DYNAMICAL ANALYSIS OF FINITE PRESTRESSED BERNOULLI-EULER BEAMS WITH GENERAL BOUNDARY CONDITIONS UNDERTRAVRLLING DISTRIBUTED LOADS

BY

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CERTIFICATION

This is to certify that this project was carried out by OGUNYEBI SEGUN NATHANIEL, Department of Mathematical Sciences, Federal University of Technology, Akure, Nigeria.

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DEDICATION

I dedicate this work to the Almighty God- the Alpha and Omega, who made me to be what I am today.

ACKNOWLEDGEMENT

My invaluable gratitude goes to my Lord Jesus Christ whom by His immeasurable grace, I am what I am today. I appreciate His kindness and love towards me in all areas of my life. To him be the glory and adoration.

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Finally, I appreciate the effort and prayer of my darling mother Mrs. Juliana Ogunyebi and others who have contributed morally and academically to the successful completion of this work. May God bless you all (Amen).

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ABSTRACT

This thesis presents the problems of dynamical analysis of finite prestressed Bernoulli-Euler beams with general boundary conditions under traveling distributed masses. The responses of the elastic structures to moving distributed forces are special cases of such dynamical problems.

The governing equation of this problem is a fourth order partial differential equation. The solution technique is based on generalized integral transforms, the use of the properties of Heaviside function $H(x-ct)$ as the generalized derivative of the Dirac delta function $\delta(x-ct)$ in the distributed sense and a modification of the asymptotic method of Struble. By the use of this technique, one is able to obtain closed form solutions for all variants of classical end conditions for this class of problems. The closed form solutions are analyzed and numerical analyses in plotted curves are presented.

The results show that as the prestress value N and foundation modulli K increases, the response amplitude of the dynamical system decreases. However, higher values of N and K are required for a more noticeable effect in the case of other boundary conditions than those of simply supported boundary condition. It is also found that for all the illustrative examples, the moving force solution is not an upper bound for the accurate solution of the

moving masses problem of a uniform Bernoulli-Euler beam under the action of a uniform distributed load. This important result also agrees with similar problems that considered the moving load as a lump mass.

Finally, in all the illustrative examples considered, for the same natural frequency, the critical speed for the moving mass problem is smaller than that of the moving force problem. Hence resonance is reached earlier in moving mass problem.

CHAPTER ONE

1.0 INTRODUCTION

In the structural dynamics, the moving-load-induced vibration problems have been the important research topic for over 100 years. Therefore, a great amount of work dealing with the dynamic analysis of structures due to moving loads can be found from existing literature[1 - 7].

Until early this century, machine and structure usually had very high mass and damping because heavy beams, castings and timbers were used in their constructions. The dynamic response of these structures and machines was low since the vibration excitation sources were often small in magnitude. However, with the development of strong lightweight materials, increased knowledge of material properties and structural loading, improved analysis and design techniques, the mass of machines and structures built to fulfill a particular function has decreased. Furthermore, the efficiency and speed of machinery have increased so that the vibration excitation with reducing machine mass and damping has continued at an increasing rate to the present day when few, if any, machine can be designed without carrying out the necessary vibration analysis.

The problem of assessing the dynamic response of elastic structures such as beams and plates to moving load is of technological importance as these structures (road-bridges, space vehicles and submarines) are constantly acted upon by moving loads. Moving loads have great effects on dynamic stresses in elastic structures and cause them to vibrate intensively at high velocities.

In highways and railways bridges, the dynamic analysis becomes complicated because the mass of a moving vehicle or train is comparatively larger than that of the bridge and suspension of the vehicles. Many scholars in the area of civil, mechanical and astronautically engineering have worked and are still investigating the behaviour of structures under the action of moving loads.

The quest for analytical solutions to the fourth order partial differential equations which have variable and singular coefficients has also drawn the attention of numerous Applied Mathematician to this research topic. Specifically, the effects of moving loads on elastic structures are dual. On one hand is the gravitational effects of the moving load while on the other hand is the inertia effects of the mass of the load on the vibrating solid bodies. These solid bodies which could be elastic, inelastic or viscoelastic include beams, plate, shells e.t.c. Thus this thesis is concerned primarily with elastic solid bodies.

Dynamical problems involving moving loads can be generally grouped into the following three classes:

(a) In the first case, the mass of the moving load is considered much smaller than the mass of the structure it is traversing.

(b) The second class comprises of the system for which the mass of the structure is assumed to be much smaller than that of the moving load and thirdly we have

(c) the case in which both the mass of the structure and that of the moving load are of comparable magnitude.

The first case is much simpler than the second and the third. Infact, the first is the commonest problem treated in literature. In this problem, the inertia effects of the moving load are assumed negligible and only the force effects of the moving load are taking into consideration. Thus, this type of problem is termed the **"moving force"** problem. Though, the problem, on the assumption, had been greatly simplified, the following question arises: how safe is a design based on this assumption? The justification of this assumption would have established had the solution of this approximate model been proved to be an upper bound for the actual deflection of the elastic system.

The most difficult of all the three types of the problem is the third, while both the second and third problems involve not only the consideration of the force effects of the moving load but also its inertia effects, the moving load in the former does not have mass commensurable with the mass of the structure. The third type of problem may be termed **"moving mass"** problem. It is remarked at this juncture that in most of the existing literature in dynamics of structure under moving loads, moving loads have been idealized as moving concentrated loads which acts at a certain point on the structure and along a single line in space [8]. That is, the moving load is modeled as a lumped load.

However, in practice, it is known that loads are actually distributed over a small segment or over the entire length of the structural member as they traverse the structure. Such moving loads are termed uniform distributed loads. Concentrated forces are mere mathematical idealization, but cannot be found in the real world, where all forces are either body forces acting over an area.

In what follows, the definition of moving loads, concentrated moving loads and Uniform distributed moving loads with their graphical representations are given.

(i) **Moving loads:** These are forces acting on a structure and continually changing position. Common examples of these include train, moving car, truck, cranes etc.

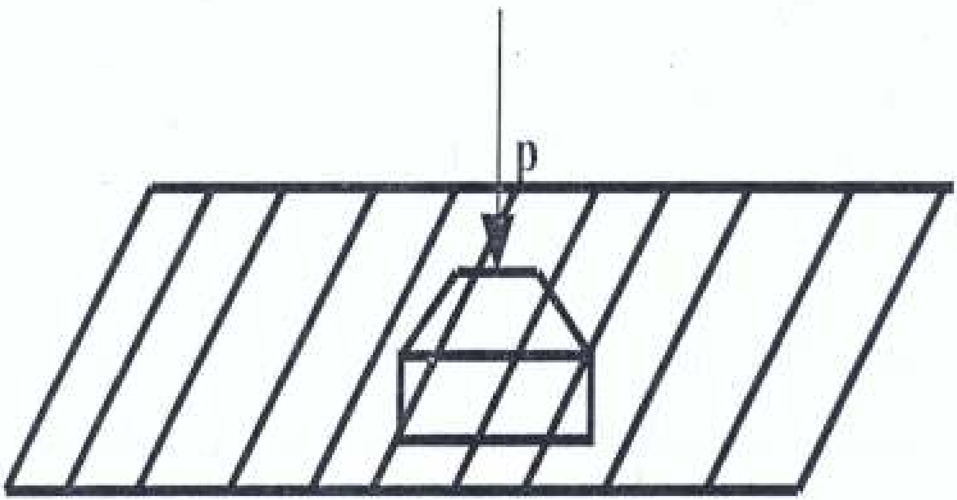

Fig. 1 Moving load

(ii) **Concentrated moving load:** A concentrated load p is a simple force applied at a certain point of the structure AB of length L. The graphical representation of this load is a straight line with an arrow, indicating the action and the sense. All concentrated loads are actually distributed loads over a small segment of the structure.

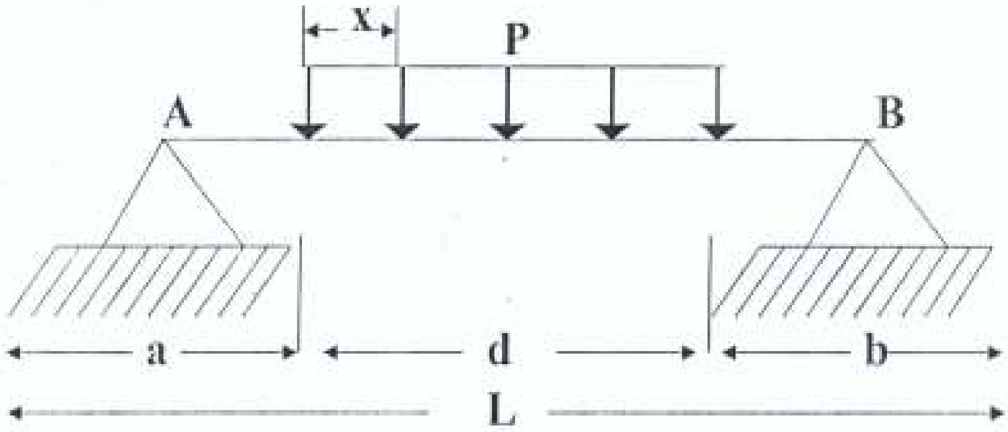

Fig. 3 Simple beam with uniform distributed load

The concern of researchers in the field of structural Dynamics is the search for reliable methods in accurately determining the response of an elastic structure under the action of moving loads. The problems often encountered arise from (i) the type of specified end–conditions; (ii) problems of expressing the boundary conditions [9] ; (iii) the complexity in the governing partial differential equation (for example, non-constant coefficients, singular coefficients, complex space-time dependencies owing to varying speeds of the load e.t.c.)

In general, the moving load problems are mathematically complex when the inertia effect of the moving load is taken into consideration. Thus most of the research works available in the literature are those in which this effect has been neglected. This is due, at least in part, to the great amount of computational labour, which is required both to set up and to solve the necessary equation. One important problem that arises when the inertia effect of the masses are considered is the singularity which occurs in the inertia terms of the governing equation of motion.

In this thesis, we shall be concerned with the problem of assessing the dynamical response to moving uniform distributed loads of finite prestressed Bernoulli-Euler beams with general boundary conditions. This work incorporates the inertia effect of the moving uniform-distributed loads and the effects of prestress and foundation moduli in the governing differential equation of the dynamical problem and sets at solving it for all variants of classical boundary conditions. The analyses of the effect of these parameters on the response of the beams when they are being traversed by moving uniform distributed loads shall be carried out.

In most of the previous investigations, the problem of assessing the dynamic behaviour of structural members carrying moving loads, work has been restricted to the case of when the loads are simplified as moving concentrated forces[10]. To the best of author's knowledge, this class of problems where the moving loads are modelled as uniform distributed loads are scarce. The problem of determining the dynamic response of elastic structure subjected to moving loads have been of theoretical and practical interest in the field of Applied mathematics, Physics and Engineering. The problem of moving load was first tackled for the case in which the structural was considered small against the mass of a single, constant load. Willis et al [11] considered the problem of elastic beam under the action of moving loads. He assumed the mass of the beam to be smaller than the mass of the load and obtained an approximate solution to the problem. This is followed by the other extreme case i.e that of the load mass small against the beam mass. In particular, the dynamic response of a simply supported beam, traverse by a constant force moving at a uniform speed was first studied by Krylov [12]. His results were obtained by using the method of expansion of eigenfunctions. He assumed that the mass of the load is smaller than that of the beam.

The problem of transverse oscillations of beams under the action of moving loads for the general case of any arbitrary prescribed law of motion was solved by Lowan [13] using Green's functions. Furthermore, Timoshenko [14] and Timoshenko et al [15] considered when a pulsating force is moving along a beam with a constant velocity. The principle of virtual work is used in solving the problem. The problem is an example of moving load problems involving time-variable concentrated force. The response of finite simply supported Euler-Bernoulli beam to a unit force moving at a uniform velocity was investigated by Steel [16]. The effects of this moving force on beams with or without an elastic foundation were analyzed. In the same trend was the work of Yoshida [17] who studied the vibration of a beam subjected to moving concentrated load using finite element method. In all the aforementioned investigations, problems have been largely restricted to the case when the inertia effects of the moving load have been neglected. The more complicated case in which consideration is given to the inertia effects of the moving load have been neglected. However, recently, the advent of long highway bridges, together with increased velocity and mass of automobiles, has forced a renewal of consideration of this problem.

In general, problem of this type are mathematically complex. A major breakthrough in this field is the work of Stanisic et al [18] who solved the problem of simply supported non-mindling plate under a multi-mass moving system by making use of an approximation of Dirac-delta function. Only the inertia term which measures the effect of local acceleration in the direction of the deflection was considered. The method of solution was based on Fourier sine transform technique suitable only for simply supported boundary conditions. A one dimensional analogue of this problem was taken up later by Milormir et al [19] who developed a theory describing the response of a beam under an arbitrary number of moving masses. The theory is based on the Fourier technique and shows that, for a simply supported beam, the resonance frequency is lower with no corresponding decrease in maximum amplitude when the inertia is considered. This work was later extended by Stanisic et al [20]to include all the components of the inertia terms. They considered only the Bernoulli-Euler beam with only simply supported end conditions. It is remarked at this juncture that in all these aforementioned investigations, method of solution have been suitable only for simply supported end conditions. To address this problem, an interesting attempt was made by Sadiku and Leipholz [21] who developed an elegant technique capable of solving Bernoulli-Euler moving load problems for all variants of classical boundary conditions. The technique involves transforming the differential equation governing the moving mass problem into integro-differential equation by using the Green's function of the associated moving force problem. Although this work is impressive, its application is limited only to the case of beams executing flexural vibrations according to the simple Bernoulli-Euler beam of flexure. The extension of the method to flexural vibrations of structure with height –span ratio larger than 1/10 has not been effected. Nonetheless, it is known that during vibration, a typical element of a beam not only performs translatory motion but also rotates, Timoshenko et al [22]. Thus, there is the need to consider the beam whose motion is not governed by Bernoulli-Euler beam theory (thin beam). More recently, Oni [7] and Gbadeyan and Oni [23] presented a theory for determining the response of a finite Rayleigh beam (thick beam) under an arbitrary number of moving concentrated masses. The theory advanced involves the development of an analytical versatile technique which is based on the modified generalized finite integral transform and the modified Struble's method. An important feature of this technique is that it is applicable to all classical end conditions, as well as both thin and thick beam moving load problems. This technique was also extended in [23] and to solve the problem of dynamic response of an elastic plate (which is two dimentional) traversed by several moving concentrated masses. In particular, a two-dimensional analogue of the analytical method in [7] was developed. The extension was carried out in a way that the technique is also suitable for all variants of classical boundary conditions.

It is remarked at this juncture that all the mentioned authors so far in this thesis modeled the moving load by concentrated moving mass or moving lumped mass. However, in practice, moving loads are in the form of moving distributed masses rather than that of moving lumped mass. For this reason, Esmailzadez and Ghorashi [24] further studied the moving-load-induced vibration problem using a moving uniform distributed mass model instead of the moving lumped mass model. They solved the problem by means of the conventional analytical approach, which is only suitable for the simple horizontal beam and will suffer much difficulty if the structures are complicated. Also, it is noted that Esmailzadeh and Gorashi [24] considered only the vertical inertia effects of the distributed mass moving along a horizontal pinned-pinned beam. This vertical inertia is called inertia force. He neglected both coriolis force and centrifugal force of the inertia term in the governing differential equation.

Wu [25] on the other hand studied the vibration analysis of a pinned-pinned beam and that of partial frame under the action of a moving uniformly distributed mass using finite element method and Newmark integration method. Other recent work that used uniformly distributed moving mass model include Dada [26] who worked on the vibration analysis of elastic plates under uniform partially distributed loads and Adetunde [27] who studied the dynamic response of Rayleigh beam carrying an added mass and traversed by uniform partially distributed moving loads. However their method of solution are only suitable for simply supported end conditions. Thus, this study presents the problems of dynamical analysis of finite prestressed Bernoulli-Euler beams with general boundary conditions under traveling distributed loads. All the components of inertia terms are considered in the analysis and the prestressed beam is assumed to lie in a Winkler foundation model.

1.2 OBJECTIVES OF THE RESEARCH

The specific objectives of this work are to

(a) obtain closed form solution of the fourth order differential equation, with variable and singular coefficients governing the motion of uniform Bermolli-Euler beams carrying moving uniform distributed masses for all variants of classical boundary conditions.

(b) classify the effect of the elastic winkler foundation on the transverse displacement response of uniform Bermolli-Euler beams for all variants of classical boundary conditions.

(c) determine and classify the axial force influence on the response to moving distributed masses of uniform Bernoulli-Euler beams resting on elastic foundation.

(d) Indicate the reliability of the moving force solution as a safe approximation to the moving mass problem.

(e) Establish the resonance conditions for both moving force and moving mass problems and the effect of axial force and foundation moduli on the resonance conditions.

1.3 DERIVATION OF GOVERNING DIFFERENTIAL EQUATION

Let us consider the motion of a straight, non-uniform beam as shown below in the diagram (fig 1a).

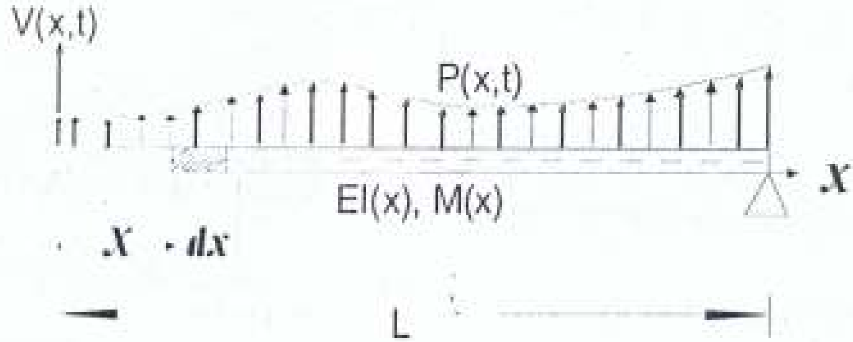

Fig. 4 Beam properties and coordinates

The significant physical properties of the beam: flexural stiffness $EI(x)$ and the mass per unit length $M(x)$, both vary arbitrarily with position x along the span L. The transverse load $P(x,t)$ is assumed to vary arbitrarily with position and time and the transverse displacement response $V(x,t)$ also is function of these variables. The end-support conditions for the beam are arbitrary, although they are pictured as simple supports for illustrative purposes. Consider a free body diagram shown below (fig 5).

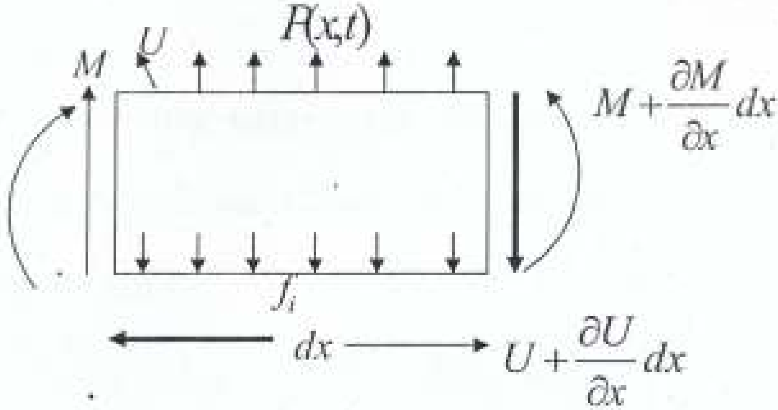

Fig. 5 Forces acting on differential element.

The equation of motion of this simple system can readily be derived by considering the equilibrium of forces acting on the differential element of the beam shown above in fig 5. Summing all forces acting vertically leads to the first dynamic-equilibrium relationship.

$$U + Pdx - \left(U + \frac{\partial U}{\partial x} dx \right) - f_i dx = 0 \qquad (1.1)$$

in which $f_i dx$ represents the distributed transverse inertia force and is given by the product of the differential mass and the local acceleration:

$$f_i dx = mdx \frac{\partial^2 V}{\partial t^2} \qquad (1.2)$$

Substituting (1.2) into equation (1.1) and simplifying yields

$$\frac{\partial U}{\partial x} = P - m \frac{\partial^2 V}{\partial t^2} \qquad (1.3)$$

which may be recognized as the standard relationship between shear force and transverse load but with the transverse load now including the inertia force of the accelerating beam. The second equilibrium relationship is obtained by summing moments about the elastic axis at the right hand face of the segment as follows:

$$M + Udx - \left(M + \frac{\partial M}{\partial x} dx \right) = 0 \tag{1.4}$$

where it has been noted that the distributed lateral force makes only a second order contribution to the moment. Simplifying (1.4) one arrives at

$$\frac{\partial M}{\partial x} = U \tag{1.5}$$

No inertia force contributes in this case to the moment equilibrium. Differentiating (1.5) w.r.t x and substituting (1.3) yields, after rearrangement,

$$\frac{\partial^2 M}{\partial x^2} + m\frac{\partial^2 V}{\partial t^2} = P \tag{1.6}$$

Finally, introducing the basic moment-curvature relationship of elementary beam theory that

$$M = EJ(x)\frac{\partial^2 V}{\partial t^2} \tag{1.7}$$

leads to the partial differential equation of motion for elementary case of beam flexure.

$$\frac{\partial^2}{\partial x^2}\left(EJ(x)\frac{\partial^2 V}{\partial x^2} \right) + M\frac{\partial^2 V}{\partial t^2} = P(x,t) \tag{1.8}$$

Now, if this beam is subjected to a force parallel to its axis in addition to the lateral loading, the beam in fig. 5 becomes

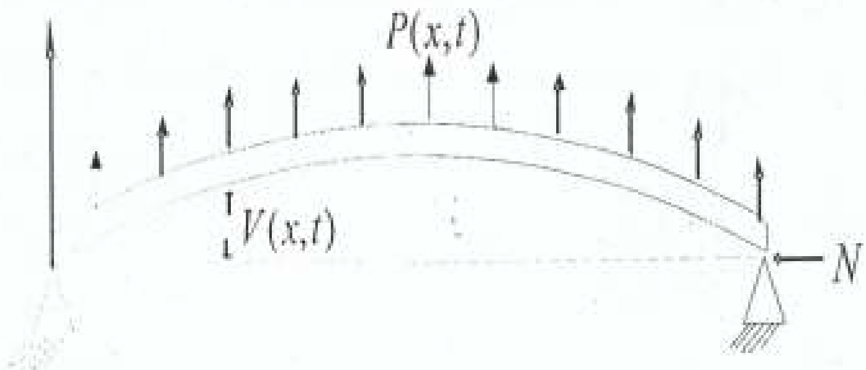

Fig. 6 beam deflected due to loading

In this case, the local equilibrium of forces is altered because the axial force interacts with the lateral displacements to produce an additional term in the moment-equilibrium expression. A free body diagram of figure 7 is presented below.

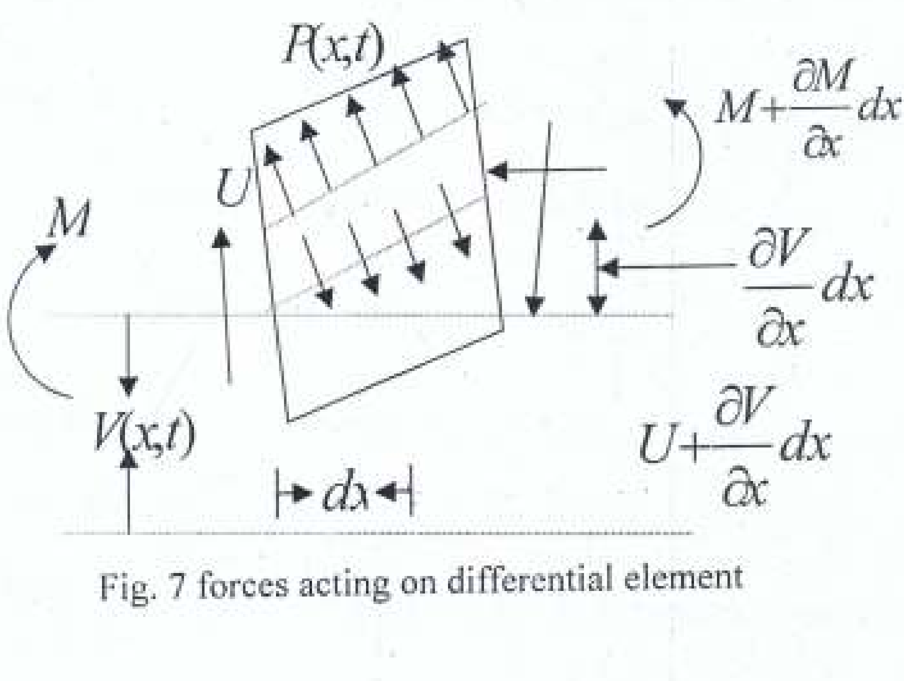

Fig. 7 forces acting on differential element

It is apparent in fig 1d that the transverse equilibrium is not affected by the axial force because its direction does not change with the beam deflection; then equation (1.3) is still valid. However, the point of application of the axial force changes with the beam deflection so that the moment-equilibrium equation now becomes

$$M + U\,dx - N(x)\frac{\partial V}{\partial x} - \left(M + \frac{\partial M}{\partial x}dx\right) = 0 \tag{1.9}$$

this implies

$$U = N(x)\frac{\partial V}{\partial x} + \frac{\partial M}{\partial x} \tag{1.10}$$

Substituting the modified expression for U into equation (1.3) and proceeding as before gives the final equation of motion including the effects of axial force as

$$\frac{\partial^2}{\partial x^2}\left(EJ(x)\frac{\partial^2 V}{\partial x^2}\right) + \frac{\partial}{\partial x}\left(N(x)\frac{\partial V}{\partial x}\right) + M(x)\frac{\partial^2 V}{\partial t^2} = P(x,t) \tag{1.11}$$

When $EI(x)$, $M(x)$ and axial force $N(x)$ are assumed to be constant with respect to time and position, equation (1.11) becomes

$$EJ\frac{\partial^4 V}{\partial x^4} + N\frac{\partial^2 V}{\partial x^2} + M\frac{\partial^2 V}{\partial t^2} = P(x,t) \tag{1.12}$$

In equation (1.12) , two important factors are neglected which may influence the dynamic response if the span-depth ratio of the beam is relatively small. They are (i) deformation due to shear forces and (ii) inertia resistance acceleration of the beam cross section (rotatory inertia). This is shown in the figure 8 below

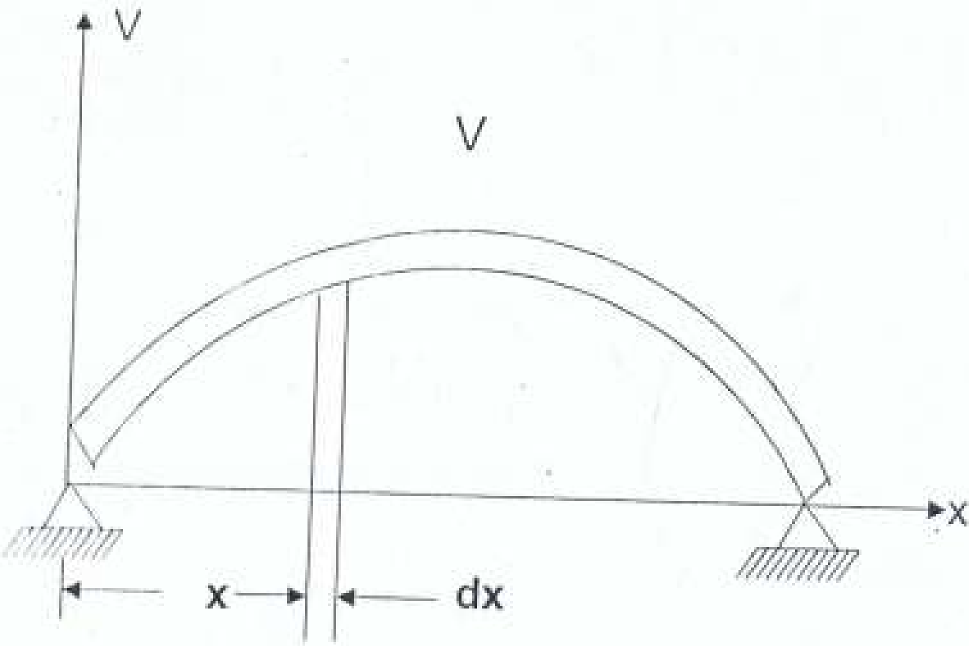

Fig. 8 dynamic beam deflection

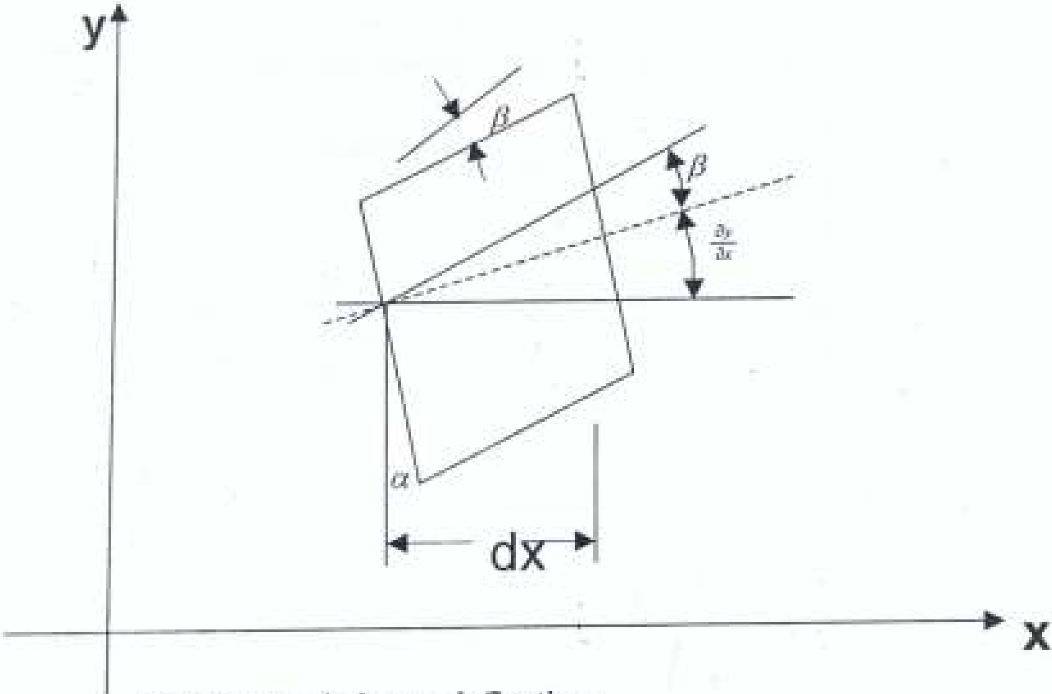

Fig.9 Dynamic beam deflection

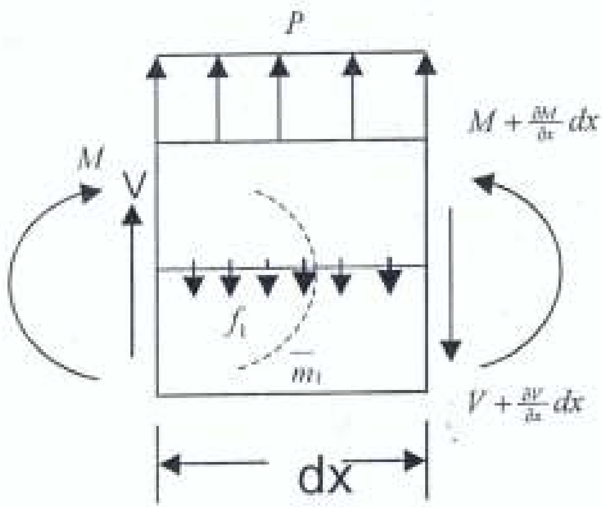

Fig.10 Forces acting on differential element

In figure 10, the forces acting on the differential elements are shown and axial forces are neglected. The rotatory inertia per unit length m, contributes directly to the moment-equilibrium relationship, then

$$M + Udx + \overline{m}_1 dx - \left(m + \frac{\partial M}{\partial x} dx \right) = 0 \qquad (1.13)$$

Where the product of the mass moment of inertia and the angular acceleration gives the rotational inertia as

$$\overline{m}_1 = \rho J \frac{\partial^2 \alpha}{\partial t^2} \qquad (1.41)$$

If ρ is the mass per unit volume ($\rho = {}^{m}/_{A}$) and J is the moment of inertia of the rotational area, we have

$$\overline{m}_1 = m \frac{J}{A} \frac{\partial^2 \alpha}{\partial t^2} = mr^2 \frac{\partial^2 \alpha}{\partial t^2} \qquad (1.15)$$

Also if $r^2 = \frac{J}{A}$ (radius of gyration of the cross section) then, equation (1.15) after simplification yields

$$\frac{\partial M}{\partial x} = U + mr^2 \frac{\partial^2 \alpha}{\partial t^2} \qquad (1.16)$$

The shear force acting on the cross section is related to the angular rotation of the elastic axis β as follows;

$$U + K'AG\beta \qquad (1.17)$$

where $K'A$ is the effective shear area of the section.

Differentiating (1.17) and substitute in equation (1.13) one obtains,

$$\frac{\partial}{\partial}(K'AG\beta) = P - m \frac{\partial^2 V}{\partial t^2} \qquad (1.18)$$

Also the moment-curvature relationship, expressed in term of cross-sectional angle α , is

$$M = EJ\frac{\partial\alpha}{\partial x} \tag{1.19}$$

Differentiating (1.19) and substituted it together with the shear-distortion expression into equation (1.3), one obtains,

$$\frac{\partial}{\partial x}\left(EJ\frac{\partial\alpha}{\partial x}\right) = K'AG\beta + mr^2\frac{\partial^2\alpha}{\partial t^2} \tag{1.20}$$

Substituting the kinematics relationship between the various rotation angle i.e

$$\beta = \alpha - \frac{\partial V}{\partial x} \tag{1.21}$$

Into the two relationship to give

$$\frac{\partial}{\partial x}\left[K'AG\left(\alpha - \frac{\partial V}{\partial x}\right)\right] = P - m\frac{\partial^2 V}{\partial t^2} \tag{1.22}$$

Then,

$$\frac{\partial}{\partial x}\left(EJ\frac{\partial\alpha}{\partial x}\right) = \left[K'AG\left(\alpha - \frac{\partial V}{\partial x}\right)\right] + mr^2\frac{\partial^2\alpha}{\partial t^2} \tag{1.23}$$

Evaluating α (rotation angle) from equation (1.22) i.e for $\frac{\partial\alpha}{\partial x}$ (that is the physical properties of the beam do not vary along its length) then,

$$\frac{\partial \alpha}{\partial x} = \frac{\partial^2 V}{\partial x^2} + \frac{1}{K'AG}\left(p - \bar{m}\frac{\partial^2 V}{\partial t^2}\right) \tag{1.24}$$

Equation (1.23) is then differentiated with respect to x and substituted for the appropriate derivatives of $\frac{\partial \alpha}{\partial x}$ from equation (1.24) which finally gives,

$$\underbrace{EJ\frac{\partial^4 V}{\partial x^4} - \left(P - \bar{m}\frac{\partial^2 V}{\partial t^2}\right)}_{\text{Elementary case}} \underbrace{- \bar{m}r^2\frac{\partial^4 V}{\partial x^2 \partial t^2}}_{\text{Rotatory inertia}} + \underbrace{\frac{EJ}{K'AG}\frac{\partial^2}{\partial x^2}\left(P - \bar{m}\frac{\partial^2 V}{\partial t^2}\right)}_{\text{Shear distortion}} - \underbrace{\frac{\bar{m}r^2}{K'AG}\frac{\partial^2}{\partial t^2}\left(P - \bar{m}\frac{\partial^2 V}{\partial t^2}\right)}_{\text{Combined shear distortion and rotatory inertia}} = 0 \tag{1.25}$$

CHAPTER TWO

UNIFORM BERNOULLI-EULER BEAM RESTING ON ELASTIC FOUNDATION AND UNDER THE ACTION OF MOVING DISTRIBUTED LOAD

2.1 THE GOVERNING EQUATION

In this chapter, the problem of the dynamic response to a distributed load moving at uniform speed on a uniform elastic beam resting on elastic foundation is considered using (1.11) the governing equation is the fourth order partial differential equation given by

$$\frac{\partial^2}{\partial x^2}\left[EJ\frac{\partial^2 W(x,t)}{\partial x^2}\right]-N\frac{\partial^2 W(x,t)}{\partial x^2}+\mu\frac{\partial^2 W(x,t)}{\partial t^2}+K(x)W(x,t)=P(x,t) \quad (2.1)$$

where x is the spatial coordinate t is the time, $W(x,t)$ is the transverse displacement E is Young's Modulus, J is the Moment of inertia, μ is the mass per unit length of the beam, N is the axial force and K is the elastic foundation and $p(x,t)$ is the uniform distributed load acting on the beam. The distributed load moving on the beam under consideration has mass commensurable with the mass of the beam. Thus, the distributed load $p(x,t)$ takes the form Timoshenko (1922)

$$P(x,t)=P_f(x,t)\left[1-\frac{1}{g}\frac{d^2W(x,t)}{dt^2}\right] \quad (2.2)$$

where $P_f(x,t)$ is the continuous moving force acting on the beam model with the properties,

$$\text{(i)} \quad \frac{d}{dx}\{H(x-ct)\} = \delta(x-ct) \tag{2.7}$$

$$\text{(ii)} \quad f(x)H(x-ct) = \begin{cases} 0, & for\ x < ct \\ f(x), & for\ x \geq ct \end{cases} \tag{2.8}$$

where $\delta(x-ct)$ represents the Dirac delta function and $H(x-ct)$ is a typical engineering function made to measure engineering applications which often involved functions that are either "off" or "on".

Below are the graphs showing the switching function of the Heaviside function. Fig.(12) shows the special case $H(x)$ which has a jump at zero and Fig.(13) is the general case $H(x-ct)$. Also, ct is the arbitrary positive number.

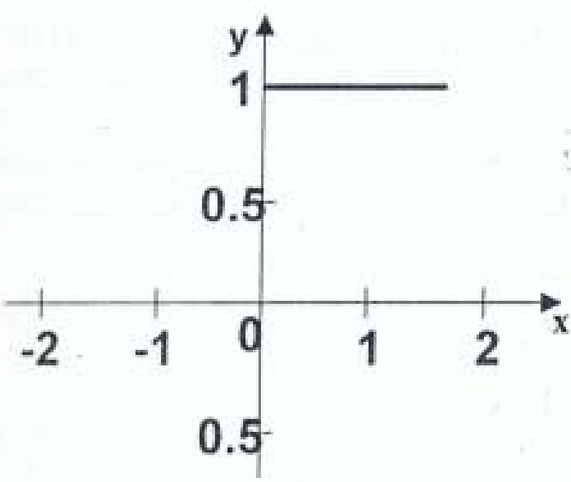

Fig .12 Special case ($H(x)$)

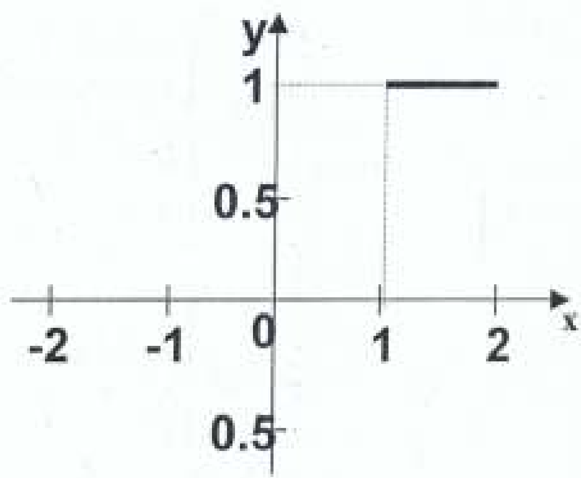

Fig .13 General case ($H(x-ct)$)

In mechanics, the Heaviside function is a function that maintain a zero value for all values of x up to $x = ct$ and a unit value for $x = ct$ and all values of $x \geq ct$. In this thesis, the Bernoulli-Euler beam under consideration is assumed to be uniform, that is, the beam properties, Young's modulus E, the moment of inertia J and the mass per unit length μ of the beam do not vary along the span of the beam. Substituting (2.2), (2.3), (2.4), (2.5) and (2.9) into (2.1), one obtains,

$$EI\frac{\partial^4 W(x,t)}{\partial x^4} - N\frac{\partial^2 W(x,t)}{\partial x^2} + \mu\frac{\partial^2 W(x,t)}{\partial t^2} + K(x)W(x,t)$$
$$+ MH(x-ct)\left[\frac{\partial^2 W(x,t)}{\partial x^2} + 2c\frac{\partial^2 W(x,t)}{\partial x \partial t} + c^2\frac{\partial^2 W(x,t)}{\partial t^2}\right] = MgH(x-ct) \quad (2.10)$$

The boundary conditions of the above problem are assumed to be arbitrary, that is, it can take any form of the classical boundary conditions. The initial conditions without any loss of generality is given by

$$W(x,t)\Big|_{t=0} = 0 = \frac{\partial W(x,t)}{\partial t}\Big|_{t=0} \quad (2.11)$$

2.2 METHOD OF SOLUTION

In this section, a general approach used in [2] is employed in order to solve the initial-value problem in equation (2.11).The approach involves in the first instance, the use of the generalized integral transformation technique. The resulting coupled second order ordinary differential equation is then treated using the modified asymptotic method of Struble and other integral transformation techniques.

2.3 THE GENERALIZED INTEGRAL TRANSFORM METHOD

In order to solve equation (2.10) subject to the conditions (2.11), first, the transformation procedure called generalized integral transformation technique is employed.

This integral transformation technique is given by

$$\overline{W}(m,t) = \int_0^L W(x,t)U_m(x)dx \quad (2.12)$$

with the inverse

$$W(x,t) = \sum_{m=1}^{\infty} \frac{\mu}{W_m}\overline{W}(m,t)U_m(x) \quad (2.13)$$

where

$$W_m = \int_0^L \mu U_m^2(x)dx \quad (2.14)$$

$U_m(x)$ is any function chosen such that the pertinent boundary conditions are satisfied. An appropriate selection of function for the beam problems are beam mode shapes. Thus, the mth normal mode of vibration of a uniform beam

$$U_m(x) = \sin\frac{\lambda_m x}{L} + A_m \cos\frac{\lambda_m x}{L} + B_m \sinh\frac{\lambda_m x}{L} + C_m \cosh\frac{\lambda_m x}{L} \qquad (2.15)$$

is chosen as a suitable kernel of the integral transform (2.12) where, λ_m is the mode frequency, A_m, B_m, C_m are constants. The parameters λ_m, A_m, B_m, and C_m are obtained by substituting (2.15) into the appropriate boundary conditions.

2.4 TRANSFORMATION OF EQUATION

Applying the generalized integral transform (2.12), equation (2.1) can be written as

$$F_1 G(0,L,t) + F_1 G_A(t) - F_2 G_B(t) + \overline{W}_{tt}(m,t) + F_3 \overline{W}(m,t) + G_C(t) + G_D(t) + G_E(t) = MgH(x-ct) \qquad (2.16)$$

where

$$F_1 = \frac{EJ}{\mu} \quad , F_2 = \frac{N}{\mu} \quad , F_3 = \frac{K}{\mu} \qquad (2.17)$$

$$G(0,L,t) = \left[\frac{\partial^3 W(x,t)}{\partial x^3} U_m(x) - \frac{\partial^2 W(x,t)}{\partial x^2} \frac{d}{dx} U_m(x) + \frac{\partial W(x,t)}{\partial x} \frac{d^2}{dx^2} U_m(x) - W(x,t) \frac{d^3}{dx^3} U_m(x) \right]_0^L \qquad (2.18)$$

$$G_A(t) = \int_0^L W(x,t) \frac{d^4 U_m(x)}{dx^4} dx \qquad (2.19)$$

$$G_B(t) = \int_0^L \frac{\partial^2 W(x,t)}{\partial x^2} U_m(x) dx \qquad (2.20)$$

$$G_C(t) = \int_0^L MH(x-ct) \frac{\partial^2 W(x,t)}{\partial t^2} U_m(x) dx \qquad (2.21)$$

$$G_D(t) = \int_0^L MH(x-ct)\frac{\partial^2 W(x,t)}{\partial x \partial t}U_m(x)dx \tag{2.22}$$

$$G_E(t) = \int_0^L MH(x-ct)\frac{\partial^2 W(x,t)}{\partial x^2}U_m(x)dx \tag{2.23}$$

It is generally known that the natural modes

$$U_m(x) = \sin\frac{\lambda_m x}{L} + A_m \cos\frac{\lambda_m x}{L} + B_m \sinh\frac{\lambda_m x}{L} + C_m \cosh\frac{\lambda_m x}{L}$$

satisfy the homogeneous differential equation

$$EJ\frac{d^4 U_m(x)}{dx^4} - \mu\Omega_m^2(x) = 0 \tag{2.24}$$

For the Euler beam, the parameter Ω_m is the natural circular frequency defined by

$$\Omega_m^2 = \frac{\lambda_m^4}{L^4}\frac{EJ}{\mu} \tag{2.25}$$

From equation (2.24) we have

$$\int_0^L W(x,t)\frac{d^4 U_m(x)}{dx^4}dx = \frac{\mu}{EJ}\Omega_m^2 \int_0^L W(x,t)U_m(x)dx \tag{2.26}$$

Thus, by (2.12)

$$G_A(t) = \frac{\mu}{EJ}\overline{W}(m,t) \tag{2.27}$$

Noting that $\overline{W}(k,t)$ is just the co-efficient of the generalized integral transform,

$$\overline{W}(x,t) = \sum_{k=1}^{\infty}\frac{\mu}{W_k}\overline{W}(k,t)U_k(x) \tag{2.28}$$

thus,

$$\frac{\partial^2}{\partial x^2}W(x,t)=\sum_{k=1}^{\infty}\frac{\mu}{W_k}\overline{W}(k,t)\frac{d^2}{dx^2}U_k(x) \tag{2.29}$$

so that integral (2.20) becomes,

$$G_B(t)=\sum_{k=1}^{\infty}\overline{W}(k,t)\int_0^L\frac{d^2U_k(x)}{dx^2}U_m(x)dx \tag{2.30}$$

From [23], the Fourier cosine transform of $\delta(x-ct)$ is given by

$$\delta(x-ct)=\frac{1}{L}+\frac{2}{L}\sum_{n=1}^{\infty}Cos\frac{n\pi x}{L}Cos\frac{n\pi x}{L} \tag{2.31}$$

Using (2.27),

it is straight forward to show that

$$H(x-ct)=\frac{x}{L}+\frac{2}{n\pi}\sum_{n=1}^{\infty}Sin\frac{n\pi cx}{L}Cos\frac{n\pi ct}{L}+C'' \tag{2.33}$$

When use is made of equation (2.31) and (2.28), one obtains,

$$G_C(t)=\frac{1}{\alpha_k(x)}\sum_{k=1}^{\infty}M\overline{W}_{tt}(k,t)\left[\frac{x}{L}\int_0^L U_k(x)U_m(x)dx\right.$$

$$\left.+\frac{2}{n\pi}\sum_{k=1}^{\infty}Cos\frac{n\pi ct}{L}\int_0^L Cos\frac{n\pi x}{L}U_k(x)U_m(x)dx+C''\int_0^L U_k(x)U_m(x)dx\right] \tag{2.34}$$

using similar argument in (2.28)and (2.31), it is straightforward to show that,

$$G_D(t)=\frac{1}{\alpha_k(x)}\sum_{k=1}^{\infty}MC\overline{W}_t(k,t)\left[\frac{2x}{L}\int_0^L\frac{dU_k(x)}{dx}U_m(x)dx\right.$$

$$+\frac{2}{n\pi}\sum_{k=1}^{\infty}Cos\frac{n\pi ct}{L}\int_0^L Cos\frac{n\pi x}{L}\frac{dU_k(x)}{dx}U_m(x)dx+2C''\int_0^L\frac{dU_k(x)}{dx}U_m(x)dx\,]$$

(2.35)

and

$$G_c(t)=\frac{1}{\alpha_k(x)}\sum_{k=1}^{\infty}\;Mc^2\overline{W}(k,t)\left[\frac{x}{L}\int_0^L\frac{d^2U_k(x)}{dx^2}U_m(x)dx\right.$$

$$+\frac{2}{n\pi}\sum_{k=1}^{\infty}Cos\frac{n\pi ct}{L}\int_0^L Cos\frac{n\pi x}{L}\frac{d^2U_k(x)}{dx^2}U_m(x)dx+C''\int_0^L\frac{d^2U_k(x)}{dx^2}U_m(x)dx\,] \qquad (2.36)$$

Substituting (2.26), (2.30), (2.34), (2.35) and (2.36) into (2.16) after some simplifications, and rearrangement yields,

$$\overline{W}_{tt}(m,t)+\left[\Omega_m^2+\frac{K}{\mu}\right]\overline{W}(m,t)-\frac{N}{\mu}\sum_{k=1}^{\infty}\;\overline{W}(k,t)S_1\;(k,m)$$

$$+\varepsilon_0\left\{\sum_{k=1}^{\infty}\;\overline{W}_{tt}(k,t)S_2(k,m)+\frac{2}{n\pi n}\sum_{n=1}^{\infty}\;\sum_{k=1}^{\infty}\overline{W}_{tt}(k,t)Cos\frac{n\pi ct}{L}S_3(k,m,n)\right.$$

$$+LC''\sum_{k=1}^{\infty}\;\overline{W}_{tt}(k,t)S_4(k,m)+2c\sum_{k=1}^{\infty}\overline{W}_t(k,t)S_5(k,m)$$

$$+\frac{4c}{n\pi}\frac{1}{\alpha_k(x)}\sum_{n=1}^{\infty}\;\sum_{k=1}^{\infty}\overline{W}_t(k,t)Cos\frac{n\pi ct}{L}S_6(k,m,n)+2cC''\sum_{k=1}^{\infty}\;\overline{W}_t(k,t)S_7(k,m)$$

$$+\frac{Mc^2}{\mu L}\sum_{k=1}^{\infty}\overline{W}(k,t)S_8(k,m)+\frac{2c^2}{n\pi}\sum_{n=1}^{\infty}\;\sum_{k=1}^{\infty}\overline{W}(k,t)Cos\frac{n\pi ct}{L}S_9(k,m,n)$$

$$+\;\frac{LMc^2C''}{\mu L}\sum_{k=1}^{\infty}\overline{W}(k,t)S_{10}(k,m)$$

$$=\frac{PL}{\mu\lambda_m}[-Cos\,\lambda_m+A_m Sin\,\lambda_m+B_m Cosh\,\lambda_m+C_m Sinh\,\lambda_m$$

$$+Cos\frac{\lambda_m ct}{L}-A_m Sin\frac{\lambda_m ct}{L}-B_m Cosh\frac{\lambda_m ct}{L}-C_m Sinh\frac{\lambda_m ct}{L}\;] \qquad (2.37)$$

where,

$$\varepsilon_0 = \frac{M}{\mu L} \tag{2.38}$$

and

$$S_1(k,m) = \frac{1}{\alpha_k(x)} \int_0^L U_k''(x) U_m(x) dx \tag{2.39}$$

$$S_2(k,m) = \frac{1}{\alpha_k(x)} \int_0^L X U(x) U_m(x) dx \tag{2.40}$$

$$S_3(k,m,n) = \frac{1}{\alpha_k(x)} \int_0^L U_k(x) U_m(x) Sin \frac{n\pi x}{L} dx \tag{2.41}$$

$$S_4(k,m) = \frac{1}{\alpha_k(x)} \int_0^L U(x) U_m(x) dx \tag{2.42}$$

$$S_5(k,m) = \frac{1}{\alpha_k(x)} \int_0^L X U_k'(x) U_m(x) dx \tag{2.43}$$

$$S_6(k,m,n) = \frac{1}{\alpha_k(x)} \int_0^L U_k'(x) U_m(x) Sin \frac{n\pi x}{L} dx \tag{2.44}$$

$$S_7(k,m) = \frac{1}{\alpha_k(x)} \int_0^L U_k'(x) U_m(x) dx \tag{2.45}$$

$$S_8(k,m) = \frac{1}{\alpha_k(x)} \int_0^L X U_k''(x) U_m(x) dx \tag{2.46}$$

$$S_9(k,m,n) = \frac{1}{\alpha_k(x)} \int_0^L U_k''(x) U_m(x) Sin\frac{n\pi x}{L} dx \qquad (2.47)$$

$$S_{10}(k,m,) = \frac{1}{\alpha_k(x)} \int_0^L U_k''(x) U_m(x) dx \qquad (2.48)$$

Using (2.15) and its derivatives in integrals (2.39) to (2.48) one obtains,

$$S_1(k,m) = \frac{\lambda_k^2}{\alpha_k(x) L^2} [-I_1 - A_m I_2 - B_m I_3 - C_m I_4 - A_k I_5 - A_k A_m I_6 - A_k B_m I_7 - A_k C_m I_8 + B_k I_9 + B_k A_m I_{10} + B_k B_m I_{11} + B_k C_m I_{12} + C_k I_{13} + C_k A_m I_{14} + C_k B_m I_{15} + C_k C_m I_{16}] \qquad (2.49)$$

$$S_2(k,m) = \frac{1}{\alpha_k(x)} [I_{17} + A_m I_{18} + B_m I_{19} + C_m I_{20} + A_k I_{21} + A_k A_m I_{22} + A_k B_m I_{23} + A_k C_m I_{24} + B_k I_{25} + B_k A_m I_{26} + B_k B_m I_{27} + B_k C_m I_{28} + C_k I_{29} + C_k A_m I_{30} + C_k B_m I_{31} + C_k C_m I_{32}] \qquad (2.50)$$

$$S_3(k,m,n) = \frac{1}{\alpha_k(x)} [I_{33} + A_m I_{34} + B_m I_{35} + C_m I_{36} + A_k I_{37} + A_k A_m I_{38} + A_k B_m I_{39} + A_k C_m I_{40} + B_k I_{41} + B_k A_m I_{42} + B_k B_m I_{43} + B_k C_m I_{44} + C_k I_{45} + C_k A_m I_{46} + C_k B_m I_{47} + C_k C_m I_{48}] \qquad (2.51)$$

$$S_4(k,m) = \frac{1}{\alpha_k(x)} [I_1 + A_m I_2 + B_m I_3 + C_m I_4 + A_k I_5 + A_k A_m I_6 + A_k B_m I_7 + A_k C_m I_8 + B_k I_9 + B_k A_m I_{10} + B_k B_m I_{11} + B_k C_m I_{12} + C_k I_{13} + C_k A_m I_{14} + C_k B_m I_{15} + C_k C_m I_{16}] \qquad (2.52)$$

$$S_5(k,m) = \frac{\lambda_k}{\alpha_k(x)L}\Big[\; I_{17} + A_m I_{18} + B_m I_{19} + C_m I_{20} - A_k I_{21} - A_k A_m I_{22} - A_k B_m I_{23} - A_k C_m I_{24} + B_k I_{25}$$
$$+ B_k A_m I_{26} + B_k B_m I_{27} + B_k C_m I_{28} + C_k I_{29} + C_k A_m I_{30} + C_k B_m I_{31} + C_k C_m I_{32} \;\Big]$$

(2.53)

$$S_6(k,m,n) = \frac{\lambda_k}{\alpha_k(x)L}\Big[\; I_{17} + A_m I_{18} + B_m I_{19} + C_m I_{20} - A_k I_{21} - A_k A_m I_{22} - A_k B_m I_{23} - A_k C_m I_{24} + B_k I_{25}$$
$$+ B_k A_m I_{26} + B_k B_m I_{27} + B_k C_m I_{28} + C_k I_{29} + C_k A_m I_{30} + C_k B_m I_{31} + C_k C_m I_{32} \;\Big]$$

(2.54)

$$S_7(k,m) = \frac{1}{\alpha_k(x)}\Big[\; I_1 + A_m I_2 + B_m I_3 + C_m I_4 - A_k I_5 - A_k A_m I_6 - A_k B_m I_7 - A_k C_m I_8 + B_k I_9$$
$$+ B_k A_m I_{10} + B_k B_m I_{11} + B_k C_m I_{12} + C_k I_{13} + C_k A_m I_{14} + C_k B_m I_{15} + C_k C_m I_{16} \;\Big]$$

(2.55)

$$S_8(k,m,n) = \frac{\lambda_k^2}{\alpha_k(x)L^2}\Big[\; -I_{17} - A_m I_{18} - B_m I_{19} - C_m I_{20} - A_k I_{21} - A_k A_m I_{22} - A_k B_m I_{23} - A_k C_m I_{24} + B_k I_{25}$$
$$+ B_k A_m I_{26} + B_k B_m I_{27} + B_k C_m I_{28} + C_k I_{29} + C_k A_m I_{30} + C_k B_m I_{31} + C_k C_m I_{32} \;\Big]$$

(2.56)

$$S_9(k,m,n) = \frac{\lambda_k^2}{\alpha_k(x)L^2}\Big[\; -I_{33} - A_m I_{34} - B_m I_{35} - C_m I_{36} - A_k I_{37} - A_k A_m I_{38} - A_k B_m I_{39} - A_k C_m I_{40} + B_k I_{41}$$
$$+ B_k A_m I_{42} + B_k B_m I_{43} + B_k C_m I_{44} + C_k I_{45} + C_k A_m I_{46} + C_k B_m I_{47} + C_k C_m I_{48} \;\Big]$$

(2.57)

$$S_{10}(k,m) = \frac{\lambda_k^2}{\alpha_k(x)L^2}\Big[\; -I_1 - A_m I_2 - B_m I_3 - C_m I_4 - A_k I_5 - A_k A_m I_6 - A_k B_m I_7 - A_k C_m I_8 + B_k I_9$$
$$+ B_k A_m I_{10} + B_k B_m I_{11} + B_k C_m I_{12} + C_k I_{13} + C_k A_m I_{14} + C_k B_m I_{15} + C_k C_m I_{16} \;\Big]$$

(2.58)

and

$$\alpha_m(x) = \Big[\, I_{49} + A_m I_{50} + B_m I_{51} + C_m I_{52} + A_k I_{53} + A_k A_m I_{54} + A_k B_m I_{55} + A_k C_m I_{56} + B_k I_{57}$$
$$+ B_k A_m I_{58} + B_k B_m I_{59} + B_k C_m I_{60} + C_k I_{61} + C_k A_m I_{62} + C_k B_m I_{63} + C_k C_m I_{64} \,\Big]$$

(2.59)

where,

$$I_1 = \int_0^L Sin\frac{\lambda_k x}{L} Sin\frac{\lambda_m x}{L} dx, \qquad I_2 = \int_0^L Sin\frac{\lambda_k x}{L} Cos\frac{\lambda_m x}{L} dx,$$

$$I_3 = \int_0^L Sin\frac{\lambda_k x}{L} Sinh\frac{\lambda_m x}{L} dx, \qquad I_4 = \int_0^L Sin\frac{\lambda_k x}{L} Cosh\frac{\lambda_m x}{L} dx,$$

$$I_5 = \int_0^L Cos\frac{\lambda_k x}{L} Sin\frac{\lambda_m x}{L} dx, \qquad I_6 = \int_0^L Cos\frac{\lambda_k x}{L} Cos\frac{\lambda_m x}{L} dx,$$

$$I_7 = \int_0^L Cos\frac{\lambda_k x}{L} Sinh\frac{\lambda_m x}{L} dx, \qquad I_8 = \int_0^L Cos\frac{\lambda_k x}{L} Cosh\frac{\lambda_m x}{L} dx,$$

$$I_9 = \int_0^L Sinh\frac{\lambda_k x}{L} Sin\frac{\lambda_m x}{L} dx, \qquad I_{10} = \int_0^L Sinh\frac{\lambda_k x}{L} Cos\frac{\lambda_m x}{L} dx,$$

$$I_{11} = \int_0^L Sinh\frac{\lambda_k x}{L} Sinh\frac{\lambda_m x}{L} dx, \qquad I_{12} = \int_0^L Sinh\frac{\lambda_k x}{L} Cosh\frac{\lambda_m x}{L} dx,$$

$$I_{13} = \int_0^L Cosh\frac{\lambda_k x}{L} Sin\frac{\lambda_m x}{L} dx, \qquad I_{14} = \int_0^L Cosh\frac{\lambda_k x}{L} Cos\frac{\lambda_m x}{L} dx,$$

$$I_{15} = \int_0^L Cosh\frac{\lambda_k x}{L} Sinh\frac{\lambda_m x}{L} dx, \qquad I_{16} = \int_0^L Cosh\frac{\lambda_k x}{L} Cosh\frac{\lambda_m x}{L} dx,$$

$$I_{17} = \int_0^L X Sin\frac{\lambda_k x}{L} Sin\frac{\lambda_m x}{L} dx, \qquad I_{18} = \int_0^L X Sin\frac{\lambda_k x}{L} Cos\frac{\lambda_m x}{L} dx,$$

$$I_{19} = \int_0^L X Sin\frac{\lambda_k x}{L} Sinh\frac{\lambda_m X}{L}\, dx,$$

$$I_{20} = \int_0^L X Sin\frac{\lambda_k x}{L} Cosh\frac{\lambda_m X}{L}\, dx,$$

$$I_{21} = \int_0^L X Cos\frac{\lambda_k x}{L} Sin\frac{\lambda_m x}{L}\, dx,$$

$$I_{22} = \int_0^L X Cos\frac{\lambda_k x}{L} Cos\frac{\lambda_m x}{L}\, dx,$$

$$I_{23} = \int_0^L X Cos\frac{\lambda_k x}{L} Sinh\frac{\lambda_m x}{L}\, dx,$$

$$I_{24} = \int_0^L X Cos\frac{\lambda_k x}{L} Cosh\frac{\lambda_m x}{L}\, dx,$$

$$I_{25} = \int_0^L X Sinh\frac{\lambda_k x}{L} Sin\frac{\lambda_m x}{L}\, dx,$$

$$I_{26} = \int_0^L X Sinh\frac{\lambda_k x}{L} Cos\frac{\lambda_m x}{L}\, dx,$$

$$I_{27} = \int_0^L X Sinh\frac{\lambda_k x}{L} Sinh\frac{\lambda_m x}{L}\, dx,$$

$$I_{28} = \int_0^L X Sinh\frac{\lambda_k x}{L} Cosh\frac{\lambda_m x}{L}\, dx,$$

$$I_{29} = \int_0^L X Cosh\frac{\lambda_k x}{L} Sin\frac{\lambda_m x}{L}\, dx,$$

$$I_{30} = \int_0^L X Cosh\frac{\lambda_k x}{L} Cos\frac{\lambda_m x}{L}\, dx,$$

$$I_{31} = \int_0^L X Cosh\frac{\lambda_k x}{L} Sinh\frac{\lambda_m x}{L}\, dx,$$

$$I_{32} = \int_0^L X Cosh\frac{\lambda_k x}{L} Cosh\frac{\lambda_m x}{L}\, dx,$$

$$I_{33} = \int_0^L Sin\frac{\lambda_k x}{L} Sin\frac{\lambda_m x}{L} Sin\frac{n\pi x}{L}\, dx,$$

$$I_{34} = \int_0^L Sin\frac{\lambda_k x}{L} Cos\frac{\lambda_m x}{L} Sin\frac{n\pi x}{L}\, dx,$$

$$I_{35} = \int_0^L Sin\frac{\lambda_k x}{L} Sinh\frac{\lambda_m x}{L} Sin\frac{n\pi x}{L}\, dx,$$

$$I_{36} = \int_0^L Sin\frac{\lambda_k x}{L} Cosh\frac{\lambda_m x}{L} Sin\frac{n\pi x}{L}\, dx,$$

$$I_{37} = \int_0^L Cos\frac{\lambda_k x}{L} Sin\frac{\lambda_m x}{L} Sin\frac{n\pi x}{L}\, dx,$$

$$I_{38} = \int_0^L Cos\frac{\lambda_k x}{L} Cos\frac{\lambda_m x}{L} Sin\frac{n\pi x}{L}\, dx,$$

$$I_{39} = \int_0^L Cos\frac{\lambda_k x}{L} Sinh\frac{\lambda_m x}{L} Sin\frac{n\pi x}{L}\, dx,$$

$$I_{40} = \int_0^L Cos\frac{\lambda_k x}{L} Cosh\frac{\lambda_m x}{L} Sin\frac{n\pi x}{L}\, dx,$$

$$I_{41} = \int_0^L Sinh\frac{\lambda_k x}{L} Sin\frac{\lambda_m x}{L} Sin\frac{n\pi x}{L}\, dx,$$

$$I_{42} = \int_0^L Sinh\frac{\lambda_k x}{L} Cos\frac{\lambda_m x}{L} Sin\frac{n\pi x}{L}\, dx,$$

$$I_{43} = \int_0^L Sinh\frac{\lambda_k x}{L} Sinh\frac{\lambda_m x}{L} Sin\frac{n\pi x}{L}\, dx,$$

$$I_{44} = \int_0^L Sinh\frac{\lambda_k x}{L} Cosh\frac{\lambda_m x}{L} Sin\frac{n\pi x}{L}\, dx,$$

$$I_{45} = \int_0^L Cosh\frac{\lambda_k x}{L} Sin\frac{\lambda_m x}{L} Sin\frac{n\pi x}{L} dx, \qquad I_{46} = \int_0^L Cosh\frac{\lambda_k x}{L} Cos\frac{\lambda_m x}{L} Sin\frac{n\pi x}{L} dx,$$

$$I_{47} = \int_0^L Cosh\frac{\lambda_k x}{L} Sinh\frac{\lambda_m x}{L} Sin\frac{n\pi x}{L} dx, \qquad I_{48} = \int_0^L Cosh\frac{\lambda_k x}{L} Cosh\frac{\lambda_m x}{L} Sin\frac{n\pi x}{L} dx,$$

$$I_{49} = \int_0^L Sin\frac{\lambda_m x}{L} Sin\frac{\lambda_m x}{L} dx, \qquad I_{50} = \int_0^L Sin\frac{\lambda_m x}{L} Cos\frac{\lambda_m x}{L} dx,$$

$$I_{51} = \int_0^L Sin\frac{\lambda_m x}{L} Sinh\frac{\lambda_m x}{L} dx, \qquad I_{52} = \int_0^L Sin\frac{\lambda_m x}{L} Cosh\frac{\lambda_m x}{L} dx,$$

$$I_{53} = \int_0^L Cos\frac{\lambda_m x}{L} Sin\frac{\lambda_m x}{L} dx, \qquad I_{54} = \int_0^L Cos\frac{\lambda_m x}{L} Cos\frac{\lambda_m x}{L} dx,$$

$$I_{55} = \int_0^L Cos\frac{\lambda_m x}{L} Sinh\frac{\lambda_m x}{L} dx, \qquad I_{56} = \int_0^L Cos\frac{\lambda_m x}{L} Cosh\frac{\lambda_m x}{L} dx,$$

$$I_{57} = \int_0^L Sinh\frac{\lambda_m x}{L} Sin\frac{\lambda_m x}{L} dx, \qquad I_{58} = \int_0^L Sinh\frac{\lambda_m x}{L} Cos\frac{\lambda_m x}{L} dx,$$

$$I_{59} = \int_0^L Sinh\frac{\lambda_m x}{L} Sinh\frac{\lambda_m x}{L} dx, \qquad I_{60} = \int_0^L Sinh\frac{\lambda_m x}{L} Cosh\frac{\lambda_m x}{L} dx,$$

$$I_{61} = \int_0^L Cosh\frac{\lambda_m x}{L} Sin\frac{\lambda_m x}{L} dx, \qquad I_{62} = \int_0^L Cosh\frac{\lambda_m x}{L} Cos\frac{\lambda_m x}{L} dx,$$

$$I_{63} = \int_0^L Cosh\frac{\lambda_m x}{L} Sinh\frac{\lambda_m x}{L} dx, \qquad I_{64} = \int_0^L Cosh\frac{\lambda_m x}{L} Cosh\frac{\lambda_m x}{L} dx,$$

(2.60)

The solutions to these integrals are listed under the appendix.

Equation (2.37) is the transformed equation governing the problem of a uniform Bernoulli-Euler beam on a constant elastic foundation when under the action of a traversing distributed load. This coupled non-homogeneous second order differential equation holds for all variant of the classical boundary conditions. In what follows, two special cases of the equation (2.37), namely, the **moving force** and the **moving mass** problems are discussed.

2.5 SOLUTION OF THE TRANSFORMED EQUATION

2.5.1 Bernoulli-Euler beam traversed by moving force

In this section, an approximate model of the differential equation describing the response of a uniform Bernoulli-Euler beam resting on an elastic foundation and under the action of a moving distributed force may be obtained from (2.37) by setting $\varepsilon_0 = 0$. Thus, setting $\varepsilon_0 = 0$, equation (2.37) reduces to

$$\begin{aligned} \overline{W}_{tt}(m,t) = \left[\Omega_m^2 + \frac{K}{\mu}\right]\overline{W}(m,t) - \frac{N}{\mu}\sum_{k=1}^{\infty} \overline{W}(k,t)S_1\ (k,m) \\ = \frac{PL}{\mu\lambda_m}[- Cos\,\lambda_m + A_m Sin\,\lambda_m + B_m Cosh\,\lambda_m + C_m Sinh\,\lambda_m \\ + Cos\frac{\lambda_m ct}{L} - A_m Sin\frac{\lambda_m ct}{L} - B_m Cosh\frac{\lambda_m ct}{L} - C_m Sinh\frac{\lambda_m ct}{L}\ ] \end{aligned} \tag{2.61}$$

This is an approximate model, which assumes the inertial effect of the moving mass as negligible. Evidently, an exact solution to this equation is not possible. Though the equation yields readily to numerical technique, an analytical approximate method is desirable as the solution so obtained often shed light on the vital information about the vibrating system. Therefore, we are going to use a modification of the asymptotic method due to Struble often used in treating weakly homogeneous and non-homogeneous non-linear oscillatory system. To this end, equation (2.61) is rearranged to take the form

$$\overline{W}_{tt}(m,t)+\left[\gamma_{mf}^{2}-\Gamma_0 S_1(m,m)\right]\overline{W}(m,t)-\Gamma_0\sum_{\substack{k=1\\k\neq m}}^{\infty}\overline{W}(k,t)S_1(k,m)$$

$$=\frac{PL}{\mu\lambda_m}[-Cos\,\lambda_m+A_m Sin\,\lambda_m+B_m Cosh\,\lambda_m+C_m Sinh\,\lambda_m$$

$$+Cos\frac{\lambda_m ct}{L}-A_m Sin\frac{\lambda_m ct}{L}-B_m Cosh\frac{\lambda_m ct}{L}-C_m Sinh\frac{\lambda_m ct}{L}\ ]$$

(2.62)

where,

$$\gamma_{mf}=\gamma_n^2+\frac{K}{L}\ \text{and}\quad \Gamma_0=\frac{N}{\mu} \tag{2.63}$$

By this technique, one seeks the modified frequency corresponding to the frequency of the free system due to the presence of the effect of the axial force N. An equivalent free system operator defined by the modified frequency then replaces equation (2.62). Thus, the right hand side of equation (2.62) is set to zero, we then consider a parameter $\lambda<1$ for any arbitrary ratio λ defined as

$$\lambda = \frac{\Gamma_0}{1+\Gamma_0} \tag{2.64}$$

so that

$$\Gamma_0 = \lambda + 0(\lambda) \tag{2.65}$$

Substituting equation (2.65) into the homogeneous part of equation (2.62) one obtains

$$\frac{d^2}{dt^2}\overline{W}(m,t) + \left[\gamma_{nl}^2 - \lambda S_1(m,m)\right]\overline{W}(m,t) - \lambda\sum_{\substack{k=1 \\ k\neq m}}^{\infty}\overline{W}(k,t)S_1(k,m) = 0 \tag{2.66}$$

Setting λ to zero in equation (2.66) a situation corresponding to the case in which the axial force effect is regarded as negligible is obtained, then the solution of (2.66) becomes,

$$\overline{W}_{nl}(m,t) = C_{nl}\,Cos\left[\gamma_{nl}t - \phi_{nl}\right] \tag{2.67}$$

where $C_{nl}, \gamma_{nl}, and\,\phi_{nl}$ are constants

Furthermore as $\lambda < 1$, Struble's technique requires that the asymptotic solution of the homogeneous part of the equation (2.62) be the form

$$\overline{W}(m,t) = \beta(m,t)Cos\left[\gamma_{nl}t - \phi(m,t)\right] + \lambda\phi_1 + 0(\lambda^2) \tag{2.68}$$

where $\beta(m,t)$ and $\phi(m,t)$ are slowly varying functions of time or equivalently,

$$\frac{d\beta(m,t)}{dt} \to 0(\lambda) \quad ; \quad \frac{d^2\beta(m,t)}{dt^2} \to 0(\lambda)$$

$$\frac{d\phi(m,t)}{dt} \to 0(\lambda) \quad ; \quad \frac{d^2\phi(m,t)}{dt^2} \to 0(\lambda)$$

(2.69)

where $\rightarrow$ implies "is of"

in view of (2.68), it can be shown that

$$\frac{d\overline{W}(m,t)}{dt} = \dot{\beta}(m,t)Cos[\gamma_{nf}t - \phi(m,t)] + \beta(m,t)\phi(m,t)Sin[\gamma_{nf}t - \phi(m,t)]$$
$$+ \beta(m,t)\gamma_{nf}Sin[\gamma_{nf}t - \phi(m,t)] + \lambda\overline{W}\ (m,t)$$

(2.70)

and

$$\frac{d^2W(m,t)}{dt^2} = \ddot{\beta}(m,t)Cos[\gamma_{nf}t - \phi(m,t)] + \dot{\beta}(m,t)\dot{\phi}(m,t)Sin[\gamma_{nf}t - \phi(m,t)]$$
$$- \dot{\beta}(m,t)\gamma_{nf}Sin[\gamma_{nf}t - \phi(m,t)] + \dot{\beta}(m,t)\dot{\phi}(m,t)Sin[\gamma_{nf}t - \phi(m,t)]$$
$$+ \beta(m,t)\ddot{\phi}(m,t)Sin[\gamma_{nf}t - \phi(m,t)] - \beta(m,t)\{\dot{\phi}(m,t)\}^2\phi(m,t)Cos[\gamma_{nf}t - \phi(m,t)]$$
$$+ \beta(m,t)\dot{\phi}(m,t)Cos[\gamma_{nf}t - \phi(m,t)] \quad - \dot{\beta}(m,t)\gamma_{nf}Sin[\gamma_{nf}t - \phi(m,t)]$$

$$+ \beta(m,t)\dot{\phi}(m,t)Cos[\gamma_{nf}t - \phi(m,t)] + \beta(m,t)\gamma_{nf}^2(m,t)Cos[\gamma_{nf}t - \phi(m,t)] + \lambda\overline{W}(1,m)$$

(2.71)

To obtain the modified frequency, equation (2.68), (2.70) and (2.71) are substituted into the homogeneous part of equation (2.62) . Subsequently, the variational part of the equation describing the axial force effect on the beam is extracted. Thus, substituting equations (2.68), (2.70) and (2.71) into the homogeneous part of the equation (2.62) one obtains,

$$2\beta(m,t)\gamma_{nf}\dot{\phi}(m,t)Cos[\gamma_{nf}t - \phi(m,t)] - 2\dot{\beta}(m,t)\gamma_{nf}Sin[\gamma_{nf}t - \phi(m,t)]$$
$$+ \lambda S_1(m,m)\beta(m,t)Cos[\gamma_{nf}t - \phi(m,t)] = 0$$

(2.72)

retaining terms to $0(\lambda)$ only.

The variational equation are obtained by equating the coefficient of

$Sin[\gamma_{nf}t-\phi(m,t)]$ and $Cos[\gamma_{nf}t-\phi(m,t)]$ on both sides of the equation (2.72).

Thus,

$$-2\dot{\beta}(m,t)\gamma_{nf}=0 \tag{2.73}$$

and

$$2\beta(m,t)\gamma_{nf}\dot{\phi}(m,t)\ \lambda S_1(m,m)\beta(m,t)=0 \tag{2.74}$$

Rearranging equation (2.73) and (2.74) yields,

$$\dot{\beta}(m,t)=0 \tag{2.75}$$

and

$$\dot{\phi}(m,t)=\frac{\lambda S_1(m,m)}{2\gamma_{nf}} \tag{2.76}$$

Solving equations (2.73) and (2.74) respectively gives

$$\beta(m,t)=C_m^0 \tag{2.77}$$

$$\phi(m,t)=\frac{\lambda S_1(m,m)}{2\gamma_{nf}}t+\omega_{nf} \tag{2.78}$$

where

C_m^0 and ω_{nf} are constants.

Therefore when the effect of the axial force is considered, the first approximation to the homogeneous system is

$$\overline{W}(m,t)=C_m^0 Cos[\theta_{am}t-\omega_{nf}] \tag{2.79}$$

where

$$\theta_{am}=\gamma_{nf}\left[1-\frac{\lambda S_1(m,m)}{2\gamma_{nf}}\right] \tag{2.80}$$

Equation (2.80) represents the modified natural frequency due to the effect of axial force N. It is observed that when $\lambda = 0$, we recover the frequency of the moving force problem when the axial force effect of the beam is neglected. Thus to solve the non-homogeneous equation (2.62), the differential operator which act on $\overline{W}(m,t)$ and $\overline{W}(m,t)$ is replaced by the modified frequency θ_{om}. Using equation (2.80) the homogeneous part of equation (2.62) can be written as

$$\frac{d^2}{dt^2}\overline{W}(m,t)+\theta_{om}^2\overline{W}(m,t)=0 \qquad (2.81)$$

Hence, the entire equation (2.62) takes the form

$$\frac{d^2}{dt^2}\overline{W}(m,t)+\theta_{om}\overline{W}(m,t) = \frac{PL}{\mu\lambda_m}[-Cos\,\lambda_m + A_m Sin\,\lambda_m + B_m Cosh\,\lambda_m + C_m Sinh\,\lambda_m$$

$$+Cos\frac{\lambda_m ct}{L} - A_m Sin\frac{\lambda_m ct}{L} - B_m Cosh\frac{\lambda_m ct}{L} - C_m Sinh\frac{\lambda_m ct}{L}\ ]$$

(2.82)

To obtain the solution of (2.82), it is subjected to a Laplace transform defined as

$$(\sim) = \int_0^\infty (.)e^{-st}dt \qquad (2.83)$$

where s is the Laplace parameter.

Applying the initial conditions (2.11), one obtains the simple algebraic equations given by

$$W(m,t)\left[S^2+\theta_{am}^2\right]=\frac{PL}{\mu\lambda_m}\left[-\frac{H(m,c)}{S}+\frac{S}{S^2+\alpha_k^2}-\frac{A_m\alpha_k}{S^2+\alpha_k^2}-\frac{B_mS}{S^2-\alpha_k^2}-\frac{C_m\alpha_k}{S^2-\alpha_k^2}\right]$$

(2.84)

where

$$\alpha_k=\frac{\lambda_m c}{L} \tag{2.85}$$

and

$$H(m,c)=\left[Cos\lambda_m+A_mSin\lambda_m+B_mCosh\lambda_m+C_mSinh\lambda_m\right] \tag{2.86}$$

Further rearrangement of equation (2.81) yields,

$$\overline{W}(m,t)=\frac{PL}{\mu\lambda_m}\left[-R_1+R_2-R_3-R_4-R_5\right] \tag{2.87}$$

where

$$R_1=\frac{\mathrm{H}(m,c)}{S}\times\frac{1}{S^2+\theta_{am}^2} \tag{2.88}$$

$$R_2=\frac{S}{S^2+\alpha_k^2}\times\frac{1}{S^2+\theta_{am}^2} \tag{2.89}$$

$$R_3=\frac{A_m\alpha_k}{S^2+\alpha_k^2}\times\frac{1}{S^2+\theta_{am}^2} \tag{2.90}$$

$$R_4=\frac{B_mS}{S^2-\alpha_k^2}\times\frac{1}{S^2+\theta_{am}^2} \tag{2.91}$$

$$R_5=\frac{C_m\alpha_k}{S^2-\alpha_k^2}\times\frac{1}{S^2+\theta_{am}^2} \tag{2.92}$$

Thus, the problem reduces to that of finding the Laplace inversion of (2.87).

To do this, we adopt the following representation;

$$f(s) = \frac{1}{S^2 + \theta_{am}^2} \tag{2.93}$$

$$g(s) = \left[-\frac{H(m,c)}{S} + \frac{S}{S^2 + \alpha_k^2} - \frac{A_n \alpha_k}{S^2 + \alpha_k^2} - \frac{B_n S}{S^2 - \alpha_k^2} - \frac{C_n \alpha_k}{S^2 - \alpha_k^2} \right] \tag{2.94}$$

so that the Laplace inverse of $\overline{W}(m,t)$ is the convolution of $f(s)$ and $g(s)$ defined as

$$f(s) * g(s) = \int_0 f(t-u) g(u) du \tag{2.95}$$

Using (2.95) $\overline{W}(m,t)$ is easily expressed as;

$$\overline{W}(m,t) = \frac{PL}{\mu \lambda_m} \left[-(Z_1 - Z_2) + Z_3 - Z_4 + Z_5 - Z_6 + Z_7 - Z_8 + Z_9 - Z_{10} \right] \tag{2.96}$$

where

$$Z_1 = \frac{Sin\theta_{am} t}{\theta_{am}} \int_0 Cos\theta_{am} u du \tag{2.97}$$

$$Z_2 = \frac{Cos\theta_{am} t}{\theta_{am}} \int_0 Sin\theta_{am} u du \tag{2.98}$$

$$Z_3 = \frac{Sin\theta_{am} t}{\theta_{am}} \int_0 Cos\theta_{am} u Cos\alpha_k u du \tag{2.99}$$

$$Z_4 = \frac{Cos\theta_{am} t}{\theta_{am}} \int_0 Sin\theta_{am} u Cos\alpha_k u du \tag{2.100}$$

$$Z_5 = \frac{A_m Sin\theta_{am} t}{\theta_{am}} \int_0 Cos\theta_{am} u Sin\alpha_k u du \tag{2.101}$$

$$Z_6 = \frac{A_m Cos\theta_{am}t}{\theta_{am}} \int_0^t Sin\theta_{am}uSin\alpha_k u du \tag{2.102}$$

$$Z_7 = \frac{B_m Sin\theta_{am}t}{\theta_{am}} \int_0^t Cos\theta_{am}uCosh\alpha_k u du \tag{2.103}$$

$$Z_8 = \frac{B_m Cos\theta_{am}t}{\theta_{am}} \int_0^t Sin\theta_{am}uCosh\alpha_k u du \tag{2.104}$$

$$Z_9 = \frac{C_m Sin\theta_{am}t}{\theta_{am}} \int_0^t Cos\theta_{am}uCosh\alpha_k u du \tag{2.105}$$

$$Z_{10} = \frac{C_m Cos\theta_{am}t}{\theta_{am}} \int_0^t Sin\theta_{am}uSinh\alpha_k u du \tag{2.106}$$

It is readily shown that

$$Z_1 = \frac{H(m,c)}{\theta_{am}}\left[Sin^2\theta_{am}t\right] \tag{2.106}$$

$$Z_2 = -\frac{H(m,c)}{\theta_{am}}\left[Cos^2\theta_{am}t - Cos\theta_{am}\right] \tag{2.107}$$

$$Z_3 = \frac{Sin\theta_{am}t}{2\theta_{am}}\left[\frac{Sin(\theta_{am}+\alpha_k)t}{\theta_{am}+\alpha_k} + \frac{Sin(\theta_{am}-\alpha_k)t}{\theta_{am}-\alpha_k}\right] \tag{2.108}$$

$$Z_4 = \frac{Cos\theta_{am}t}{2\theta_{am}}\left[\frac{Cos(\theta_{am}+\alpha_k)t}{\theta_{am}+\alpha_k} + \frac{Cos(\theta_{am}-\alpha_k)t}{\theta_{am}-\alpha_k} - \frac{2\theta_{am}}{\theta_{am}^2-\alpha_k^2}\right] \tag{2.109}$$

$$Z_5 = -\frac{A_m Sin\theta_{am}t}{2\theta_{am}}\left[\frac{Cos(\theta_{am}+\alpha_k)t}{\theta_{am}+\alpha_k} + \frac{Cos(\theta_{am}-\alpha_k)t}{\theta_{am}-\alpha_k} + \frac{2\theta_{am}Sin\theta_{am}}{\theta_{am}^2-\alpha_k^2}\right]$$

(2.110)

$$Z_6 = \frac{A_m Cos\theta_{am}t}{2\theta_{am}}\left[\frac{Sin(\theta_{am}-\alpha_k)t}{\theta_{am}-\alpha_k} - \frac{Sin(\theta_{am}+\alpha_k)t}{\theta_{am}+\alpha_k}\right] \tag{2.111}$$

$$Z_7 = \frac{B_m Sin\theta_{am} t}{\theta_{am}\left[\alpha_k^2 - \theta_{am}\right]}\left[\alpha_k Cos\theta_{am} t Sinh\alpha_k t - \theta_{am} Sin\theta_{am} t Cosh\alpha_k t\right] \quad (2.112)$$

$$Z_8 = \frac{B_m Cos\theta_{am} t}{\theta_{am}\left[\alpha_k^2 - \theta_{am}\right]}\left[\alpha_k Sin\theta_{am} t Sinh\alpha_k t - \theta_{am} Cos\theta_{am} t Cosh\alpha_k t - \frac{\theta_{am}}{\alpha_k^2 - \theta_{am}^2}\right]$$

(2.113)

$$Z_9 = \frac{C_m Sin\theta_{am} t}{\theta_{am}\left[\alpha_k^2 + \theta_{am}\right]}\left[\alpha_k Cos\theta_{am} t Cosh\alpha_k t + \theta_{am} Sin\theta_{am} t Cosh\alpha_k t - \frac{\alpha_k}{\alpha_k^2 + \theta_{am}^2}\right]$$

(2.114)

$$Z_{10} = \frac{C_m Cos\theta_{am} t}{\theta_{am}\left[\alpha_k^2 - \theta_{am}^2\right]}\left[\alpha_k Sin\theta_{am} t Cosh\alpha_k t + \theta_{am} Cos\theta_{am} t Sinh\alpha_k t\right]$$

(2.115)

when Z_1 to Z_{10} are substituted into equation (2.96) after some rearrangements one obtains,

$$\overline{W}(m,t) = \frac{PL}{\mu\lambda_m}\left[\frac{H(m,t)(1 - Cos\theta_{am} t)}{\theta_{am} t} - \frac{Cos\alpha_k t - Cos\theta_{am} t}{\theta_{am}^2 - \alpha_k^2}\right.$$

$$+ \frac{A_m\left(Sin\alpha_k t + Sin\theta_{am} t\right)}{\theta_{am}^2 - \alpha_k^2} + \frac{4B_m\theta_{am}\alpha_k Sin\theta_{am} t Sinh\alpha_k t}{\alpha_k^4 - \theta_{am}^4}$$

$$+ \frac{2B_m\alpha_k^2 Cos\theta_{am} t Cosh\alpha_k t}{\alpha_k^4 - \theta_{am}^4} + \frac{B_m\theta_{am} Cosh\alpha_k t + B_m\left(\alpha_k^2 + \theta_{am}^2\right)}{\alpha_k^4 - \theta_{am}^4}$$

$$+ \frac{4C_m\theta_{am}^2\alpha_k Sin\theta_{am} t Cosh\alpha_k t}{\alpha_k^4 - \theta_{am}^4} + \frac{2C_m\alpha_k^2 Cos\theta_{am} t Sinh\alpha_k t}{\alpha_k^4 - \theta_{am}^4}$$

$$\left.+ \frac{C_m\theta_{am}^2 Sin\alpha_k t}{\alpha_k^4 - \theta_{am}^4} + \frac{C_m\alpha_k Sin\theta_{am} t\left(\alpha_k^2 - \theta_{am}^2\right)}{\alpha_k^4 - \theta_{am}^4}\right]$$

(2.116)

which on inversion yields,

$$W(x,t)=\frac{1}{\alpha_m(x)}\sum_{m=1}^{\infty}\frac{PL}{\mu\lambda_m}\left[\frac{H(m,t)(1-Cos\theta_{am}t)}{\theta_{am}}-\frac{Cos\alpha_k t-Cos\theta_{am}t}{\theta_{am}^2-\alpha_k^2}\right.$$

$$+\frac{A_m(\ Sin\alpha_k t+Sin\theta_{am}t\ )}{\theta_{am}^2-\alpha_k^2}+\frac{4B_m\theta_{am}\alpha_k Sin\theta_{am}tSinh\alpha_k t}{\alpha_k^4-\theta_{am}^4}$$

$$+\frac{2B_m\alpha_k^2Cos\theta_{am}tCosh\alpha_k t}{\alpha_k^4-\theta_{am}^4}+\frac{B_m\theta_{am}Cosh\alpha_k t+B_m\left(\alpha_k^2+\theta_{am}^2\right)}{\alpha_k^4-\theta_{am}^4}$$

$$+\frac{4C_m\theta_{am}^2\alpha_k Sin\theta_{am}tCosh\alpha_k t}{\alpha_k^4-\theta_{am}^4}+\frac{2C_m\alpha_k^2Cos\theta_{am}tSinh\alpha_k t}{\alpha_k^4-\theta_{am}^4}$$

$$\left.+\frac{C_m\theta_{am}^2Sin\alpha_k t}{\alpha_k^4-\theta_{am}^4}+\frac{C_m\alpha_k Sin\theta_{am}t\left(\alpha_k^2-\theta_{am}^2\right)}{\alpha_k^4-\theta_{am}^4}\right]$$

$$\times\left(Sin\frac{\lambda_m x}{L}+A_m\cos\frac{\lambda_m x}{L}+B_m\sinh\frac{\lambda_m x}{L}+C_m\cosh\frac{\lambda_m x}{L}\right) \qquad (2.117)$$

Equation (2.117) represents the transverse response to a moving force moving at constant velocity of a uniform Bernoulli-Euler beam resting on elastic foundation and having arbitrary support end conditions.

2.5.2 Bernoulli-Euler beam traversed by a moving mass

In this section, the solution to the entire equation (2.37) is sought when no terms of the coupled differential equation is neglected. As in moving force problem, in section (2.5.1), an exact analytical solution to equation (2.37) does not exist. Thus, the approximate analytical solution discussed in section (2.5.1) namely, modified Struble's asymptotic method is employed. It is noted that, neglecting the terms representing the inertia effect of the moving mass equation (2.37) one obtains (2.61).

Consequently, the homogenous part of equation (2.61) can be replaced by a free system operator defined by the modified frequency due to the presence of axial force N. Thus equation (2.62) can be rewritten in the form

$$
\begin{aligned}
&\overline{W}_{tt}(m,t)+\theta_{mm}^2\overline{W}(m,t)+\varepsilon_0\Bigg\{\sum_{k=1}^{\infty}\overline{W}_{tt}(k,t)S_2(k,m)+\frac{2}{n\pi n}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\overline{W}_{tt}(k,t)Cos\frac{n\pi ct}{L}S_3(k,m,n)\\
&+LC''\sum_{k=1}^{\infty}\overline{W}_{tt}(k,t)S_4(k,m)+2c\sum_{k=1}^{\infty}\overline{W}_t(k,t)S_5(k,m)\\
&+\frac{4c}{n\pi}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\overline{W}_t(k,t)Cos\frac{n\pi ct}{L}S_6(k,m,n)+2cC'\sum_{k=1}^{\infty}\overline{W}_t(k,t)S_7(k,m)\\
&+\frac{Mc^2}{\mu L}\sum_{k=1}^{\infty}\overline{W}(k,t)S_8(k,m)+\frac{2c^2}{n\pi}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\overline{W}(k,t)Cos\frac{n\pi ct}{L}S_9(k,m,n)\\
&+\frac{LMc^2C''}{\mu L}\sum_{k=1}^{\infty}\overline{W}(k,t)S_{10}(k,m)\\
&=\frac{PL}{\mu\lambda_m}[-Cos\,\lambda_m+A_m Sin\,\lambda_m+B_m Cosh\,\lambda_m+C_m Sinh\,\lambda_m\\
&+Cos\frac{\lambda_m ct}{L}-A_m Sin\frac{\lambda_m ct}{L}-B_m Cosh\frac{\lambda_m ct}{L}-C_m Sinh\frac{\lambda_m ct}{L}\;]
\end{aligned}
$$

(2.119)

Further arrangement of equation of equation (2.119) yields

$$\overline{W}_{tt}(m,t)\frac{\varepsilon_0\left[2cS_5(m,m)+\sum_{n=1}^{\infty}Cos\frac{n\pi ct}{L}S_6(m,m,n)+2cC^{o}S_7(m,m)\right]}{\left[\left(1+\varepsilon_0\left(S_2(m,m)\right)+\frac{2}{n\pi m}\sum_{n=1}^{\infty}Cos\frac{n\pi ct}{L}S_3(m,m,n)+LC^{o}S_4(m,m)\right)\right]}\overline{W}t(m,t)$$

$$+\frac{\left[\theta_{am}^2+\varepsilon_0\left(c^2S_8(m,m)\right)+\frac{2c^2}{n\pi}\sum_{n=1}^{\infty}Cos\frac{n\pi ct}{L}S_9(m,m,n)+Lc^2C^{o}S_{10}(m,m)\right]}{\left[\left(1+\varepsilon_0\left(S_2(m,m)\right)+\frac{2}{n\pi m}\sum_{n=1}^{\infty}Cos\frac{n\pi ct}{L}S_3(m,m,n)+LC^{o}S_4(m,m)\right)\right]}\overline{W}t(m,t)$$

$$+\sum_{\substack{k=1\\k\neq m}}^{\infty}\varepsilon_0\left\{\frac{\left[\left(S_2(k,m)\right)+\frac{2}{n\pi m}\sum_{n=1}^{\infty}Cos\frac{n\pi ct}{L}S_3(m,m,n)+LC^{o}S_4(k,m)\right]}{\left[\left(1+\varepsilon_0\left(S_2(m,m)\right)+\frac{2}{n\pi m}\sum_{n=1}^{\infty}Cos\frac{n\pi ct}{L}S_3(m,m,n)+LC^{o}S_4(m,n)\right)\right]}\overline{W}_{tt}(k,t)\right.$$

$$+\frac{\varepsilon_0\left[2cS_5(k,m)+\sum_{n=1}^{\infty}Cos\frac{n\pi ct}{L}S_6(k,m,n)+2cC^{o}S_7(k,m)\right]}{\left[\left(1+\varepsilon_0\left(S_2(m,m)\right)+\frac{2}{n\pi m}\sum_{n=1}^{\infty}Cos\frac{n\pi ct}{L}S_3(m,m,n)+LC^{o}S_4(m,m)\right)\right]}\overline{W}t(k,t)$$

$$+\frac{\left[c^2S_8(k,m)+\frac{2c^2}{n\pi}\sum_{n=1}^{\infty}Cos\frac{n\pi ct}{L}S_9(k,m,n)+Lc^2C^{o}S_{10}(k,m)\right]}{\left[\left(1+\varepsilon_0\left(S_2(m,m)\right)+\frac{2}{n\pi m}\sum_{n=1}^{\infty}Cos\frac{n\pi ct}{L}S_3(m,m,n)+LC^{o}S_4(m,m)\right)\right]}\overline{W}(k,t)$$

$$=\frac{\frac{\varepsilon_0L^2g}{\lambda_m}\left[-\mathrm{H}(m,t)+Cos\frac{\lambda_m ct}{L}-A_mSin\frac{\lambda_m ct}{L}-B_mCosh\frac{\lambda_m ct}{L}-C_mSinh\frac{\lambda_m ct}{L}\right]}{\left[\left(1+\varepsilon_0\left(S_2(m,m)\right)+\frac{2}{n\pi m}\sum_{n=1}^{\infty}Cos\frac{n\pi ct}{L}S_3(m,m,n)+LC^{o}S_4(m,m)\right)\right]}$$

(2.120)

where,

$$S_2(m,m)=S_2(k,m)\big|_{k=m} \qquad S_7(m,m)=S_7(k,m)\big|_{k=m}$$

$$S_3(m,m)=S_3(k,m,m)\big|_{k=m} \qquad S_8(m,m)=S_8(k,m)\big|_{k=m}$$

$$S_4(m,m) = S_4(k,m)\big|_{k=m} \qquad S_9(m,m) = S_9(k,m)\big|_{k=m}$$

$$S_5(m,m) = S_5(k,m)\big|_{k=m} \qquad S_{10}(m,m) = S_{10}(k,m)\big|_{k=m}$$

$$S_6(m,m) = S_6(k,m)\big|_{k=m}$$

(2.121)

Like in the previous case the homogeneous part of equation (2.120) is considered and a modified frequency corresponding to the frequency of the free system due To the presence of the moving mass M is sought. An equivalent free system operator defined by the modified frequency then replaces equation (2.120).Thus a parameter $\eta < 1$ for arbitrary mass ratio defined by

$$\eta = \frac{\varepsilon_0}{1+\varepsilon_0}$$

(2.122)

is considered.

It can be shown that

$$\varepsilon_0 = \eta + 0(\eta)^2$$

(2.123)

and

$$\frac{1}{\left[1+\eta(S_2(m,m)) + \frac{2}{n\pi m}\sum_{n=1}^{\infty} Cos\frac{n\pi ct}{L} S_3(m,m,n) + LC''S_4(m,m)\right]}$$

$$= 1-\eta\left(S_2(m,m)\right)+\frac{2}{n\pi m}\sum_{n=1}^{\infty}Cos\frac{n\pi ct}{L}S_3(m,m,n)+LC''S_4(m,m)+0(\eta^2)$$

(2.124)

where

$$\left|\eta\left(S_2(m,m)\right)+\frac{2}{n\pi m}\sum_{n=1}^{\infty}Cos\frac{n\pi ct}{L}S_3(m,m,n)+LC''S_4(m,m)+0(\eta^2)\right|<1$$

(2.125)

Now using (2.123) and (2.124) in (2.120) one obtains,

$$\overline{W}_{tt}(m,t)+\eta\left[2cS_5(m,m)+\sum_{n=1}^{\infty}Cos\frac{n\pi ct}{L}S_6(m,m,n)+2cC''S_7(m,m)\right]\overline{W}_t(m,t)$$

$$+\left[\theta_{om}^2+\eta\left(c^2S_8(m,m)+\frac{2c^2}{n\pi}\sum_{n=1}^{\infty}Cos\frac{n\pi ct}{L}S_9(m,m,n)+Lc^2C''S_{10}(m,m)\right)\right]\overline{W}(m,t)$$

$$-\left[\eta\theta_{om}^2D(m,m,n)\right]\overline{W}(m,t)+\sum_{\substack{k=1\\k\neq m}}^{\infty}\eta\left[\left[1-\eta\left(D(m,m,n)\right)\right]\times\left[D(m,m,n)\right]\overline{W}_{tt}(k,t)\right.$$

$$+\left[1-\left(D(k,m,n)\right)\right]\times\left[2cS_5(k,m)+\sum_{n=1}^{\infty}Cos\frac{n\pi ct}{L}S_6(k,m,n)+2cC''S_7(k,m)\right]\overline{W}_t(k,t)$$

$$+\left[1-\left(D(k,m,n)\right)\right]\times\left[c^2S_8(k,m)+\frac{2c^2}{n\pi}\sum_{n=1}^{\infty}Cos\frac{n\pi ct}{L}S_9(k,m,n)+Lc^2C''S_{10}(k,m)\right]\overline{W}(k,t)$$

$$=\left[1-\eta\left(D(m,m,n)\right)\right]\times\frac{\eta L^2g}{\lambda_m}\left[-H(m,t)+Cos\frac{\lambda_m ct}{L}-A_mSin\frac{\lambda_m ct}{L}-B_mCosh\frac{\lambda_m ct}{L}-C_mSinh\frac{\lambda_m ct}{L}\right]$$

(2.126)

to $0(\eta)$ only

where,

$$D(m,m,n)=\left[(S_2(m,m))+\frac{2}{n\pi m}\sum_{n=1}^{\infty}Cos\frac{n\pi ct}{L}S_3(m,m,n)+LC''S_4(m,m)\right]$$

(2.127)

$$D(k,m,n)=\left[(S_2(k,m))+\frac{2}{n\pi m}\sum_{n=1}^{\infty}Cos\frac{n\pi ct}{L}S_3(k,m,n)+LC''S_4(k,m)\right]$$

(2.128)

when $\eta=0$ a case corresponding to the case when the inertia effect of the mass of the system is neglected, then the solution of (2.126) can be written as

$$\overline{W}(m,t)=C_m^0 Cos[\theta_{am}t-\Phi(m,t)]$$

(2.129)

where θ_{ω} is as defined previously and C_m^0 and $\Phi(m,t)$ are constants.

However, since for any arbitrary mass ratio ε_0 we always have $\eta<1$, the solution

For the homogeneous part of equation (2.126) can be written as

$$\overline{W}(m,t)=\Delta(m,t)Cos[\theta_{am}t-\Phi(m,t)]+\eta_1\overline{W}(m,t)+0(\eta)$$

(2.130)

As in the previous section, equation (2.130) and its derivatives are substituted into the homogeneous part of equation(2.126). Subsequently, only the variational part of the equation describing the behavior of $\Delta(m,t)$ and $\Phi(m,t)$ during the motion of the mass is extracted. Thus, substituting equations (2.130) and its derivatives into the homogeneous part of equation (2.126) and taking into account (2.126), one obtains

$$-2\dot{\Delta}(m,t)\theta_{am}Sin[\theta_{am}t-\Phi(m,t)]+2\Delta(m,t)\theta_{am}\dot{\Phi}(m,t)Cos[\theta_{am}t-\Phi(m,t)]$$

$$-2c\eta\Delta(m,t)\theta_{am}S_5(m,m)Sin[\theta_{am}t-\Phi(m,t)]-\frac{4c}{n\pi}\eta\sum_{n=1}^{\infty}Cos\frac{n\pi ct}{L}S_6(m,m,n)\Delta(m,t)\theta_{am}Cos[\theta_{am}t-\Phi(m,t)]$$

$$-2cC''S_7(m,m)\Delta(m,t)\theta_{am}Cos[\theta_{am}t-\Phi(m,t)]-c^2\Delta(m,t)S_8(m,m)Cos[\theta_{am}t-\Phi(m,t)]$$

$$-\frac{2c^2}{n\pi}\eta\sum_{n=1}^{\infty}Cos\frac{n\pi ct}{L}\Delta(m,t)S_9(m,m,n)Cos[\theta_{am}t-\Phi(m,t)]$$

$$-L^2C''S_{10}(m,m)\Delta(m,m)\theta_{am}Cos[\theta_{am}t-\Phi(m,t)]$$

$$-\theta_{am}^2\eta S_2(m,m)\Delta(m,t)Cos[\theta_{am}t-\Phi(m,t)]$$

$$-\theta_{am}^2\frac{2}{n\pi}\eta\sum_{n=1}^{\infty}Cos\frac{n\pi ct}{L}\Delta(m,t)S_3(m,m,n)Cos[\theta_{am}t-\Phi(m,t)]$$

$$-\theta_{am}^2LC''\eta\Delta(m,t)S_4(m,m)Cos[\theta_{am}t-\Phi(m,t)]$$

$$+\sum_{\substack{k=1\\k\neq l}}^{\infty}\eta\left\{\left[S_2(k,m)+\frac{2}{n\pi}\sum_{k=1}^{\infty}Cos\frac{n\pi ct}{L}S(_3k,m,n)+LC''S_4(k,m)\right](\Delta_{11}(k,t))\right.$$

$$+\left[2cS_5(k,m)+\frac{4c}{n\pi}\sum_{k=1}^{\infty}Cos\frac{n\pi ct}{L}S_6(k,m,n)+2cC''S_7S(k,m)\right](\Delta_{12}(k,t))$$

$$+\left[c^2 S_8(k,m) + \frac{2c^2}{n\pi}\sum_{k=1}^{\infty} Cos\frac{n\pi ct}{L} S_9(k,m,n) + L^2 C^o S_{10}(k,m)\right](\Delta_{13}(k,t))\Big\} = 0$$

(2.131)

where

$$\Delta_{11}(k,t) = \ddot{\Delta}(k,t)Cos[\theta_{am}t - \Phi(m,t)] - 2\dot{\Delta}(k,t)\theta_{am}Sin[\theta_{am}t - \Phi(m,t)]$$

$$+\dot{\Delta}(k,t)\dot{\Phi}(m,t)Cos[\theta_{am}t - \Phi(m,t)] - \Delta(k,t)\{\dot{\Phi}(m,t)\}^2 Cos[\theta_{am}t - \Phi(m,t)]$$

$$-\Delta(k,t)\theta_{am}^2 Cos[\theta_{am}t - \Phi(m,t)]$$

$$\Delta_{12}(k,t) = \dot{\Delta}(k,t)Cos[\theta_{am}t - \Phi(m,t)] + \Delta(k,t)\dot{\Phi}(m,t)Sin[\theta_{am}t - \Phi(m,t)]$$

$$-\Delta(k,t)\theta_{am}Sin[\theta_{am}t - \Phi(m,t)]$$

$$\Delta_{13}(k,t) = \Delta(k,t)Cos[\theta_{am}t - \Phi(m,t)]$$

(2.132)

retaining term to $0(\eta)$

Setting the coefficient of $Sin[\theta_{am} - \Phi(m,t)]$ and $Cos[\theta_{am} - \Phi(m,t)]$

To Zero, we have,

$$-2\dot{\Delta}(m,t)\theta_{am} - 2c\eta\Delta(m,t)\theta_{am}S_5(m,m) - 2\eta cC^o S_7(m,m)\Delta(m,t) = 0$$

(2.133)

and

$$\Delta(m,t)\theta_{am}\dot{\Phi}(m,t) + \eta Lc^2\Delta(m,t)S_{10}(m,m) + c^2\eta\Delta(m,t)S_8(m,m)$$

$$-\eta\theta_{am}^2\Delta(m,t)S_2(m,m) - \theta_{am}^2 LC^o\eta\Delta(m,t)S_4(m,m) = 0 \quad (2.134)$$

Rearranging (2.133) and (2.134) one,

$$\Delta(m,t) = \frac{-\left[c\eta S_5(m,m) + cC''\eta\theta_{am}S_7(m,m)\right]}{\theta_{am}}\Delta(m,t) \tag{2.135}$$

and

$$\Phi(m,t) = \frac{\left[\eta\theta_{am}^2 S_2(m,m) + \theta_{am}^2 LC''\eta S_4(m,m) - Lc^2C''\eta S_{10}(m,m) - c^2\eta S_8(m,m)\right]}{2\theta_{am}}$$

(2.136)

Solving equations (2.134) and (2.135) respectively, we have,

$$\Delta(m,t) = C_0 e^{-\alpha_m t} \tag{2.137}$$

where

$$\alpha_m = \frac{\left[c\eta S_5(m,m) + cC''\eta\theta_{am}S_7(m,m)\right]}{\theta_{am}}$$

(2.138)

and

$$\Phi(m,t) = \frac{\left[\eta\theta_{am}^2 S_2(m,m) + \theta_{am}^2 LC''\eta S_4(m,m) - Lc^2C''\eta S_{10}(m,m) - c^2\eta S_8(m,m)\right]}{2\theta_{am}} t + \omega^*_m$$

(2.139)

where C_0 and ω^*_m are constants

Therefore when the effects of the particle is considered, the first approximation to the homogeneous system is given as,

$$\overline{W}(m,t) = C_m^0 Cos[\theta_{bt}t - \Phi(m,t)] \tag{2.140}$$

where

$$\theta_{bt} = \theta_m \left[1 - \frac{\eta}{2} \left\{ \left(S_2(m,m) + LC^o S_4(m,m) \right) - \frac{\left(Lc^2 C^o S_{10}(m,m) + c^2 S_8(m,m) \right)}{\theta_m^2} \right\} \right] \tag{2.141}$$

is called the modified frequency corresponding to the frequency of the free system due to the presence of the moving mass. Thus, the homogeneous part of (2.139) can be written as,

$$\frac{d^2}{dt^2}\overline{W}(m,t) + \theta_{bt}^2 \overline{W}(m,t) = 0 \tag{2.142a}$$

Hence the entire equation (2.126) taking into account (2.69) takes the form,

$$\frac{d^2}{dt^2}\overline{W}(m,t) + \theta_{bt}^2 \overline{W}(m,t) = \frac{\eta L^2 g}{\lambda_m}\left[-H(m,t) + Cos\frac{\lambda_m ct}{L} - A_m Sin\frac{\lambda_m ct}{L} - B_m Cosh\frac{\lambda_m ct}{L} - C_m Sinh\frac{\lambda_m ct}{L} \right] \tag{2.142b}$$

This is analogous to equation (2.82).This, using similar argument as in the previous section, the solution to equation (2.142b)is given by

$$\overline{W}(m,t) = \frac{PL}{\mu\lambda_m}\left[\frac{H(m,t)(1 - Cos\theta_{bt}t)}{\theta_{bt}} - \frac{Cos\alpha_k t - Cos\theta_{bt}t}{\theta_{bt}^2 - \alpha_k^2} \right.$$

$$+ \frac{A_m(Sin\alpha_k t + Sin\theta_{bt}t)}{\theta_{bt}^2 - \alpha_k^2} + \frac{4B_m\theta_{bt}\alpha_k Sin\theta_{bt}tSinh\alpha_k t}{\alpha_k^4 - \theta_{bt}^4}$$

$$+ \frac{2B_m\alpha_k^2 Cos\theta_{bt}tCosh\alpha_k t}{\alpha_k^4 - \theta_{bt}^4} + \frac{B_m\theta_L Cosh\alpha_k t + B_m(\alpha_k^2 + \theta_{bt}^2)}{\alpha_k^4 - \theta_{bt}^4}$$

$$+ \frac{4C_m\theta_{bt}^2\alpha_k Sin\theta_{bt}tCosh\alpha_k t}{\alpha_k^4 - \theta_{bt}^4} + \frac{2C_m\alpha_k^2 Cos\theta_{bt}tSinh\alpha_k t}{\alpha_k^4 - \theta_{bt}^4}$$

$$+\frac{C_m\theta_{bm}^2 Sin\alpha_k t}{\alpha_k^4-\theta_{bm}^4}+\frac{C_m\alpha_k Sin\theta_{bm}t\left(\alpha_k^2-\theta_{bm}^2\right)}{\alpha_k^4-\theta_{bm}^4}\Bigg]$$

(2.143)

which on inversion yields,

$$\overline{W}(m,t)=\frac{1}{\alpha_m(x)}\sum_{m=1}^{\infty}\frac{\varepsilon_0 Lg}{\mu\lambda_m}\Bigg[\frac{H(m,t)(1-Cos\theta_{bm}t)}{\theta_{bm}}-\frac{Cos\alpha_k t-Cos\theta_{bm}t}{\theta_{bm}^2-\alpha_k^2}$$

$$+\frac{A_m(\ Sin\alpha_k t+Sin\theta_{bm}t\ )}{\theta_{bm}^2-\alpha_k^2}+\frac{4B_m\theta_{bm}\alpha_k Sin\theta_{bm}tSinh\alpha_k t}{\alpha_k^4-\theta_{bm}^4}$$

$$+\frac{2B_m\alpha_k^2 Cos\theta_{bm}tCosh\alpha_k t}{\alpha_k^4-\theta_{bm}^4}+\frac{B_m\theta_{bm}Cosh\alpha_k t+B_m\left(\alpha_k^2+\theta_{bm}^2\right)}{\alpha_k^4-\theta_{bm}^4}$$

$$+\frac{4C_m\theta_{bm}^2\alpha_k Sin\theta_{bm}tCosh\alpha_k t}{\alpha_k^4-\theta_{bm}^4}+\frac{2C_m\alpha_k^2 Cos\theta_{bm}tSinh\alpha_k t}{\alpha_k^4-\theta_{bm}^4}$$

$$+\frac{C_m\theta_{bm}^2 Sin\alpha_k t}{\alpha_k^4-\theta_{bm}^4}+\frac{C_m\alpha_k Sin\theta_{bm}t\left(\alpha_k^2-\theta_{bm}^2\right)}{\alpha_k^4-\theta_{bm}^4}\Bigg]$$

$$\times\left(Sin\frac{\lambda_m x}{L}+A_m\cos\frac{\lambda_m x}{L}+B_m\sinh\frac{\lambda_m x}{L}+C_m\cosh\frac{\lambda_m x}{L}\right)$$

(2.144)

Equation (2.144) represents the transverse response to a moving mass moving at constant velocity of a uniform Bernoulli-Euler beam resting on elastic foundation and having arbitrary support end conditions.

In the chapter that follows, the theory discussed in this thesis is illustrated using some end conditions of practical interest in engineering design and construction.

CHAPTER THREE

ILLUSRIATIVE EXAMPLES, NUMERICAL CALCULATIONS AND DISCUSSION OF RESULTS (UNIFORM BERNOULLI-EULER BEAM)

3.1.0 ILLUSRIATIVE EXAMPLES

In this chapter, some examples of classical boundary conditions are selected to illustrate the analyses presented in this work. Three commonly encountered end conditions selected are (i) simply supported boundary conditions (ii) Clamped-clamped ends conditions and (iii) Clamped-free ends conditions (Cantilever beam)

3.1.1 Simply supported Boundary Conditions

In this case, the uniform Bernoulli-Euler beam has simple support at ends $x=0$ and $x=L$. The displacement and the bending moment vanish at simply supported ends. Thus, the conditions are expressed as

$$W(0,t)=0=W(L,t) \quad , \quad \frac{\partial^2 W(0,t)}{\partial x^2}=0=\frac{\partial^2 W(L,t)}{\partial x^2} \tag{3.1}$$

and hence for normal modes

$$U_m(0)=0=U_m(L) \quad , \quad \frac{\partial^2 U_m(0)}{\partial x^2}=0=\frac{\partial^2 U_m(L)}{\partial x^2} \tag{3.2}$$

which implies that

$$U_k(0)=0=U_k(L) \quad , \quad \frac{\partial^2 U_k(0)}{\partial x^2}=0=\frac{\partial^2 U_k(L)}{\partial x^2} \tag{3.3}$$

Thus, making use the of boundary conditions, it can be shown that

$$A_m = 0,\ B_m = 0,\ C_m = 0 \text{ and } \lambda_m = m\pi \tag{3.4}$$

$$A_k = 0,\ B_k = 0,\ C_k = 0 \text{ and } \lambda_k = k\pi \tag{3.5}$$

and the frequency equation becomes

$$Sin\lambda_m = Sin\lambda_k = 0 \tag{3.6}$$

which implies

$$\lambda_m = m\pi \quad \text{and} \quad \lambda_k = k\pi \tag{3.7}$$

Substituting (3.4) and (3.5) into the transformed equation (2.56) to obtain the transformed equation for Bernoulli-Euler beam resting on elastic foundation, having supports at both edges. That is,

$$\overline{W}_{tt}(m,t) + \left[\Omega_m^2 + \frac{K}{\mu}\right]\overline{W}(m,t) - \frac{N}{\mu}\sum_{k=1}^{\infty}\overline{W}(k,t)\left(\frac{k^2\pi^2}{L^2}\right)I_1^*(k,m)$$

$$+\varepsilon_0\left\{\sum_{k=1}^{\infty}\overline{W}_{tt}(k,t)I_2^* + \frac{2}{n\pi m}\frac{2}{L}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\overline{W}_{tt}(k,t)Cos\frac{n\pi ct}{L}I_3^*(k,m,n)\right.$$

$$+LC^o\frac{2}{L}\sum_{k=1}^{\infty}\overline{W}_{tt}(k,t)I_4^*(k,m) + 2c\frac{2}{L}\sum_{k=1}^{\infty}\overline{W}_t\left(\frac{k\pi}{L}\right)(k,t)I_5^*(k,m)$$

$$+\frac{2c}{n\pi}\frac{2}{L}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\overline{W}_t(k,t)Cos\frac{n\pi ct}{L}\frac{k\pi}{L}I_6^*(k,m,n) + 2cC^o\frac{2}{L}\sum_{k=1}^{\infty}\overline{W}_t(k,t)\left(\frac{k\pi}{L}\right)I_7^*(k,m)$$

$$+\frac{2c^2}{L}\frac{2}{L}\sum_{k=1}^{\infty}\overline{W}(k,t)\left(-\frac{k^2\pi^2}{L^2}\right)I_8^*(k,m)$$

$$+\frac{2c^2}{n\pi}\frac{2}{L}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\overline{W}(k,t)Cos\frac{n\pi ct}{L}\left(-\frac{k^2\pi^2}{L^2}\right)I_9^*(k,m,n) + c^2C^o\sum_{k=1}^{\infty}\overline{W}(k,t)\frac{k^2\pi^2}{L^2}I_{10}^*$$

$$= \frac{PL}{\mu m\pi}\left[-(-1)^m + Cos\frac{m\pi ct}{L}\right] \tag{3.8}$$

where,

$$I_1^* = \int_0^L Sin\frac{k\pi x}{L}Sin\frac{m\pi x}{L}dx \tag{3.9}$$

$$I_2^* = \int_0^L xSin\frac{k\pi x}{L}Sin\frac{m\pi x}{L}dx \tag{3.10}$$

$$I_3^* = \int_0^L Sin\frac{k\pi x}{L}Sin\frac{m\pi x}{L}Sin\frac{n\pi x}{L}dx \tag{3.11}$$

$$I_4^* = I_1^* \tag{3.12}$$

$$I_5^* = \int_0^L xCos\frac{k\pi x}{L}Sin\frac{m\pi x}{L}dx \tag{3.13}$$

$$I_6^* = \int_0^L Cos\frac{k\pi x}{L}Sin\frac{m\pi x}{L}Sin\frac{n\pi x}{L}dx \tag{3.14}$$

$$I_7^* = \int_0^L Cos\frac{k\pi x}{L}Sin\frac{m\pi x}{L}dx \tag{3.15}$$

$$I_8^* = I_2^* \tag{3.16}$$

$$I_9^* = I_3^* \tag{3.17}$$

$$I_{10}^* = I_1^* \tag{3.18}$$

On evaluating the integrals (3.9) to (3.18) the solutions are presented as follows:

$$I_1^* = \begin{cases} 0\ , & if\ k \neq m \\ L/2\ , & if\ \mathrm{k} = m \end{cases} \tag{3.19}$$

$$I_2^* = \frac{L^2}{\pi^2}\left[\frac{1}{(k+m)^2} - \frac{1}{(k-m)^2}\right] \tag{3.20}$$

$$I_3^* = \frac{L}{4}\left[\sum_{n=1}^{\infty} Sin\frac{m\pi ct}{L}\, Sin\frac{k\pi ct}{L}\right] \tag{3.21}$$

$$I_4^* = I_1^* \tag{3.22}$$

$$I_5^* == -\frac{L^2}{2\pi}\left[-\frac{1}{k+m} - \frac{1}{k-m}\right] \tag{3.23}$$

$$I_6^* = \frac{-L2m\pi^3\left[n^2 + k^2 - m^2\right]}{\left[(n+k)^2\pi^2 - m^2\pi^2\right]\left[(n-k)^2\pi^2 - m^2\pi^2\right]} \tag{3.24}$$

$$I_7^* = \frac{-2Lm\pi}{k^2\pi^2 - m^2\pi^2} \tag{3.25}$$

$$I_8^* = I_2^* \tag{3.26}$$

$$I_9^* = I_3^* \tag{3.27}$$

$$I_{10}^* = I_1^* \tag{3.28}$$

on substituting equation (3.19) to (3.28) into (3.8) after some rearrangement, one obtains

$$\overline{W}_{tt}(m,t) + \left[\frac{EI}{\mu}\left(\frac{m\pi}{L}\right)^4 + \frac{k}{\mu} + \frac{N}{\mu}\left(\frac{m^2\pi^2}{2L}\right)\right]\overline{W}(m,t)$$

$$+\varepsilon_0\left\{\sum_{k=1}^{\infty}\left[\frac{L^2}{\pi}\left[-\frac{1}{(k+m)^2}-\frac{1}{(k-m)^2}\right]\right]\overline{W}_{tt}(k,t)+\frac{1}{n\pi m}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}Sin\frac{m\pi ct}{L}Sin\frac{k\pi ct}{L}\overline{W}_{tt}(k,t)\right.$$

$$+LC''\sum_{k=1}^{\infty}\overline{W}_{tt}(k,t)-\frac{4c}{L}\sum_{k=1}^{\infty}\left(\frac{k\pi}{L}\right)\left[\frac{L^2}{2\pi}\left(\frac{k^2}{k^2-m^2}\right)\right]\overline{W}_t(k,t)$$

$$-\frac{c}{n\pi}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}Cos\frac{n\pi ct}{L}\left(\frac{k\pi}{L}\right)G_{km}\overline{W}_t(k,t)-cC''\sum_{k=1}^{\infty}\left(\frac{k\pi}{L}\right)\left[\frac{8Lm\pi}{k^2\pi^2-m^2\pi^2}\right]\overline{W}_t(k,t)$$

$$+4c^2\sum_{k=1}^{\infty}\left(\frac{k^2\pi^2}{L^2}\right)\left[\frac{L^2}{\pi}\left[-\frac{1}{(k+m)^2}-\frac{1}{(k-m)^2}\right]\right]\overline{W}_t(k,t)$$

$$-\frac{c^2}{n\pi}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\frac{k^2\pi^2}{L^2}\right)\left[Sin\frac{m\pi ct}{L}Sin\frac{k\pi ct}{L}\right]\overline{W}(k,t)-Lc^2C''\sum_{k=1}^{\infty}\left(\frac{k^2\pi^2}{2L}\right)\overline{W}(k,t)$$

$$=\frac{PL}{\mu m\pi}\left[-(-1)^m+Cos\frac{m\pi ct}{L}\right] \tag{3.29}$$

$$G_{km}=\frac{16m^2\pi^3c}{nL\{[(n+m)^2-m^2][(n-m)^2-m^2]\}} \tag{3.30}$$

Equation (3.29) is now the fundamental equation of our problem when the Bernoulli-Euler beam has simple support at all edges. In what follows, we shall discuss two cases of the equation.

SIMPLY SUPPORTED BERNOULLI-EULER BEAM TRAVERSED BY MOVING FORCE

As defined previously, an appropriate model of the system when the inertia of moving mass M is neglected i.e. when ε_0 is set to zero, is the moving force problem associated with the system. Thus, the differential equation is given by

$$\frac{d^2}{dt^2}\overline{W}(m,t)+\theta_{mf}^2\overline{W}(m,t)=\frac{PL}{\mu m\pi}\left[-(-1)^m+Cos\frac{m\pi ct}{L}\right] \tag{3.32}$$

Where

$$\theta_{mf}^2=\left\{\frac{EJ}{\mu}\left(\frac{m\pi}{L}\right)^4+\frac{N}{\mu}\left(\frac{m\pi}{L}\right)^2+\frac{K}{\mu}\right\} \tag{3.33}$$

To obtain the solution to equation (3.2), it is subjected to a Laplace transform defined as,

$$(\cdot)=\int_0^\infty e^{-st}dt \tag{3.34}$$

Where s is the Laplace parameter. Applying the initial condition (2.11), one obtains the simple algebraic equation given by

$$\overline{W}(m,t)=\frac{P_m}{(S^2+\theta_{mf}^2)}\left[\frac{S}{S^2+\omega_m^2}-\frac{(-1)^m}{S}\right] \tag{3.35}$$

Where $P_m=\dfrac{PL}{\mu m\pi}$ (3.36a)

$$\omega_n=\frac{m\pi ct}{L} \tag{3.36b}$$

$$\left.\begin{aligned}
V_{m1} &= \frac{Sin\theta_{mf}t}{2\theta_{mf}}\left[\frac{Sin(\theta_{mf}+\omega_m)t}{\theta_{mf}+\omega_m}+\frac{Sin(\theta_{mf}-\omega_m)t}{\theta_{mf}-\omega_m}\right] \\
-V_{m2} &= \frac{Cos\theta_{mf}t}{2\theta_{mf}}\left[\frac{Cos(\theta_{mf}+\omega_m)t}{\theta_{mf}+\omega_m}+\frac{Cos(\theta_{mf}-\omega_m)t}{\theta_{mf}-\omega_m}-\frac{2\theta_{mf}}{\theta_{mf}^2-\omega_m^2}\right] \\
V_{m3} &= \frac{(-1)^m Sin^2\theta_{mf}t}{\theta_{mf}^2} \\
V_{m4} &= \frac{(-1)^m Cos^2\theta_{mf}t - Cos\theta_{mf}t}{\theta_{mf}^2}
\end{aligned}\right\} \tag{3.42}$$

On substituting $V_{m1} - V_{m4}$ into equation (3.40), after some rearrangements, one obtains,

$$\overline{W}(m,t) = \frac{PL}{\mu m\pi}\left[\frac{-(-1)^m}{\theta_{mf}^2}-\frac{Cos\omega_m t - Cos\theta_{mf}t}{\theta_{mf}^2-\omega_m^2}\right] \tag{3.43}$$

Which on inversion yields

$$W(x,t) = \frac{2}{L}\sum_{m=1}^{\infty}\frac{PL}{\mu m\pi}\left[\frac{-(-1)^m}{\theta_{mf}^2}-\frac{Cos\omega_m t - Cos\theta_{mf}t}{\theta_{mf}^2-\omega_m^2}\right] \tag{3.44}$$

Equation (3.44) above represents the transverse displacement response to a moving force moving at a constant velocity of a simply supported Bernoulli-Euler Beam resting on elastic foundation.

SIMPLY SUPPORTED BERNOULLI-EULER BEAM TRAVERSED BY MOVING MASS

This section attempts to solve equation (3.8) subjected to the initial conditions (2.11) when all the inertia terms are considered. It is observed again, that no exact analytical solution exists for this equation. The modified Struble's asymptotic method discussed in the previous section shall therefore be employed. To this end, equation (3.8) is rearranged to take the form

$$\overline{W}_{tt}(m,t)+\theta_{m}^{2}\overline{W}(m,t)$$

$$+\varepsilon_0\left\{\sum_{k=1}^{\infty}\left[\frac{L^2}{\pi}\left[-\frac{1}{(k+m)^2}-\frac{1}{(k-m)^2}\right]\right]\overline{W}_{tt}(k,t)+\frac{1}{n\pi n}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}Sin\frac{m\pi ct}{L}Sin\frac{k\pi ct}{L}\overline{W}_{tt}(k,t)\right.$$

$$+LC^{*}\sum_{k=1}^{\infty}\overline{W}_{tt}(k,t)-\frac{4c}{L}\sum_{k=1}^{\infty}\left(\frac{k\pi}{L}\right)\left[\frac{L^2}{2\pi}\left(\frac{k^2}{k^2-m^2}\right)\right]\overline{W}_{t}(k,t)$$

$$-\frac{c}{n\pi}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}Cos\frac{n\pi ct}{L}\left(\frac{k\pi}{L}\right)G_{1m}\overline{W}_{t}(k,t)-cC^{*}\sum_{k=1}^{\infty}\left(\frac{k\pi}{L}\right)\left[\frac{8Lm\pi}{k^2\pi^2-m^2\pi^2}\right]\overline{W}_{t}(k,t)$$

$$+4c^2\sum_{k=1}^{\infty}\left(\frac{k^2\pi^2}{L^2}\right)\left[\frac{L^2}{\pi}\left[-\frac{1}{(k+m)^2}-\frac{1}{(k-m)^2}\right]\right]\overline{W}_{t}(k,t)$$

$$\left.-\frac{c^2}{n\pi}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\left(\frac{k^2\pi^2}{L^2}\right)\left[Sin\frac{m\pi ct}{L}Sin\frac{k\pi ct}{L}\right]\overline{W}(k,t)-Lc^2C^{*}\sum_{k=1}^{\infty}\left(\frac{k^2\pi^2}{2L}\right)\overline{W}(k,t)\right\}$$

$$=\frac{PL}{\mu m\pi}\left[-(-1)^n+Cos\frac{m\pi ct}{L}\right] \tag{3.45}$$

From equation (3.45), it can be shown that

$$\left[1+\varepsilon_0\left(\frac{L}{2m^2\pi^2}-\frac{1}{n\pi m}\sum_{n=1}^{\infty}\left(1-Cos\frac{m\pi ct}{L}\right)\overline{W}_{tt}(m,t)+LC^{o}\right)\right]$$

$$+\left[\varepsilon_0(c-G_{1m})\right]\overline{W}_t(m,t)$$

$$+\left[\theta_{m}^{2}-\varepsilon_0\left(-\frac{c^2}{2L}+\frac{2c^2}{n^2\pi^2m}\sum_{n=1}^{\infty}\left(1-Cos\frac{m\pi ct}{L}\right)-\frac{m^2\pi^2c^2C^{o}}{2L}\right)\right]\overline{W}(m,t)$$

$$+\sum_{\substack{k=1\\k\neq1}}^{\infty}\varepsilon_0\left\{\left(\frac{8kLm}{\pi^2\left[k^2-2k^2m^2+m^2\right]}+\frac{2}{n\pi m}\sum_{n=1}^{\infty}Cos\frac{m\pi ct}{L}+LC^{o}\right)\overline{W}_{tt}(k,t)\right.$$

$$+\left(\frac{4kcm}{k^2-m^2}-G_{2m}-\frac{8C^{o}m^2c}{L(k^2-m^2)}\right)\overline{W}_t(k,t)$$

$$\left.+\left(\frac{8km^2c^2}{L\left[k^2-2k^2m^2+m^2\right]}-\frac{m^2c^2\pi}{nL^2}\sum_{n=1}^{\infty}Cos\frac{m\pi ct}{L}-\frac{m^2c^2\pi^2C^{o}}{L}\overline{W}(k,t)\right\}\right)\right\}$$

$$=\frac{\varepsilon_0L^2g}{m\pi}\left[-(-1)^{m}-Cos\frac{m\pi ct}{L}\right]$$

(3.46)

A further rearrangement of equation (3.46) gives.

$$\overline{W}_{tt}(m,t)+\frac{\left[\varepsilon_0(c-G_{1m})\right]}{\left[1+\varepsilon_0\left(\frac{L}{2m^2\pi^2}-\frac{1}{n\pi m}\sum_{n=1}^{\infty}\left(1-Cos\frac{m\pi ct}{L}\right)\overline{W}_{tt}(m,t)-LC^{o}\right)\right]}\overline{W}_t(m,t)$$

$$+\frac{\left[\theta_{m}^{2}-\varepsilon_0\left(-\frac{c^2}{2L}+\frac{2c^2}{n^2\pi^2m}\sum_{n=1}^{\infty}\left(1-Cos\frac{m\pi ct}{L}\right)-\frac{m^2\pi^2c^2C^{o}}{2L}\right)\right]}{\left[1+\varepsilon_0\left(\frac{L}{2m^2\pi^2}-\frac{1}{n\pi m}\sum_{n=1}^{\infty}\left(1-Cos\frac{m\pi ct}{L}\right)\overline{W}_{tt}(m,t)+LC^{o}\right)\right]}\overline{W}(m,t)$$

and

$$\frac{1}{\left[1+\varepsilon_0\left(\frac{L}{2m^2\pi^2}-\frac{1}{n\pi m}\sum_{n=1}^{\infty}\left(1-Cos\frac{m\pi ct}{L}\right)+LC''\right)\right]}=1-\lambda\left[1+\varepsilon_0\left(\frac{L}{2m^2\pi^2}-\frac{1}{n\pi m}\sum_{n=1}^{\infty}\left(1-Cos\frac{m\pi ct}{L}\right)+LC''\right)\right]$$

(3.50)

Applying (3.49) and (3.50) in (3.47), one obtains

$$\overline{W}_{tt}(m,t)+[\lambda(c-G_{1w})]\overline{W}_{t}(m,t)$$

$$+\left[\theta_{mf}^2-\lambda\left(-\frac{c^2}{2L}+\frac{2c^2}{n^2\pi^2 m}\sum_{n=1}^{\infty}\left(1-Cos\frac{m\pi ct}{L}\right)-\frac{m^2\pi^2c^2C''}{2L}\right)\right]\overline{W}(m,t)-\theta_{mf}^2\lambda N(m,m)\overline{W}(m,t)$$

$$+\sum_{\substack{k=1\\k\neq m}}^{\infty}\lambda\{[1-N(m,m)]\overline{W}_{tt}(k,t)\times\left[\left(\frac{4kcm}{k^2-m^2}-G_{2w}-\frac{8C''m^2c}{L(k^2-m^2)}\right)\right]\overline{W}_{t}(k,t)$$

$$+[1-Z(m,m)]\overline{W}_{tt}(k,t)\times\left[\frac{8km^3c^2}{L[k^2-2k^2m^2+m^2]}-\frac{m^2c^2\pi}{nL^2}\sum_{n=1}^{\infty}Cos\frac{m\pi ct}{L}-\frac{m^2c^2\pi^2C''}{L}\right]\overline{W}(k,t)$$

$$=[1-N(m,m)]\overline{W}_{tt}(k,t)\times\frac{\lambda L^2 g}{m\pi}\left[-(-1)^m+Cos\frac{m\pi ct}{L}\right]$$

(3.51)

to $0(\lambda)$ only.

when $\lambda=0$ in equation (3.51) a case corresponding to the case when the inertia effect of the mass of the system is obtained, then the solution of (3.51) can be written in the form

$$\overline{W}(m,t)=C_m Cos[\theta_{mf}t-\psi_m] \tag{3.52}$$

where θ_{mf} is as previously defined and C_m and ψ_m are constants.

ever, since for any arbitrary mass ratio ε_0 we always have $\lambda < 1$, the ion for the homogeneous part of equation (3.51) can be written as

$$\overline{W}(m,t) = \Omega(m,t)Cos\left[\theta_{mf} - \phi(m,t) + \lambda_1\overline{W}(m,t) + 0(\lambda^2)\right] \qquad (3.53)$$

der to obtain the modified frequency, equation (3.53) and its derivatives substituted into the homogeneous part of equation (3.51). Subsequently, the variational part of the equation describing the behaviour of $\Omega(m,t)$ $\phi(m,t)$ during the motion of the mass is extracted. Thus, substituting ation (3.53) and its derivatives into the homogeneous part of equation 51) and taking into account (2.66) one obtains

$$\Omega(m,t)\theta_{mf} Sin\left[\theta_{mf}t - \phi(m,t)\right] + 2\Omega(m,t)\dot{\phi}(m,t)\theta_{mf} Cos\left[\theta_{mf}t - \phi(m,t)\right]$$

$$\Omega(m,t)\theta_{mf} Sin\left[\theta_{mf}t - \phi(m,t)\right] + \frac{\lambda 16m^2\pi^2 c}{nL\left\{\left[(n+m)^2 - m^2\right]\left[(n-m)^2 - m^2\right]\right\}}\Omega(m,t)\theta_{mf} Sin\left[\theta_{mf}t - \phi(m,t)\right]$$

$$\sum \frac{\lambda c^2}{L}\Omega(m,t)Cos\left[\theta_{mf}t - \phi(m,t)\right] + \frac{1}{2}\sum_{n=1}^{\infty}\frac{\lambda c^2}{L}\Omega(m,t)Cos\frac{m\pi ct}{L}Cos\left[\theta_{mf}t - \phi(m,t)\right]$$

$$\frac{m^2\pi^2c^2C''}{L}\Omega(m,t)Cos\left[\theta_{mf}t - \phi(m,t)\right] - \lambda\theta_{mf}^2\frac{L}{2m^2\pi^2}\Omega(m,t)Cos\left[\theta_{mf}t - \phi(m,t)\right] + \lambda\theta_{mf}^2 LC''\Omega(m,t)Cos\left[\theta_{mf}\right.$$

$$+\sum_{n=1}^{\infty} Cos\frac{n\pi ct}{L}\frac{\lambda 16m^4c^2}{n^2\left[(n^2 - 4m^2)\right]}\Omega(m,t)Cos\left[\theta_{mf}t - \phi(m,t)\right] + \lambda\theta_{mf} LC''\Omega(m,t)Cos\left[\theta_{mf}t - \phi(m,t)\right]$$

$$+\sum_{\substack{k=1\\k\neq m}}^{\infty}\lambda\left[1 - Z(m,m)\right]\overline{W}_u(k,t)\times\left[\left[\lambda\left(\frac{8kLm}{\pi^2\left[k^2 - 2k^2m^2 + m^2\right]} + \frac{2}{n\pi m}\sum_{n=1}^{\infty}Cos\frac{m\pi ct}{L} + LC''\right)\right]\right]\overline{W}_t(k,t)$$

$$+\left[1 - Z(m,m)\right]\overline{W}_u(k,t)\times\left[\left(\frac{4kcm}{k^2 - m^2} - G_{2m} - \frac{8C''m^2c}{L(k^2 - m^2)}\right)\right]\overline{W}(k,t)\Bigg\} = 0$$

(3.54)

where terms higher than $0(\lambda)$ have been neglected.

To obtain variational equation, we equate the coefficient of $Sin[\theta_{mf}t-\phi(m,t)]$ and $Cos[\theta_{mf}t-\phi(m,t)]$ terms on both sides of the equation. To do this we note the trigonometric identity below

$$Cos[\theta_{mf}t-\phi(m,t)]Cos\frac{m\pi ct}{L}=\frac{1}{2}\left\{Cos\left[\theta_{mf}t-\phi(m,t)+\frac{m\pi ct}{L}\right]+Cos\left[\theta_{mf}t-\phi(m,t)-\frac{m\pi ct}{L}\right]\right\}$$

(3.55)

Since only the terms involving $Sin[\theta_{mf}t-\phi(m,t)]$ and $Cos[\theta_{mf}t-\phi(m,t)]$ contribute to the variational equations describing the behaviour of $\Omega(m,t)$ and $\phi(m,t)$, in view of the identities (3.55), equation (3.54) reduces to

$$-2\dot{\Omega}(m,t)\theta_{mf}Sin[\theta_{mf}t-\phi(m,t)]-2\Omega(m,t)\dot{\phi}(m,t)\theta_{mf}Cos[\theta_{mf}t-\phi(m,t)]$$

$$-\frac{\lambda 16m^2\pi^2c}{nL\{[(n+m)^2-m^2][(n-m)^2-m^2]\}}\Omega(m,t)\theta_{mf}Sin[\theta_{mf}t-\phi(m,t)]+\frac{1}{2}\sum_{n=1}^{\infty}\frac{\lambda c^2}{L}\Omega(m,t)Cos[\theta_{mf}t-\phi(m,t)]$$

$$\frac{1}{2}\sum_{n=1}^{\infty}\frac{\lambda c^2}{L}\Omega(m,t)Cos\frac{m\pi ct}{L}Cos[\theta_{mf}t-\phi(m,t)]+\frac{\lambda m^2\pi^2c^2C''}{L}\Omega(m,t)Cos[\theta_{mf}t-\phi(m,t)]$$

$$-\lambda\theta_{mf}^2\frac{L}{2m^2\pi^2}\Omega(m,t)Cos[\theta_{mf}t-\phi(m,t)]+\sum_{n=1}^{\infty}\frac{\lambda\theta_{mf}^2}{n\pi m}\Omega(m,t)Cos[\theta_{mf}t-\phi(m,t)]$$

$$+\sum_{n=1}^{\infty}\frac{\lambda\theta_{mf}^2}{n\pi m}Cos\frac{m\pi ct}{L}\Omega(m,t)Cos[\theta_{mf}t-\phi(m,t)]+\lambda\theta_{mf}^2LC''\Omega(m,t)Cos[\theta_{mf}t-\phi(m,t)]=0$$

(3.56)

Setting coefficient of $Sin[\theta_{mf}t-\phi(m,t)]$ and $Cos[\theta_{mf}t-\phi(m,t)]$ on both sides to zero, one obtains

$$-2\theta_{nf}\dot{\Omega}(m,t)\theta_{nf} - \lambda c\theta_{nf}\Omega(m,t) - \frac{\lambda 16 m^2\pi^2 c}{nL\left\{\left[(n+m)^2 - m^2\right]\left[(n-m)^2 - m^2\right]\right\}}\theta_{nf}\Omega(m,t) = 0$$

(3.57)

and

$$2\Omega(m,t)\dot{\phi}(m,t)\theta_{nf} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{\lambda c^2}{L}\Omega(m,t) + \frac{\lambda m^2\pi^2 c^2 C^{o}}{L}\Omega(m,t) - \sum_{n=1}^{\infty}\frac{\lambda\theta_{nf}^2}{n\pi m}\Omega(m,t)$$
$$\lambda\theta_{nf}^2 L C^{o}\Omega(m,t) = 0$$

(3.58)

Further arrangement of (3.57) and (3.58) gives

$$\frac{\dot{\Omega}(m,t)}{\Omega(m,t)} = \left[\frac{-\lambda c - \dfrac{\lambda 16 m^2\pi^2 c}{nL\left\{\left[(n+m)^2 - m^2\right]\left[(n-m)^2 - m^2\right]\right\}}}{2}\right]$$

(3.59)

and

$$\dot{\phi}(m,t) = \left[-\frac{\lambda c^2 C^{o} m^2\pi^2}{L} - \frac{\lambda m^2\pi^2 c^2}{2L} - \lambda\theta_{nf}^2 Lc - \lambda\theta_{nf}^2\frac{L}{2}\right] \quad (3.60)$$

Solving equations (3.59) and (3.60) respectively yields

$$\Omega(m,t) = \psi e^{-C^{*}_{m}} \quad (3.61)$$

and

$$\phi(m,t) = \left[-\frac{\lambda c^2 C^{o} m^2\pi^2}{L} - \frac{\lambda m^2\pi^2 c^2}{2L} - \lambda\theta_{nf}^2 Lc - \lambda\theta_{nf}^2\frac{L}{2}\right]t + \psi^{*} \quad (3.62)$$

Where

$$C_m^* = \frac{-\lambda c - \dfrac{\lambda 16 m^2 \pi^2 c}{nL\left\{\left[(n+m)^2 - m^2\right]\left[(n-m)^2 - m^2\right]\right\}}}{2} \tag{3.63}$$

and ψ, ψ^* and C_m^* are constants

Therefore, when the mass effect of the particle is considered, the first approximation to the homogeneous system is given by

$$\overline{W}(m,t) = C_m^* Cos\left[\theta_{mm} t - \phi(m,t)\right] \tag{3.64}$$

where

$$\theta_{mm} = \theta_{mt}\left[1 - \frac{\lambda}{2}\left\{\theta_{mt}\left(\frac{L}{2m^2\pi^2} + LC''\right) - \sum_{n=1}^{\infty}\left(\frac{\theta_{mt}}{n\pi m} - \frac{c^2}{\theta_{mt} L}\right) + \frac{m^2\pi^2 c^2}{\theta_{mt} L}\right\}\right]$$

(3.65)

Equation (3.65) is called the modified frequency corresponding to the frequency of the simply supported system due to the presence of moving mass. Thus, the homogeneous part of (3.51) can be expressed as,

$$\frac{d^2}{dt^2}\overline{W}(m,t) + \theta_{mm}^2 \overline{W}(m,t) = 0 \tag{3.66}$$

Hence, the entire equation (3.45) takes the form,

$$\frac{d^2}{dt^2}\overline{W}(m,t) + \theta_{mm}^2 \overline{W}(m,t) = \frac{PL}{\mu m\pi}\left[-(-1)^m + Cos\frac{m\pi ct}{L}\right] \tag{3.67}$$

Clearly, equation (3.67) is analogous to equation (3.32) and solving yields

$$\overline{W}(m,t) = \frac{PL}{\mu m\pi}\left[\frac{(-1)^m(1 - Cos\theta_{mm} t)}{\theta_{mm}} - \frac{Cos\omega_k t - Cos\theta_{mm} t}{\theta_{mm}^2 - \omega_m^2}\right] \tag{3.68}$$

which on inversion yields,

$$W(x,t) = 2\sum_{m=1}^{\infty} \frac{\lambda L^2 g}{m\pi}\left[\frac{(-1)^m(1-Cos\theta_{mm})}{\theta_{mm}} - \frac{(Cos\omega_k t - Cos\theta_{mm})}{\theta_{mm}^2 - \omega_m^2}\right] \times Sin\frac{m\pi x}{L} \qquad (3.69)$$

Equation (3.69) represents the transverse-displacement response to a distributed mass moving with constant speed of simply supported uniform Bernoulli-Euler beam resting on elastic foundation.

3.1.2 Clamped/Fixed Ends Condition

At clamped-clamped ends, both deflection and slope vanish. Thus when the long thin beam is clamped at $x = 0$ and $x = L$, the condition are expressed as,

$$W(0,t) = 0 = W(L,t) \quad \text{and} \quad \frac{\partial}{\partial x}W(0,t) = 0 = \frac{\partial}{\partial x}W(L,t) \qquad (3.70)$$

And for normal modes,

$$U_m(0) = 0 = U_m(L) \quad \text{and} \quad \frac{\partial}{\partial x}U_m(0) = 0 = \frac{\partial}{\partial x}U_m(L) \qquad (3.71)$$

which implies that,

$$U_k(0) = 0 = U_k(L) \quad \text{and} \quad \frac{\partial}{\partial x}U_k(0) = 0 = \frac{\partial}{\partial x}U_k(L) \qquad (3.72)$$

Thus it can be shown that

$$A_m = \frac{Sinh\lambda_m - Sin\lambda_m}{Cos\lambda_m - Cosh\lambda_m} = \frac{Cos\lambda_m - Cosh\lambda_m}{Sin\lambda_m + Sinh\lambda_m} = -C_m \quad \text{and} \quad B_m = -1 \qquad (3.73)$$

In view of (3.73), the frequency equation is given as

$$Cos\lambda_m Cosh\lambda_m = 1 \qquad (3.74)$$

It follows from equation (3.74) that

$$\lambda_1 = 4.73004, \qquad \lambda_2 = 7.85320, \qquad \lambda_3 = 10.99561, \qquad (3.75)$$

Substituting (3.73) and (3.75) into equations (2.117) and (2.144) one obtains the displacement response respectively to a moving force and a moving mass of a Clamped/fixed long thin elastic beam resting on elastic foundation.

3.1.3 One End Clamped and One End free Condition-Cantilever beam

Next at $x = 0$ the beam is taken to be clamped at the end $L = 0$, the beam is free. Thus, the boundary conditions of the Bernoulli-Euler beam can be written as,

$$W(0,t) = 0 = \frac{\partial}{\partial x}W(0,t) \quad \text{and} \quad \frac{\partial^2}{\partial x^2}W(L,t) = 0 = \frac{\partial^3}{\partial x^3}W(L,t) \tag{3.76}$$

And for normal modes,

$$U_m(0) = 0 = \frac{d}{dx}U_m(0) \quad \text{and} \quad \frac{d^2}{\partial x^2}U_m(L) = 0 = \frac{d^3}{dx^3}U_m(L) \tag{3.77}$$

which implies that,

$$U_k(0) = 0 = \frac{d}{dx}U_k(0) \text{ and } \frac{d^2}{dx^2}U_k(L) = 0 = \frac{d^3}{dx^3}U_k(L) \tag{3.78}$$

Using (3.78), we can show that at $x = 0$,

$$A_m = C_m \quad \text{and} \quad B_m = -1 \tag{3.79}$$

$$A_m = -\frac{Sin\lambda_m - Sinh\lambda_m}{Cos\lambda_m + Cosh\lambda_m} = \frac{Cos\lambda_m - Cosh\lambda_m}{Sinh\lambda_m - Sin\lambda_m} = -C_m \text{ and } B_m = -1 \tag{3.80}$$

and the frequency equation for both end conditions is

$$Cos\lambda_m Cosh\lambda_m = -1 \tag{3.81}$$

such that

$$\lambda_1 = 1.875, \qquad \lambda_2 = 4.694, \qquad \lambda_3 = 7.855, \tag{3.82}$$

Using (3.79), (3.80) and (3.82) in equation (2.117) and (2.144), one obtains the displacement response respectively to a moving force and a moving mass of a uniform clamped-free ends of Bernoulli-Euler beam resting on elastic foundation.

3.2.0 DISCUSSION OF THE ANALYTICAL SOLUTIONS

If the undamped system such as this is studied, it is desirable to examine the response amplitude of the dynamical system which may grow without bound. Thus we called this resonance conditions. Equation (3.44) clearly shows that the simply supported elastic beam resting on elastic foundation and traversed by moving force reaches a state of resonance whenever

$$\theta_{mf} = \frac{m\pi c}{L} \tag{3.83}$$

while equation (3.69) indicates that the same result beam under the action of a moving mass experiences resonance effect when

$$\theta_{mf} = \frac{m\pi c}{L} \tag{3.84}$$

from equation(3.65).

$$\theta_{mm} = \theta_{mf}\left\{1-\frac{\lambda}{2}\left\{\left[\left(\frac{L}{2m^2\pi^2}+LC^{o}\right)-\sum_{n=1}^{\infty}\left(\frac{\theta_{mf}}{n\pi m}-\frac{c^2}{\theta_{mf}L}\right)+\frac{m^2\pi^2c^2}{\theta_{mf}L}\right]\right\}\right\} \qquad (3.85)$$

which implies

$$\theta_{mm} = \frac{\frac{m\pi c}{L}}{\left[1-\frac{\lambda}{2}\left\{\left[\left(\frac{L}{2m^2\pi^2}+LC^{o}\right)-\sum_{n=1}^{\infty}\left(\frac{\theta_{mf}}{n\pi m}-\frac{c^2}{\theta_{mf}L}\right)+\frac{m^2\pi^2c^2}{\theta_{mf}L}\right]\right\}\right]} \qquad (3.86)$$

From equation (3.86 it is deduced that for the same natural frequency, the critical speed for the system consisting of a simply supported elastic beam resting on an elastic foundation and traversed by a force moving with a uniform speed is greater than that of the moving mass problem. Thus, for the same natural frequency of an elastic beam, resonance is reached earlier in the moving mass system than in the moving force system.

For the resonance conditions of other classical boundary conditions, equation (2.117) clearly shows that the uniform elastic beam resting on an elastic foundation and traversed by a force moving with a constant speed reaches a state of resonance whenever

$$\theta_{mf} = \frac{m\pi c}{L} \qquad (3.87)$$

while equation (2.144) shows that same beam under the action of a moving mass experiences resonance effect whenever

$$\theta_{bm} = \frac{m\pi c}{L} \tag{3.88}$$

From equation (2.141)

$$\theta_{bm} = \theta_{am}\left[1-\frac{\eta}{2}\left\{\left(S_2(m,m)+LC^{\circ}S_4(m,m)\right)-\frac{\left(Lc^2C^{\circ}S_{10}(m,m)+c^2S_8(m,m)\right)}{\theta_{am}^2}\right\}\right] \tag{3.89}$$

which implies

$$\theta_{am} = \frac{m\pi c/L}{1-\frac{\eta}{2}\left\{\left(S_2(m,m)+LC^{\circ}S_4(m,m)\right)-\frac{\left(Lc^2C^{\circ}S_{10}(m,m)+c^2S_8(m,m)\right)}{\theta_{am}^2}\right\}} \tag{3.90}$$

Evidently, from (3.88) and (3.89), the same results and analyses obtained in the case of a simply supported Bernoulli-Euler beam are obtained for all other examples of classical boundary conditions.

3.3.0 NUMERICAL CALCULATION AND DISCUSSION OF RESULTS

For the purpose of Numerical analysis of our dynamical system, the uniform beam of length 12.192m is considered. Also, $\frac{EI}{\mu} = 2200m^4/s^2$, speed of the mass is 8.128 m/s and the ratio of the mass of the load to the beam is 0.25. The transverse deflections of the beam are calculated and plotted against time for various values of axial force N and subgrade K. Values of N between 0 and 20,000,000 were used while the values of K were varied between 0 N/m^3 and 400,000 N/m^3. The results are as shown on the various graphs below for the various classical boundary conditions considered.

3.3.1 Simply Supported Ends

Figure 3.1 displays transverse displacement response of a simply supported uniform beam under the action of distributed forces moving at variable velocities for various values of axial force N for fixed values of foundation moduli K=40,000. The figure shows that as N increases the deflection of the uniform beam decreases. The same result is obtained when the simply supported is traversed by a distributed masses moving at constant speed as shown in figure 3.3. Also, for various time t, the deflection profile of the beam for various values of foundation moduli K and for fixed axial force N are shows in figure 3.2. It is observed that higher values of foundation moduli reduce the deflection profile of the beam. The same behaviour characterizes the deflection profile of the simply supported beam under the action of distributed masses moving at constant velocity for various values of foundation moduli as shown in figure 3.4.

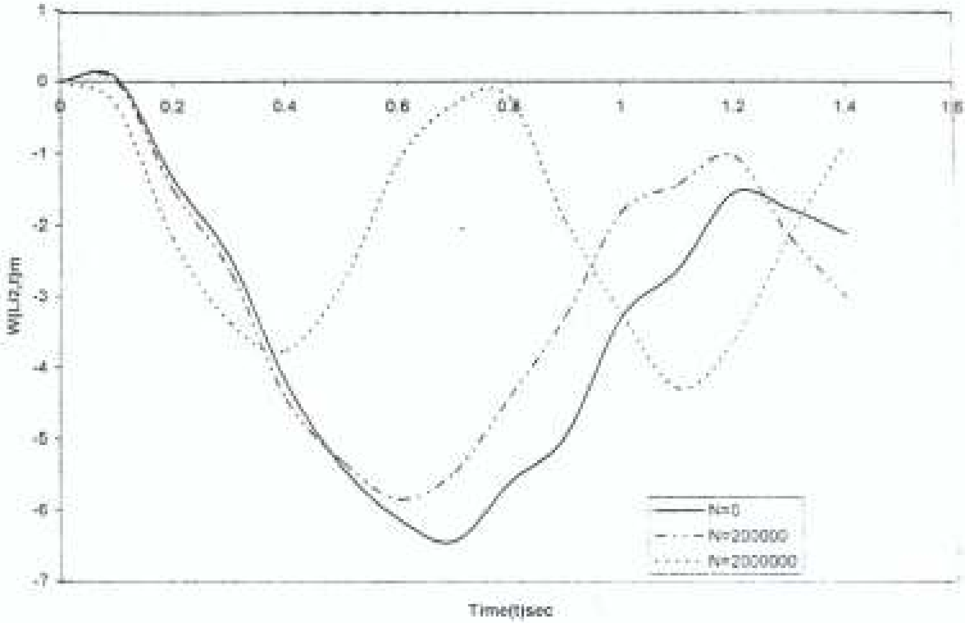

Fig 3.1: Transverse displacement of the simply supported beam under the action of forces moving at constant velocity for various values of axial force N for fixed value of foundation moduli K (40000).

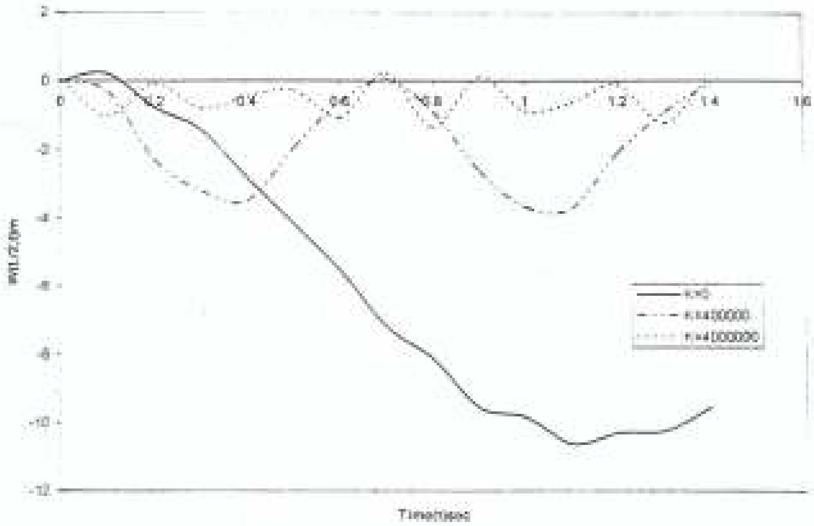

Fig 3.2: Deflection profile of the simply supported beam under the action of force moving at constant velocity for various values of foundation moduli K for fixed value of axial force N (20000).

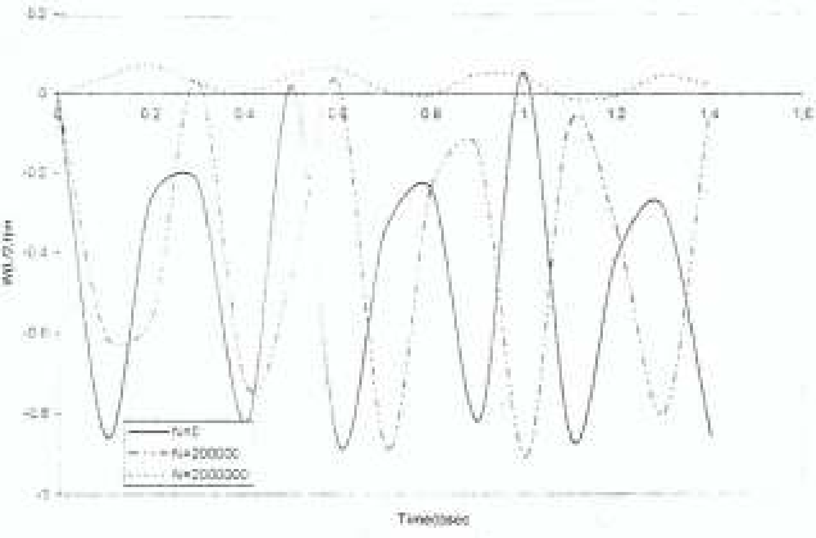

Fig 3.3: Transverse displacement of the simply supported beam under the action of distributed masses moving at constant velocity for various values of axial force N for fixed value of foundation moduli K (40000).

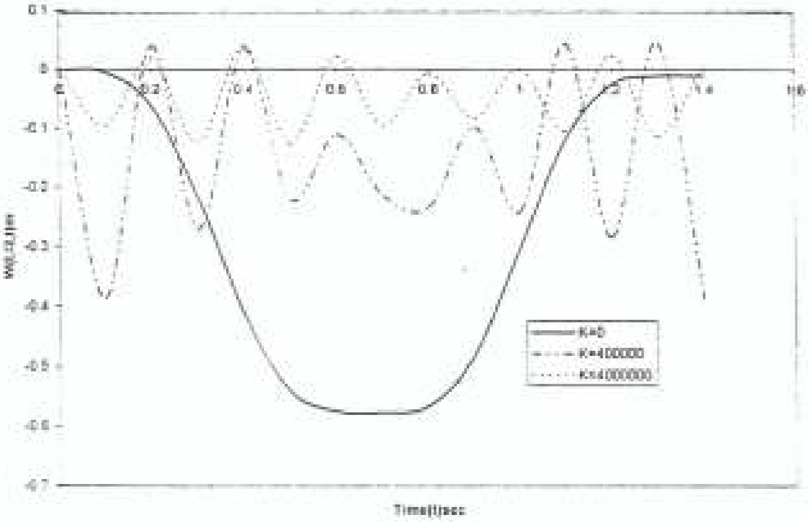

Fig 3.4: Deflection profile of the simply supported beam under the action of distributed masses moving at constant velocity for various values foundation moduli K for fixed value of axial force N (200000).

Finally, figure 3.5 depicts the comparison of the transverse displacement of moving force and moving mass cases for simply supported uniform beam traversed by a moving distributed load moving at constant velocities for N=200,000 and K=40,000.Clearly, the response amplitude of moving distributed mass is higher than that of the moving distributed force. This important results agrees with the existing result for cases when the traveling load is moving at variable velocities.

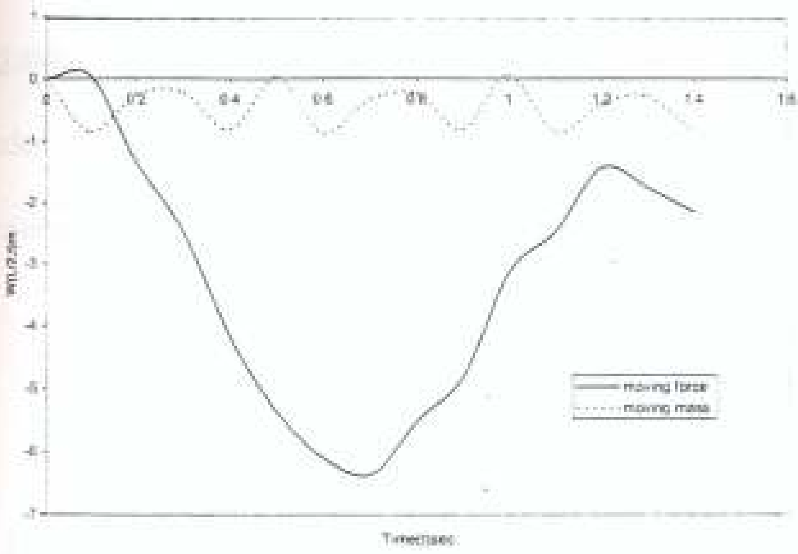

Fig 3.5: Comparison of the displacement response of moving force and moving mass cases for simply supported beam for N=200000 and K=40000

3.3.2 Clamped Ends

Figure 3.6 displays transverse displacement response of a clamped-clamped uniform beam under the action of distributed forces moving at constant velocity for various values of axial force N for fixed value of foundation moduli K=40,000. The figure shows that as N increases the deflection of the uniform beam decreases. The same results is obtained when the simply supported beam is traversed by a distributed mass moving at constant speed as shown in figure3.8. Also, for various time t, the deflection profile of the beam for various values of foundation moduli K and for fixed axial force N are shown in figure 3.7. It is shown that higher values of foundation moduli reduce the deflection profile of the beam. The same behaviour characterizes the deflection profile of the clamped-clamped beam under the action of distributed masses moving at constant velocity for various values of foundation moduli as shown in figure 3.9.

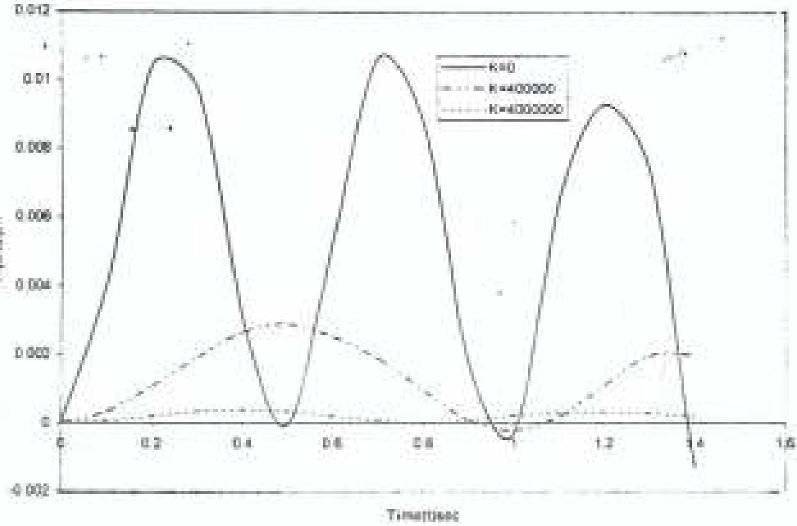

Fig 3.6:Deflection profile of the clamped-clamped uniform beam under the action of distributed forces moving at constant velocity for various values of foundation moduli K and for fixed value of axial force N (200000).

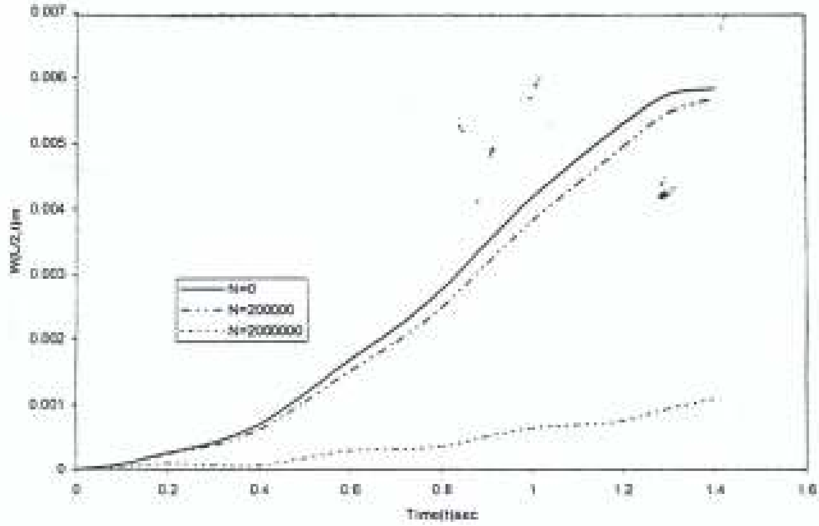

Fig 3.7: Transverse displacement of the clamped-clamped uniform beam under the action distributed masses moving at constant velocity for various values of axial force N and for fixed value of foundation moduli K (40000).

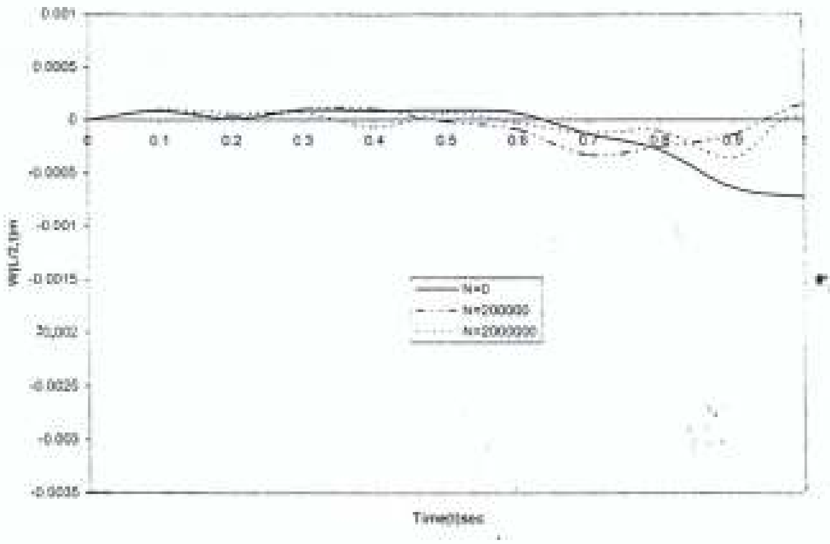

Fig 3.8: Deflection profile of the clamped-clamped uniform beam under the action of distributed masses moving at constant velocity for various values of foundation moduli K and for fixed value of axial force N (200000).

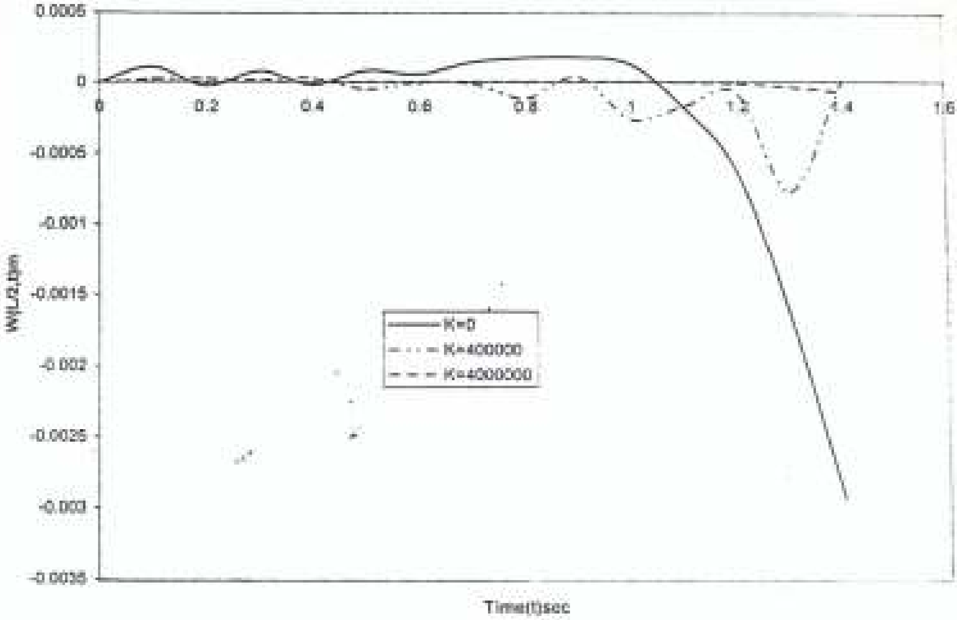

Fig 3.9:Deflection profile of the clamped-clamped uniform beam under the action of distributed masses moving at constant velocity for various values of foundation moduli K and for fixed value of axial force N (200000).

Finally, figure 3.10 depicts the comparison of the transverse displacement of moving force and moving mass cases for clamped-clamped uniform beam traversed by a moving distributed load moving at variable velocity for fixed N=200,000 and K=40,000. Clearly, the response amplitude of moving mass is higher than that of the moving force. This is in conformity with what obtained in [6,10,15] that relying on the moving force problem as a good approximation to a moving mass problem is not only misleading but tragic.

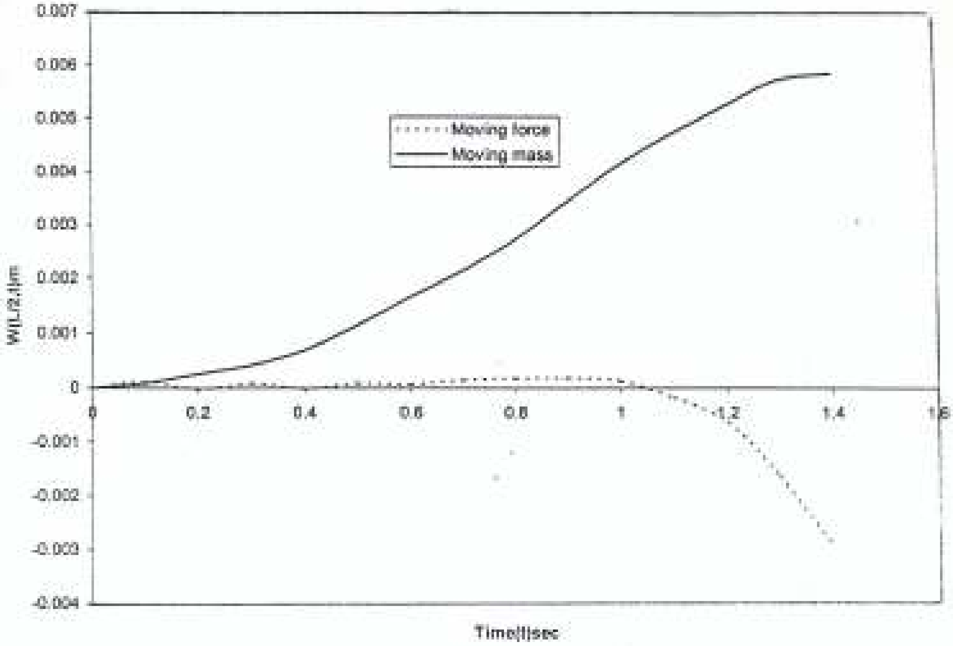

Fig 3.10: Comparison of the displacement response of moving distributed force and moving distributed mass cases for clamped-clamped beam for N=200000 and K=40000.

3.3.3 One End Clamped and One End Free.

In figure 3.11, the transverse response of a cantilever uniform beam under the action of distributed force moving at constant velocity for various values of axial force N and for fixed value of foundation modulus K=40,000 is displayed. The figure shows that as N increases, the deflection of the uniform beam decreases.

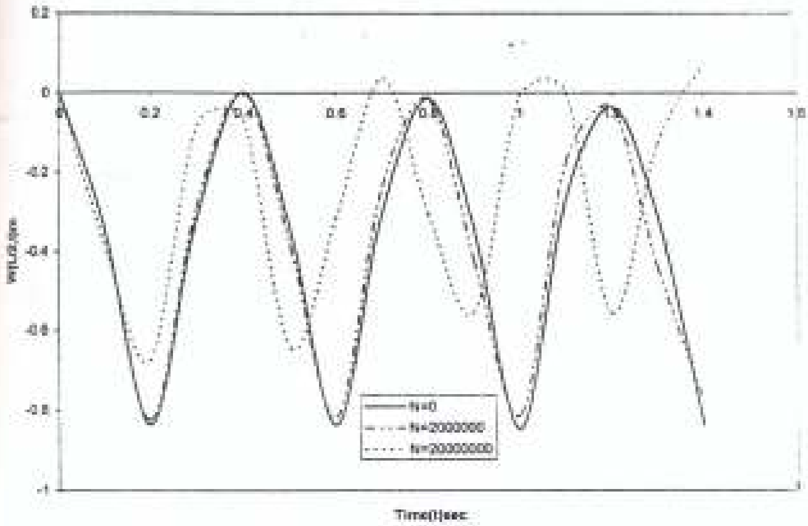

Fig 3.11: Transverse displacement of the clamped-free uniform beam under the action of distributed forces moving at constant velocity for various values of axial force N for fixed value of foundation moduli K (40000).

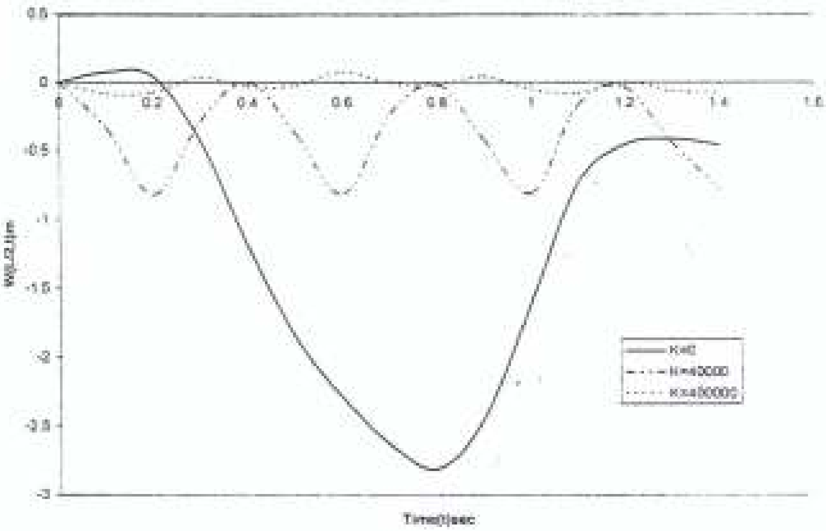

Fig 3.12:Deflection profile of the clamped-free uniform beam under the action of distributed forces moving at constant velocity for various values foundation moduli K and for fixed value of axial force N (200000).

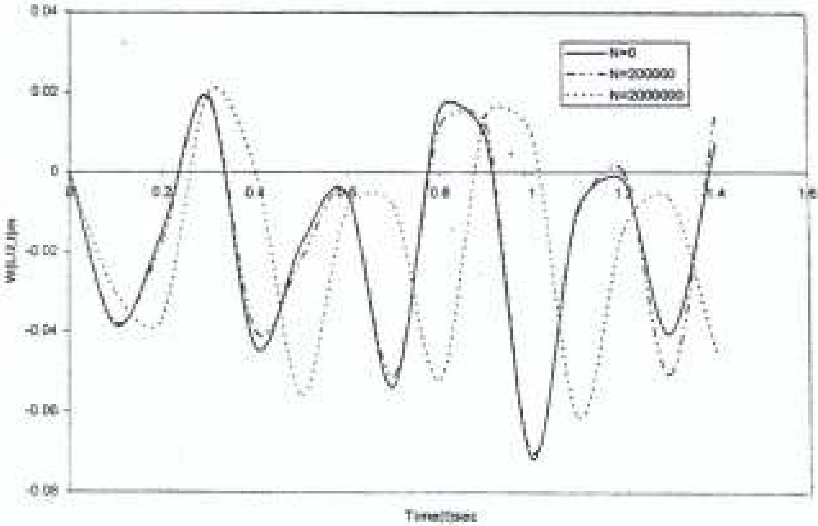

Fig 3.13: Transverse displacement of the clamped-free Uniform beam under the action distributed masses moving at constant velocity for various values of axial force N for fixed value of foundation moduli K (40000).

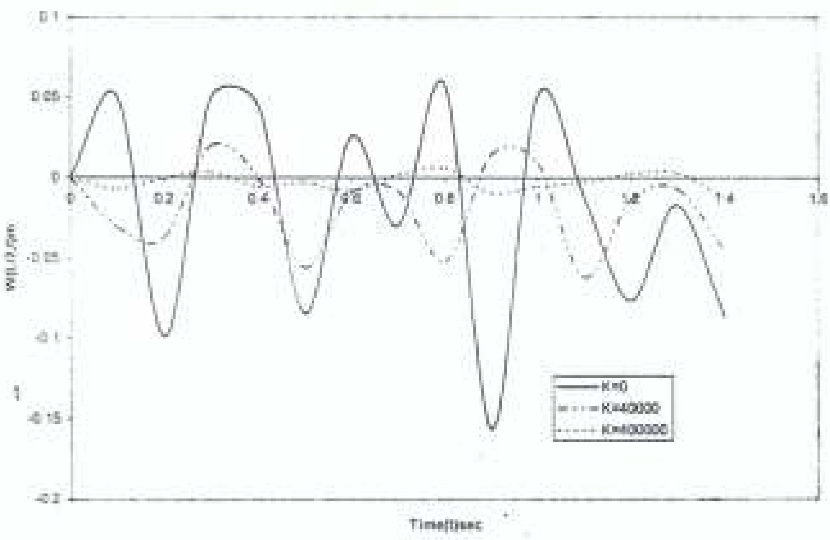

Fig 3.14: Deflection profile of the clamped-free uniform beam under the action of distributed masses moving at constant velocity for various values of foundation moduli K and for fixed value of axial force N (200000).

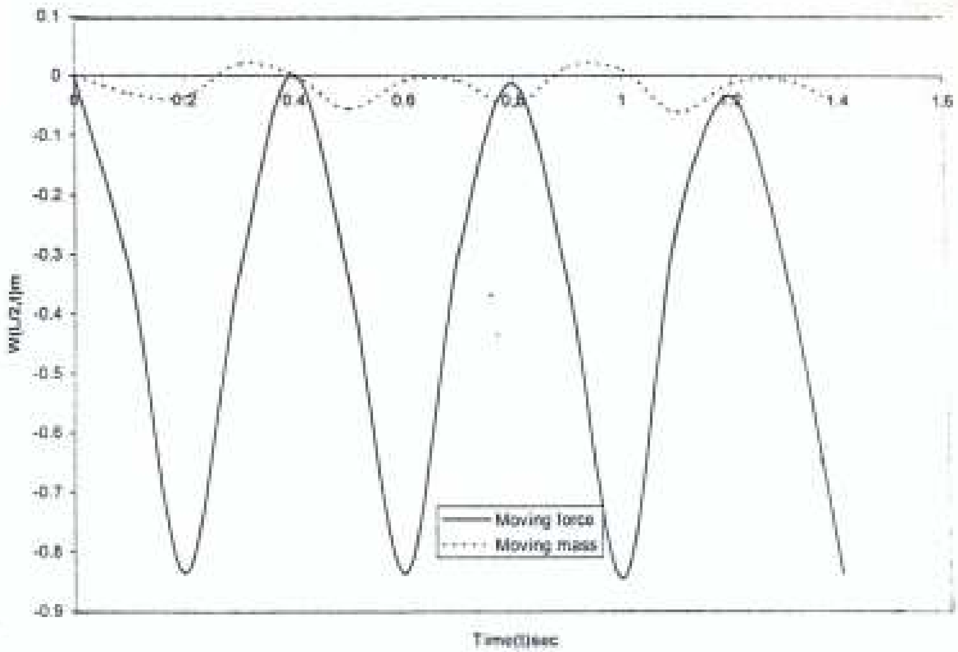

Fig 3.15: Comparison of the displacement response of moving force and moving mass cases for clamped-free beam for N=200000 and K=40000

The same result is obtained when the cantilever beam is traversed by distributed masses moving at constant speed as shown in figure 3.12. Also, for various time t, the deflection profiles of the beam for various values of foundation modulus K and for fixed axial force N are shown in figure 3.13. It is shown that as foundation modulus increases, the deflection profile of the beam decreases. The same behaviour characterizes the deflection profile of the cantilever beam under the action of a distributed mass moving at constant velocity for various values of foundation moduli as shown in figure 3.14.

Finally, figure 3.15 depicts the comparison of the transverse displacement of moving force and moving mass cases for cantilever uniform beam traversed by moving distributed load moving at constant velocity for N=200,000 and K=40,000. Clearly, the response amplitude of moving mass is higher than that of the moving force.

CHAPTER FOUR

GENERAL CONCLUSION

4.1 SUMMARY OF RESEARCH WORK

The problem of dynamical analysis of finite prestressed Bernoulli-Euler beam with general boundary conditions when it is under the action of transverse travelling loads is considered in this thesis. The governing equation is a non-homogeneous fourth order partial differential equation with variable and singular coefficients. At the right hand side is the so-called Heaviside function which describes the arrival of a continuous load distributed along the beam. The main objective is to obtain a closed form solution valid for all variants of classical boundary conditions to the cumbersome partial differential equations. To this end, the Heaviside function is expressed in series form and the partial differential equation subjected to generalized integral transformation. The resulting second order differential equation is then solved using the modified Struble's Asymptotic technique and the method of integral transformations.

The theory is then illustrated using some examples of Classical boundary conditions commonly encountered in engineering practice. These are

(i) Simply supported end conditions

(ii) Clamped-Clamped end conditions

(iii) Cantilever or Clamped-free end conditions

Analysis of the closed form solutions obtained is carried out for all the illustrative examples. The resonance conditions for the various end conditions are obtained.

The influence of the axial force and foundation moduli on the dynamic response to moving force and moving mass of prestressed Bernouli-Euler beams when it is under the actions of moving distributed loads is investigated. The transverse displacement for all the illustrative examples are calculated and presented in ploted curves.

The major findings from the various analyses are summarized as follows:

(i) For all the four illustrative examples considered, the moving force solution is not an upper bound for the accurate solution of the moving mass solution for uniform Bernoulli-Euler beams problems.

(ii) As the axial force N increases, the amplitude of uniform Bernoulli-Euler beams under the action of moving loads moving with constant velocities decreases.

(iii) When the axial force N is fixed, the displacement of a uniform Bernoulli-Euler beam resting on elastic foundation and traversed by distributed masses travelling with constant speed decreases as the foundation moduli increases for all variants of the boundary conditions.

(iv) For fixed axial force and foundation modulus, the response amplitude for the moving mass problem is greater than that of the moving force problem for all illustrative end conditions considered when the beam is uniform.

(v) Higher values of axial force N and foundation modulus K are required for a more noticeable effect in the case of other boundary conditions than those of simply supported boundary conditions for both moving mass problem of uniform beams.

(iv)In all the illustrative examples considered, for the same natural frequency, the critical speed for moving mass problem is smaller than that of the moving force problem. Hence, resonance is reached earlier in moving mass problem.

Finally, this work has suggested valuable method of analytical solutions for this category of problems for all variants of classical boundary conditions.

6.2 CONTRIBUTION TO KNOWLEDGE

Analytical solutions have been provided for problems of finite prestressed Bernuolli- Euler Beams under travelling distributed loads for all variants of classical boundary condition and analysis have indicated

(i) the resonance conditions for both moving force and moving mass problems of the Bernoulli-Euler beam model.

(ii) the effect of axial force on uniform Bernoulli-Euler beam when under the actions of distributed masses or forces moving at constant speed.

(iii) the influence of the foundation modulus on the displacement response of uniform Bernoulli-Euler beam model.

(iv)the reliability of the moving force solution as a safe approximation to the moving mass problem for all variants of classical boundary conditions.

6.3 LIMITATION TO THE STUDY AND RECOMMENDATION FOR FURTHER RESEARCH

The main objective of this study is the dynamical analysis of a prestressed Bernoulli-Euler beam resting on an elastic foundation and traversed by distributed masses moving at constant velocities.

Illustrative examples have been limited to classical boundary conditions only. Non-classical boundary conditions such as (i) elastically supported end conditions. (ii)time dependent boundary conditions are not considered and as such are suggested for further research. The two-dimensional analogue of the theory developed in this thesis is left for further research.

Structures such as beam or plate on the other foundation models are left for further research. Other beam model under the action of moving loads such as shear beams, Rayleigh beams and Timoshenko beams are not considered in this study.

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APPENDIX

This appendix presents the solutions of the definite integrals listed in chapters two of this thesis.

$$I_1 = \begin{cases} \frac{L}{2}\left[\frac{Sin(\lambda_k - \lambda_m)}{\lambda_k - \lambda_m} - \frac{Sin(\lambda_k + \lambda_m)}{\lambda_k + \lambda_m}\right] & ,\lambda_k \neq \lambda_m \\ \frac{L}{2}\left[1 - \frac{Sin2\lambda_m}{2\lambda_m}\right] & ,\lambda_k = \lambda_m \end{cases}$$

$$I_2 = \begin{cases} \frac{-L}{2}\left[\frac{(\lambda_k - \lambda_m)(Cos(\lambda_k + \lambda_m) - 1) + (\lambda_k + \lambda_m)(Cos(\lambda_k - \lambda_m) - 1)}{\lambda_k^2 - \lambda_m^2}\right] & ,\lambda_k \neq \lambda_m \\ \frac{-L}{2}\left[\frac{Cos2\lambda_m - 1}{2\lambda_m}\right] & ,\lambda_k = \lambda_m \end{cases}$$

$$I_3 = \frac{\lambda_m L}{\lambda_m^2 + \lambda_k^2}\left[Sin\lambda_k Cosh\lambda_m - \frac{\lambda_k}{\lambda_m} Cos\lambda_k Sinh\lambda_m\right]$$

$$I_4 = \frac{\lambda_m L}{\lambda_m^2 + \lambda_k^2}\left[Sin\lambda_k Sinh\lambda_m - \frac{\lambda_k}{\lambda_m}(Cos\lambda_k Cosh\lambda_m - 1)\right]$$

$$I_5 = \begin{cases} \frac{L}{2}\left[\frac{(\lambda_k + \lambda_m)(Cos(\lambda_k - \lambda_m) - 1) + (\lambda_k - \lambda_m)(1 - Cos(\lambda_k + \lambda_m))}{\lambda_k^2 - \lambda_m^2}\right] & ,\lambda_k \neq \lambda_m \\ \frac{-L}{2}\left[\frac{Cos2\lambda_m - 1}{2\lambda_m}\right] & ,\lambda_k = \lambda_m \end{cases}$$

$$I_6 = \begin{cases} \frac{L}{2}\left[\frac{Sin(\lambda_k + \lambda_m)}{\lambda_k + \lambda_m} + \frac{Sin(\lambda_k - \lambda_m)}{\lambda_k - \lambda_m}\right] & ,\lambda_k \neq \lambda_m \\ \frac{L}{2}\left[1 + \frac{Sin2\lambda_m}{2\lambda_m}\right] & ,\lambda_k = \lambda_m \end{cases}$$

$$I_7 = \frac{\lambda_m L}{\lambda_m^2 + \lambda_k^2}\left[Cos\lambda_k Cosh\lambda_m + \frac{\lambda_k}{\lambda_m} Sin\lambda_k Sinh\lambda_m - 1\right]$$

$$I_8 = \frac{\lambda_m L}{\lambda_m^2 + \lambda_k^2}\left[Cos\lambda_k Sinh\lambda_m + \frac{\lambda_k}{\lambda_m} Sin\lambda_k Cosh\lambda_m\right]$$

$$I_9 = \frac{\lambda_k L}{\lambda_k^2 + \lambda_m^2}\left[Sin\lambda_m Cosh\lambda_k - \frac{\lambda_m}{\lambda_k} Cos\lambda_m Sinh\lambda_k\right]$$

$$I_{10} = \frac{\lambda_k L}{\lambda_k^2 + \lambda_m^2}\left[Cos\lambda_m Cosh\lambda_k + \frac{\lambda_m}{\lambda_k} Sin\lambda_m Sinh\lambda_k - 1\right]$$

$$I_{11} = \begin{cases} \frac{L}{2}\left[\frac{Sinh(\lambda_k + \lambda_m)}{\lambda_k + \lambda_m} + \frac{Sinh(\lambda_k - \lambda_m)}{\lambda_k - \lambda_m}\right] & ,\lambda_k \neq \lambda_m \\ \frac{L}{2}\left[\frac{Sinh 2\lambda_m}{2\lambda_m} - 1\right] & ,\lambda_k = \lambda_m \end{cases}$$

$$I_{12} = \begin{cases} \frac{L}{2}\left[\frac{(\lambda_k - \lambda_m)(Cosh(\lambda_k + \lambda_m) - 1) + (\lambda_k + \lambda_m)(Cosh(\lambda_k - \lambda_m) - 1)}{\lambda_k^2 + \lambda_m^2}\right] & ,\lambda_k \neq \lambda_m \\ \frac{L}{2}\left[\frac{Cosh 2\lambda_m - 1}{2\lambda_m}\right] & ,\lambda_k = \lambda_m \end{cases}$$

$$I_{13} = \frac{\lambda_k L}{\lambda_k^2 + \lambda_m^2}\left[Sin\lambda_m Sinh\lambda_k - \frac{\lambda_m}{\lambda_k}(Cos\lambda_m Cosh\lambda_k - 1)\right]$$

$$I_{14} = \frac{\lambda_k L}{\lambda_k^2 + \lambda_m^2}\left[Cos\lambda_m Sinh\lambda_k + \frac{\lambda_m}{\lambda_k} Sin\lambda_m Cosh\lambda_k\right]$$

$$I_{15} = \begin{cases} \frac{L}{2}\left[\frac{(\lambda_m - \lambda_k)(Cosh(\lambda_m + \lambda_k) - 1) + (\lambda_m + \lambda_k)(Cosh(\lambda_m - \lambda_k) - 1)}{\lambda_m^2 + \lambda_k^2}\right] \\ \frac{L}{2}\left[\frac{Cosh 2\lambda_m - 1}{2\lambda_m}\right] \end{cases}$$

$$I_{16}=\begin{cases}\dfrac{L}{2}\left[\dfrac{Sinh(\lambda_k+\lambda_m)}{\lambda_k+\lambda_m}+\dfrac{Sinh(\lambda_k-\lambda_m)}{\lambda_k-\lambda_m}\right] & ,\lambda_k\neq\lambda_m\\ \dfrac{L}{2}\left[\dfrac{Sinh2\lambda_m}{2\lambda_m}+1\right] & ,\lambda_k=\lambda_m\end{cases}$$

$$I_{17}=\begin{cases}\dfrac{L^2}{2(\lambda_k-\lambda_m)^2}\left[(\lambda_k-\lambda_m)Sin(\lambda_k-\lambda_m)+(Cos(\lambda_k-\lambda_m)-1)\right] & \\ -\dfrac{L^2}{2(\lambda_k+\lambda_m)^2}\left[\lambda_k+\lambda_m)Sin(\lambda_k+\lambda_m)+(Cos(\lambda_k+\lambda_m)-1)\right] & ,\lambda_k\neq\lambda_m\\ \dfrac{L^2}{4}-\dfrac{L^2}{2(\lambda_k+\lambda_m)^2}\left[\lambda_k+\lambda_m)Sin(\lambda_k+\lambda_m)+(Cos(\lambda_k+\lambda_m)-1)\right] & ,\lambda_k=\lambda_m\end{cases}$$

$$I_{18}=\begin{cases}-\dfrac{L^2}{2}\left[\dfrac{(\lambda_k+\lambda_m)Cos(\lambda_k-\lambda_m)-Sin(\lambda_k-\lambda_m)}{(\lambda_k+\lambda_m)^2}\right.\\ \left.+\dfrac{(\lambda_k+\lambda_m)Cos(\lambda_k-\lambda_m)-Sin(\lambda_k-\lambda_m)}{(\lambda_k+\lambda_m)^2}\right] & ,\quad\lambda_k\neq\lambda_m\\ -\dfrac{L^2}{2}\left[\dfrac{2\lambda_k Cos2\lambda_k-Sin2\lambda_k}{4\lambda_k^2}\right] & ,\quad\lambda_k=\lambda_m\end{cases}$$

$$I_{19}=\frac{L^2}{(\lambda_k^2+\lambda_m^2)^2}\Big[(\lambda_k^2+\lambda_m^2)\lambda_m Sin\lambda_k Cosh\lambda_m-(\lambda_k^2+\lambda_m^2)\lambda_k Cos\lambda_k Sinh\lambda_m$$
$$+(\lambda_k^2-\lambda_m^2)\lambda_m Sin\lambda_k Sinh\lambda_m+2\lambda_m\lambda_k Cos\lambda_k Cosh\lambda_m-2\lambda_m\lambda_k\Big]$$

$$I_{20}=\frac{L^2}{(\lambda_k^2+\lambda_m^2)^2}\Big[(\lambda_k^2+\lambda_m^2)\lambda_m Sin\lambda_k Sinh\lambda_m-(\lambda_k^2+\lambda_m^2)\lambda_k Cos\lambda_k Cosh\lambda_m$$
$$-2\lambda_m\lambda_k Cos\lambda_k Sinh\lambda_m+(\lambda_k^2+\lambda_m^2)Sin\lambda_k Cosh\lambda_m\Big]$$

$$I_{21}=\frac{-L^2}{2}\left[\frac{(\lambda_m-\lambda_k)Cos(\lambda_m-\lambda_k)-Sin(\lambda_m-\lambda_k)}{(\lambda_m-\lambda_k)^2}\right.$$
$$\left.+\frac{(\lambda_m+\lambda_k)Cos(\lambda_m+\lambda_k)-Sin(\lambda_m-\lambda_k)}{(\lambda_m+\lambda_k)^2}\right]$$

$$I_{22}=\begin{cases}\dfrac{L^2}{2(\lambda_k+\lambda_m)^2}\left[(\lambda_k+\lambda_m)Sin(\lambda_k+\lambda_m)+(Cos(\lambda_k+\lambda_m)-1)\right] \\ +\dfrac{L^2}{2(\lambda_k-\lambda_m)^2}\left[\lambda_k-\lambda_m)Sin(\lambda_k-\lambda_m)+(Cos(\lambda_k-\lambda_m)-1)\right] & ,\lambda_k\neq\lambda_m \\ \dfrac{L^2}{4}+\dfrac{L^2}{2(2\lambda_k)^2}\left[2\lambda_k Sin2\lambda_k+Cos(2\lambda_k)-1)\right] & ,\lambda_k=\lambda_m\end{cases}$$

$$I_{23}=\frac{L^2}{(\lambda_k^2+\lambda_m^2)^2}\Big[(\lambda_m^2+\lambda_k^2)\lambda_k Sin\lambda_k Sinh\lambda_m+(\lambda_m^2+\lambda_k^2)\lambda_k Cos\lambda_k Cosh\lambda_m$$
$$-2\lambda_m\lambda_k Sin\lambda_k Cosh\lambda_m-(\lambda_m^2-\lambda_k^2)Cos\lambda_k Sinh\lambda_m\Big]$$

$$I_{24}=\frac{L^2}{(\lambda_m^2+\lambda_k^2)^2}\Big[(\lambda_m^2+\lambda_k^2)\lambda_k Sin\lambda_k Cosh\lambda_m+(\lambda_m^2+\lambda_k^2)\lambda_m Cos\lambda_k Sinh\lambda_m$$
$$-2\lambda_m\lambda_k Sin\lambda_k Sinh\lambda_m-(\lambda_m^2-\lambda_k^2)Cos\lambda_k Sinh\lambda_m-(\lambda_k^2-\lambda_m^2)\Big]$$

$$I_{25}=\frac{L^2}{(\lambda_m^2+\lambda_k^2)^2}\Big[(\lambda_k^2+\lambda_m^2)\lambda_k Sin\lambda_m Cosh\lambda_k-(\lambda_k^2+\lambda_m^2)\lambda_m Cos\lambda_m Sinh\lambda_k$$
$$+(\lambda_m^2-\lambda_k^2)Sin\lambda_m Sinh\lambda_k+2\lambda_m\lambda_k Cos\lambda_m Cosh\lambda_k-2\lambda_m\lambda_k\Big]$$

$$I_{26}=\frac{L^2}{(\lambda_m^2+\lambda_k^2)^2}\Big[(\lambda_m^2+\lambda_k^2)\lambda_m Sin\lambda_m Sinh\lambda_k+(\lambda_m^2+\lambda_k^2)\lambda_k Cos\lambda_k Cosh\lambda_k$$
$$-2\lambda_m\lambda_k Sin\lambda_m Cosh\lambda_k-(\lambda_k^2-\lambda_m^2)Cos\lambda_m Sinh\lambda_k\Big]$$

$$I_{27}=\begin{cases}\dfrac{L^2}{2(\lambda_k+\lambda_m)^2}\left[(\lambda_k+\lambda_m)Sinh(\lambda_k+\lambda_m)-(Cosh(\lambda_k+\lambda_m)+1)\right] \\ -\dfrac{L^2}{2(\lambda_k-\lambda_m)^2}\left[(\lambda_k-\lambda_m)Sinh(\lambda_k-\lambda_m)-(Cosh(\lambda_k-\lambda_m)+1)\right] & ,\lambda_k\neq\lambda_m \\ \dfrac{L^2}{8\lambda_m^2}\left[2\lambda_m Sinh2\lambda_m-Cosh2\lambda_k-2\lambda_m^2+1\right] & ,\lambda_k=\lambda_m\end{cases}$$

$$I_{28}=\begin{cases}\dfrac{L^2}{2(\lambda_k^2-\lambda_m^2)^2}\left[(\lambda_k-\lambda_m)Cosh(\lambda_k+\lambda_m)+(\lambda_k+\lambda_m)(Cosh(\lambda_k+\lambda_m)\right] \\ -\dfrac{L^2}{2(\lambda_k^2-\lambda_m^2)^2}\left[(\lambda_k-\lambda_m)^2 Sinh(\lambda_k+\lambda_m)+(\lambda_k+\lambda_m)^2 Sinh(\lambda_k-\lambda_m)\right] & ,\lambda_k\neq\lambda_m \\ \dfrac{L^2}{8\lambda_m^2}\left[2\lambda_m Cosh2\lambda_m-Sinh2\lambda_k\right] & ,\lambda_k=\lambda_m\end{cases}$$

$$I_{29}=\frac{L^2}{(\lambda_k^2+\lambda_m^2)^2}\Big[(\lambda_m^2+\lambda_k^2)\lambda_k Sin\lambda_k Sinh\lambda_m-(\lambda_m^2+\lambda_k^2)\lambda_m Cos\lambda_m Cosh\lambda_k$$
$$+2\lambda_m\lambda_k Sinh\lambda_k Cos\lambda_m+(\lambda_m^2-\lambda_k^2)Cosh\lambda_k Sin\lambda_m\Big]$$

$$I_{30}=\frac{L^2}{(\lambda_k^2+\lambda_m^2)^2}\Big[(\lambda_m^2+\lambda_k^2)\lambda_m Sin\lambda_m Cosh\lambda_k+(\lambda_m^2+\lambda_k^2)\lambda_k Sinh\lambda_k Cosh\lambda_m$$
$$-2\lambda_m\lambda_k Sinh\lambda_k Sin\lambda_m-(\lambda_k^2-\lambda_m^2)Cosh\lambda_k Cosh\lambda_m-(\lambda_m^2-\lambda_k^2)\Big]$$

$$I_{31}=\begin{cases}\dfrac{L^2}{2(\lambda_m^2-\lambda_k^2)^2}\big[(\lambda_m-\lambda_k)Cosh(\lambda_m+\lambda_k)+(\lambda_m+\lambda_k)(Cosh(\lambda_m-\lambda_k)\big] & \\ -\dfrac{L^2}{2(\lambda_m^2-\lambda_k^2)^2}\big[(\lambda_m-\lambda_k)^2 Sinh(\lambda_m+\lambda_k)+(\lambda_m+\lambda_k)^2 Sinh(\lambda_m-\lambda_k)\big] & ,\lambda_k\neq\lambda_m \\ \dfrac{L^2}{8\lambda_k^2}\big[2\lambda_k Cosh2\lambda_k-Sinh2\lambda_k\big] & ,\lambda_k=\lambda_m\end{cases}$$

$$I_{32}=\begin{cases}\dfrac{L^2}{2(\lambda_k^2+\lambda_m^2)^2}\big[(\lambda_k+\lambda_m)Sinh(\lambda_k+\lambda_m)-(Cosh(\lambda_k+\lambda_m)+1\big] & \\ +\dfrac{L^2}{2(\lambda_k^2-\lambda_m^2)^2}\big[(\lambda_k-\lambda_m)^2 Sinh(\lambda_k+\lambda_m)-Cosh(\lambda_k-\lambda_m)+1\big] & ,\lambda_k\neq\lambda_m \\ \dfrac{L^2}{8\lambda_m^2}\big[2\lambda_m Sinh2\lambda_m+2\lambda_m^2-Cosh2\lambda_m+1\big] & ,\lambda_k=\lambda_m\end{cases}$$

$$I_{33}=\frac{L}{4}\Bigg[\frac{Cos(\lambda_k+\lambda_m+n\pi)-1}{\lambda_k+\lambda_m+n\pi}+\frac{Cos(\lambda_k+\lambda_m-n\pi)-1}{\lambda_k+\lambda_m-n\pi}-\frac{Cos(\lambda_k+\lambda_m+n\pi)-1}{\lambda_k+\lambda_m+n\pi}-\frac{Cos(\lambda_k-\lambda_m}{\lambda_k-\lambda_m}$$

$$I_{34}=\frac{L}{4}\Bigg[\frac{Sin(\lambda_k+n\pi+\lambda_m)}{\lambda_k+n\pi+\lambda_m}+\frac{Sin(\lambda_k+n\pi-\lambda_m)}{\lambda_k+n\pi-\lambda_m}-\frac{Sin(\lambda_k-n\pi+\lambda_m)}{\lambda_k-n\pi+\lambda_m}-\frac{Sin(\lambda_k-n\pi-\lambda_m)}{\lambda_k-n\pi-\lambda_m}$$

$$I_{35}=\frac{L}{2}\Bigg[\frac{\lambda_m}{\lambda_m^2+(n\pi+\lambda_k)^2}\Big\{Cos(n\pi+\lambda_k)Cosh\lambda_m+\frac{n\pi+\lambda_k}{\lambda_m}Sinh\lambda_m Sin(n\pi+\lambda_k)\Big\}$$
$$+\frac{\lambda_m}{\lambda_m^2+(n\pi+\lambda_k)^2}\Big\{Cos(n\pi-\lambda_k)Cosh\lambda_m+\frac{n\pi-\lambda_k}{\lambda_m}Sinh\lambda_m Sin(n\pi-\lambda_k)\Big\}\Bigg]$$

$$I_{36} = \frac{L}{2}\left[\frac{\lambda_m}{\lambda_m^2 + (n\pi + \lambda_k)^2}\left\{Cos(n\pi + \lambda_k)Cosh\lambda_m + \frac{n\pi + \lambda_k}{\lambda_m}Cosh\lambda_m Sin(n\pi + \lambda_k)\right\}\right.$$

$$\left. + \frac{\lambda_m}{\lambda_m^2 + (n\pi - \lambda_k)^2}\left\{Cos(n\pi - \lambda_k)Cosh\lambda_m + \frac{n\pi - \lambda_k}{\lambda_m}Cosh\lambda_m Sin(n\pi - \lambda_k)\right\}\right]$$

$$I_{37} = \frac{L}{4}\left[\frac{Sin(\lambda_k + \lambda_m + n\pi)}{\lambda_k + \lambda_m + n\pi} - \frac{Sin(\lambda_k + \lambda_m - n\pi)}{\lambda_k + \lambda_m - n\pi} - \frac{Sin(\lambda_k - \lambda_m + n\pi)}{\lambda_k - \lambda_m + n\pi} - \frac{Sin(\lambda_k - \lambda_m - n\pi)}{\lambda_k - \lambda_m - n\pi}\right]$$

$$I_{38} = \frac{L}{4}\left[\frac{Cos(n\pi + \lambda_k + \lambda_m) - 1}{n\pi + \lambda_k + \lambda_m} + \frac{Cos(n\pi + \lambda_k - \lambda_m) - 1}{n\pi + \lambda_k - \lambda_m} - \frac{Cos(n\pi - \lambda_k + \lambda_m) - 1}{n\pi - \lambda_k + \lambda_m} - \frac{Cos(n\pi - \lambda_k - \lambda}{n\pi - \lambda_k - \lambda}\right.$$

$$I_{39} = \frac{L}{2}\left[\frac{\lambda_m}{\lambda_m^2 - (n\pi + \lambda_k)^2}\left\{Sin(n\pi + \lambda_k)Cosh\lambda_m - \frac{n\pi + \lambda_k}{\lambda_m}Sinh\lambda_m Cos(n\pi + \lambda_k)\right\}\right.$$

$$\left. + \frac{\lambda_m}{\lambda_m^2 - (n\pi - \lambda_k)^2}\left\{Sin(n\pi - \lambda_k)Cosh\lambda_m - \frac{n\pi - \lambda_k}{\lambda_m}Sinh\lambda_m Cos(n\pi - \lambda_k)\right\}\right]$$

$$I_{40} = \frac{L}{2}\left[\frac{\lambda_m}{\lambda_m^2 - (n\pi + \lambda_k)^2}\left\{Sin(n\pi + \lambda_k)Sinh\lambda_m - \frac{n\pi + \lambda_k}{\lambda_m}Cosh\lambda_m Cos(n\pi + \lambda_k)\right\}\right.$$

$$\left. + \frac{\lambda_m}{\lambda_m^2 - (n\pi - \lambda_k)^2}\left\{Sin(n\pi - \lambda_k)Sinh\lambda_m - \frac{n\pi - \lambda_k}{\lambda_m}Cosh\lambda_m Cos(n\pi - \lambda_k)\right\}\right]$$

$$I_{41} = \frac{L}{2}\left[\frac{\lambda_k}{\lambda_k^2 + (n\pi + \lambda_m)^2}\left\{Cos(n\pi + \lambda_m)Cosh\lambda_k + \frac{n\pi + \lambda_m}{\lambda_k}Sinh\lambda_k Sin(n\pi + \lambda_m)\right\}\right.$$

$$\left. + \frac{\lambda_k}{\lambda_k^2 + (n\pi + \lambda_m)^2}\left\{Cos(n\pi - \lambda_m)Cosh\lambda_k + \frac{n\pi - \lambda_m}{\lambda_k}Sinh\lambda_k Sin(n\pi - \lambda_m)\right\}\right]$$

$$I_{42} = \frac{L}{2}\left[\frac{\lambda_k}{\lambda_k^2 - (n\pi + \lambda_m)^2}\left\{Sin(n\pi + \lambda_m)Cos\lambda_k - \frac{n\pi + \lambda_m}{\lambda_k}Sin\lambda_k Cos(n\pi + \lambda_m)\right\}\right.$$

$$+\frac{\lambda_k}{\lambda_k^2-(n\pi-\lambda_m)^2}\left\{Sin(n\pi-\lambda_m)Cos\lambda_k-\frac{n\pi-\lambda_m}{\lambda_k}Sin\lambda_k Cos(n\pi-\lambda_m)\right\}\Bigg]$$

$$I_{43}=\frac{L}{2}\left[\frac{n\pi Cosh(\lambda_k+\lambda_m)Cosn\pi}{(n\pi+\lambda_m)^2+n^2\pi^2}-1-\frac{n\pi Cosh(\lambda_k-\lambda_m)Cosn\pi}{(n\pi-\lambda_m)^2+n^2\pi^2}-1\right]$$

$$I_{44}=\frac{L}{2}\Bigg[\frac{n\pi Sinh(\lambda_k+\lambda_m)Cosn\pi}{(n\pi-\lambda_m)^2-n^2\pi^2}+\frac{(\lambda_k+\lambda_m)Cosh(\lambda_k+\lambda_m)Sinn\pi}{(n\pi-\lambda_m)^2-n^2\pi^2}$$

$$+\frac{n\pi Sinh(\lambda_k-\lambda_m)Cosn\pi}{(n\pi-+_m)^2+n^2\pi^2}+\frac{(\lambda_k-\lambda_m)Cosh(\lambda_k-\lambda_m)Sinn\pi}{(n\pi+\lambda_m)^2+n^2\pi^2}\Bigg]$$

$$I_{45}=\frac{L}{2}\Bigg[\frac{\lambda_k}{\lambda_k^2+(n\pi+\lambda_m)^2}\left\{Cos(n\pi+\lambda_m)Cosh\lambda_k+\frac{n\pi+\lambda_m}{\lambda_k}Cosh\lambda_k Sin(n\pi+\lambda_m)\right\}$$

$$+\frac{\lambda_k}{\lambda_k^2+(n\pi+\lambda_m)^2}\left\{Cos(n\pi-\lambda_m)Cosh\lambda_k+\frac{n\pi-\lambda_m}{\lambda_k}Cosh\lambda_k Sin(n\pi-\lambda_m)\right\}\Bigg]$$

$$I_{46}=\frac{L}{2}\Bigg[\frac{\lambda_k}{\lambda_k^2-(n\pi+\lambda_m)^2}\left\{Sin(n\pi+\lambda_m)Sinh\lambda_k-\frac{n\pi-\lambda_m}{\lambda_k}Cosh\lambda_k Cos(n\pi+\lambda_m)\right\}$$

$$+\frac{\lambda_k}{\lambda_k^2-(n\pi-\lambda_m)^2}\left\{Sin(n\pi-\lambda_m)Sinh\lambda_k-\frac{n\pi-\lambda_m}{\lambda_k}Cosh\lambda_k Cos(n\pi-\lambda_m)\right\}\Bigg]$$

$$I_{47}=\frac{L}{2}\Bigg[\frac{n\pi Sinh(\lambda_m-\lambda_k)Cosn\pi}{(\lambda_m-\lambda_k)^2-n^2\pi^2}+\frac{(\lambda_m+\lambda_k)Cosh(\lambda_k+\lambda_m)Sinn\pi}{(\lambda_m-\lambda_k)^2-n^2\pi^2}$$

$$-\frac{n\pi Sinh(\lambda_m-\lambda_k)Coshn\pi}{(n\pi+\lambda_m)^2+n^2\pi^2}+\frac{(\lambda_m-\lambda_k)Cosh(\lambda_m-\lambda_k)Sinn\pi}{(\lambda_m-\lambda_k)^2+n^2\pi^2}\Bigg]$$

$$I_{48}=\frac{L}{2}\left[\frac{n\pi Cosh(\lambda_k+\lambda_m)Cosn\pi}{(\lambda_k+\lambda_m)^2+n^2\pi^2}-1-\frac{n\pi Cosh(\lambda_k-\lambda_m)Cosn\pi}{(\lambda_k-\lambda_m)^2+n^2\pi^2}-1\right]$$

$$I_{49} = \frac{L}{2}\left[1 - \frac{Sin2\lambda_m}{2\lambda_m}\right]$$

$$I_{50} = -\frac{A_m L}{2}\left[\frac{Cos2\lambda_m}{2\lambda_m} - 1\right]$$

$$I_{51} = \frac{B_m L}{2\lambda_m}\left[Sin\lambda_m Cosh\lambda_m - Cos\lambda_m Sinh\lambda_m\right]$$

$$I_{52} = \frac{C_m L}{2\lambda_m}\left[Sin\lambda_m Sinh\lambda_m - (Cos\lambda_m Cosh\lambda_m - 1)\right]$$

$$I_{53} = -\frac{A_m L}{2}\left[\frac{Cos2\lambda_m}{2\lambda_m} - 1\right]$$

$$I_{54} = \frac{A_m^2 L}{2}\left[1 + \frac{Sin2\lambda_m}{2\lambda_m}\right]$$

$$I_{55} = \frac{A_m B_m L}{2\lambda_m}\left[(Cos\lambda_m Cosh\lambda_m - 1) + Sin\lambda_m Sinh\lambda_m\right]$$

$$I_{56} = \frac{A_m C_m L}{2\lambda_m}\left[Sin\lambda_m Cosh\lambda_m + Cos\lambda_m Sinh\lambda_m\right]$$

$$I_{57} = \frac{B_m L}{2\lambda_m}\left[Sin\lambda_m Cosh\lambda_m - Cos\lambda_m Sinh\lambda_m\right]$$

$$I_{58} = \frac{A_m B_m L}{2\lambda_m}\left[(Cos\lambda_m Cosh\lambda_m - 1) + Sin\lambda_m Sinh\lambda_m\right]$$

$$I_{59} = \frac{B_m^2 L}{2}\left[\frac{Sinh2\lambda_m}{2\lambda_m} - 1\right]$$

$$I_{60} = \frac{B_m C_m L}{2}\left[\frac{Cos2\lambda_m}{2\lambda_m} - 1\right]$$

$$I_{61} = \frac{C_m L}{2\lambda_m}\left[Sin\lambda_m Sinh\lambda_m - (Cos\lambda_m Cosh\lambda_m - 1)\right]$$

$$I_{62} = \frac{A_m C_m L}{2\lambda_m}\left[Sin\lambda_m Cosh\lambda_m + Cos\lambda_m Sinh\lambda_m\right]$$

$$I_{63} = \frac{C_m B_m L}{2}\left[\frac{Cosh2\lambda_m}{2\lambda_m} - 1\right]$$

$$I_{64} = \frac{C_m^2 L}{2}\left[1 + \frac{Sinh2\lambda_m}{2\lambda_m}\right]$$

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CLS
REM   THIS PROGRAM IS WRITTEN BY OGUNYEBI SEGUN N.
REM   IT IS WRITTEN TO EVALUATE THE TRANSVERSE DISPLACEMENT OF
REM   MOVING FORCE UNIFORM BEAM FOR OTHER BOUNDARY CONDITION.
10    DIM WB(3), LM(3), LK(3)
20    OPEN "CFMFAF.BAS" FOR OUTPUT AS #1
30    FOR mm = 1 TO 5
40    PRINT "Enter the value of AF N", mm
50    INPUT AF
60    L = 12.192
70    E = 2.10924E+09
80    P = 8407.27 * 9.81
90    MI = 2758.291
100   MIU = 2758.291
110   IN = .00287698#
120   M1 = 8407.27
130   pi = 22 / 7
140   FM = 40000
150   'AF = 20000
160   X = L / 2
170   C = 8.128
180   CO = 5
190   PRINT #1, "TIME(t)", SPC(2); "DEFLECTION V(x,t)"
200   PRINT #1,
210   LM(1) = 1.875
220   LM(2) = 4.694
230   LM(3) = 7.855
240   n = 1
250   LK(1) = 1.875
260   LK(2) = 4.694
270   LK(3) = 7.855
280   PRINT #1, "This is The Result for N", AF
290   FOR t = 0 TO 1.5 STEP .1
300   FOR m = 1 TO 3
310   k = m
320   AM1 = SIN(LM(m) - (EXP(LM(m)) - EXP(-LM(m))) / 2)
330   AM2 = COS(LM(m)) + ((EXP(LM(m)) + EXP(-LM(m))) / 2)
340   AM = AM2 / AM1
350   BM = 1
360   CM = AM
370   AK1 = SIN(LK(k) - (EXP(LK(k)) - EXP(-LK(k))) / 2)
380   AK2 = COS(LK(k)) + ((EXP(LK(k)) + EXP(-LK(k))) / 2)
390   AK = AK2 / AK1
400   BK = 1
410   CK = AK
420   COSHLM = (EXP(LM(m)) + EXP(-LM(m))) / 2
430   SINHLM = (EXP(LM(m)) - EXP(-LM(m))) / 2
440   COSHLK = (EXP(LK(m)) + EXP(-LK(m))) / 2
450   SINHLK = (EXP(LK(m)) - EXP(-LK(m))) / 2
460   COSH1 = (EXP(LM(m) + LK(m)) + EXP(-(LM(m) + LK(m)))) / 2
470   SINH1 = (EXP(LM(m) + LK(m)) - EXP(-(LM(m) + LK(m)))) / 2
480   COSH2 = (EXP(LK(m) - LM(m)) + EXP(-(LK(m) - LM(m)))) / 2
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490   SINH2 = (EXP(LK(m) - LM(m)) - EXP(-(LK(m) - LM(m)))) / 2
500   COSH3 = (EXP(LM(m) - LK(m)) + EXP(-(LM(m) - LK(m)))) / 2
510   SINH3 = (EXP(LM(m) - LK(m)) - EXP(-(LM(m) - LK(m)))) / 2
520   COSH4 = (EXP(2 * LM(m)) + EXP(-2 * LM(m))) / 2
530   SINH4 = (EXP(2 * LM(m)) - EXP(-2 * LM(m))) / 2
540   COSH5 = (EXP(2 * LK(m)) + EXP(-2 * LK(m))) / 2
550   SINH5 = (EXP(2 * LK(m)) - EXP(-2 * LK(m))) / 2
560   G1 = L / 2 * (1 - SIN(2 * LM(m)) / 2 * LM(m))
570   G2a = AM * L / 2
580   G2b = (COS(2 * LM(m)) / 2 * LM(m)) - 1
590   G2 = -G2a * G2b
600   G3a = (BM * L) / 2 * LM(m)
610   G3b = SIN(LM(m)) * COSHLM - COS(LM(m)) * SINHLM
620   G3 = G3a * G3b
630   G4a = (CM * L) / 2 * LM(m)
640   G4b = SIN(LM(m)) * SINHLM - (COS(LM(m)) * COSHLM - 1)
650   G4 = G4a * G4b
660   G5a = AM * L / 2
670   G5b = (COS(2 * LM(m)) / 2 * LM(m)) - 1
680   G5 = -G5a * G5b
690   G6a = AM ^ 2 * L / 2
700   G6b = 1 + (SIN(2 * LM(m)) / 2 * LM(m))
710   G6 = G6a * G6b
720   G7a = (AM * BM * L) / 2 * LM(m)
730   G7b = (COS(LM(m)) * COSHLM - 1) + SIN(LM(m)) * SINHLM
740   G7 = G7a * G7b
750   G8a = (AM * CM * L) / 2 * LM(m)
760   G8b = (COS(LM(m)) * SINHLM) + (SIN(LM(m)) * COSHLM)
770   G8 = G8a * G8b
780   G9a = (BM * L / 2 * LM(m))
790   G9b = SIN(LM(m)) * COSHLM - COS(LM(m)) * SINHLM
800   G9 = G9a * G9b
810   G10a = (AM * CM * L) / 2 * LM(m)
820   G10b = (COS(LM(m)) * COSHLM - 1) + (SIN(LM(m)) * SNHLM)
830   G10 = G10a * G10b
840   G11a = BM ^ 2 * L / 2
850   G11b = (SIN(2 * LM(m)) / 2 * LM(m))
860   G11 = G11a * G11b
570   G12a = (BM * CM * LM(m)) / 2
580   G12b = (COS(2 * LM(m)) / 2 * LM(m)) - 1
590   G12 = G12a * G12b
600   G13a = (CM * L) / 2 * LM(m)
610   G13b = SIN(LM(m)) * SINHLM - (COS(LM(m)) * COSHLM - 1)
620   G13 = G13a * G13b
630   G14a = (AM * CM * L) / 2 * LM(m)
640   G14b = COS(LM(m)) * SINHLM + SIN(LM(m)) * COSHLM
650   G14 = G14a * G14b
660   G15a = CM * BM * L / 2
670   G15b = (COSH4 / 2 * LM(m)) - 1
680   G15 = G15a * G15b
690   G16a = CM ^ 2 * L / 2
700   G16b = 1 + (SINH4 / 2 * LM(m))
```

```
710  G16 = G16a * G16b
720  afAM1 = G1 + G2 + G3 + G4 + G5 + G6 + G7 + G8 + G7 + G8
730  afAM2 = G9 + G10 + G11 + G12 + G13 + G14 + G15 + G16
740  afam = afAM1 + afAM2
750  IF m = k THEN
760  I1a = SIN(2 * LM(m)) / (2 * LM(m))
770  I1 = L / 2 * (1 - I1a)
780  ELSE
790  I1a = SIN(LK(k) - LM(m)) / (LK(k) - LM(m))
800  I1b = SIN(LK(k) + LM(m)) / (LK(k) + LM(m))
810  I1 = L / 2 * (I1a - I1b)
820  END IF
830  IF k = m THEN
840  I2a = (COS(2 * LM(m)) - 1) / (2 * LM(m))
850  I2 = -L / 2 * (I2a)
860  ELSE
870  I2a = (LK(k) - LM(m)) * (COS(LK(k) + LM(m)) - 1)
880  I2b = (LK(k) + LM(m)) * (COS(LK(k) - LM(m)) - 1)
890  I2c = -(L / 2) / (LK(k) ^ 2 - LM(m) ^ 2)
900  I2 = I2c * (I2a + I2b)
910  END IF
920  I3a = SIN(LK(k)) * COSHLM
930  I3b = (LK(k) / LM(m)) * (COS(LK(k)) * SINHLM)
940  I3c = (LM(m) * L) / (LM(m) ^ 2 + LK(k) ^ 2)
950  I3 = I3c * (I3a - I3b)
960  I4a = SIN(LK(k)) * SINHLM
970  I4b = (LK(k) / LM(m)) * ((COS(LK(k)) * COSHLM) - 1)
980  I4c = (LM(m) * L) / (LM(m) ^ 2 + LK(k) ^ 2)
990  I4 = I4c * (I4a - I4b)
1000 IF k = m THEN
1010 I5a = (COS(2 * LM(m)) - 1) / (2 * LM(m))
1020 I5 = -L / 2 * I5a
1030 ELSE
1040 I5a = (LK(k) + LM(m)) * (COS(LK(k) - LM(m)) - 1)
1050 I5b = (LK(k) - LM(m)) * (1 - COS(LK(k) + LM(m)))
1060 I5c = (L / 2) / (LK(k) ^ 2 - LM(m) ^ 2)
1070 I5 = I5c * (I5a + I5b)
1080 END IF
1090 IF k = m THEN
1100 I6a = SIN(2 * LM(m)) / (2 * LM(m))
1110 I6 = L / 2 * (I6a + 1)
1120 ELSE
1130 I6a = SIN(LK(k) + LM(m)) / (LK(k) + LM(m))
1140 I6b = SIN(LK(k) - LM(m)) / (LK(k) - LM(m))
1150 I6 = L / 2 * (I6a + I6b)
1160 END IF
1170 I7a = COS(LK(k)) * COSHLM
1180 I7b = (LK(k) / LM(m)) * ((SIN(LK(k)) * SINHLM) - 1)
1190 I7c = (LM(m) * L) / (LM(m) ^ 2 + LK(k) ^ 2)
1200 I7 = I7c * (I7a + I7b)
1210 I8a = COS(LK(k)) * SINHLM
1220 I8b = (LK(k) / LM(m)) * (SIN(LK(k)) * COSHLM)
```

```
1230 I8c = (LM(m) * L) / (LM(m) ^ 2 + LK(k) ^ 2)
1240 I8 = I8c * (I8a + I8b)
1250 I9a = SIN(LM(m)) * COSHLK
1260 I9b = (LM(m) / LK(k)) * (COS(LM(m)) * SINHLK)
1270 I9c = (LK(k) * L) / (LK(k) ^ 2 + LM(m) ^ 2)
1280 I9 = I9c * (I9a - I9b)
1290 I10a = COS(LM(m)) * COSHLK
1300 I10b = (LM(m) / LK(m)) * (SIN(LM(m)) * SINHLK - 1)
1310 I10c = (LK * L) / (LK(k) ^ 2 + LM(m) ^ 2)
1320 I10 = I10c * (I10a + I10b)
1330 IF k = m THEN
1340 I11a = (SINH4 / (2 * LM(m)))
1350 I11 = L / 2 * (I11a - 1)
1360 ELSE
1370 I11a = SINH1 / (LK(k) + LM(m))
1380 I11b = SINH2 / (LK(k) - LM(m))
1390 I11 = L / 2 * (I11a - I11b)
1400 END IF
1410 IF k = m THEN
1420 I12a = (COS(2 * LM(m)) - 1) / (2 * LM(m))
1430 I12 = L / 2 * (I12a)
1440 ELSE
1450 I12a = (LK(k) - LM(m)) * (COSH1 - 1)
1460 I12b = (LK(k) + LM(m)) * (COSH2 - 1)
1470 I12c = (L / 2) / (LK(k) ^ 2 + LM(m) ^ 2)
1480 I12 = I12c * (I12a + I12b)
1490 END IF
1500 I13a = SIN(LM(m)) * SINHLK
1510 I13b = (LK(k) / LM(m)) * ((COS(LM(m)) * COSHLK(k)) - 1)
1520 I13c = (LK(k) * L) / (LK(k) ^ 2 + LM(m) ^ 2)
1530 I13 = I13c * (I13a - I13b)
1540 I14a = COS(LM(m)) * SINHLK
1550 I14b = (LK / LM(m)) * (SIN(LM(m)) * COSHLK)
1560 I14c = (LK * L) / (LK(k) ^ 2 + LM(m) ^ 2)
1570 I14 = I14c * (I14a + I14b)
1580 IF k = m THEN
1590 I15a = (COS(2 * LM(m)) - 1) / (2 * LM(m))
1600 I15 = L / 2 * (I15a - 1)
1610 ELSE
1620 I15a = (LM(m) - LK(k)) * (COSH1 - 1)
1630 I15b = (LM(m) + LK(k)) * (COSH2 - 1)
1640 I15c = (L / 2) / (LM(m) ^ 2 + LK(k) ^ 2)
1650 I15 = I15c * (I15a + I15b)
1660 END IF
1670 IF k = m THEN
1680 I16a = (SINH4 / (2 * LM(m)))
1690 I16 = L / 2 * (I16a + 1)
1700 ELSE
1710 I16a = SINH1 / (LK(k) + LM(m))
1720 I16b = SINH2 / (LK(k) - LM(m))
1730 I16 = L / 2 * (I16a + I16b)
1740 END IF
```

```
1750 IF k = m THEN
1760 I17a = (LK(k) + LM(m)) * SIN(LK(k) + LM(m))
1780 I17b = (COS(LK(k) + LM(m)) - 1)
1790 I17c = L ^ 2 / 2 * (LK(k) + LM(m)) ^ 2
1800 I17 = (L ^ 2 / 4) - (I17c * I17a) + I17b
1810 ELSE
1820 I17a = (LK(k) - LM(m)) * SIN(LK(k) - LM(m))
1830 I17b = (COS(LK(k) - LM(m)) - 1)
1840 I17c = (LK(k) + LM(m)) * SIN(LK(k) + LM(m))
1850 I17d = (COS(LK(k) + LM(m)) - 1)
1860 I17e = (L ^ 2 / 2 * (LK(k) - LM(m)) ^ 2) * (I17a + I17b)
1870 I17f = (L ^ 2 / 2 * (LK(k) + LM(m)) ^ 2) * (I17c + I17d)
1880 I17 = I17e - I17f
1890 END IF
1900 IF k = m THEN
1910 I18a = (2 * LK(k) * COS(2 * LK(k)) - SIN(2 * LK(k))) / 4 * LK(k)
1920 I18 = -(L ^ 2 / 2) * (I18a)
1930 ELSE
1940 I18a = ((LK(k) - LM(m)) * COS(LK(k) - LM(m)) - SIN(LK(k)
1950 I18b = ((LK(k) + LM(m)) * COS(LK(k) + LM(m)) - SIN(LK(k)
1960 I18 = -L ^ 2 / 2 * (I18a + I18b)
1970 END IF
1980 I19a = (LK(k) ^ 2 + LM(m) ^ 2) * LM(m) * SIN(LK(k)) * COSHLM
1990 I19b = (LK(k) ^ 2 + LM(m) ^ 2) * LK(k) * COSHLK * SINHLM
2000 I19c = (LK(k) ^ 2 - LM(m) ^ 2) * SIN(LK(k)) * SINHLM
2010 I19d = 2 * LM(m) * LK(k) * COSLK(k) * COSHLM - 2 * LM(m) * LK(k)
2020 I19e = L ^ 2 / (LK(k) ^ 2 + LM(m) ^ 2) ^ 2
2030 I19 = I19e * (I19a - I19b + I19c + I19d)
2040 I20a = (LM(m) ^ 2 + LK(k) ^ 2) * LM(m) * SIN(LK(k)) * SINHLM
2050 I20b = (LK(k) ^ 2 + LM(m) ^ 2) * LK(k) * COS(LK(k)) * COSHLM
2060 I20c = 2 * LK(k) * LM(m) * COS(LK(k)) * SINHLM
2070 I20d = (LK(k) ^ 2 - LM(m) ^ 2) * SIN(LK(k)) * COSHLM
2080 I20e = L ^ 2 / (LK(k) ^ 2 + LM(m) ^ 2) ^ 2
2090 I20 = I20e * (I20a - I20b + I20c + I20d)
2100 IF k = m THEN
2110 I21a = L ^ 2 / (2 * LM(m))
2120 I21b = COS(LM(m)) - (SIN(LM(m)) / (2 * LM(m)))
2130 I21 = I21a * I21b
2140 ELSE
2150 I21a = (LM(m) - LK(k)) * COS(LM(m) - LK(k)) - SIN(LM(m) - LK(k))
2160 I21b = (LM(m) + LK(k)) * COS(LM(m) + LK(k)) - SIN(LM(m) + LK(k))
2170 I21 = -L ^ 2 / 2 * (I21a + I21b)
2180 END IF
2190 IF k = m THEN
2200 I22a = 2 * LK(k) * SIN(2 * LK(k))
2210 I22b = (COS(2 * LK(k)) - 1)
2220 I22c = L ^ 2 / 2 * (2 * LK(k)) ^ 2
2230 I22 = (L ^ 2 / 4) + (I22c * I22a) + I22b
2240 ELSE
2250 I22a = (LK(k) + LM(m)) * SIN(LK(k) + LM(m))
2260 I22b = (COS(LK(k) + LM(m)) - 1)
2270 I22c = (LK(k) - LM(m)) * SIN(LK(k) - LM(m))
```

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2280 I22d = (COS(LK(k) - LM(m)) - 1)
2290 I22e = (L ^ 2 / 2 * (LK(k) + LM(m)) ^ 2) * (I22a + I22b)
2300 I22f = (L ^ 2 / 2 * (LK(k) - LM(m)) ^ 2) * (I22c + I22d)
2310 I22 = I22e + I22f
2320 END IF
2330 I23a = (LM(m) ^ 2 + LK(k) ^ 2) * LK(k) * SIN(LK(k)) * SINHLM
2340 I23b = (LK(k) ^ 2 + LM(m) ^ 2) * LM(m) * COS(LK(k)) * COSHLM
2350 I23c = 2 * LK(k) * LM(m) * SIN(LK(k)) * COSHLM
2360 I23d = (LM(m) ^ 2 - LK(k) ^ 2) * COS(LK(k)) * SINHLM
2370 I23e = L ^ 2 / (LM(m) ^ 2 + LK(k) ^ 2) ^ 2
2380 I23 = I23e * (I23a + I23b - I23c - I23d)
2390 I24a = (LM(m) ^ 2 + LK(k) ^ 2) * LK(k) * SIN(LK(k)) * COSHLM
2400 I24b = (LK(k) ^ 2 + LM(m) ^ 2) * LM(m) * COS(LK(k)) * SINHLM
2410 I24c = 2 * LK(k) * LM(m) * SIN(LK(k)) * SINHLM
2420 I24d = (LM(m) ^ 2 - LK(k) ^ 2) * COS(LK(k)) * COSHLM - (LK(k)
2430 I24e = L ^ 2 / (LM(m) ^ 2 + LK(k) ^ 2) ^ 2
2440 I24 = I24e * (I24a + I24b - I24c - I24d)
2450 I25a = (LK(k) ^ 2 + LM(m) ^ 2) * LK(k) * SIN(LM(m)) * COSHLK
2460 I25b = (LK ^ 2 + LM(m) ^ 2) * LM(m) * COS(LM(m)) * SINHLK
2470 I25c = (LM(m) ^ 2 - LK(k) ^ 2) * SIN(LM(m)) * SINHLK(k)
2480 I25d = 2 * LM(m) * LK(k) * COS(LM(m)) * COSHLK - 2 * LM(m) * LK
2490 I25e = L ^ 2 / (LK(k) ^ 2 + LM(m) ^ 2) ^ 2
2500 I25 = I25e * (I25a - I25b + I25c + I25d)
2510 I26a = (LM(m) ^ 2 + LK(k) ^ 2) * LM(m) * SIN(LM(m)) * SINHLK
2520 I26b = (LK ^ 2 + LM(m) ^ 2) * LK(k) * COS(LM(m)) * COSHLK(k)
2530 I26c = 2 * LK(k) * LM(m) * SIN(LM(m)) * COSHLK(k)
2540 I26d = (LM(m) ^ 2 - LK(k) ^ 2) * COS(LM(m)) * SINHLK
2550 I26e = L ^ 2 / (LM(m) ^ 2 + LK ^ 2) ^ 2
2560 I26 = I26e * (I26a + I26b - I26c - I26d)
2570 IF k = m THEN
2580 I27a = 2 * LM(m) * SIN(2 * LM(m)) - COSH4 - 2 * LM(m) ^ 2 + 1
2590 I27 = (L ^ 2 / 8 * LM(m) ^ 2) * (I27a)
2600 ELSE
2610 I27a = (LK(k) + LM(m)) * SINH1
2620 I27b = (COS(LK(k) + LM(m)) + 1)
2630 I27c = (LK(k) - LM(m)) * SINH3
2640 I27d = ((COSH2) + 1)
2650 I27e = (L ^ 2 / 2 * (LK(k) + LM(m)) ^ 2) * (I27a - I27b)
2660 I27f = (L ^ 2 / 2 * (LK(k) - LM(m)) ^ 2) * (I27c - I27d)
2670 I27 = I27e - I27f
2680 END IF
2690 IF k = m THEN
2700 I28a = 2 * LM(m) * COSH4 - SINH4
2710 I28 = L ^ 2 / 8 * LM(m) ^ 2 * (I28a)
2720 ELSE
2730 I28a = (LK(k) - LM(m)) * COSH1
2740 I28b = (LK(k) + LM(m)) * (COSH2)
2750 I28c = (LK(k) - LM(m)) ^ 2 * SINH1
2760 I28d = (LK(k) + LM(m)) ^ 2 * (SINH2)
2770 I28e = (L ^ 2 / 2 * (LK(k) ^ 2 - LM(m) ^ 2)) * (I28a + I28b)
2780 I28f = (L ^ 2 / 2 * (LK(k) ^ 2 - LM(m) ^ 2)) * (I28c + I28d)
2790 I28 = I28e - I28f
```

```
800  END IF
810  I29a = (LM(m) ^ 2 + LM(m) ^ 2) * LK(k) * SIN(LM(m)) * SINHLK
820  I29b = (LK(k) ^ 2 + LM(m) ^ 2) * LM(m) * COS(LM(m)) * COSHLK
830  I29c = 2 * LK(k) * LM(m) * COS(LM(m)) * SINHLK
840  I29d = (LM(m) ^ 2 - LK ^ 2) * SIN(LM(m)) * COSHLK
850  I29e = L ^ 2 / (LK(k) ^ 2 + LM(m) ^ 2) ^ 2
860  I29 = I29e * (I29a - I29b + I29c + I29d)
870  I30a = (LM(m) ^ 2 + LK(k) ^ 2) * LM(m) * SIN(LM(m)) * COSHLK
880  I30b = (LM(m) ^ 2 + LK ^ 2) * LK(k) * COSHLM * SINHLK(k)
890  I30c = 2 * LK(k) * LM(m) * SIN(LM(m)) * SINHLK
900  I30d = (LK(k) ^ 2 - LM(m) ^ 2) * COSHLM * COSHLK - (LM(m) ^ 2
910  I30e = L ^ 2 / (LM(m) ^ 2 + LK(k) ^ 2) ^ 2
920  I30 = I30e * (I30a + I30b - I30c - I30d)
930  IF m = k THEN
940  I31a = 2 * LK(k) * COSH5 - SINH5
950  I31 = L ^ 2 / 8 * LK(k) ^ 2 * (I31a)
960  ELSE
970  I31a = (LM(m) - LK(k)) * COSH1
980  I31b = (LM(m) + LK(k)) * COSH
990  I32 = I32e + I32f
000  END IF
010  I33a = (COS(LK(k) - LM(m) + n * pi) - 1) / (LK(k) - LM(m)
020  I33b = (COS(LK(k) - LM(m) + n * pi) - 1) / (LK(k) - LM(m)
030  I33c = (COS(LK(k) + LM(m) + n * pi) - 1) / (LK(k) + LM(m)
040  I33d = (COS(LK(k) - LM(m) - n * pi) - 1) / (LK(k) - LM(m)
050  I33 = L / 4 * (I33a + I33b - I33c - I33d)
060  I34a = (SIN(LK(k) + n * pi + LM(m))) / (LK(k) + n * pi + LM(m))
070  I34b = (SIN(LK(k) + n * pi - LM(m))) / (LK(k) + n * pi - LM(m))
080  I34c = (SIN(LK(k) - n * pi + LM(m))) / (LK(k) - n * pi + LM(m))
090  I34d = (SIN(LK(k) - n * pi - LM(m))) / (LK(k) - n * pi - LM(m))
100  I34 = L / 4 * (I34a - I34b - I34c + I34d)
110  I35a = LM(m) / (LM(m) ^ 2 + (n * pi + LK(k)) ^ 2)
120  I35b = LM(m) / (LM(m) ^ 2 + (n * pi - LK(k)) ^ 2)
130  I35c = COS(n * pi + LK(k)) * COSHLM
140  I35d = (n * pi + LK(k)) / LM(m) * SINHLM * SIN(n * pi + LK(k))
150  I35e = COS(n * pi - LK(k)) * COSHLM
160  I35f = (n * pi - LK(k)) / LM(m) * SINHLM * SIN(n * pi - LK(k))
170  I35 = L / 2 * ((I35a * (I35c + I35d) + (I35b * (I35e + I35f))))
180  I36a = LM(m) / (LM(m) ^ 2 + (n * pi + LK(k)) ^ 2)
190  I36b = LM(m) / (LM(m) ^ 2 + (n * pi - LK(k)) ^ 2)
200  I36c = COS(n * pi + LK(k)) * COSHLM
210  I36d = (n * pi + LK(k)) / LM(m) * COSHLM * SIN(n * pi + LK(k))
220  I36e = COS(n * pi - LK(k)) * COSHLM
230  I36f = (n * pi - LK(k)) / LM(m) * COSHLM * SIN(n * pi - LK(k))
240  I36 = L / 2 * ((I36a * (I36c + I36d) + (I36b * (I36e + I36f))))
250  I37a = (SIN(n * pi + LK(k) + LM(m))) / n * pi + LK(k) + LM(m)
260  I37b = (SIN(n * pi + LK(k) - LM(m))) / n * pi + LK(k) - LM(m)
270  I37c = (SIN(n * pi - LK(k) + LM(m))) / n * pi - LK(k) + LM(m)
280  I37d = (SIN(n * pi - LK(k) - LM(m))) / n * pi - LK(k) - LM(m)
290  I37 = L / 4 * (I37a - I37b + I37c - I37d)
00   I38a = (COS(n * pi + LK(k) + LM(m)) - 1) / (n * pi + LK(k)
10   I38b = (COS(n * pi + LK(k) - LM(m)) - 1) / (n * pi + LK(k)
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3320 I38c = (COS(n * pi - LK(k) + LM(m)) - 1) / (n * pi - LK(k)
3330 I38d = (COS(n * pi - LM(m) - LM(m)) - 1) / (n * pi - LK(k)
3340 I38 = L / 4 * (I38a + I38b - I38c - I38d)
3350 I39a = LM(m) / (LM(m) ^ 2 - (n * pi + LK(k)) ^ 2)
3360 I39b = LM(m) / (LM(m) ^ 2 - (n * pi - LK(k)) ^ 2)
3370 I39c = SIN(n * pi + LK(k)) * COSHLM
3380 I39d = (n * pi + LK(k)) / LM(m) * COS(n * pi + LK(k)) * SINHLM
3390 I39e = SIN(n * pi - LK(k)) * COSHLM
3400 I39f = (n * pi - LK(k)) / LM(m) * COS(n * pi - LK(k)) * SINHLM
3410 I39 = L / 2 * ((I39a * (I39c - I39d) + (I39b * (I39e - I39f))))
3420 I40a = LM(m) / (LM(m) ^ 2 - (n * pi + LK(k)) ^ 2)
3430 I40b = LM(m) / (LM(m) ^ 2 - (n * pi - LK(k)) ^ 2)
3440 I40c = SIN(n * pi + LK(k)) * SINHLM
3450 I40d = (n * pi + LK(k)) / LM(m) * COS(n * pi + LK(k)) * COSHLM
3460 I40e = SIN(n * pi - LK(k)) * SINHLM
3470 I40f = (n * pi - LK(k)) / LM(m) * COS(n * pi - LK(k)) * COSHLM
3480 I40 = L / 2 * ((I40a * (I40c - I40d) + (I40b * (I40e - I40f))))
3490 I41a = LK(k) / (LK(k) ^ 2 + (n * pi + LM(m)) ^ 2)
3500 I41b = LK(k) / (LK(k) ^ 2 + (n * pi - LM(m)) ^ 2)
3510 I41c = COS(n * pi + LM(m)) * COSHLK
3520 I41d = (n * pi + LM(m)) / LK(k) * SIN(n * pi + LM(m)) * SINHLK
3530 I41e = COS(n * pi - LM(m)) * COSHLK
3540 I41f = (n * pi - LM(m)) / LK(k) * SIN(n * pi + LM(m)) * SIHNLK
3550 I41 = L / 2 * ((I41a * (I41c + I41d) + (I41b * (I41e + I41f))))
3560 I42a = LK(k) / (LK(k) ^ 2 - (n * pi + LM(m)) ^ 2)
3570 I42b = LK(k) / (LK(k) ^ 2 - (n * pi - LM(m)) ^ 2)
3580 I42c = SIN(n * pi + LM(m)) * COSLK(k)
3590 I42d = (n * pi + LM(m)) / LK(k) * COS(n * pi + LM(m)) * SINLK(k)
3600 I42e = SIN(n * pi - LM(m)) * COSLK(k)
3610 I42f = (n * pi - LM(m)) / LK(k) * COS(n * pi - LM(m)) * SINLK(k)
3620 I42 = L / 2 * ((I42a * (I42c - I42d) + (I42b * (I42e - I42f))))
3630 I43a = 1 / ((LK(k) + LM(m)) ^ 2 + (n ^ 2 * pi ^ 2))
3640 I43b = 1 / ((LK(k) - LM(m)) ^ 2 + (n ^ 2 * pi ^ 2))
3650 I43c = (n * pi * COSH1 * COS(n * pi)) - 1
3660 I43d = (n * pi * COSH2) * COS(n * pi) - 1
3670 I43 = L / 2 * ((I43a * I43c) - (I43b * I43d))
3680 I44a = 1 / ((LK(k) - LM(m)) ^ 2 - (n ^ 2 * pi ^ 2))
3690 I44b = 1 / ((LK(k) + LM(m)) ^ 2 + (n ^ 2 * pi ^ 2))
3700 I44c = (n * pi * SINH1 * COS(n * pi))
3710 I44d = (LK(k) + LM(m)) * COSH1 * SIN(n * pi)
3720 I44e = (n * pi) * SINH2 * COS(n * pi)
3730 I44f = (LK(k) - LM(m)) * COS(LK(k) - LM(m)) * SIN(n * pi)
3740 I44g = (I44a * I44c) + (I44a * I44d)
3750 I44h = (I44b * I44e) + (I44b * I44f)
3760 I44 = L / 2 * (I44g + I44h)
3770 I45a = LK(k) / (LK(k) ^ 2 + (n * pi + LM(m)) ^ 2)
3780 I45b = LK(k) / (LK(k) ^ 2 + (n * pi - LM(m)) ^ 2)
3790 I45c = COS(n * pi + LM(m)) * COSHLK
3800 I45d = (n * pi + LM(m)) / LK(k) * COSHLK * SIN(n * pi + LM(m))
3810 I45e = COS(n * pi - LM(m)) * COSHLK
3820 I45f = (n * pi - LM(m)) / LK(k) * COSHLK * SIN(n * pi - LM(m))
3830 I45 = L / 2 * ((I45a * (I45c + I45d) + (I45b * (I45e + I45f))))
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```
3840 I46a = LK(k) / (LK(k) ^ 2 - (n * pi + LM(m)) ^ 2)
3850 I46b = LK(k) / (LK(k) ^ 2 - (n * pi - LM(m)) ^ 2)
3860 I46c = SIN(n * pi + LM(m)) * SINHLK
3870 I46d = (n * pi + LM(m)) / LK(k) * COS(n * pi + LM(m))
3880 I46e = SIN(n * pi - LM(m)) * SINHLK
3890 I46f = (n.* pi - LM(m)) / LK(k) * COS(n * pi - LM(m)) * COSHLK
3900 I46 = L / 2 * ((I46a * (I46c - I46d) + (I46b * (I46e - I46f))))
3910 I47a = 1 / ((LM(m) - LK(k)) ^ 2 - (n ^ 2 * pi ^ 2))
3920 I47b = 1 / ((LM(m) + LK(k)) ^ 2 + (n ^ 2 * pi ^ 2))
3930 I47c = (n * pi * SINH1 * COS(n * pi))
3940 I47d = (LM(m) + LK(k)) * COSH1 * SIN(n * pi)
3950 I47e = (n * pi) * SINH3 * COS(n * pi)
3960 I47f = (LM(m) - LK(k)) * COSH3 * SIN(n * pi)
3970 I47g = (I47a * I47c) + (I47a * I47d)
3980 I47h = (I47b * I47e) + (I47b * I47f)
3990 I47 = L / 2 * (I47g + I47h)
4000 I48a = 1 / ((LK(k) + LM(m)) ^ 2 + (n ^ 2 * pi ^ 2))
4010 I48b = 1 / ((LK(k) - LM(m)) ^ 2 + (n ^ 2 * pi ^ 2))
4020 I48c = (n * pi * COS(LK(k) + LM(m)) * COS(n * pi)) - 1
4030 I48d = (n * pi * COS(LK(k) - LM(m)) * COS(n * pi)) - 1
4040 I48 = L / 2 * ((I48a * I48c) + (I48b * I48d))
4050 S1A = -G1 - G2 - G3 - G4 - G5 - G6 - G7 - G8
4060 S1B = G9 + G10 + G11 + G12 + G13 + G14 + G15 + G16
4070 S1 = ((LM(m) / L) ^ 2 * (S1A + S1B)) / afam
4080 F17 = (L ^ 2 / (8 * LM(m) ^ 2)) * ((2 * LM * SIN(2 * LM(m)))
4090 COS(2 * LM(m)) - 2 * LM(m) ^ 2 - 1)
4100 F18 = -(L ^ 2 / (8 * LM(m) ^ 2)) * ((2 * LM(m) * (COS(2 * LM(m))
4110 F19a = (SIN(LM(m)) * COSHLM) - (COS(LM(m)) * SINHL
4120 F19b = (SIN(LM(m)) * SINHLM) - ((COSLM(m) * COSHLM) - 1)
4130 F19c = ((COS(LM(m)) * COSHLM) - 1) + (SIN(LM(m)) * SINHLM)
4140 F19 = ((L ^ 2 / (2 * LM(m))) * F19a) - ((L ^ 2 / (4 * LM(m)
4150 F20a = SIN(LM(m)) * SINHLM - (COS(LM(m)) * COSHLM - 1)
4160 F20b = SIN(LM(m)) * COSHLM - COS(LM(m)) * SINHLM
4170 F20c = COS(LM(m)) * SINHLM - SIN(LM(m)) * COSHLM
4180 F20 = ((L ^ 2 / (2 * LM(m))) * F20a) - ((L ^ 2 / (4 * LM(m)
4190 F21 = ((-L / (4 * LM(m))) * (COS(LM(m)) - 1)) + ((L ^ 2 /
4200 F22 = (L ^ 2 / 4) + ((L ^ 2 / (4 * LM(m) ^ 2)) * ((2 * LM(m)
4210 F23a = (SIN(LM(m)) * SINHLM) + ((COS(LM(m)) * COSHLM) - 1)
4220 F23b = (SIN(LM(m)) * COSHLM) + (COS(LM(m)) * SINHLM)
4230 F23c = (SIN(LM(m)) * COSHLM) - (COS(LM(m)) * SINHLM)
4240 F23 = ((L.^ 2 / (2 * LM(m))) * F23a) - ((L ^ 2 / (2 * LM(m)
4250 F24a = (SIN(LM(m)) * COSHLM) + (COS(LM(m)) * SINHLM)
4260 F24b = ((COS(LM(m)) * COSHLM) - 1) + (SIN(LM(m)) * SINHLM)
4270 F24c = (SIN(LM(m)) * SINHLM) - ((COS(LM(m)) * COSHLM) - 1)
4280 F24 = ((L ^ 2 / (2 * LM(m))) * F24a) - ((L ^ 2 / (4 * LM(m)
4290 F25a = (SIN(LM(m)) * COSHLM) - (COS(LM(m)) * SINHLM)
4300 F25b = (SIN(LM(m)) * SINHLM) - ((COS(LM(m)) * COSHLM) - 1)
4310 F25c = ((COS(LM(m)) * COSHLM) - 1) + (SIN(LM(m)) * SINHLM)
4320 F25 = ((L ^ 2 / (2 * LM(m))) * F25a) - ((L ^ 2 / (4 * LM(m)
4330 F26a = (SIN(LM(m)) * SINHLM) + ((COS(LM(m)) * COSHLM) - 1)
4340 F26b = (SIN(LM(m)) * COSHLM) + (COS(LM(m)) * SINHLM)
4350 F26c = (SIN(LM(m)) * COSHLM) - (COS(LM(m)) * SINHLM)
```

```
4360 F26 = ((L ^ 2 / (2 * LM(m))) * F26a) - ((L ^ 2 / (2 * LM(m)
4370 F27 = (L ^ 2 / (8 * LM(m) ^ 2)) * ((2 * LM * SINH4) + COSH4 - 2
4380 F18 = -(L ^ 2 / (8 * LM(m) ^ 2)) * ((2 * LM(m) * (COSH4 - 1))
4390 F29 = F20
4400 F30 = F24
4410 F31 = ((-L / (4 * LM(m))) * (COSHLM - 1)) + ((L ^ 2 / (8 * LM(m)
4420 F32 = (L ^ 2 / 4) + ((L ^ 2 / (4 * LM(m) ^ 2)) * ((2 * LM(m)
4430 S21 = F17 + AM * F18 + BM * F19 + CM * F20 + AM * F21 + AM ^ 2
4440 F22 + AM * BM * F23 + AM * CM * F24
4450 S22 = BM * F25 + BM * AM * F26 + BM ^ 2 * F27 + BM * CM * F28
4460 S2 = (S21 + S22) / afam
4470 S41 = G1 + AM * G2 + BM * G3 + CM * G4 + AM * G5 + AM ^ 2 * G6
4480 S42 = BM * AM * G10 + BM ^ 2 * G11 + BM * CM * G12 + CM ^ 2
4490 S4 = (S41 + S42) / afam
4500 S81 = -F17 - AM * F18 - BM * F19 - CM * F20 - AM * F21 - AM ^ 2
4510 S82 = -AM * CM * F24 + BM * F25 + BM * AM * F26 + BM ^ 2 * F27
4520 S8 = ((LM(m) ^ 2 / L ^ 2) * (S81 + S82)) / afam
4530 S101 = -G1 - AM * G2 - BM * G3 - CM * G4 - AM * G5 - AM ^ 2 * G6
4540 S102 = BM * AM * G10 + BM ^ 2 * G11 + BM * CM * G12 + CM * G13
4550 S10 = (LM(m) ^ 2 / L ^ 2 * (S101 + S102)) / afam
4560 LAM = AF / MIU
4570 OME = ((E * IN / MIU) * ((LM(m)) / L) ^ 4) + (FM / MIU)
4580 TAI1 = (LAM * S1) / (2 * OME ^ 2)
4590 TAI2 = 1 - TAI1
4600 TAI = OME * TAI2
4610 AK = LM(m) * C / L
4620 A = -COS(LM(m)) + (AM * SIN(LM(m))) + (BM * COSHLM)
4630 PRINT TAI, A
4640 T1 = (A * (1 - COS(TAI * t))) / (TAI)
4650 T2a = (COS(AK * t) - COS(TAI * t))
4660 T2b = TAI ^ 2 - AK ^ 2
4670 T2 = T2a / T2b
4680 T3a = AM * (TAI * SIN(AK * t) + AK * SIN(TAI * t))
4690 T3b = TAI * (TAI ^ 2 - AK ^ 2)
4700 T3 = T3a / T3b
4710 T4a = AK * BM * SIN(TAI * t) * COS(TAI * t) * (EXP(AK * t)
4720 EXP(-AK * t))
4730 T4b = 2 * TAI * (AK ^ 2 + TAI ^ 2)
4740 T4 = T4a / T4b
4750 T5a = BM * TAI * (SIN(TAI * t)) ^ 2 * (EXP(AK * t)
4760 T5b = 2 * TAI * (AK ^ 2 + TAI ^ 2)
4770 T5 = T5a / T5b
4780 T6a = BM * AK * SIN(TAI * t) * COS(TAI * t) * (EXP(AK * t)
4790 EXP(-AK * t))
4800 T6b = 2 * TAI * (AK ^ 2 - TAI ^ 2)
4810 T6 = T6a / T6b
4820 T7a = BM * TAI * COS(TAI * t) * (COS(TAI * t) * ((EXP(AK * t)
4830 EXP(-AK * t)) / 2) - 1)
4840 T7b = (AK ^ 2 - TAI ^ 2)
4850 T7 = T7a / T7b
4860 T8a = CM * AK * SIN(TAI * t) * (COS(TAI * t) * ((EXP(AK * t)
4870 EXP(-AK * t)) / 2) - 1)
```

```
4880 T8b = TAI * (AK ^ 2 + TAI ^ 2)
4890 T8 = T8a / T8b
4900 T9a = CM * TAI * (SIN(TAI * t)) ^ 2 * (EXP(AK * t)
4910 T9b = 2 * TAI * (AK ^ 2 + TAI ^ 2)
4920 T9 = T9a / T9b
4930 T10a = CM * AK * SIN(TAI * t) * COS(TAI * t) * (EXP(AK * t)
4940 T10b = 2 * TAI * (AK ^ 2 + TAI ^ 2)
4950 T10 = T10a / T10b
4960 T11a = CM * TAI * (COS(TAI * t)) ^ 2 * (EXP(AK * t)
4970 T11b = 2 * (AK ^ 2 - TAI ^ 2)
4980 T11 = T11a / T11b
4990 'T12 = ((BM * SIN(TAI * t)) / (AK ^ 2 - TAI ^ 2))
5000 M11 = SIN(LM(m) * X / L) + AM * COS(LM(m) * X / L)
5010 M12 = BM * (EXP(LM(m) * X / L) - EXP(-LM(m) * X / L)) + CM
5020 wa = (T1 + T2 - T3 - T4 - T5 + T6 - T7 - T8 - T9 + T10 + T11
5030 T12) * (M11 + (M12 / 2))
5040 WB(m) = wa * ((P * L) / (MIU * LM(m) * afam))
5050 NEXT m
5060 W = WB(1) + WB(2) + WB(3)
5070 PRINT #1, t, W
5080 'PRINT WB(1), WB(2), WB(3)
5090 PRINT t, W
5100 'SLEEP
5110 NEXT t
5120 PRINT #1,.
5130 NEXT mm

CLS
REM  THIS PROGRAM IS WRITTEN BY OGUNYEBI SEGUN N.
REM  IT IS WRITTEN TO EVALUATE THE TRANSVERSE DISPLACEMENT OF
REM  MOVING FORCE UNIFORM BEAM FOR OTHER BOUNDARY CONDITION.
10    DIM WB(3), LM(3), LK(3)
20    OPEN "CFMMAF.BAS" FOR OUTPUT AS #1
30    FOR mm = 1 TO 5
40    PRINT "Enter the value of AF N", mm
50    INPUT AF
60    L = 12.192
70    E = 2.10924E+09
80    P = 8407.27 * 9.81
90    GRA = 9.81
100   MI = 2758.291
110   MIU = 2758.291
120   IN = .00287698#
130   M1 = 8407.27
140   pi = 22 / 7
150   FM = 40000
160   'AF = 20000
170   X = L / 2
180   C = 8.128
190   CO = 5
200   PRINT #1, "TIME(t)", SPC(2); "DEFLECTION V(x,t)"
210   PRINT #1,
220   LM(1) = 1.875
```

```
230   LM(2) = 4.694
240   LM(3) = 7.855
250   n = 1
260   LK(1) = 1.875
270   LK(2) = 4.694
280   LK(3) = 7.855
290   PRINT #1, "This is The Result for N", AF
300   FOR t = 0 TO 1.5 STEP .1
310   FOR m = 1 TO 3
320   k = m
330   AM1 = SIN(LM(m) - (EXP(LM(m)) - EXP(-LM(m))) / 2)
340   AM2 = COS(LM(m)) + ((EXP(LM(m)) + EXP(-LM(m))) / 2)
350   AM = AM2 / AM1
360   BM = 1
370   CM = AM
380   AK1 = SIN(LK(k)) - (EXP(LK(k)) - EXP(-LK(k)) / 2)
390   AK2 = COS(LK(k)) + ((EXP(LK(k)) + EXP(-LK(k))) / 2)
400   AK = AK2 / AK1
410   BK = 1
420   CK = AK
430   COSHLM = (EXP(LM(m)) + EXP(-LM(m))) / 2
440   SINHLM = (EXP(LM(m)) - EXP(-LM(m))) / 2
450   COSHLK = (EXP(LK(m)) + EXP(-LK(m))) / 2
460   SINHLK = (EXP(LK(m)) - EXP(-LK(m))) / 2
470   COSH1 = (EXP(LM(m) + LK(m)) + EXP(-(LM(m) + LK(m)))) / 2
480   SINH1 = (EXP(LM(m) + LK(m)) - EXP(-(LM(m) + LK(m)))) / 2
490   COSH2 = (EXP(LK(m) - LM(m)) + EXP(-(LK(m) - LM(m)))) / 2
500   SINH2 = (EXP(LK(m) - LM(m)) - EXP(-(LK(m) - LM(m)))) / 2
510   COSH3 = (EXP(LM(m) - LK(m)) + EXP(-(LM(m) - LK(m)))) / 2
520   SINH3 = (EXP(LM(m) - LK(m)) - EXP(-(LM(m) - LK(m)))) / 2
530   COSH4 = (EXP(2 * LM(m)) + EXP(-2 * LM(m))) / 2
540   SINH4 = (EXP(2 * LM(m)) - EXP(-2 * LM(m))) / 2
550   COSH5 = (EXP(2 * LK(m)) + EXP(-2 * LK(m))) / 2
560   SINH5 = (EXP(2 * LK(m)) - EXP(-2 * LK(m))) / 2
570   G1 = L / 2 * (1 - SIN(2 * LM(m)) / 2 * LM(m))
580   G2a = AM * L / 2
590   G2b = (COS(2 * LM(m)) / 2 * LM(m)) - 1
600   G2 = -G2a * G2b
610   G3a = (BM * L) / 2 * LM(m)
620   G3b = SIN(LM(m)) * COSHLM - COS(LM(m)) * SINHLM
630   G3 = G3a * G3b
640   G4a = (CM * L) / 2 * LM(m)
650   G4b = SIN(LM(m)) * SINHLM - (COS(LM(m)) * COSHLM - 1)
660   G4 = G4a * G4b
670   G5a = AM * L / 2
680   G5b = (COS(2 * LM(m)) / 2 * LM(m)) - 1
690   G5 = -G5a * G5b
700   G6a = AM ^ 2 * L / 2
710   G6b = 1 + (SIN(2 * LM(m)) / 2 * LM(m))
720   G6 = G6a * G6b
730   G7a = (AM * BM * L) / 2 * LM(m)
740   G7b = (COS(LM(m)) * COSHLM - 1) + SIN(LM(m)) * SINHLM
750   G7 = G7a * G7b
760   G8a = (AM * CM * L) / 2 * LM(m)
770   G8b = (COS(LM(m)) * SINHLM) + (SIN(LM(m)) * COSHLM)
780   G8 = G8a * G8b
790   G9a = (BM * L / 2 * LM(m))
```

```
800   G9b = SIN(LM(m)) * COSHLM - COS(LM(m)) * SINHLM
810   G9 = G9a * G9b
820   G10a = (AM * CM * L) / 2 * LM(m)
830   G10b = (COS(LM(m)) * COSHLM - 1) + (SIN(LM(m)) * SNHLM)
840   G10 = G10a * G10b
850   G11a = BM ^ 2 * L / 2
860   G11b = (SIN(2 * LM(m)) / 2 * LM(m))
870   G11 = G11a * G11b
880   G12a = (BM * CM * LM(m)) / 2
890   G12b = (COS(2 * LM(m)) / 2 * LM(m)) - 1
900   G12 = G12a * G12b
910   G13a = (CM * L) / 2 * LM(m)
920   G13b = SIN(LM(m)) * SINHLM - (COS(LM(m)) * COSHLM - 1)
930   G13 = G13a * G13b
940   G14a = (AM * CM * L) / 2 * LM(m)
950   G14b = COS(LM(m)) * SINHLM + SIN(LM(m)) * COSHLM
960   G14 = G14a * G14b
970   G15a = CM * BM * L / 2
980   G15b = (COSH4 / 2 * LM(m)) - 1
990   G15 = G15a * G15b
1000  G16a = CM ^ 2 * L / 2
1010  G16b = 1 + (SINH4 / 2 * LM(m))
1020  G16 = G16a * G16b
1030  afAM1 = G1 + G2 + G3 + G4 + G5 + G6 + G7 + G8 + G7 + G8
1040  afAM2 = G9 + G10 + G11 + G12 + G13 + G14 + G15 + G16
1050  afam = afAM1 + afAM2
1060  IF m = k THEN
1070  I1a = SIN(2 * LM(m)) / (2 * LM(m))
1080  I1 = L / 2 * (1 - I1a)
1090  ELSE
1100  I1a = SIN(LK(k) - LM(m)) / (LK(k) - LM(m))
1110  1b = SIN(LK(k) + LM(m)) / (LK(k) + LM(m))
1120  I1 = L / 2 * (I1a - I1b)
1130  END IF
1140  IF k = m THEN
1150  I2a = (COS(2 * LM(m)) - 1) / (2 * LM(m))
1160  I2 = -L / 2 * (I2a)
1170  ELSE
1180  I2a = (LK(k) - LM(m)) * (COS(LK(k) + LM(m)) - 1)
1190  I2b = (LK(k) + LM(m)) * (COS(LK(k) - LM(m)) - 1)
1200  I2c = -(L / 2) / (LK(k) ^ 2 - LM(m) ^ 2)
1210  I2 = I2c * (I2a + I2b)
1220  END IF
1230  I3a = SIN(LK(k)) * COSHLM
1240  I3b = (LK(k) / LM(m)) * (COS(LK(k)) * SINHLM)
1250  I3c = (LM(m) * L) / (LM(m) ^ 2 + LK(k) ^ 2)
1260  I3 = I3c * (I3a - I3b)
1270  I4a = SIN(LK(k)) * SINHLM
1280  I4b = (LK(k) / LM(m)) * ((COS(LK(k)) * COSHLM) - 1)
1290  I4c = (LM(m) * L) / (LM(m) ^ 2 + LK(k) ^ 2)
1300  I4 = I4c * (I4a - I4b)
1310  IF k = m THEN
1320  I5a = (COS(2 * LM(m)) - 1) / (2 * LM(m))
1330  I5 = -L / 2 * I5a
1340  ELSE
1350  I5a = (LK(k) + LM(m)) * (COS(LK(k) - LM(m)) - 1)
1360  I5b = (LK(k) - LM(m)) * (1 - COS(LK(k) + LM(m)))
```

```
1370  I5c = (L / 2) / (LK(k) ^ 2 - LM(m) ^ 2)
1380  I5 = I5c * (I5a + I5b)
1390  END IF
1400  IF k = m THEN
1410  I6a = SIN(2 * LM(m)) / (2 * LM(m))
1420  I6 = L / 2 * (I6a + 1)
1430  ELSE
1440  I6a = SIN(LK(k) + LM(m)) / (LK(k) + LM(m))
1450  I6b = SIN(LK(k) - LM(m)) / (LK(k) - LM(m))
1460  I6 = L / 2 * (I6a + I6b)
1470  END IF
1480  I7a = COS(LK(k)) * COSHLM
1490  I7b = (LK(k) / LM(m)) * ((SIN(LK(k)) * SINHLM) - 1)
1500  I7c = (LM(m) * L) / (LM(m) ^ 2 + LK(k) ^ 2)
1510  I7 = I7c * (I7a + I7b)
1520  I8a = COS(LK(k)) * SINHLM
1530  I8b = (LK(k) / LM(m)) * (SIN(LK(k)) * COSHLM)
1540  I8c = (LM(m) * L) / (LM(m) ^ 2 + LK(k) ^ 2)
1550  I8 = I8c * (I8a + I8b)
1560  I9a = SIN(LM(m)) * COSHLK
1570  I9b = (LM(m) / LK(k)) * (COS(LM(m)) * SINHLK)
1580  I9c = (LK(k) * L) / (LK(k) ^ 2 + LM(m) ^ 2)
1590  I9 = I9c * (I9a - I9b)
1600  I10a = COS(LM(m)) * COSHLK
1610  I10b = (LM(m) / LK(m)) * (SIN(LM(m)) * SINHLK - 1)
1620  I10c = (LK * L) / (LK(k) ^ 2 + LM(m) ^ 2)
1630  I10 = I10c * (I10a + I10b)
1640  IF k = m THEN
1650  I11a = (SINH4 / (2 * LM(m)))
1660  I11 = L / 2 * (I11a - 1)
1670  ELSE
1680  I11a = SINH1 / (LK(k) + LM(m))
1690  I11b = SINH2 / (LK(k) - LM(m))
1700  I11 = L / 2 * (I11a - I11b)
1710  END IF
1720  IF k = m THEN
1730  I12a = (COS(2 * LM(m)) - 1) / (2 * LM(m))
1740  I12 = L / 2 * (I12a)
1750  ELSE
1760  I12a = (LK(k) - LM(m)) * (COSH1 - 1)
1770  I12b = (LK(k) + LM(m)) * (COSH2 - 1)
1780  I12c = (L / 2) / (LK(k) ^ 2 + LM(m) ^ 2)
1790  I12 = I12c * (I12a + I12b)
1800  END IF
1810  I13a = SIN(LM(m)) * SINHLK
1820  I13b = (LK(k) / LM(m)) * ((COS(LM(m)) * COSHLK(k)) - 1)
1830  I13c = (LK(k) * L) / (LK(k) ^ 2 + LM(m) ^ 2)
1840  I13 = I13c * (I13a - I13b)
1850  I14a = COS(LM(m)) * SINHLK
1860  I14b = (LK / LM(m)) * (SIN(LM(m)) * COSHLK)
1870  I14c = (LK * L) / (LK(k) ^ 2 + LM(m) ^ 2)
1880  I14 = I14c * (I14a + I14b)
1890  IF k = m THEN
1900  I15a = (COS(2 * LM(m)) - 1) / (2 * LM(m))
1910  I15 = L / 2 * (I15a - 1)
1920  ELSE
1930  I15a = (LM(m) - LK(k)) * (COSH1 - 1)
```

```
1940  I15b = (LM(m) + LK(k)) * (COSH2 - 1)
1950  I15c = (L / 2) / (LM(m) ^ 2 + LK(k) ^ 2)
1960  I15 = I15c * (I15a + I15b)
1970  END IF
1980  IF k = m THEN
1990  I16a = (SINH4 / (2 * LM(m)))
2000  I16 = L / 2 * (I16a + 1)
2010  ELSE
2020  I16a = SINH1 / (LK(k) + LM(m))
2030  I16b = SINH2 / (LK(k) - LM(m))
2040  I16 = L / 2 * (I16a + I16b)
2050  END IF
2060  IF k = m THEN
2070  I17a = (LK(k) + LM(m)) * SIN(LK(k) + LM(m))
2080  I17b = (COS(LK(k) + LM(m)) - 1)
2090  I17c = L ^ 2 / 2 * (LK(k) + LM(m)) ^ 2
2100  I17 = (L ^ 2 / 4) - (I17c * I17a) + I17b
2110  ELSE
2120  I17a = (LK(k) - LM(m)) * SIN(LK(k) - LM(m))
2130  I17b = (COS(LK(k) - LM(m)) - 1)
2140  I17c = (LK(k) + LM(m)) * SIN(LK(k) + LM(m))
2150  I17d = (COS(LK(k) + LM(m)) - 1)
2160  I17e = (L ^ 2 / 2 * (LK(k) - LM(m)) ^ 2) * (I17a + I17b)
2170  I17f = (L ^ 2 / 2 * (LK(k) + LM(m)) ^ 2) * (I17c + I17d)
2180  I17 = I17e - I17f
2190  END IF
2200  IF k = m THEN
2210  I18a = (2 * LK(k) * COS(2 * LK(k)) - SIN(2 * LK(k))) / 4 * LK(k) ^ 2
2220  I18 = -(L ^ 2 / 2) * (I18a)
2230  ELSE
2240  I18a = ((LK(k) - LM(m)) * COS(LK(k) - LM(m)) - SIN(LK(k) - LM(m))) /(LK(k
2260  I18b = ((LK(k) + LM(m)) * COS(LK(k) + LM(m)) - SIN(LK(k) + LM(m))) /
2270  I18 = -L ^ 2 / 2 * (I18a + I18b)
2280  END IF
2290  I19a = (LK(k) ^ 2 + LM(m) ^ 2) * LM(m) * SIN(LK(k)) * COSHLM
2300  I19b = (LK(k) ^ 2 + LM(m) ^ 2) * LK(k) * COSHLK * SINHLM
2310  I19c = (LK(k) ^ 2 - LM(m) ^ 2) * SIN(LK(k)) * SINHLM
2320  I19d = 2 * LM(m) * LK(k) * COSLK(k) * COSHLM - 2 * LM(m) * LK(k)
2330  I19e = L ^ 2 / (LK(k) ^ 2 + LM(m) ^ 2) ^ 2
2340  I19 = I19e * (I19a - I19b + I19c + I19d)
2350  I20a = (LM(m) ^ 2 + LK(k) ^ 2) * LM(m) * SIN(LK(k)) * SINHLM
2360  I20b = (LK(k) ^ 2 + LM(m) ^ 2) * LK(k) * COS(LK(k)) * COSHLM
2370  I20c = 2 * LK(k) * LM(m) * COS(LK(k)) * SINHLM
2380  I20d = (LK(k) ^ 2 - LM(m) ^ 2) * SIN(LK(k)) * COSHLM
2390  I20e = L ^ 2 / (LK(k) ^ 2 + LM(m) ^ 2) ^ 2
2400  I20 = I20e * (I20a - I20b + I20c + I20d)
2410  IF k = m THEN
2420  I21a = L ^ 2 / (2 * LM(m))
2430  I21b = COS(LM(m)) - (SIN(LM(m)) / (2 * LM(m)))
2440  I21 = I21a * I21b
2450  ELSE
2460  I21a = (LM(m) - LK(k)) * COS(LM(m) - LK(k)) - SIN(LM(m) - LK(k)) / (LM(m)
2470  LK(k)) ^ 2
2780  I21b = (LM(m) + LK(k)) * COS(LM(m) + LK(k)) - SIN(LM(m) + LK(k)) / (LM(m)
2790  I21 = -L ^ 2 / 2 * (I21a + I21b)
2800  END IF
2810  IF k = m THEN
```

```
2820  I22a = 2 * LK(k) * SIN(2 * LK(k))
2830  I22b = (COS(2 * LK(k)) - 1)
2840  I22c = L ^ 2 / 2 * (2 * LK(k)) ^ 2
2850  I22 = (L ^ 2 / 4) + (I22c * I22a) + I22b
2860  ELSE
2870  I22a = (LK(k) + LM(m)) * SIN(LK(k) + LM(m))
2880  I22b = (COS(LK(k) + LM(m)) - 1)
2890  I22c = (LK(k) - LM(m)) * SIN(LK(k) - LM(m))
2900  I22d = (COS(LK(k) - LM(m)) - 1)
2910  I22e = (L ^ 2 / 2 * (LK(k) + LM(m)) ^ 2) * (I22a + I22b)
2920  I22f = (L ^ 2 / 2 * (LK(k) - LM(m)) ^ 2) * (I22c + I22d)
2930  I22 = I22e + I22f
2940  END IF
2950  I23a = (LM(m) ^ 2 + LK(k) ^ 2) * LK(k) * SIN(LK(k)) * SINHLM
2960  I23b = (LK(k) ^ 2 + LM(m) ^ 2) * LM(m) * COS(LK(k)) * COSHLM
2970  I23c = 2 * LK(k) * LM(m) * SIN(LK(k)) * COSHLM
2980  I23d = (LM(m) ^ 2 - LK(k) ^ 2) * COS(LK(k)) * SINHLM
2990  I23e = L ^ 2 / (LM(m) ^ 2 + LK(k) ^ 2) ^ 2
3000  I23 = I23e * (I23a + I23b - I23c - I23d)
3010  I24a = (LM(m) ^ 2 + LK(k) ^ 2) * LK(k) * SIN(LK(k)) * COSHLM
3020  I24b = (LK(k) ^ 2 + LM(m) ^ 2) * LM(m) * COS(LK(k)) * SINHLM
3030  I24c = 2 * LK(k) * LM(m) * SIN(LK(k)) * SINHLM
3040  I24d = (LM(m) ^ 2 - LK(k) ^ 2) * COS(LK(k)) * COSHLM - (LK(k) ^ 2 -
3050  I24e = L ^ 2 / (LM(m) ^ 2 + LK(k) ^ 2) ^ 2
3060  I24 = I24e * (I24a + I24b - I24c - I24d)
3070  I25a = (LK(k) ^ 2 + LM(m) ^ 2) * LK(k) * SIN(LM(m)) * COSHLK
3080  I25b = (LK ^ 2 + LM(m) ^ 2) * LM(m) * COS(LM(m)) * SINHLK
3090  I25c = (LM(m) ^ 2 - LK(k) ^ 2) * SIN(LM(m)) * SINHLK(k)
3100  I25d = 2 * LM(m) * LK(k) * COS(LM(m)) * COSHLK - 2 * LM(m) * LK
3110  I25e = L ^ 2 / (LK(k) ^ 2 + LM(m) ^ 2) ^ 2
3120  I25 = I25e * (I25a - I25b + I25c + I25d)
3130  I26a = (LM(m) ^ 2 + LK(k) ^ 2) * LM(m) * SIN(LM(m)) * SINHLK
3140  I26b = (LK ^ 2 + LM(m) ^ 2) .* LK(k) * COS(LM(m)) * COSHLK(k)
3150  I26c = 2 * LK(k) * LM(m) * SIN(LM(m)) * COSHLK(k)
3160  I26d = (LM(m) ^ 2 - LK(k) ^ 2) * COS(LM(m)) * SINHLK
3170  I26e = L ^.2 / (LM(m) ^ 2 + LK ^ 2) ^ 2
3180  I26 = I26e * (I26a + I26b - I26c - I26d)
3190  IF k = m THEN
3200  I27a = 2 * LM(m) * SIN(2 * LM(m)) - COSH4 - 2 * LM(m) ^ 2 + 1
3210  I27 = (L ^ 2 / 8 * LM(m) ^ 2) * (I27a)
3220  ELSE
3230  I27a = (LK(k) + LM(m)) * SINH1
3240  I27b = (COS(LK(k) + LM(m)) + 1)
3250  I27c = (LK(k) - LM(m)) * SINH3
3260  I27d = ((COSH2) + 1)
3270  I27e = (L ^ 2 / 2 * (LK(k) + LM(m)) ^ 2) * (I27a - I27b)
3280  I27f = (L ^ 2 / 2 * (LK(k) - LM(m)) ^ 2) * (I27c - I27d)
3290  I27 = I27e - I27f
3300  END IF
3310  IF k = m THEN
3320  I28a = 2 * LM(m) * COSH4 - SINH4
3330  I28 = L ^ 2 / 8 * LM(m) ^ 2 .* (I28a)
3340  ELSE
3350  I28a = (LK(k) - LM(m)) * COSH1
3360  I28b = (LK(k) + LM(m)) * (COSH2)
3370  I28c = (LK(k) - LM(m)) ^ 2 * SINH1
3380  I28d = (LK(k) + LM(m)) ^ 2 * (SINH2)
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3390  I28e = (L ^ 2 / 2 * (LK(k) ^ 2 - LM(m) ^ 2)) * (I28a + I28b)
3400  I28f = (L ^ 2 / 2 * (LK(k) ^ 2 - LM(m) ^ 2)) * (I28c + I28d)
3410  I28 = I28e - I28f
3420  END IF
3430  I29a = (LM(m) ^ 2 + LM(m) ^ 2) * LK(k) * SIN(LM(m)) * SINHLK
3440  I29b = (LK(k) ^ 2 + LM(m) ^ 2) * LM(m) * COS(LM(m)) * COSHLK
3450  I29c = 2 * LK(k) * LM(m) * COS(LM(m)) * SINHLK
3460  I29d = (LM(m) ^ 2 - LK ^ 2) * SIN(LM(m)) * COSHLK
3470  I29e = L ^ 2 / (LK(k) ^ 2 + LM(m) ^ 2) ^ 2
3480  I29 = I29e * (I29a - I29b + I29c + I29d)
3490  I30a = (LM(m) ^ 2 + LK(k) ^ 2) * LM(m) * SIN(LM(m)) * COSHLK
3500  I30b = (LM(m) ^ 2 + LK ^ 2) * LK(k) * COSHLM * SINHLK(k)
3510  I30c = 2 * LK(k) * LM(m) * SIN(LM(m)) * SINHLK
3520  I30d = (LK(k) ^ 2 - LM(m) ^ 2) * COSHLM * COSHLK - (LM(m) ^ 2 - LK ^ 2
3530  I30e = L ^ 2 / (LM(m) ^ 2 + LK(k) ^ 2) ^ 2
3540  I30 = I30e * (I30a + I30b - I30c - I30d)
3550  IF m = k THEN
3560  I31a = 2 * LK(k) * COSH5 - SINH5
3570  I31 = L ^ 2 / 8 * LK(k) ^ 2 * (I31a)
3580  ELSE
3590  I31a = (LM(m) - LK(k)) * COSH1
3600  I31b = (LM(m) + LK(k)) * COSH
3610  I32 = I32e + I32f
3620  END IF
3630  I33a = (COS(LK(k) - LM(m) + n * pi) - 1) / (LK(k) - LM(m) + n * pi)
3640  I33b = (COS(LK(k) - LM(m) + n * pi) - 1) / (LK(k) - LM(m) + n * pi)
3650  I33c = (COS(LK(k) + LM(m) + n * pi) - 1) / (LK(k) + LM(m) + n * pi)
3660  I33d = (COS(LK(k) - LM(m) - n * pi) - 1) / (LK(k) - LM(m) - n * pi)
3670  I33 = L / 4 * (I33a + I33b - I33c - I33d)
3680  I34a = (SIN(LK(k) + n * pi + LM(m))) / (LK(k) + n * pi + LM(m))
3690  I34b = (SIN(LK(k) + n * pi - LM(m))) / (LK(k) + n * pi - LM(m))
3700  I34c = (SIN(LK(k) - n * pi + LM(m))) / (LK(k) - n * pi + LM(m))
3710  I34d = (SIN(LK(k) - n * pi - LM(m))) / (LK(k) - n * pi - LM(m))
3720  I34 = L / 4 * (I34a - I34b - I34c + I34d)
3730  I35a = LM(m) / (LM(m) ^ 2 + (n * pi + LK(k)) ^ 2)
3740  I35b = LM(m) / (LM(m) ^ 2 + (n * pi - LK(k)) ^ 2)
3750  I35c = COS(n * pi + LK(k)) * COSHLM
3760  I35d = (n * pi + LK(k)) / LM(m) * SINHLM * SIN(n * pi + LK(k))
3770  I35e = COS(n * pi - LK(k)) * COSHLM
3780  I35f = (n * pi - LK(k)) / LM(m) * SINHLM * SIN(n * pi - LK(k))
3790  I35 = L / 2 * ((I35a * (I35c + I35d) + (I35b * (I35e + I35f))))
3800  I36a = LM(m) / (LM(m) ^ 2 + (n * pi + LK(k)) ^ 2)
3810  I36b = LM(m) / (LM(m) ^ 2 + (n * pi - LK(k)) ^ 2)
3820  I36c = COS(n * pi + LK(k)) * COSHLM
3830  I36d = (n * pi + LK(k)) / LM(m) * COSHLM * SIN(n * pi + LK(k))
3840  I36e = COS(n * pi - LK(k)) * COSHLM
3850  I36f = (n * pi - LK(k)) / LM(m) * COSHLM * SIN(n * pi - LK(k))
3860  I36 = L / 2 * ((I36a * (I36c + I36d) + (I36b * (I36e + I36f))))
3870  I37a = (SIN(n * pi + LK(k) + LM(m))) / n * pi + LK(k) + LM(m)
3880  I37b = (SIN(n * pi + LK(k) - LM(m))) / n * pi + LK(k) - LM(m)
3890  I37c = (SIN(n * pi - LK(k) + LM(m))) / n * pi - LK(k) + LM(m)
3900  I37d = (SIN(n * pi - LK(k) - LM(m))) / n * pi - LK(k) - LM(m)
3910  I37 = L / 4 * (I37a - I37b + I37c - I37d)
3920  I38a = (COS(n * pi + LK(k) + LM(m)) - 1) / (n * pi + LK(k) + LM(m))
3930  I38b = (COS(n * pi + LK(k) - LM(m)) - 1) / (n * pi + LK(k) - LM(m))
3940  I38c = (COS(n * pi - LK(k) + LM(m)) - 1) / (n * pi - LK(k) - LM(m))
3950  I38d = (COS(n * pi - LM(m) - LM(m)) - 1) / (n * pi - LK(k) - LM(m))
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3960  I38 = L / 4 * (I38a + I38b - I38c - I38d)
3970  I39a = LM(m) / (LM(m) ^ 2 - (n * pi + LK(k)) ^ 2)
3980  I39b = LM(m) / (LM(m) ^ 2 - (n * pi - LK(k)) ^ 2)
3990  I39c = SIN(n * pi + LK(k)) * COSHLM
4000  I39d = (n * pi + LK(k)) / LM(m) * COS(n * pi + LK(k)) * SINHLM
4010  I39e = SIN(n * pi - LK(k)) * COSHLM
4020  I39f = (n * pi - LK(k)) / LM(m) * COS(n * pi - LK(k)) * SINHLM
4030  I39 = L / 2 * ((I39a * (I39c - I39d) + (I39b * (I39e - I39f))))
4040  I40a = LM(m) / (LM(m) ^ 2 - (n * pi + LK(k)) ^ 2)
4050  I40b = LM(m) / (LM(m) ^ 2 - (n * pi - LK(k)) ^ 2)
4060  I40c = SIN(n * pi + LK(k)) * SINHLM
4070  I40d = (n * pi + LK(k)) / LM(m) * COS(n * pi + LK(k)) * COSHLM
4080  I40e = SIN(n * pi - LK(k)) * SINHLM
4090  I40f = (n * pi - LK(k)) / LM(m) * COS(n * pi - LK(k)) * COSHLM
4100  I40 = L / 2 * ((I40a * (I40c - I40d) + (I40b * (I40e - I40f))))
4110  I41a = LK(k) / (LK(k) ^ 2 + (n * pi + LM(m)) ^ 2)
4120  I41b = LK(k) / (LK(k) ^ 2 + (n * pi - LM(m)) ^ 2)
4130  I41c = COS(n * pi + LM(m)) * COSHLK
4140  I41d = (n * pi + LM(m)) / LK(k) * SIN(n * pi + LM(m)) * SINHLK
4150  I41e = COS(n * pi - LM(m)) * COSHLK
4160  I41f = (n * pi - LM(m)) / LK(k) * SIN(n * pi + LM(m)) * SIHNLK
4170  I41 = L / 2 * ((I41a * (I41c + I41d) + (I41b * (I41e + I41f))))
4180  I42a = LK(k) / (LK(k) ^ 2 - (n * pi + LM(m)) ^ 2)
4190  I42b = LK(k) / (LK(k) ^ 2 - (n * pi - LM(m)) ^ 2)
4200  I42c = SIN(n * pi + LM(m)) * COSLK(k)
4210  I42d = (n * pi + LM(m)) / LK(k) * COS(n * pi + LM(m)) * SINLK(k)
4220  I42e = SIN(n * pi - LM(m)) * COSLK(k)
4230  I42f = (n * pi - LM(m)) / LK(k) * COS(n * pi - LM(m)) * SINLK(k)
4240  I42 = L / 2 * ((I42a * (I42c - I42d) + (I42b * (I42e - I42f))))
4250  I43a = 1 / ((LK(k) + LM(m)) ^ 2 + (n ^ 2 * pi ^ 2))
4260  I43b = 1 / ((LK(k) - LM(m)) ^ 2 + (n ^ 2 * pi ^ 2))
4270  I43c = (n * pi * COSH1 * COS(n * pi)) - 1
4280  I43d = (n * pi * COSH2) * COS(n * pi) - 1
4290  I43 = L / 2 * ((I43a * I43c) - (I43b * I43d))
4300  I44a = 1 / ((LK(k) - LM(m)) ^ 2 - (n ^ 2 * pi ^ 2))
4310  I44b = 1 / ((LK(k) + LM(m)) ^ 2 + (n ^ 2 * pi ^ 2))
4320  I44c = (n * pi * SINH1 * COS(n * pi))
4330  I44d = (LK(k) + LM(m)) * COSH1 * SIN(n * pi)
4340  I44e = (n * pi) * SINH2 * COS(n * pi)
4350  I44f = (LK(k) - LM(m)) * COS(LK(k) - LM(m)) * SIN(n * pi)
4360  I44g = (I44a * I44c) + (I44a * I44d)
4370  I44h = (I44b * I44e) + (I44b * I44f)
4380  I44 = L / 2 * (I44g + I44h)
4390  I45a = LK(k) / (LK(k) ^ 2 + (n * pi + LM(m)) ^ 2)
4400  I45b = LK(k) / (LK(k) ^ 2 + (n * pi - LM(m)) ^ 2)
4410  I45c = COS(n * pi + LM(m)) * COSHLK
4420  I45d = (n * pi + LM(m)) / LK(k) * COSHLK * SIN(n * pi + LM(m))
4430  I45e = COS(n * pi - LM(m)) * COSHLK
4440  I45f = (n * pi - LM(m)) / LK(k) * COSHLK * SIN(n * pi - LM(m))
4450  I45 = L / 2 * ((I45a * (I45c + I45d) + (I45b * (I45e + I45f))))
4460  I46a = LK(k) / (LK(k) ^ 2 - (n * pi + LM(m)) ^ 2)
4470  I46b = LK(k) / (LK(k) ^ 2 - (n * pi - LM(m)) ^ 2)
4480  I46c = SIN(n * pi + LM(m)) * SINHLK
4490  I46d = (n * pi + LM(m)) / LK(k) * COS(n * pi + LM(m)) * COSHLK(k)
4500  I46e = SIN(n * pi - LM(m)) * SINHLK
4510  5I46f = (n * pi - LM(m)) / LK(k) * COS(n * pi - LM(m)) * COSHLK
4520  I46 = L / 2 * ((I46a * (I46c - I46d) + (I46b * (I46e - I46f))))
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4530  I47a = 1 / ((LM(m) - LK(k)) ^ 2 - (n ^ 2 * pi ^ 2))
4540  I47b = 1 / ((LM(m) + LK(k)) ^ 2 + (n ^ 2 * pi ^ 2))
4550  I47c = (n * pi * SINH1 * COS(n * pi))
4560  I47d = (LM(m) + LK(k)) * COSH1 * SIN(n * pi)
4570  I47e = (n * pi) * SINH3 * COS(n * pi)
4580  I47f = (LM(m) - LK(k)) * COSH3 * SIN(n * pi)
4590  I47g = (I47a * I47c) + (I47a * I47d)
4600  I47h = (I47b * I47e) + (I47b * I47f)
4610  I47 = L / 2 * (I47g + I47h)
4620  I48a = 1 / ((LK(k) + LM(m)) ^ 2 + (n ^ 2 * pi ^ 2))
4630  I48b = 1 / ((LK(k) - LM(m)) ^ 2 + (n ^ 2 * pi ^ 2))
4640  I48c = (n * pi * COS(LK(k) + LM(m)) * COS(n * pi)) - 1
4650  I48d = (n * pi * COS(LK(k) - LM(m)) * COS(n * pi)) - 1
4660  I48 = L / 2 * ((I48a * I48c) + (I48b * I48d))
4670  S1A = -G1 - G2 - G3 - G4 - G5 - G6 - G7 - G8
4680  S1B = G9 + G10 + G11 + G12 + G13 + G14 + G15 + G16
4690  S1 = ((LM(m) / L) ^ 2 * (S1A + S1B)) / afam
4700  F17 = (L ^ 2 / (8 * LM(m) ^ 2)) * ((2 * LM * SIN(2 * LM(m))) + COS(2 * 
4710  F18 = -(L ^ 2 / (8 * LM(m) ^ 2)) * ((2 * LM(m) * (COS(2 * LM(m)) - 1))
4720  F19a = (SIN(LM(m)) * COSHLM) - (COS(LM(m)) * SINHLM)
4730  F19b = (SIN(LM(m)) * SINHLM) - ((COSLM(m) * COSHLM) - 1)
4740  F19c = ((COS(LM(m)) * COSHLM) - 1) + (SIN(LM(m)) * SINHLM)
4750  F19 = ((L ^ 2 / (2 * LM(m))) * F19a) - ((L ^ 2 / (4 * LM(m) ^ 2))
4760  F20a = SIN(LM(m)) * SINHLM - (COS(LM(m)) * COSHLM - 1)
4770  F20b = SIN(LM(m)) * COSHLM - COS(LM(m)) * SINHLM
4780  F20c = COS(LM(m)) * SINHLM - SIN(LM(m)) * COSHLM
4790  F20 = ((L ^ 2 / (2 * LM(m))) * F20a) - ((L ^ 2 / (4 * LM(m) ^ 2))
4800  F21 = ((-L / (4 * LM(m))) * (COS(LM(m)) - 1)) + ((L ^ 2 / (8 * LM(m) ^ 
4810  F22 = (L ^ 2 / 4) + ((L ^ 2 / (4 * LM(m) ^ 2)) * ((2 * LM(m) * SIN(2 * 
LM(m)))
4820  F23a = (SIN(LM(m)) * SINHLM) + ((COS(LM(m)) * COSHLM) - 1)
4830  F23b = (SIN(LM(m)) * COSHLM) + (COS(LM(m)) * SINHLM)
4840  F23c = (SIN(LM(m)) * COSHLM) - (COS(LM(m)) * SINHLM)
4850  F23 = ((L ^ 2 / (2 * LM(m))) * F23a) - ((L ^ 2 / (2 * LM(m) ^ 2))
4860  F24a = (SIN(LM(m)) * COSHLM) + (COS(LM(m)) * SINHLM)
4870  F24b = ((COS(LM(m)) * COSHLM) - 1) + (SIN(LM(m)) * SINHLM)
4880  F24c = (SIN(LM(m)) * SINHLM) - ((COS(LM(m)) * COSHLM) - 1)
4890  F24 = ((L ^ 2 / (2 * LM(m))) * F24a) - ((L ^ 2 / (4 * LM(m) ^ 2))
4900  F25a = (SIN(LM(m)) * COSHLM) - (COS(LM(m)) * SINHLM)
5000  F25b = (SIN(LM(m)) * SINHLM) - ((COS(LM(m)) * COSHLM) - 1)
5010  F25c = ((COS(LM(m)) * COSHLM) - 1) + (SIN(LM(m)) * SINHLM)
5020  F25 = ((L ^ 2 / (2 * LM(m))) * F25a) - ((L ^ 2 / (4 * LM(m) ^ 2))
5030  F26a = (SIN(LM(m)) * SINHLM) + ((COS(LM(m)) * COSHLM) - 1)
5040  F26b = (SIN(LM(m)) * COSHLM) + (COS(LM(m)) * SINHLM)
5050  F26c = (SIN(LM(m)) * COSHLM) - (COS(LM(m)) * SINHLM)
5060  F26 = ((L ^ 2 / (2 * LM(m))) * F26a) - ((L ^ 2 / (2 * LM(m) ^ 2))
5070  F27 = (L ^ 2 / (8 * LM(m) ^ 2)) * ((2 * LM * SINH4) + COSH4 - 2 * LM(m)
5080  F28 = -(L ^ 2 / (8 * LM(m) ^ 2)) * ((2 * LM(m) * (COSH4 - 1)) - SINH4)
5090  F29 = F20
5100  F30 = F24
5110  F31 = ((-L / (4 * LM(m))) * (COSHLM - 1)) + ((L ^ 2 / (8 * LM(m) ^ 2))
5120  F32 = (L ^ 2 / 4) + ((L ^ 2 / (4 * LM(m) ^ 2)) * ((2 * LM(m) * SINH4)
5130  S21 = F17 + AM * F18 + BM * F19 + CM * F20 + AM * F21 + AM ^ 2 * F22 + A
5140  S22 = BM * F25 + BM * AM * F26 + BM ^ 2 * F27 + BM * CM * F28 + CM * F29
5150  S2 = (S21 + S22) / afam
5160  S41 = G1 + AM * G2 + BM * G3 + CM * G4 + AM * G5 + AM ^ 2 * G6 + AM * BM
5170  S42 = BM * AM * G10 + BM ^ 2 * G11 + BM * CM * G12 + CM ^ 2 * G13 + CM *
```

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5180  S4 = (S41 + S42) / afam
5190  S81 = -F17 - AM * F18 - BM * F19 - CM * F20 - AM * F21 - AM ^ 2 * F22 -
5200  S82 = -AM * CM * F24 + BM * F25 + BM * AM * F26 + BM ^ 2 * F27 + BM * CM
5210  S8 = ((LM(m) ^ 2 / L ^ 2) * (S81 + S82)) / afam
5220  S101 = -G1 - AM * G2 - BM * G3 - CM * G4 - AM * G5 - AM ^ 2 * G6 - AM *
5230  S102 = BM * AM * G10 + BM ^ 2 * G11 + BM * CM * G12 + CM * G13 + CM * AM
5240  S10 = (LM(m) ^ 2 / L ^ 2 * (S101 + S102)) / afam
5250  LAM = AF / MIU
5260  OME = ((E * IN / MIU) * ((LM(m)) / L) ^ 4) + (FM / MIU)
5270  TAI1 = (LAM * S1) / (2 * OME ^ 2)
5280  TAI2 = 1 - TAI1
5290  TAI = OME * TAI2
5300  LAM2 = M1 / (MIU * L)
5310  TBI1 = S2 + L * CO * S4
5320  TBI2 = (L * C ^ 2 * CO * S10 - C ^ 2 * S8) / (TAI ^ 2)
5330  TBI = TAI * (1 - ((LAM2 / 2) * (TBI1 - TBI2)))
5340  AK = LM(m) * C / L
5350  A = -COS(LM(m)) + (AM * SIN(LM(m))) + (BM * COSHLM) + (CM * SINHLM)
5360  PRINT TBI, A
5370  T1 = (A * (1 - COS(TBI * t))) / (TBI)
5380  T2a = (COS(AK * t) - COS(TBI * t))
5390  T2b = TAI ^ 2 - AK ^ 2
5400  T2 = T2a / T2b
5410  T3a = AM * (TBI * SIN(AK * t) + AK * SIN(TBI * t))
5420  T3b = TBI * (TBI ^ 2 - AK ^ 2)
5430  T3 = T3a / T3b
5440  T4a = AK * BM * SIN(TBI * t) * COS(TBI * t) * (EXP(AK * t) - EXP(-AK * t
5450  T4b = 2 * TBI * (AK ^ 2 + TBI ^ 2)
5460  T4 = T4a / T4b
5470  T5a = BM * TBI * (SIN(TBI * t)) ^ 2 * (EXP(AK * t) + EXP(-AK * t))
5480  T5b = 2 * TBI * (AK ^ 2 + TAI ^ 2)
5490  T5 = T5a / T5b
5500  T6a = BM * AK * SIN(TAI * t) * COS(TBI * t) * (EXP(AK * t)   EXP( AK * t
5510  T6b = 2 * TBI * (AK ^ 2 - TAI ^ 2)
5520  T6 = T6a / T6b
5530  T7a = BM * TBI * COS(TBI * t) * (COS(TBI * t) * ((EXP(AK * t) + EXP(-AK
5540  T7b = (AK ^ 2 - TBI ^ 2)
5550  T7 = T7a / T7b
5560  T8a = CM * AK * SIN(TBI * t) * (COS(TBI * t) * ((EXP(AK * t) + EXP(-AK *
5570  T8b = TBI * (AK ^ 2 + TBI ^ 2)
5580  T8 = T8a / T8b
5590  T9a = CM * TBI * (SIN(TBI * t)) ^ 2 * (EXP(AK * t) - EXP(-AK * t))
5600  T9b = 2 * TBI * (AK ^ 2 + TBI ^ 2)
5610  T9 = T9a / T9b
5620  T10a = CM * AK * SIN(TBI * t) * COS(TBI * t) * (EXP(AK * t) + EXP( AK *
5630  T10b = 2 * TBI * (AK ^ 2 + TBI ^ 2)
5640  T10 = T10a / T10b
5650  T11a = CM * TBI * (COS(TBI * t)) ^ 2 * (EXP(AK * t) - EXP(-AK * t))
5660  T11b = 2 * (AK ^ 2 - TBI ^ 2)
5670  T11 = T11a / T11b
5680  'T12 = ((BM * SIN(TBI * t)) / (AK ^ 2 - TBI ^ 2)) + ((CM * AK * COS(TBI
5690  M11 = SIN(LM(m) * X / L) + AM * COS(LM(m) * X / L)
5700  M12 = BM * (EXP(LM(m) * X / L) - EXP(-LM(m) * X / L)) + CM * (EXP(LM(m)
5710  wa = (T1 + T2 - T3 - T4 - T5 + T6 - T7 - T8 - T9 + T10 + T11 + T12)
5720  WB(m) = wa * ((LAM2 * GRA * L ^ 2) / (LM(m) * afam))
5730  NEXT m
5740  W = WB(1) + WB(2) + WB(3)
```

```
5750  PRINT #1, t, W
5760  'PRINT WB(1), WB(2), WB(3)
5770  PRINT t, W
5780  'SLEEP
5790  NEXT t
5800  PRINT #1,
5810  NEXT mm
```