv. 57, pp.223 - 242.

23. **Timoshenko, S. (1921):** On the correction for shear of the differential equation for transverse vibration of prismatic bars. Phil. Mag. S. 6 Vol. 41. pp. 744 - 776.

24. **Timoshenko, S. (1922):** On the transverse Vibrations of bars of Uniform Cross-Section. Philosophy Mag. Ser. 6, Vol. 43, pp. 125 - 131.

25. **Timoshenko, S, Young, D.H. and Weaver, W. (1974):** Vibration problems in Engineering. Wiley, pp 448 - 471.

26. **Wilson, J.F. (1974):** Dynamic whip of elastically restrained plate strip to rapid transit load. Transactions of the American Society of Mechanical Engineers, Series G. 96. pp. 163 - 168.

APPENDIX

SOLUTIONS OF SOME INTEGRALS

This appendix presents the solutions of the definite integrals listed in chapters two and four of this thesis.

$$I_1 = \begin{cases} \frac{L}{2}\left[\frac{\sin(\lambda_k - \lambda_m)}{\lambda_k - \lambda_m} - \frac{\sin(\lambda_k + \lambda_m)}{\lambda_k + \lambda_m}\right], & \lambda_k \neq \lambda_m \\ \frac{L}{2}\left[1 - \frac{\sin 2\lambda_m}{2\lambda_m}\right], & \lambda_k = \lambda_m \end{cases}$$

$$I_2 = \begin{cases} -\frac{L}{2}\frac{[(\lambda_k - \lambda_m)(\cos(\lambda_k+\lambda_m)-1) + (\lambda_k + \lambda_m)(\cos(\lambda_k - \lambda_m)-1)]}{\lambda_k^2 - \lambda_m^2}, & \lambda_k \neq \lambda_m \\ -\frac{L}{2}\frac{[\cos 2\lambda_m - 1]}{2\lambda_m} & \lambda_k = \lambda_m \end{cases}$$

$$I_3 = \frac{\lambda_m L}{\lambda_m^2 + \lambda_k^2}\left[\sin\lambda_k \cosh\lambda_m - \frac{\lambda_k}{\lambda_m}\cos\lambda_k \sinh\lambda_m\right]$$

$$I_4 = \frac{\lambda_m L}{\lambda_m^2 + \lambda_k^2}\left[\sin\lambda_k \sinh\lambda_m - \frac{\lambda_k}{\lambda_m}(\cos\lambda_k \cosh\lambda_m - 1)\right]$$

$$I_5 = \begin{cases} \frac{L}{2}\frac{[(\lambda_k + \lambda_m)(\cos(\lambda_k - \lambda_m)-1) + (\lambda_k - \lambda_m)(1-\cos(\lambda_k + \lambda_m))]}{\lambda_k^2 - \lambda_m^2}, & \lambda_k \neq \lambda_m \\ -\frac{L}{2}\frac{[\cos 2\lambda_m - 1]}{2\lambda_m} & \lambda_k = \lambda_m \end{cases}$$

$$I_6 = \begin{cases} \frac{L}{2}\left[\frac{\sin(\lambda_k+\lambda_m)}{\lambda_k+\lambda_m} + \frac{\sin(\lambda_k-\lambda_m)}{\lambda_k-\lambda_m}\right], & \lambda_k \neq \lambda_m \\ \frac{L}{2}\left[1 + \frac{\sin 2\lambda_m}{2\lambda_m}\right] & \lambda_k = \lambda_m \end{cases}$$

$$I_7 = \frac{\lambda_m L}{\lambda_m^2 + \lambda_k^2}\left[\cos\lambda_k \cosh\lambda_m + \frac{\lambda_k}{\lambda_m}\sin\lambda_k \sinh\lambda_m - 1\right]$$

$$I_8 = \frac{\lambda_m L}{\lambda_m^2 + \lambda_k^2}\left[\cos\lambda_k \sinh\lambda_m + \frac{\lambda_k}{\lambda_m}\sin\lambda_k \cosh\lambda_m\right]$$

$$I_9 = \frac{\lambda_k L}{\lambda_k^2 + \lambda_m^2}\left[\sin\lambda_m \cosh\lambda_k - \frac{\lambda_m}{\lambda_k}\cos\lambda_m \sinh\lambda_k\right]$$

$$I_{10} = \frac{\lambda_k L}{\lambda^2_m + \lambda^2_k}\left[\cos\lambda_m \cosh\lambda_k + \frac{\lambda_m}{\lambda_k}\sin\lambda_m \sinh\lambda_k - 1\right]$$

$$I_{11} = \begin{cases} \frac{L}{2}\left[\frac{\sinh(\lambda_k+\lambda_m)}{\lambda_k+\lambda_m} - \frac{\sinh(\lambda_k-\lambda_m)}{\lambda_k-\lambda_m}\right], & \lambda_k \neq \lambda_m \\ \frac{L}{2}\left[\frac{\sinh 2\lambda_m}{2\lambda_m} - 1\right] & \lambda_k = \lambda_m \end{cases}$$

$$I_{12} = \begin{cases} \frac{L}{2}\left[\frac{(\lambda_k-\lambda_m)(\cosh(\lambda_k+\lambda_m)-1)+(\lambda_k+\lambda_m)(\cosh(\lambda_k-\lambda_m)-1)}{\lambda^2_k+\lambda^2_m}\right] & \lambda_k \neq \lambda_m \\ \frac{L}{2}\left[\frac{\cosh 2\lambda_m}{2\lambda_m} - 1\right] & \lambda_k = \lambda_m \end{cases}$$

$$I_{13} = \frac{\lambda_k L}{\lambda^2_k + \lambda^2_m}\left[\sin\lambda_m \sinh\lambda_k - \frac{\lambda_m}{\lambda_k}(\cos\lambda_m \cosh\lambda_k - 1)\right]$$

$$I_{14} = \frac{\lambda_k L}{\lambda^2_k + \lambda^2_m}\left[\cos\lambda_m \sinh\lambda_k + \frac{\lambda_m}{\lambda_k}\sin\lambda_m \cosh\lambda_k\right]$$

$$I_{15} = \begin{cases} \frac{L}{2}\left[\frac{(\lambda_m-\lambda_k)(\cosh(\lambda_m+\lambda_k)-1)+(\lambda_m+\lambda_k)(\cosh(\lambda_m-\lambda_k)-1)}{\lambda^2_m+\lambda^2_k}\right] & \lambda_k \neq \lambda_m \\ \frac{L}{2}\left[\frac{\cosh 2\lambda_m}{2\lambda_m} - 1\right] & \lambda_k = \lambda_m \end{cases}$$

$$I_{16} = \begin{cases} \frac{L}{2}\left[\frac{\sinh(\lambda_m+\lambda_k)}{\lambda_k+\lambda_m} + \frac{\sin(\lambda_k-\lambda_m)}{\lambda_k-\lambda_m}\right], & \lambda_k \neq \lambda_m \\ \frac{L}{2}\left[\frac{\sinh 2\lambda_m}{2\lambda_m} + 1\right] & \lambda_k = \lambda_m \end{cases}$$

$$I_{17} = \begin{cases} \frac{L}{4}\left(\frac{\sin(n\pi+\lambda_k-\lambda_m)}{n\pi+\lambda_k-\lambda_m} + \frac{\sin(n\pi-\lambda_k+\lambda_m)}{n\pi-\lambda_k+\lambda_m} - \frac{\sin(n\pi+\lambda_k+\lambda_m)}{n\pi+\lambda_k+\lambda_m} - \frac{\sin(n\pi-\lambda_k-\lambda_m)}{n\pi-\lambda_k-\lambda_m}\right) \\ \frac{L}{4}\left[1 - \frac{\sin 2(n\pi+\lambda_k)}{2(n\pi+\lambda_k)}\right]\Bigg|_{\substack{\lambda_m = n\pi+\lambda_k \\ n\pi = \lambda_m - \lambda_k}} - \frac{L}{4}\left[1 - \frac{\sin 2(n\pi-\lambda_k)}{2(n\pi-\lambda_k)}\right]\Bigg|_{\substack{\lambda_m = n\pi+\lambda_k \\ n\pi = \lambda_m - \lambda_k}} \end{cases}$$

$$I_{18} = \frac{L}{4}\left[\frac{(\cos(n\pi-\lambda_k-\lambda_m)-1)}{n\pi-(\lambda_k+\lambda_m)} + \frac{(1-\cos(n\pi+\lambda_k+\lambda_m))}{n\pi+(\lambda_k+\lambda_m)} + \frac{(\cos(n\pi-\lambda_k+\lambda_m)-1)}{n\pi-(\lambda_k-\lambda_m)} + \frac{(1-\cos(n\pi+\lambda_k-\lambda_m))}{n\pi+(\lambda_k-\lambda_m)}\right]$$

$$I_{19} = \frac{L}{2}\left[\frac{\lambda_m}{\lambda^2_m + (n\pi+\lambda_k)^2}\left(\sin(n\pi+\lambda_k)\cosh\lambda_m - \frac{(n\pi+\lambda_k)}{\lambda_m}\cos(n\pi+\lambda_k)\sinh\lambda_m\right)\right.$$

$$\left. + \frac{\lambda_m}{\lambda^2_m + (n\pi-\lambda_k)^2}\left(\sin(n\pi-\lambda_k)\cosh\lambda_m - \frac{(n\pi-\lambda_k)}{\lambda_m}\cos(n\pi-\lambda_k)\sinh\lambda_m\right)\right]$$

$$I_{20} = \frac{L}{2}\left[\frac{\lambda_m}{\lambda^2_m + (n\pi+\lambda_k)^2}\left(\sin(n\pi+\lambda_k)\sinh\lambda_m - \frac{n\pi+\lambda_k}{\lambda_m}\left(\cos(n\pi+\lambda_m)\cosh\lambda_m - 1\right)\right)\right.$$

$$\left. - \frac{\lambda_m}{\lambda^2_m + (n\pi-\lambda_k)^2}\left(\sin(n\pi-\lambda_k)\sinh\lambda_m - \frac{n\pi-\lambda_k}{\lambda_m}\left(\cos(n\pi-\lambda_k)\cosh\lambda_m - 1\right)\right)\right]$$

$$I_{21} = \frac{L}{4}\left[\frac{(\cos(n\pi+\lambda_k-\lambda_m)-1)}{(n\pi+\lambda_k)-\lambda_m} + \frac{(1-\cos(n\pi+\lambda_k+\lambda_m))}{(n\pi+\lambda_k)+\lambda_m} + \frac{(\cos(n\pi-\lambda_k-\lambda_m)-1)}{(n\pi-\lambda_k)-\lambda_m}\right.$$

$$\left. + \frac{(1-\cos(n\pi-\lambda_k+\lambda_m))}{(n\pi-\lambda_k)+\lambda_m}\right]$$

$$I_{22} = \frac{L}{4}\left[\frac{\sin(n\pi+\lambda_k+\lambda_m)}{n\pi+\lambda_m+\lambda_k} + \frac{\sin(n\pi-\lambda_k-\lambda_m)}{n\pi-\lambda_k-\lambda_m} + \frac{\sin(n\pi+\lambda_k-\lambda_m)}{n\pi+\lambda_k-\lambda_m}\right.$$

$$\left. + \frac{\sin(n\pi-\lambda_k+\lambda_m)}{n\pi-\lambda_k+\lambda_m}\right]$$

$$I_{23} = \frac{L}{2}\left[\frac{\lambda_m}{\lambda^2_m + (n\pi+\lambda_k)^2}\left(\cos(n\pi+\lambda_k)\cosh\lambda_m + \frac{n\pi+\lambda_k}{\lambda_m}\sin(n\pi+\lambda_k)\sinh\lambda_m - 1\right)\right.$$

$$\left. + \frac{\lambda_m}{\lambda^2_m + (n\pi-\lambda_k)^2}\left(\cos(n\pi-\lambda_k)\cosh\lambda_m + \frac{n\pi-\lambda_k}{\lambda_m}\sin(n\pi-\lambda_k)\sinh\lambda_m - 1\right)\right]$$

$$I_{24} = \frac{L}{2}\left[\frac{\lambda_m}{\lambda^2_m + (n\pi+\lambda_k)^2}\left(\cos(n\pi+\lambda_k)\sinh\lambda_m + \frac{(n\pi+\lambda_k)}{\lambda_m}\sin(n\pi+\lambda_k)\cosh\lambda_m\right)\right.$$

$$\left. + \frac{\lambda_m}{\lambda^2_m + (n\pi+\lambda_k)^2}\left(\cos(n\pi-\lambda_k)\sinh\lambda_m + \frac{(n\pi-\lambda_k)}{\lambda_m}\sin(n\pi-\lambda_k)\cosh\lambda_m\right)\right]$$

$$I_{25} = \frac{L}{2}\left[\frac{\lambda_k}{\lambda^2_k + (n\pi+\lambda_m)^2}\left(\sin(n\pi+\lambda_m)\cosh\lambda_k - \frac{(n\pi+\lambda_m)}{\lambda_k}\cos(\lambda_m+n\pi)\sinh\lambda_k\right)\right.$$

$$\left. + \frac{\lambda_k}{\lambda^2_k + (n\pi-\lambda_m)^2}\left(\sin(n\pi-\lambda_m)\,\mathrm{coshh}\lambda_k - \frac{(n\pi-\lambda_m)}{\lambda_k}\cos(n\pi-\lambda_m)\sinh\lambda_k\right)\right]$$

$$I_{26} = \frac{L}{2}\left[\frac{\lambda_k}{\lambda_k^2+(\lambda_m+n\pi)^2}\left(\cos(n\pi+\lambda_m)\cosh\lambda_k+\frac{(n\pi+\lambda_m)}{\lambda_k}\sin(n\pi+\lambda_m)\sinh\lambda_k-1\right)\right.$$

$$\left.+\frac{\lambda_k}{\lambda_k^2+(n\pi-\lambda_m)^2}\left(\cos(n\pi-\lambda_m)\cosh\lambda_k+\frac{(n\pi-\lambda_m)}{\lambda_k}\sin(n\pi-\lambda_m)\sinh\lambda_k-1\right)\right]$$

$$I_{27} = \frac{L}{2}\left[\frac{(\lambda_k+\lambda_m)(-1)^n\sinh(\lambda_k+\lambda_m)}{(\lambda_k+\lambda_m)^2+n^2\pi^2}-\frac{(\lambda_k-\lambda_m)(-1)^n\sinh(\lambda_k-\lambda_m)}{(\lambda_k-\lambda_m)^2+n^2\pi^2}\right]$$

$$I_{28} = \frac{L}{2}\left[\frac{(\lambda_m+\lambda_k)\left((-1)^n\cosh(\lambda_k+\lambda_m)-1\right)}{(\lambda_k+\lambda_m)^2+n^2\pi^2}+\frac{(\lambda_k-\lambda_m)\left((-1)^n\cosh(\lambda_k-\lambda_m)-1\right)}{(\lambda_k-\lambda_m)^2+n^2\pi^2}\right]$$

$$I_{29} = \frac{L}{2}\left[\frac{\lambda_k}{\lambda_k^2+(n\pi+\lambda_m)^2}\left(\sin(n\pi+\lambda_m)\sinh\lambda_k-\frac{(n\pi+\lambda_m)}{\lambda_k}\left(\cos(n\pi+\lambda_m)\cosh\lambda_k-1\right)\right)\right.$$

$$\left.-\frac{\lambda_k}{\lambda_k^2+(n\pi-\lambda_m)^2}\left(\sin(n\pi-\lambda_m)\sinh\lambda_k-\frac{(n\pi-\lambda_m)}{\lambda_k}\left(\cos(n\pi-\lambda_m)\cosh\lambda_k-1\right)\right)\right]$$

$$I_{30} = \frac{L}{2}\left[\frac{\lambda_k}{\lambda_k^2+(n\pi+\lambda_m)^2}\left(\cos(n\pi+\lambda_m)\sinh\lambda_k+\frac{(n\pi+\lambda_m)}{\lambda_k}\sin(n\pi+\lambda_m)\cosh\lambda_k\right)\right.$$

$$\left.+\frac{\lambda_k}{\lambda_k^2+(n\pi-\lambda_m)^2}\left(\cos(n\pi-\lambda_m)\sinh\lambda_k+\frac{(n\pi-\lambda_m)}{\lambda_k}\sin(n\pi-\lambda_m)\cosh\lambda_k\right)\right]$$

$$I_{31} = \frac{L}{2}\left[\frac{(\lambda_k+\lambda_m)\left((-1)^n\cosh(\lambda_k+\lambda_m)-1\right)}{(\lambda_k+\lambda_m)^2+n^2\pi^2}+\frac{(\lambda_m-\lambda_k)\left((-1)^n\cosh(\lambda_m-\lambda_k)-1\right)}{(\lambda_m-\lambda_k)^2+n^2\pi^2}\right]$$

$$I_{32} = \frac{L}{2}\left[\frac{(\lambda_k+\lambda_m)(-1)^n\sinh(\lambda_k+\lambda_m)}{(\lambda_k+\lambda_m)^2+n^2\pi^2}+\frac{(\lambda_k-\lambda_m)(-1)^n\sinh(\lambda_k-\lambda_m)}{(\lambda_k-\lambda_m)^2+n^2\pi^2}\right]$$

$$I_{17A} = \begin{cases} \dfrac{L^3}{2(\lambda_k-\lambda_m)^3}\left[\left((\lambda_k-\lambda_m)^2-2\right)\sin(\lambda_k-\lambda_m)+2(\lambda_k-\lambda_m)\cos(\lambda_k-\lambda_m)\right] \\ \quad -\dfrac{L^3}{2(\lambda_k+\lambda_m)^3}\left[\left((\lambda_k+\lambda_m)^2-2\right)\sin(\lambda_k+\lambda_m)+2(\lambda_k+\lambda_m)\cos(\lambda_k+\lambda_m)\right], & m \neq k \\ \dfrac{L^3}{6}-\dfrac{L^3}{2(2\lambda_k)^3}\left[(2\lambda_k)^2\sin 2\lambda_k+4\lambda_k\cos\lambda_k-2\sin 2\lambda_k\right], & m = k \end{cases}$$

$$I_{18A} = \begin{cases} \dfrac{L^3}{2(\lambda_k+\lambda_m)^3}\left[-(\lambda_k+\lambda_m)^2\cos(\lambda_k+\lambda_m)+2(\lambda_k+\lambda_m)\sin(\lambda_k+\lambda_m)+2\cos(\lambda_k+\lambda_m)-2\right] \\ \quad +\dfrac{L^3}{2(\lambda_k-\lambda_m)^3}\left[-(\lambda_k-\lambda_m)^2\cos(\lambda_k-\lambda_m)+2(\lambda_k-\lambda_m)\sin(\lambda_k-\lambda_m)+2\cos(\lambda_k-\lambda_m)-2\right], & m \neq k \\ \dfrac{L^3}{2(2\lambda_k)^3}\left[-(2\lambda_k)^2\cos 2\lambda_k+2(2\lambda_k)\sin 2\lambda_k+2\cos 2\lambda_k-2\right], & m = k \end{cases}$$

$$I_{19}A = \frac{L^3}{(\lambda^2_m+\lambda^2_k)^3} \{[(\lambda^2_m+\lambda^2_k)^2 - 2(\lambda^2_k-\lambda^2_m) - 4\lambda^2_k]\lambda_m \sin\lambda_k \cosh\lambda_m$$

$$- [(\lambda^2_m+\lambda^2_k)^2 + 2(\lambda^2_m-\lambda^2_k) + 4\lambda^2_m]\lambda_k \cos\lambda_k \sinh\lambda_m$$
$$+ 2(\lambda^4_k-\lambda^4_m)\sin\lambda_k \sinh\lambda_m$$
$$+ 4\lambda_m\lambda_k(\lambda^2_m+\lambda^2_k)\cos\lambda_k \cosh\lambda_m \}$$

$$I_{20}A = \frac{L}{\lambda^2_m+\lambda^2_k} [\lambda_m L^2 \sin\lambda_k \sinh\lambda_m - \lambda_k L^2 \cos\lambda_k \cosh\lambda_m - 2\lambda_m I_{19c} + 2\lambda_k I_{24c}]$$

$$I_{21}A = \frac{L^3}{2(\lambda_k+\lambda_m)^3} [-(\lambda_k+\lambda_m)^2 \cos(\lambda_k+\lambda_m) + 2(\lambda_k+\lambda_m)\sin(\lambda_k+\lambda_m) + 2\cos(\lambda_k+\lambda_m) - 2]$$

$$+ \frac{L^3}{2(\lambda_m-\lambda_k)^3} [-(\lambda_m-\lambda_k)^2 \cos(\lambda_m-\lambda_k) + 2(\lambda_m-\lambda_k)\sin(\lambda_m-\lambda_k) + 2\cos(\lambda_m-\lambda_k) - 2]$$

$$I_{22}A = \frac{L^3}{2(\lambda_k+\lambda_k)^3} \{[(\lambda_k+\lambda_m)^2 - 2]\sin(\lambda_k+\lambda_m) + 2(\lambda_k+\lambda_m)\cos(\lambda_k+\lambda_m)\}$$

$$+ \frac{L^3}{2(\lambda_k-\lambda_m)^3} \{[(\lambda_k-\lambda_m)^2-2]\sin(\lambda_k-\lambda_m) + 2(\lambda_k-\lambda_m)\cos(\lambda_k-\lambda_m)\}$$

$$I_{23}A = \frac{L^3}{(\lambda^2_m+\lambda^2_k)} [\lambda_m \cos\lambda_k \cosh\lambda_m + \lambda_k \sin\lambda_k \sinh\lambda_m]$$

$$- \frac{2L}{(\lambda^2_m+\lambda^2_k)} [\lambda_m I_{24c} + \lambda_k I_{19c}]$$

$$I_{24}A = \frac{L^3}{\lambda^2_m+\lambda^2_k} [\lambda_m \cos\lambda_k \sinh\lambda_m + \lambda_k \sin\lambda_k \cosh\lambda_m]$$

$$\frac{-2L}{\lambda^2_m+\lambda^2_k} [\lambda_m I_{23c} + \lambda_k I_{20c}]$$

$$I_{25A} = -\frac{L^3}{\lambda^2_k+\lambda^2_m} [\lambda_k \sin\lambda_m \cosh\lambda_k - \lambda_m \cos\lambda_m \sinh\lambda_k]$$

$$\frac{-2L}{\lambda^2_k+\lambda^2_m} [\lambda_k I_{29c} - \lambda_m I_{26c}]$$

$$I_{26A} = \frac{L^3}{\lambda^2_m+\lambda^2_k} [\lambda_k \cos\lambda_m \cosh\lambda_k + \lambda_m \sin\lambda_m \sinh\lambda_k]$$

$$- \frac{2L}{\lambda^2_m+\lambda^2_k} [\lambda_k I_{30c} + \lambda_m I_{25c}]$$

$$
I_{27A} = \begin{cases}
\dfrac{L^3}{2}\left[\dfrac{\sinh(\lambda_k+\lambda_m)}{\lambda_k+\lambda_m} - \dfrac{\sinh(\lambda_k-\lambda_m)}{\lambda_k-\lambda_m}\right] \\
\dfrac{-L^3}{(\lambda_k+\lambda_m)^3}\left[(\lambda_k+\lambda_m)\cosh(\lambda_k+\lambda_m) - \sinh(\lambda_k+\lambda_m)\right] \\
+\dfrac{L^3}{(\lambda_k-\lambda_m)^3}\left[(\lambda_k-\lambda_m)\cosh(\lambda_k-\lambda_m) - \sinh(\lambda_k-\lambda_m)\right] & \lambda_k \neq \lambda_m \\
\dfrac{L^3}{4\lambda_m}[\sinh 2\lambda_m - 2\lambda_m] - \dfrac{L^3}{8\lambda^3_m}\left[2\lambda_m\cosh 2\lambda_m - \sinh 2\lambda_m + \dfrac{L^3}{3}\right], & \lambda_k = \lambda_m
\end{cases}
$$

$$
I_{28A} = \begin{cases}
\dfrac{L^3\left[(\lambda_k+\lambda_m)^2\cosh(\lambda_k+\lambda_m) - 2(\lambda_k+\lambda_m)\sinh(\lambda_k+\lambda_m) + 2(\cosh(\lambda_k+\lambda_m)-1)\right]}{2(\lambda_k+\lambda_m)^3} \\
+\dfrac{L^3\left[(\lambda_k-\lambda_m)^2\cosh(\lambda_k-\lambda_m) - 2(\lambda_k-\lambda_m)\sinh(\lambda_k-\lambda_m) + 2(\cosh(\lambda_k-\lambda_m)-1)\right]}{2(\lambda_k-\lambda_m)^3}, & \lambda_k \neq \lambda_m \\
\dfrac{L^3}{8\lambda^3_m}\left[2\lambda^2_m\cosh(2\lambda_m) - 2\lambda_m\sinh(2\lambda_m) + \cosh(2\lambda_m) - 1 - 8\lambda^3_m\right] & \lambda_k = \lambda_m
\end{cases}
$$

$$
I_{29}A = \frac{L}{\lambda^2_m+\lambda^2_k}\left[\lambda_k L^2\sin\lambda_m\sinh\lambda_k - \lambda_m L^2\cos\lambda_m\cosh\lambda_k - 2\lambda_k I_{25c} + 2\lambda_m I_{30c}\right]
$$

$$
I_{30A} = \frac{L^3}{\lambda^2_m+\lambda^2_k}\left[\lambda_k\cos\lambda_m\sinh\lambda_k + \lambda_m\sin\lambda_m\cosh\lambda_k\right] - \frac{2L\left[\lambda_k I_{26c} + \lambda_m I_{29c}\right]}{\lambda^2_m+\lambda^2_k}
$$

$$
I_{31A} = \begin{cases}
\dfrac{L^3\left[(\lambda_k+\lambda_m)^2\cosh(\lambda_k+\lambda_m) - 2(\lambda_k+\lambda_m)\sinh(\lambda_k+\lambda_m) + 2(\cosh(\lambda_k+\lambda_m)-1)\right]}{2(\lambda_k+\lambda_m)^3} \\
+\dfrac{L^3\left[(\lambda_m-\lambda_k)^2\cosh(\lambda_m-\lambda_k) - 2(\lambda_m-\lambda_k)\sinh(\lambda_m-\lambda_k) + 2(\cosh(\lambda_m-\lambda_k)-1)\right]}{2(\lambda_m-\lambda_k)^3}, & \lambda_k \neq \lambda_m \\
\dfrac{L^3}{8\lambda_k^3}\left[2\lambda^2_k\cosh(2\lambda_k) - 2\lambda_k\sinh(2\lambda_k) + \cosh(2\lambda_k) - 1 - 8\lambda_k^3\right] & \lambda_m = \lambda_k
\end{cases}
$$

$$I_{32A} = \begin{cases} \dfrac{L^3}{2(\lambda_k+\lambda_m)^3}\left[((\lambda_k+\lambda_m)^2+2)\sinh(\lambda_k+\lambda_m)-2(\lambda_k+\lambda_m)\cosh(\lambda_m+\lambda_k)\right] \\ + \dfrac{L^3}{2(\lambda_k-\lambda_m)^3}\left[((\lambda_k-\lambda_m)^2+2)\sinh(\lambda_k-\lambda_m)-2(\lambda_k-\lambda_m)\cosh(\lambda_k-\lambda_m)\right] & , \lambda_k \neq \lambda_m \\ \dfrac{L^3}{8\lambda_m^3}\left[(2\lambda_m^2+1)\sinh(2\lambda_m)-2\lambda_m\cosh(2\lambda_m)\right] + \dfrac{L^3}{6} & , \lambda_k = \lambda_m \end{cases}$$

$$I_{17B} = \begin{cases} \dfrac{L^4}{2(\lambda_k-\lambda_m)^4}\{(\lambda_k-\lambda_m)[(\lambda_k-\lambda_m)^2-6]\sin(\lambda_k-\lambda_m)+3[[(\lambda_k-\lambda_m)^2-2]\cos(\lambda_k-\lambda_m)+2]\} \\ \dfrac{-L^4}{2(\lambda_k+\lambda_m)^4}\{(\lambda_k+\lambda_m)[(\lambda_k+\lambda_m)^2-6]\sin(\lambda_k+\lambda_m)+3[[(\lambda_k+\lambda_m)^2-2]\cos(\lambda_k+\lambda_m)+2]\} & , \lambda_k \neq \lambda_m \\ \dfrac{L^4}{8(\lambda_k+\lambda_m)^4}[(\lambda_k+\lambda_m)^4-(4(\lambda_k+\lambda_m)^3-24(\lambda_k+\lambda_m))\sin(\lambda_k+\lambda_m)-(12(\lambda_k+\lambda_m)^2 \\ \qquad -24)\cos(\lambda_k+\lambda_m)-24] & , \lambda_k = \lambda_m \end{cases}$$

$$I_{18B} = \begin{cases} \dfrac{L^4}{2(\lambda_k+\lambda_m)^4}\{[6-(\lambda_k+\lambda_m)^2](\lambda_k+\lambda_m)\cos(\lambda_k+\lambda_m)+3[(\lambda_k+\lambda_m)^2-2]\sin(\lambda_k+\lambda_m)\} \\ + \dfrac{L^4}{2(\lambda_k-\lambda_m)^4}\{[6-(\lambda_k-\lambda_m)^2](\lambda_k-\lambda_m)\cos(\lambda_k-\lambda_m)+3[(\lambda_k-\lambda_m)^2-2]\sin(\lambda_k-\lambda_m)\} & , \lambda_k \neq \lambda_m \\ \dfrac{L^4}{2(\lambda_k+\lambda_m)}[6(\lambda_k+\lambda_m)\cos(\lambda_k+\lambda_m)+3(\lambda_k+\lambda_m)^2\sin(\lambda_k+\lambda_m)-(\lambda_k+\lambda_m)^3\cos(\lambda_k+\lambda_m) \\ \qquad -6\sin(\lambda_k+\lambda_m)] & , \lambda_k = \lambda_m \end{cases}$$

$$I_{19B} = \frac{L^4}{\lambda_m^2+\lambda_k^2}[\lambda_m \sin\lambda_k \cosh\lambda_m - \lambda_k \cos\lambda_k \sinh\lambda_m] - \frac{3L}{\lambda_m^2+\lambda_k^2}[\lambda_m I_{20A} - \lambda_k I_{23A}]$$

$$I_{20B} = \frac{L^4}{\lambda_m^2+\lambda_k^2}[\lambda_m \sin\lambda_k \sinh\lambda_m - \lambda_k \cos\lambda_k \cosh\lambda_m] - \frac{3L}{\lambda_m^2+\lambda_k^2}[\lambda_m I_{19A} - \lambda_k I_{24A}]$$

$$I_{21B} = \frac{L^4}{2(\lambda_k+\lambda_m)^4}\{[6-(\lambda_k+\lambda_m)^2](\lambda_k+\lambda_m)\cos(\lambda_k+\lambda_m)+3[(\lambda_k+\lambda_m)^2-2]\sin(\lambda_k+\lambda_m)\}$$

$$+ \frac{L^4}{2(\lambda_m-\lambda_k)^4}\{[6-(\lambda_m-\lambda_k)^2](\lambda_m-\lambda_k)\cos(\lambda_m-\lambda_k)+3[(\lambda_m-\lambda_k)^2-2]\sin(\lambda_m-\lambda_k)\}$$

$$I_{22B} = \begin{cases} \frac{L^4}{2}\left[\frac{\sin(\lambda_k+\lambda_m)}{\lambda_k+\lambda_m} + \frac{\sin(\lambda_k-\lambda_m)}{\lambda_k-\lambda_m}\right] \\ -\frac{3}{2}\left\{\frac{L^4}{(\lambda_m+\lambda_k)^4}\left[\left[2-(\lambda_k+\lambda_m)^2\right]\cos(\lambda_k+\lambda_m) + 2(\lambda_k+\lambda_m)\sin(\lambda_k+\lambda_m) - 2\right]\right. \\ \left. + \frac{L^4}{(\lambda_k-\lambda_m)^4}\left[\left[2-(\lambda_k-\lambda_m)^2\right]\cos(\lambda_k-\lambda_m) + 2(\lambda_k-\lambda_m)\sin(\lambda_k-\lambda_m) - 2\right]\right\} & \lambda_k \neq \lambda_m \\ \frac{L^4}{2}\left[1 + \frac{\sin(2\lambda_m)}{2\lambda_m}\right] - \frac{3L^4}{16\lambda_m^4}\left[2\lambda_m^4 - 2\lambda_m^2\cos 2\lambda_m + 2\lambda_m^2 + 2\lambda_m\sin 2\lambda_m \right. \\ \left. + \cos 2\lambda_m - 1\right], & \lambda_k = \lambda_m \end{cases}$$

$$I_{23B} = \frac{L^4}{\lambda_m^2+\lambda_k^2}\left[\lambda_m \cos\lambda_k \cosh\lambda_m + \lambda_k \sin\lambda_k \sinh\lambda_m\right] - \frac{3L}{\lambda_m^2+\lambda_k^2}\left[\lambda_m I_{24A} + \lambda_k I_{19A}\right]$$

$$I_{24}B = \frac{L^4}{\lambda_m^2+\lambda_k^2}\left[\lambda_m \cos\lambda_k \sinh\lambda_m + \lambda_k \sin\lambda_k \cosh\lambda_m\right] - \frac{3L}{\lambda_m^2+\lambda_k^2}\left[\lambda_m I_{23A} + \lambda_k I_{20A}\right]$$

$$I_{25}B = \frac{L^4}{\lambda_m^2+\lambda_k^2}\left[\lambda_k \sin\lambda_m \cosh\lambda_k - \lambda_m \cos\lambda_m \sinh\lambda_k\right] - \frac{3L}{\lambda_m^2+\lambda_k^2}\left[\lambda_k I_{29A} - \lambda_m I_{26A}\right]$$

$$I_{26B} = \frac{L^4}{\lambda_k^2+\lambda_m^2}\left[\lambda_k \cos\lambda_m \cosh\lambda_k + \lambda_m \sin\lambda_m \sinh\lambda_k\right] - \frac{3L}{\lambda_m^2+\lambda_k^2}\left[\lambda_k I_{30A} + \lambda_m I_{25A}\right]$$

$$I_{27B} = \begin{cases} \frac{L^4}{2(\lambda_k+\lambda_m)^4}\left[(\lambda_k+\lambda_m)^3\sinh(\lambda_k+\lambda_m) - 3(\lambda_m+\lambda_k)^2\cosh(\lambda_m+\lambda_k)\right. \\ \left. + 6(\lambda_k+\lambda_m)\sinh(\lambda_k+\lambda_m) - 6\cosh(\lambda_m+\lambda_k) + 6\right] \\ - \frac{L^4}{2(\lambda_k-\lambda_m)}\left[(\lambda_k-\lambda_m)^3\sinh(\lambda_k-\lambda_m) - 3(\lambda_k-\lambda_m)^2\cosh(\lambda_k-\lambda_m)\right. \\ \left. + 6(\lambda_k-\lambda_m)\sinh(\lambda_k-\lambda_m) - 6\cosh(\lambda_k-\lambda_m) + 6\right] & , \lambda_k \neq \lambda_m \\ \frac{L^4\left[4\lambda_m^3\sinh 2\lambda_m - 6\lambda_m^2\cosh 2\lambda_m + 6\lambda_m\sinh 2\lambda_m - 3\cosh 2\lambda_m + 3 - 2\lambda_m^4\right]}{16\lambda_m^4} & , \lambda_k = \lambda_m \end{cases}$$

$$
I_{28B} = \begin{cases}
\dfrac{L^4}{2(\lambda_k+\lambda_m)^4}\big[(\lambda_k+\lambda_m)^3\cosh(\lambda_k+\lambda_m) - 3(\lambda_k+\lambda_m)^2\sinh(\lambda_m+\lambda_k) \\
\qquad + 6(\lambda_k+\lambda_m)\cosh(\lambda_k+\lambda_m) - 6\sinh(\lambda_k+\lambda_m)\big] \\
+ \dfrac{L^4}{2(\lambda_k-\lambda_m)^4}\big[(\lambda_k-\lambda_m)^3\cosh(\lambda_k-\lambda_m) - 3(\lambda_k-\lambda_m)^2\sinh(\lambda_k-\lambda_m) \\
\qquad + 6(\lambda_k-\lambda_m)\cosh(\lambda_k-\lambda_m) - 6\sinh(\lambda_k-\lambda_m)\big] & , \lambda_k \neq \lambda_m \\
\dfrac{L^4}{16\lambda_m^4}\big[4\lambda_m^3\cosh 2\lambda_m - 6\lambda_m^2\sinh 2\lambda_m + 6\lambda_m\cosh 2\lambda_m - 3\sinh 2\lambda_m\big] & , \lambda_k = \lambda_m
\end{cases}
$$

$$
I_{29B} = \frac{L^4}{\lambda_m^2+\lambda_k^2}\left[\lambda_k \sin\lambda_m \sinh\lambda_k - \lambda_m \cos\lambda_m \cosh\lambda_k\right] - \frac{3L}{\lambda_m^2+\lambda_k^2}\left[\lambda_k I_{25A} - \lambda_m I_{30A}\right]
$$

$$
I_{30B} = \frac{L^4}{\lambda_m^2+\lambda_k^2}\left[\lambda_k \cos\lambda_m \sinh\lambda_k + \lambda_m \sin\lambda_m \cosh\lambda_k\right] - \frac{3L}{\lambda_m^2+\lambda_k^2}\left[\lambda_k I_{26A} + \lambda_m I_{29A}\right]
$$

$$
I_{31B} = \begin{cases}
\dfrac{L^4}{2(\lambda_k+\lambda_m)^4}\big[(\lambda_k+\lambda_m)^3\cosh(\lambda_k+\lambda_m) - 3(\lambda_k+\lambda_m)^2\sinh(\lambda_k+\lambda_m) \\
\qquad + 6(\lambda_k+\lambda_m)\cosh(\lambda_k+\lambda_m) - 6\sinh(\lambda_k+\lambda_m)\big] \\
+ \dfrac{L^4}{2(\lambda_m-\lambda_k)^4}\big[(\lambda_m-\lambda_k)^3\cosh(\lambda_m-\lambda_k) - 3(\lambda_m-\lambda_k)^2\sinh(\lambda_m-\lambda_k) \\
\qquad + 6(\lambda_m-\lambda_k)\cosh(\lambda_m-\lambda_k) - 6\sinh(\lambda_m-\lambda_k)\big] & , \lambda_k \neq \lambda_m \\
\dfrac{L^4}{16\lambda_k^4}\big[4\lambda_k^3\cosh 2\lambda_k - 6\lambda_k^2\sinh 2\lambda_k + 6\lambda_k\cosh 2\lambda_k - 3\sinh 2\lambda_k\big] & , \lambda_k = \lambda_m
\end{cases}
$$

$$
I_{32B} = \begin{cases}
\dfrac{L^4}{2(\lambda_k+\lambda_m)^4}\big[(\lambda_k+\lambda_m)^3\sinh(\lambda_k+\lambda_m) - 3(\lambda_k+\lambda_m)^2\cosh(\lambda_k+\lambda_m) \\
\qquad + 6(\lambda_k+\lambda_m)\sinh(\lambda_k+\lambda_m) - 6\cosh(\lambda_k+\lambda_m) + 6\big] \\
+ \dfrac{L^4}{2(\lambda_k-\lambda_m)^4}\big[(\lambda_k-\lambda_m)^3\sinh(\lambda_k-\lambda_m) - 3(\lambda_k-\lambda_m)^2\cosh(\lambda_k-\lambda_m) \\
\qquad + 6(\lambda_k-\lambda_m)\sinh(\lambda_k-\lambda_m) - 6\cosh(\lambda_k-\lambda_m) + 6\big] & , \lambda_k \neq \lambda_m \\
\dfrac{L^4}{16\lambda_m^4}\big[4\lambda_m^3\sinh 2\lambda_m - 6\lambda_m^2\cosh 2\lambda_m + 6\lambda_m\sinh 2\lambda_m - 3\cosh 2\lambda_m + 3 + 2\lambda_m^4\big] & , \lambda_k = \lambda_m
\end{cases}
$$

$$I_{17c} = \begin{cases} \dfrac{L^2}{2(\lambda_k-\lambda_m)^2}\left\{(\lambda_k-\lambda_m)\sin(\lambda_k-\lambda_m) + [\cos(\lambda_k-\lambda_m)-1]\right\} \\ -\dfrac{L^2}{2(\lambda_k+\lambda_m)^2}\left\{(\lambda_k+\lambda_m)\sin(\lambda_k+\lambda_m) + [\cos(\lambda_k+\lambda_m)-1]\right\} & m \neq k \\ \dfrac{L^2}{4} - \dfrac{L^2}{2(\lambda_k+\lambda_m)^2}\left\{(\lambda_k+\lambda_m)\sin(\lambda_k+\lambda_m)+\cos(\lambda_k+\lambda_m)-1\right\} & m = k \end{cases}$$

$$I_{18c} = \begin{cases} \dfrac{-L^2}{2}\left\{\dfrac{(\lambda_k-\lambda_m)\cos(\lambda_k-\lambda_m) - \sin(\lambda_k-\lambda_m)}{(\lambda_k-\lambda_m)^2} + \dfrac{(\lambda_k+\lambda_m)\cos(\lambda_k+\lambda_m) - \sin(\lambda_k+\lambda_m)}{(\lambda_k+\lambda_m)^2}\right\} & m \neq k \\ -\dfrac{L^2}{2}\left[\dfrac{2\lambda_k\cos 2\lambda_k - \sin 2\lambda_k}{4\lambda_k^2}\right] & m = k \end{cases}$$

$$I_{19C} = \frac{L^2}{(\lambda^2_k+\lambda^2_m)^2}\Big[(\lambda^2_k+\lambda^2_m)\lambda_m \sin\lambda_k \cosh\lambda_m - (\lambda^2_k+\lambda^2_m)\lambda_k \cos\lambda_k \sinh\lambda_m + (\lambda^2_k-\lambda^2_m)\sin\lambda_k \sinh\lambda_m + 2\lambda_m\lambda_k \cos\lambda_k \cosh\lambda_m - 2\lambda_m\lambda_k\Big]$$

$$I_{20C} = \frac{L^2}{(\lambda^2_m+\lambda^2_k)^2}\Big\{(\lambda^2_m+\lambda^2_k)\lambda_m \sin\lambda_k \sinh\lambda_m - (\lambda^2_k+\lambda^2_m)\lambda_k \cos\lambda_k \cosh\lambda_m + 2\lambda_k\lambda_m \cos\lambda_k \sinh\lambda_m + (\lambda^2_k-\lambda^2_m)\sin\lambda_k \cosh\lambda_m\Big\}$$

$$I_{21C} = -\frac{L^2}{2}\left\{\frac{(\lambda_m-\lambda_k)\cos(\lambda_m-\lambda_k) - \sin(\lambda_m-\lambda_k)}{(\lambda_m-\lambda_k)^2} + \frac{(\lambda_m+\lambda_k)\cos(\lambda_m+\lambda_k) - \sin(\lambda_m-\lambda_k)}{(\lambda_m+\lambda_k)^2}\right\}$$

$$I_{22}C = \begin{cases} \dfrac{L^2}{2(\lambda_k+\lambda_m)^2}\left\{(\lambda_k+\lambda_m)\sin(\lambda_k+\lambda_m) + \cos(\lambda_k+\lambda_m)-1\right\} \\ + \dfrac{L^2}{2(\lambda_k-\lambda_m)^2}\left\{(\lambda_k-\lambda_m)\sin(\lambda_k-\lambda_m) + \cos(\lambda_k-\lambda_m)-1\right\} & m \neq k \\ \dfrac{L^2}{4} + \dfrac{L^2}{2(2\lambda_k)^2}\left\{2\lambda_k\sin 2\lambda_k + \cos(2\lambda_k)-1\right\} & m = k \end{cases}$$

$$I_{23C} = \frac{L^2}{(\lambda^2_m+\lambda^2_k)^2}\Big[\lambda_k(\lambda^2_m+\lambda^2_k)\sin\lambda_k \sinh\lambda_m + \lambda_m(\lambda^2_m+\lambda^2_k)\cos\lambda_k \cosh\lambda_m - 2\lambda_k\lambda_m \sin\lambda_k \cosh\lambda_m - (\lambda^2_m-\lambda^2_k)\cos\lambda_k \sinh\lambda_m\Big]$$

$$I_{24C} = \frac{L^2}{(\lambda^2_m+\lambda^2_k)^2}[\lambda_k(\lambda^2_m+\lambda^2_k)\sin\lambda_k\cosh\lambda_m + \lambda_m(\lambda^2_m+\lambda^2_k)\cos\lambda_k\sinh\lambda_m - 2\lambda_k\lambda_m\sin\lambda_k\sinh\lambda_m - (\lambda^2_m-\lambda^2_k)\cos\lambda_k\cosh\lambda_m - (\lambda^2_k-\lambda^2_m)]$$

$$I_{25C} = \frac{L^2}{(\lambda^2_m+\lambda^2_k)^2}[(\lambda^2_k+\lambda^2_m)\lambda_k\sin\lambda_m\cosh\lambda_k - (\lambda^2_k+\lambda^2_m)\lambda_m\cos\lambda_m\sinh\lambda_k + (\lambda^2_m-\lambda^2_k)\sin\lambda_m\sinh\lambda_k + 2\lambda_m\lambda_k\cos\lambda_m\cosh\lambda_k - 2\lambda_m\lambda_k]$$

$$I_{26C} = \frac{L^2}{(\lambda^2_m+\lambda^2_k)^2}[\lambda_m(\lambda^2_m+\lambda^2_k)\sin\lambda_m\sinh\lambda_k + \lambda_k(\lambda^2_m+\lambda^2_k)\cos\lambda_m\cosh\lambda_k - 2\lambda_k\lambda_m\sin\lambda_m\cosh\lambda_k - (\lambda^2_k-\lambda^2_m)\cos\lambda_m\sinh\lambda_k]$$

$$I_{27C} = \begin{cases} \frac{L^2}{2(\lambda_k+\lambda_m)^2}[(\lambda_k+\lambda_m)\sinh(\lambda_k+\lambda_m) - \cosh(\lambda_k+\lambda_m) + 1] \\ \quad - \frac{L^2}{2(\lambda_k-\lambda_m)^2}[(\lambda_k-\lambda_m)\sinh(\lambda_k-\lambda_m) - \cosh(\lambda_k-\lambda_m) + 1], & \lambda_k \neq \lambda_m \\ \frac{L^2}{8\lambda^2_m}[2\lambda_m\sinh 2\lambda_m - \cosh 2\lambda_m - 2\lambda^2_m + 1], & \lambda_k = \lambda_m \end{cases}$$

$$I_{28C} = \begin{cases} \frac{L^2}{2(\lambda^2_k-\lambda^2_m)}[(\lambda_k-\lambda_m)\cosh(\lambda_k+\lambda_m) + (\lambda_k+\lambda_m)\cosh(\lambda_k-\lambda_m)] \\ \frac{-L^2}{2(\lambda^2_k-\lambda^2_m)}[(\lambda_k-\lambda_m)^2\sinh(\lambda_k+\lambda_m) + (\lambda_k+\lambda_m)^2\sinh(\lambda_k-\lambda_m)], & \lambda_k \neq \lambda_m \\ \frac{L^2}{8\lambda^2_m}[2\lambda_m\cosh 2\lambda_m - \sinh 2\lambda_m], & \lambda_k = \lambda_m \end{cases}$$

$$I_{29C} = \frac{L^2}{(\lambda^2_m+\lambda^2_k)^2}[(\lambda^2_m+\lambda^2_k)\lambda_k\sin\lambda_m\sinh\lambda_k - (\lambda^2_m+\lambda^2_k)\lambda_m\cos\lambda_m\cosh\lambda_k + 2\lambda_k\lambda_m\cos\lambda_m\sinh\lambda_k + (\lambda^2_m-\lambda^2_k)\sin\lambda_m\cosh\lambda_k]$$

$$I_{30C} = \frac{L^2}{(\lambda^2_m+\lambda^2_k)^2}[\lambda_m(\lambda^2_m+\lambda^2_k)\sin\lambda_m\cosh\lambda_k + \lambda_k(\lambda^2_m+\lambda^2_k)\cos\lambda_m\sinh\lambda_k - 2\lambda_k\lambda_m\sin\lambda_m\sinh\lambda_k - (\lambda^2_k-\lambda^2_m)\cos\lambda_m\cosh\lambda_k - (\lambda^2_m-\lambda^2_k)]$$

$$
I_{31}c = \begin{cases} \dfrac{L^2}{2(\lambda^2_m-\lambda^2_k)} \left[(\lambda_m-\lambda_k)\cosh(\lambda_m+\lambda_k) + (\lambda_m+\lambda_k)\cosh(\lambda_m-\lambda_k) \right] \\ \dfrac{-L^2}{2(\lambda^2_m-\lambda^2_k)^2} \left[(\lambda_m-\lambda_k)^2 \sinh(\lambda_m+\lambda_k) + (\lambda_m+\lambda_k)^2 \sinh(\lambda_m-\lambda_k) \right], & \lambda_m \neq \lambda_k \\ \dfrac{L^2}{8\lambda^2_k} \left[2\lambda_k \cosh 2\lambda_k - \sinh 2\lambda_k \right], & \lambda_m = \lambda_k \end{cases}
$$

$$
I_{32}c = \begin{cases} \dfrac{L^2}{2(\lambda_k+\lambda_m)^2} \left[(\lambda_k+\lambda_m)\sinh(\lambda_m+\lambda_k) - \cosh(\lambda_m+\lambda_k) + 1 \right] \\ + \dfrac{L^2}{2(\lambda_k-\lambda_m)^2} \left[(\lambda_k-\lambda_m)\sinh(\lambda_k-\lambda_m) - \cosh(\lambda_k-\lambda_m) + 1 \right], & \lambda_k \neq \lambda_m \\ \dfrac{L^2}{8\lambda^2_m} \left[2\lambda_m \sinh 2\lambda_m + 2\lambda^2_m - \cosh 2\lambda_m + 1 \right], & \lambda_k = \lambda_m \end{cases}
$$

$$I_{33} = \frac{L}{4}\left\{ \left[\frac{\cos(n\pi + \lambda_k + \lambda_m) - 1}{n\pi + \lambda_k + \lambda_m} \right] - \left[\frac{\cos(n\pi + \lambda_k - \lambda_m) - 1}{n\pi + \lambda_k - \lambda_m} \right] + \left[\frac{\cos(n\pi - \lambda_k - \lambda_m) - 1}{n\pi - \lambda_k - \lambda_m} \right] - \left[\frac{\cos(n\pi - \lambda_k + \lambda_m) - 1}{n\pi - \lambda_k + \lambda_m} \right] \right\}$$

$$I_{34} = \frac{L}{4}\left\{ \frac{\sin(n\pi - \lambda_k - \lambda_m)}{n\pi - \lambda_k - \lambda_m} - \frac{\sin(n\pi + \lambda_k + \lambda_m)}{n\pi + \lambda_k + \lambda_m} + \frac{\sin(n\pi - \lambda_k + \lambda_m)}{n\pi - \lambda_k + \lambda_m} - \frac{\sin(n\pi + \lambda_k - \lambda_m)}{n\pi + \lambda_k - \lambda_m} \right\}$$

$$I_{35} = \tfrac{1}{2}\left\{ \frac{\lambda_m L}{\lambda^2_m + (n\pi - \lambda_k)^2}\left[\cos(n\pi - \lambda_k)\cosh\lambda_m + \frac{(n\pi - \lambda_k)}{\lambda_m}\sin(n\pi - \lambda_k)\sinh\lambda_m) - 1\right] - \frac{\lambda_m L}{\lambda^2_m + (n\pi + \lambda_k)^2}\left[\cos(n\pi + \lambda_k)\cosh\lambda_m + \frac{(n\pi + \lambda_k)}{\lambda_m}\sin(n\pi + \lambda_k)\sinh\lambda_m - 1\right] \right\}$$

$$I_{36} = \tfrac{1}{2}\left\{ \frac{\lambda_m L}{\lambda^2_m + (n\pi - \lambda_k)^2}\left[\cos(n\pi - \lambda_k)\sinh\lambda_m + \frac{(n\pi - \lambda_k)}{\lambda_m}\sin(n\pi - \lambda_k)\cosh\lambda_m\right] - \frac{\lambda_m L}{\lambda^2_m + (n\pi + \lambda_k)^2}\left[\cos(n\pi + \lambda_k)\sinh\lambda_m + \frac{(n\pi + \lambda_k)}{\lambda_m}\sin(n\pi + \lambda_k)\cosh\lambda_m\right] \right\}$$

$$I_{37} = \frac{L}{4}\left\{ \frac{\sin(n\pi - \lambda_m - \lambda_k)}{n\pi - \lambda_m - \lambda_k} - \frac{\sin(n\pi + \lambda_m + \lambda_k)}{n\pi + \lambda_m + \lambda_k} + \frac{\sin(n\pi - \lambda_m + \lambda_k)}{n\pi - \lambda_m + \lambda_k} - \frac{\sin(n\pi + \lambda_m - \lambda_k)}{n\pi + \lambda_m - \lambda_k} \right\}$$

$$I_{38} = -\frac{L}{4}\left\{ \left[\frac{\cos(n\pi + \lambda_k + \lambda_m) - 1}{n\pi + \lambda_k + \lambda_m} \right] + \left[\frac{\cos(n\pi - \lambda_k - \lambda_m) - 1}{n\pi - \lambda_k - \lambda_m} \right] + \left[\frac{\cos(n\pi + \lambda_k - \lambda_m) - 1}{n\pi + \lambda_k - \lambda_m} \right] + \left[\frac{\cos(n\pi - \lambda_k + \lambda_m) - 1}{n\pi - \lambda_k + \lambda_m} \right] \right\}$$

$$I_{39} = \tfrac{1}{2}\left\{ \frac{\lambda_m L}{\lambda^2_m + (n\pi + \lambda_k)^2}\left[\sin(n\pi + \lambda_k)\cosh\lambda_m - \frac{(n\pi + \lambda_k)}{\lambda_m}\cos(n\pi + \lambda_k)\sinh\lambda_m)\right] \right.$$

$$+ \frac{\lambda_m L}{\lambda^2_m + (n\pi - \lambda_k)^2} \left[\sin(n\pi - \lambda_k) \cosh\lambda_m - \frac{(n\pi - \lambda_k)}{\lambda_m} \cos(n\pi - \lambda_k) \sinh\lambda_m \right] \Bigg\}$$

$$I_{40} = ½ \Bigg\{ \frac{\lambda_m L}{\lambda^2_m + (n\pi + \lambda_k)^2} \left[\sin(n\pi + \lambda_k) \sinh\lambda_m - \frac{(n\pi + \lambda_k)}{\lambda_m} (\cos(n\pi + \lambda_k) \cosh\lambda_m - 1) \right]$$

$$+ \frac{\lambda_m L}{\lambda^2_m + (n\pi - \lambda_k)^2} \left[\sin(n\pi - \lambda_k) \sinh\lambda_m - \frac{(n\pi - \lambda_k)}{\lambda_m} (\cos(n\pi - \lambda_k) \cosh\lambda_m - 1) \right] \Bigg\}$$

$$I_{41} = ½ \Bigg\{ \frac{\lambda_k L}{\lambda^2_k + (n\pi - \lambda_m)^2} \left[\cos(n\pi - \lambda_m) \cosh\lambda_k + \frac{(n\pi - \lambda_m)}{\lambda_k} \sin(n\pi - \lambda_m) \sinh\lambda_k - 1 \right]$$

$$- \frac{\lambda_k L}{\lambda^2_k + (n\pi + \lambda_m)^2} \left[\cos(n\pi + \lambda_m) \cosh\lambda_k + \frac{(n\pi + \lambda_m)}{\lambda_k} \sin(n\pi + \lambda_m) \sinh\lambda_k - 1 \right] \Bigg\}$$

$$I_{42} = ½ \Bigg\{ \frac{\lambda_k L}{\lambda^2_k + (n\pi + \lambda_m)^2} \left[\sin(n\pi + \lambda_m) \cosh\lambda_k - \frac{(n\pi + \lambda_m)}{\lambda_k} \cos(n\pi + \lambda_m) \sinh\lambda_k \right]$$

$$+ \frac{\lambda_k L}{\lambda^2_k + (n\pi - \lambda_m)^2} \left[\sin(n\pi - \lambda_m) \cosh\lambda_k - \frac{(n\pi - \lambda_m)}{\lambda_k} \cos(n\pi - \lambda_m) \sinh\lambda_k \right] \Bigg\}$$

$$I_{43} = ½ \Bigg\{ \frac{(\lambda_k + \lambda_m) L}{(\lambda_k + \lambda_m)^2 + (n\pi)^2} \left[\sin(n\pi) \sinh(\lambda_k + \lambda_m) - \frac{n\pi}{\lambda_k + \lambda_m} (\cos(n\pi) \cosh(\lambda_k + \lambda_m) - 1) \right]$$

$$- \frac{(\lambda_k - \lambda_m) L}{(\lambda_k - \lambda_m)^2 + (n\pi)^2} \left[\sin(n\pi) \sinh(\lambda_k - \lambda_m) - \frac{n\pi}{\lambda_k - \lambda_m} (\cos(n\pi) \cosh(\lambda_k - \lambda_m) - 1) \right] \Bigg\}$$

$$I_{44} = ½ \Bigg\{ \frac{(\lambda_k + \lambda_m) L}{(\lambda_k + \lambda_m)^2 + (n\pi)^2} \left[\sin(n\pi) \cosh(\lambda_k + \lambda_m) - \frac{n\pi}{\lambda_k + \lambda_m} \cos(n\pi) \sinh(\lambda_k + \lambda_m) \right]$$

$$+ \frac{(\lambda_k - \lambda_m) L}{(\lambda_k - \lambda_m)^2 + (n\pi)^2} \left[\sin(n\pi) \cosh(\lambda_k - \lambda_m) - \frac{n\pi}{\lambda_k - \lambda_m} \cos(n\pi) \sinh(\lambda_k - \lambda_m) \right] \Bigg\}$$

$$I_{45} = ½ \Bigg\{ \frac{\lambda_k L}{\lambda^2_k + (n\pi - \lambda_m)^2} \left[\cos(n\pi - \lambda_m) \sinh\lambda_k + \frac{(n\pi - \lambda_m)}{\lambda_k} \sin(n\pi - \lambda_m) \cosh\lambda_k \right]$$

$$- \frac{\lambda_k L}{\lambda_k^2 + (n\pi + \lambda_m)^2}\left[\cos(n\pi + \lambda_m)\sinh\lambda_k + \frac{(n\pi + \lambda_m)}{\lambda_k}\sin(n\pi + \lambda_m)\cosh\lambda_k\right]\Bigg\}$$

$$I_{46} = ½ \Bigg\{ \frac{\lambda_k L}{\lambda_k^2 + (n\pi + \lambda_m)^2}\left[\sin(n\pi + \lambda_m)\sinh\lambda_k - \frac{(n\pi + \lambda_m)}{\lambda_k}(\cos(n\pi + \lambda_m)\cosh\lambda_k - 1)\right]$$

$$+ \frac{\lambda_k L}{\lambda_k^2 + (n\pi - \lambda_m)^2}\left[\sin(n\pi - \lambda_m)\sinh\lambda_k - \frac{(n\pi - \lambda_m)}{\lambda_k}(\cos(n\pi - \lambda_m)\cosh\lambda_k - 1)\right]\Bigg\}$$

$$I_{47} = ½ \Bigg\{ \frac{(\lambda_m + \lambda_k)L}{(\lambda_m + \lambda_k)^2 + (n\pi)^2}\left[\sin(n\pi)\cosh(\lambda_m + \lambda_k) - \frac{n\pi}{\lambda_m + \lambda_k}\cos(n\pi)\sinh(\lambda_m + \lambda_k)\right]$$

$$+ \frac{(\lambda_m - \lambda_k)L}{(\lambda_m - \lambda_k)^2 + (n\pi)^2}\left[\sin(n\pi)\cosh(\lambda_m - \lambda_k) - \frac{n\pi}{\lambda_m - \lambda_k}\cos(n\pi)\sinh(\lambda_m - \lambda_k)\right]\Bigg\}$$

$$I_{48} = ½ \Bigg\{ \frac{(\lambda_k + \lambda_m)L}{(\lambda_k + \lambda_m)^2 + (n\pi)^2}\left[\sin(n\pi)\sinh(\lambda_k + \lambda_m) - \frac{n\pi}{\lambda_k + \lambda_m}(\cos(n\pi)\cosh(\lambda_k + \lambda_m) - 1)\right]$$

$$+ \frac{(\lambda_k - \lambda_m)L}{(\lambda_k - \lambda_m)^2 + (n\pi)^2}\left[\sin(n\pi)\sinh(\lambda_k - \lambda_m) - \frac{n\pi}{\lambda_k - \lambda_m}(\cos(n\pi)\cosh(\lambda_k - \lambda_m) - 1)\right]\Bigg\}$$