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**IDENTIFICATION OF OPTIMAL AUTOREGRESSIVE INTEGRATED
MOVING AVERAGE MODEL FOR METEOROLOGICAL DATA**

BY

OLUSOLA SAMUEL MAKINDE

IMT/03/2839

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JUNE, 2011

CERTIFICATION

This is to certify that this thesis was carried out by Mr MAKINDE, Olusola Samuel, in the Department of Mathematical Sciences, in partial fulfilment of the award of Master of Technology in Statistics of Federal University of Technology, Akure, Nigeria and has not been presented elsewhere for the award of a degree, or any other purpose.

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DEDICATION

This work is dedicated to almighty God



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ABSTRACT

Autoregressive Integrated Moving Average (ARIMA) model of various orders was presented, with a view to identifying the optimal model. In this study, ARIMA (p, d, q) model of various orders was presented in both causal and inverted forms, behavioural patterns of both causality and invertibility parameters were investigated, parameters of the models were estimated using Ordinary Least Square approach and optimal model from a class of models was identified using Normalized Bayesian Information Criterion (BIC) as model selection criterion. It was deduced that behaviour of causality parameter, ψ_i and invertibility parameter, π_i depend on positive and negative values of autoregressive parameter ϕ and moving average parameter θ respectively. ψ_i is skewed to right and sinusoidal for positive and negative values of ϕ respectively. $|\psi_i|$ of ARMA (1, q) model increases as the value of q increases for positive values of ϕ . Also, ψ_i increases as ϕ increases as shown in Figure 2.4 for positive values of ϕ . Similarly, π_i of ARIMA ($p, d, 1$) model for various integer values of d are sinusoidal for positive values of θ irrespective of ϕ_p 's arrangement, $|\pi_i|$ increases as d increases and the lower the integer value of d , the faster $|\pi_i|$ converges to zero. For negative integer values of θ , π_i of ARIMA ($p, d, 1$) model oscillates as the value of d increases with $|\pi_i|_{d=k} < |\pi_i|_{d=k+2}$ for all even integer value $k > 0$ provided $i \geq 3$ and $|\pi_i|_{d=k} > |\pi_i|_{d=k+2}$ for all odd integer value $k > 0$. Similarly, π_i of ARIMA ($p, d, 1$) model was found to be sinusoidal for $p = 1, 2, 3$ and 4 and $d = 0$ and 1 with $|\pi_i|_{p=4} > |\pi_i|_{p=3} > |\pi_i|_{p=2} > |\pi_i|_{p=1}$ for all positive values of θ . For all negative values of θ , $|\pi_i|_{p=k} < |\pi_i|_{p=k+2}$ for odd integer value $k > 0$ and $|\pi_i|_{p=k} < |\pi_i|_{p=k+2}$ for even integer value $k > 0$. ARIMA (p, d, q) model was formulated for daily maximum meteorological temperature data of Ondo, Nigeria ($07^{\circ}06'27''N, 04^{\circ}50'19''E$) and Zaira, Nigeria ($11^{\circ}04'N, 07^{\circ}42'E$), daily rain amount of Calabar, Nigeria ($07^{\circ}06'27''N, 08^{\circ}20'E$) and Ondo, Nigeria ($07^{\circ}06'27''N, 04^{\circ}50'19''E$) and daily cloud of Warri, Nigeria ($05^{\circ}30'N, 05^{\circ}41'E$) and Benin City, Nigeria ($07^{\circ}06'27''N, 05^{\circ}31'E$) from January 1995 to December 2005. The choice of ARIMA model of orders p and q was intended to retain any persistence in natural process. To determine the performance of models, Normalized BIC was adopted. ARIMA (1, 1, 1) model was adequate for modelling daily maximum temperature of Ondo, Nigeria and Zaira, Nigeria, parameters of the model was estimated and redundant variable was removed for each of the cities. Similarly for modelling daily rain amount of Calabar, Nigeria and Ondo, Nigeria, and daily cloud of Warri, Nigeria and Benin City, Nigeria, ARIMA (1, 0, 1) model was found adequate and parameters of the model were estimated.

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CHAPTER ONE

INTRODUCTION



1.1 INTRODUCTION

Atmospheric dispersion models require meteorological data on winds; air temperature, atmospheric stability, mixed layer heights, e.t.c. to transport and disperse pollutants. Weather forecasting is the application of science and technology to predict the state of the atmosphere for a future time and a given location. Human beings have attempted to predict the weather informally for millennia, and formally since at least the nineteenth century. Weather forecasts are made by collecting quantitative data about the current state of the atmosphere and using scientific understanding of atmospheric processes to project how the atmosphere will evolve. Forecasts based on temperature and precipitation are important to agriculture, and therefore to traders within commodity markets. Temperature forecasts are used by utility companies to estimate demand over coming days. Ancient weather forecasting methods usually relied on observed patterns of events, also termed pattern recognition. For example, it might be observed that if the sunset was particularly red, the following day often brought fair weather. This experience accumulated over the generations to produce weather lore. However, not all of these predictions prove reliable, and many of them have since been found not to stand up to rigorous statistical testing.

Models are initialized using the observed data. As proposed by Edward Lorenz in 1963, long range forecasts, those made at a range of two weeks or more, are impossible to definitively predict the state of the atmosphere, owing to the chaotic nature of the mechanism involved. Forecast Models are now used to determine future conditions. Human input is still required to pick the best possible forecast model to base the forecast upon, which involves pattern recognition skills, teleconnections, knowledge of model performance, and knowledge of model biases. The chaotic nature of the atmosphere, the massive computational power required to solve the equations that describe the atmosphere, error involved in measuring the conditions and an incomplete understanding of atmospheric processes mean that forecasts become less accurate as the difference in current time and the time for which the forecast is being made (the range of the forecast) increases. The use of ensembles and model consensus help narrow the error and pick the most likely outcome. The sequence of meteorological observations taken sequentially at time points follows time-series.

A time series of T successive observations $y_t, t = 1, 2, \dots, T$ is regarded as a sample from an infinite population of such time-series that could have been generated by the stochastic process under study. A powerful way of extracting useful information on the underlying process solely on the basis of the past behaviour of the time series itself is the univariate Box-Jenkins approach, named after the statisticians who proposed it during the 1960s and 1970s. Although originally developed for forecasting purposes (Box and Jenkins 1976, Nelson 1976), Box-Jenkins models are useful tools for describing the time dependent structure of stationary and non-stationary time series.

In statistics and econometrics, and in particular time series analysis, an Autoregressive Integrated Moving Average (ARIMA) model is a generalization of an autoregressive moving average (ARMA) model. These models are fitted to time series data either to better understand the data or to predict future points in the series (forecasting). They are applied in some cases where data show evidence of non-stationarity, where an initial differencing step (corresponding to the "integrated" part of the model) can be applied to remove the non-stationarity. The model is generally referred to as an ARIMA (p, d, q) model where p, d , and q are integers greater than or equal to zero and refer to the order of the autoregressive, integrated, and moving average parts of the model respectively. ARIMA models form an important part of the Box-Jenkins approach to time-series modelling.

Box-Jenkins models for stationary time series, or ARMA models, have been applied in many areas of researches, for example, in tree-ring chronologies by Meko (1981), evolution of unemployment rate by Dobre and Alexandru (2008), etc.

1.2 AIM AND OBJECTIVES

Aim

To identify optimal model from a class of models and determine the behavioural pattern of causality and invertibility parameters of ARIMA (p, d, q) model.

Objectives

The objectives include to

- (i) investigate the performance of ARIMA model under various orders.

- (ii) examine the behavioural pattern of causality and invertibility parameter of ARIMA (p, d, q) model for some values of p, d, and q.
- (iii) evaluate parameters of the models in (i) and identify optimal model in a class of ARIMA models.

1.3 MOTIVATION

Many stochastic models have been proposed for time series data based on the trends of the series. This study is motivated by the need to identify the optimal autoregressive integrated moving average model for various meteorological outcomes in some stations and also to determine the behavioural pattern of causality and invertibility parameter of ARIMA (p, d, q) model.

1.4 LITERATURE REVIEW

The modelling and forecasting procedures needed in identifying patterns in Time Series data involves knowledge about mathematical model of the process. However, in real life research and practice, patterns of the data are unclear, individual observations involve considerable error, and need not only to uncover the hidden patterns in the data but also to forecast. The ARIMA methodology developed by Box and Jenkins (1976) allows us to do that.

In considering estimates of parameters of time series model, Pham-Dinh (1978) computed exact log likelihood of time series model, proposed and justified an asymptotic approximation of it.

Granger and Andersen(1978) proposed a generalized definition of invertibility and applied it to linear, non-linear and bilinear models. It was shown that some recently studied non-linear models are not invertible, but conditions for invertibility can be achieved for the other models.

Anderson (1978) deduced conditions for the general Moving Average process, of order q , to be invertible or borderline non-invertible. He termed the conditions as acceptability conditions. It turned out that they depended on the magnitude of the final moving average parameter, θ_q . If $|\theta_q| > 1$, the process is not acceptable. Should $|\theta_q| = 1$, the conditions, for

any particular q , follow simply - if use is made of the remainder theorem. When $|\theta_q| < 1$, an appeal was made to ROUCH* E'S theorem, to establish the conditions. Analogous stationarity results immediately follow for autoregressive processes. Tiao and Tsay (1983) presented a unified treatment of the consistency properties of the ordinary least squares estimates in an autoregressive fitting of time series from non-stationary or stationary time autoregressive moving average models. For a given model, the orders of autoregressions which produce consistent estimates were obtained, limiting values and biases of the estimates of other autoregressions were investigated.

Mikosch et al (1995) considered ARMA process of the form $\phi(L)X_t = \theta(L)Z_t$, where the innovations Z_t belong to the domain of attraction of a stable law, so that neither the X_t have a finite variance. They estimated the coefficients of ϕ and θ using Whittle estimator based on sample periodogram of X sequence. They showed that their estimators were consistent, obtained their asymptotic distributions and showed that they converged to the true values faster than in the usual $\mathcal{L}^2$ case.

Triacca (2004) analyzed the relationship between feedback in stochastic systems, Granger causality and a measure of dissimilarity between ARMA models in bivariate vector autoregression. In particular, he considered a bivariate vector autoregressive processes of order p (a bivariate VAR(p) process) and argued that if the distance between the univariate ARMA models implied by the VAR representation is greater than a certain number that is a function of p , then Granger causality must exist in at least one direction in the variables.

Bobba et al (2006) formulated a stochastic model simulating trends in hydrological and meteorological variables. Their choice of ARMA model of orders p and q was intended to retain any persistence in the natural process. They claimed that ARMA (1, 1) model was adequate for modelling three variables of temperature, precipitation and stream flow on a season basis in North East Pond River Watershed.

Dobre and Alexandru (2008) modelled the evolution of unemployment rate using the Box-Jenkins methodology during the period 1998-2007 monthly data. They claimed that the most adequate model for the unemployment rate in Romania was ARIMA (2, 1, 2). Using the model, they forecast the values of unemployment rate for January and February 2008.

Ojo (2008) investigated optimal models in higher order of integrated autoregressive models and autoregressive integrated moving average models in the presence of $2^k - 1$ subsets. He

used Marquardt algorithm and Newton-Raphson iterative method to estimate the parameters of the model. Ojo (2009) compared subset autoregressive integrated moving average models with full autoregressive integrated moving average models. The parameters of these models were estimated and the statistical properties of the derived estimates were investigated. An algorithm was proposed to eliminate redundant parameters from the full order autoregressive integrated moving average models. To determine the performance of the models, residual variance, Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) were adopted. He revealed that the residual variance attached to the subset autoregressive integrated moving average models is smaller than the residual variance attached to the full autoregressive integrated moving average models. Subset autoregressive integrated moving average models performed better than the full autoregressive integrated moving average models.

Shittu and Yaya (2009) studied forecast performance of autoregressive integrated moving average and autoregressive fractionally integrated moving average models for stationary time series that exhibit long memory properties. They analyzed UK Pound/US Dollar exchange rate data using root mean square forecast error and mean absolute percentage forecast error as measurement criteria. They found out that ARFIMA model was better than ARIMA model as indicated by diagnostic tools.

In this work, ARIMA (p, d, q) model of various orders was presented. ARIMA (p, d, q) model was investigated in both causal and inverted forms for various values of p, d and q . invertibility and causality conditions were presented, behavioural patterns of causality and invertibility parameters were examined and the parameters of ARIMA (p, d, q) model were evaluated for various values of p and d using ordinary least squares method and Crammer's rule. The optimal model in a class of models for some meteorological data were identified and redundancy of parameters were checked using t-test of significance.

1.5 RESEARCH METHODOLOGY

Autoregressive integrated moving average (ARIMA) model is a generalisation of autoregressive moving average model. It consists of Autoregressive model (AR), Integrated part (I; which is differencing term) and moving average model (MA). ARIMA model is employed when time series is non stationary.

The governing equation is



$$y_t = c + \sum_{i=1}^p \phi_i y_{t-i} + \varepsilon_t + \sum_{i=1}^q \theta_i \varepsilon_{t-i} \quad (1.01)$$

where c is a constant and ϕ_i and θ_i are parameters of AR and MA parts respectively. Causality of Equation (1.01) above was considered for some values of p and q . The equation could be expressed as

$$y_t = F + \sum_{i=0}^{\infty} \psi_i \varepsilon_{t-i} \quad (1.02)$$

where $F = F(c, \phi)$ and $\psi = \psi(\phi_i, \theta_i), i = 0, 1, 2, 3, \dots$ from which expectation of the ARIMA model could be derived as $E(y_t) = E(F) = F(c, \phi)$. Invertibility of y_t was also considered to be of the form

$$\varepsilon_t = F + \sum_{i=0}^{\infty} \pi_i y_{t-i} \quad (1.03)$$

where $F = F(c, \theta)$ and $\pi = \pi(\phi_i, \theta_i), i = 0, 1, 2, 3, \dots$. Causality and invertibility conditions were also established. The causality and invertibility parameters of ARIMA (p, d, q) model were investigated and the behavioural pattern of causality parameter ψ_i and invertibility parameter, π_i were examined for various integer values of p, d and q .

To estimate the parameters of the governing equation for various values of p and d , the Ordinary Least Squares technique was used and the resulting equations were then solved using the Crammer's rule.

The ARIMA model of various orders p, d , and q were subjected to Normalized Bayesian Information Criterion as model selection criterion to obtain optimal models for some meteorological information. Parameters of optimal models were analysed and t-test of significance was used to check for redundancy. Forecast was carried out for the subsequent days in month of December, 2005 and compared with observed values.

1.7 ORGANISATION OF THESIS

This thesis has five chapters and it is organised as follows: Chapter one contains introduction to the research work, the motivation, aim and objectives, literature review and expected contribution to knowledge. Chapter two gives the theoretical framework, derivation of causality and invertibility of ARIMA models, application of causality and invertibility theorem for ARIMA models and investigation on behavioural pattern of causality and invertibility parameters of ARIMA model. In chapter three, parameters of ARIMA model of various orders are evaluated using Ordinary Least Square (OLS) when the model contains a constant term ($c = 0$) and when it does not contain a constant term ($c \neq 0$). Chapter four contains analyses of some meteorological information of some cities in Nigeria. The results are presented and discussed. Lastly, chapter five presents the summary of the work done and the conclusion.

1.8 SOURCES OF DATA

The meteorological data got from Nigerian Meteorological Agency, Oshodi, Lagos State, Nigeria was used in this work. The data reports daily maximum meteorological temperature of Ondo city, Nigeria ($07^{\circ}06'27''N$, $04^{\circ}50'19''E$) and Zaira, Nigeria ($11^{\circ}04'N$, $07^{\circ}42'E$), daily rain amount of Calabar, Nigeria ($07^{\circ}06'27''N$, $08^{\circ}20'E$) and Ondo, Nigeria ($07^{\circ}06'27''N$, $04^{\circ}50'19''E$) and daily cloud of Warri, Nigeria ($05^{\circ}30'N$, $05^{\circ}41'E$) and Benin City, Nigeria ($07^{\circ}06'27''N$, $05^{\circ}31'E$) from January 1995 to December 2005. Benin City is a capital in the Southern Nigeria, capital of Edo State, on the Benin River. Calabar is a city and seaport in South-eastern Nigeria, capital of Cross River State, on an estuary of the Gulf of Guinea. Ondo City is situated in the south-western belt of Nigerian rainforest. Warri is a city in the South-western Nigeria, located in Delta State. Warri is a major port city in the Niger Delta and serves as the cargo transfer point between the Niger River and the Atlantic Ocean. Zaria is a major city in Kaduna state in Northern Nigeria.

CHAPTER TWO

THEORETICAL FRAMEWORK



2.1 STATIONARITY

A type of stochastic process that has received a great deal of attention and scrutiny by time series analysts is the so-called stationary stochastic process. When discussing stationarity, we look into two forms it takes. They are covariance stationarity and strict stationarity. A covariance stationary stochastic process is known as a weakly stationary or second-order stationary stochastic process if the mean, variance and autocovariance do not grow over time. To explain weak stationarity, let y_t be a stochastic time series with these properties:

$$\text{Mean, } E(y_t) = E(y_{t-1}) = E(y_{t-2}) = \dots = \mu \quad (2.01)$$

$$\text{Variance, } E(y_t - \mu)^2 = E(y_{t-1} - \mu)^2 = E(y_{t-2} - \mu)^2 = \dots = \sigma^2 \quad (2.02)$$

$$\text{Autocovariance, } E(y_t - \mu)(y_{t-s} - \mu) = E(y_{t-j} - \mu)(y_{t-j-s} - \mu) = \gamma_s \quad (2.03)$$

A process is said to be strictly stationary if for any value of $j_1, j_2, \dots, j_n$, the joint distribution of $(y_t, y_{t+j_1}, y_{t+j_2}, \dots, y_{t+j_n})$ depends only on the interval separating the dates $(j_1, j_2, \dots, j_n)$ and not on the date (t) itself. If a process is strictly stationary with finite second moments, then it must be covariance stationary (Hamilton [1994]).

In short, if a time series is stationary, its mean, variance, and autocovariance (at various lags) remain the same no matter at what point we measure them; that is, they are time invariant.

2.1.1 Test of stationarity

Although there are several tests of stationarity, we will discuss only three tests which are:

- (1) graphical analysis; (2) the correlogram and (3) unit root test.

(1) Graphical Analysis

Before one pursues formal tests, it is always advisable to plot the time series under study for the data. Such a plot gives an initial clue about the likely nature of the time series.

(2) Autocorrelation function and correlogram

Stationarity can be tested by the correlogram of a time series, which is a graph of autocorrelation at various lags. For stationary time series, the correlogram tapers off quickly, whereas for non-stationary time series it dies off gradually.

This test of stationarity is based on the so-called autocorrelation function (ACF). The ACF at lag k , denoted by ρ_k , is defined as $\rho_k = \frac{\gamma_k}{\gamma_0}$. Since both covariance and variance are measured in the same units of measurement, ρ_k is a unitless, or pure number. It lies between -1 and $+1$, as any correlation coefficient does. If we plot ρ_k against k , the graph we obtain is known as the population correlogram. Since in practice we only have a realization (i.e., sample) of a stochastic process, we can only compute the sample autocorrelation function (SAFC), r_k . To compute this, we must first compute the sample covariance at lag k , γ_k , and the sample variance, γ_0 .

For a purely white noise process, the autocorrelations at various lags hover around zero. This is the picture of a correlogram of a stationary time series. Thus, if the correlogram of an actual (economic) time series resembles the correlogram of a white noise time series, we can say that time series is probably stationary.

The statistical significance of any r_k can be judged by its standard error. Bartlett has shown that if a time series is purely random, that is, it exhibits white noise, the sample autocorrelation coefficients r_k are approximately

$$r_k \sim N(0, 1/n) \quad (2.04)$$

that is, in large samples the sample autocorrelation coefficients are normally distributed with zero mean and variance equals to one over the sample size.

If autocorrelations start high and decline slowly, then series is nonstationary, and should be differenced. ARMA process is said to be stationary if autocorrelation function and partial autocorrelation function decay exponentially after a few lags. In this work, use of correlogram was adopted to test for stationarity of meteorological data.

(3) Unit Root Test

At the formal level, stationarity can be checked by finding out if the time series contains a unit root. The Dickey-Fuller (DF) and augmented Dickey-Fuller (ADF) tests can be used for this purpose.

2.2 TEST OF ADEQUACY OF THE MODEL

To test for adequacy of the model, we use a statistic developed by Greta M. Ljung and George E. P. Box.

The Ljung-Box test is a type of statistical test of whether any of a group of autocorrelations of a time series is different from zero. Instead of testing randomness at each distinct lag, it tests the "overall" randomness based on a number of lags, and is therefore a portmanteau test.

Definition:

The Ljung-Box test test can be defined as follows:

H_0 : The data is random (i.e. the correlations in the population from which the sample is taken are 0, so that any observed correlations in the data result from randomness of the sampling process).

H_2 : The data is not random.

The test statistic is:

$$Q(\hat{r}) = n(n+2) \sum_{k=1}^h \frac{\hat{r}_k^2}{n-k} \quad (2.05)$$

where n is the sample size, $\hat{r}_k$ is the sample autocorrelation at lag k , and h is the number of lags being tested. For significance level α , the critical region for rejection of the hypothesis of randomness is

$$Q > \chi_{1-\alpha, h}^2 \quad (2.06)$$

where $\chi_{1-\alpha, h}^2$ is the α -quantile of the chi-square distribution with h degrees of freedom. The Ljung-Box test is commonly used in autoregressive integrated moving average (ARIMA) modelling. Note that it is applied to the residuals of a fitted ARIMA model, not the original series.

2.3 TEST OF NORMALITY

Although several tests of normality are discussed in the literature, we will consider just three:

(1) histogram of residuals

(2) normal probability plot (NPP), a graphical device

(3) the Jarque-Bera test.

(1) **Histogram of Residuals:** A histogram of residuals is a simple graphic device that is used to learn something about the shape of the probability density function (PDF) of a random variable. On the horizontal axis, we divide the values of the variable of interest (e.g. OLS residuals) into suitable intervals, and in each class interval we erect rectangles equal in height to the number of observations (that is, frequency) in that class interval. If you mentally superimpose the bell-shaped normal distribution curve on the histogram, you will get some idea as to whether normal (PDF) approximation may be appropriate. It is always a good practice to plot the histogram of the residuals as a rough and ready method of testing for the normality assumption.

(2) **Normal Probability Plot:** A comparatively simple graphical device to study the shape of the probability density function (PDF) of a random variable is the normal probability plot (NPP) which makes use of normal probability paper, a specially designed graph paper. On the horizontal axis, we plot values of the variable of interest (say, OLS residuals, $\hat{\epsilon}_i$), and on the vertical axis, we show the expected value of this variable if it were normally distributed. Therefore, if the variable is, in fact, from the normal population, the NPP will be approximately a straight line. If the fitted line in the NPP is approximately a straight line, one can conclude that the variable of interest is normally distributed.

(3) **Jarque-Bera (JB) Test of Normality:** The JB test of normality is an asymptotic or large-sample test. It is also based on the OLS residuals. To carry out this test, we first compute the skewness and kurtosis, measures of the OLS residuals and uses the following test statistic:

$$JB = n \left[\frac{S^2}{6} + \frac{(K - 3)^2}{24} \right] \quad (2.07)$$

where n = sample size, S = skewness coefficient, and K = kurtosis coefficient.

For a normally distributed variable, $S = 0$ and $K = 3$. Therefore, the JB test of normality is a test of the joint hypothesis that S and K are 0 and 3, respectively. In that case the value of the JB statistic is expected to be 0.

Under the null hypothesis that the residuals are normally distributed, Jarque and Bera showed that asymptotically (i.e. in large samples) the JB statistic given in equation (2.07) follows the chi-square distribution with 2 degrees of freedom. If the computed p value of the JB statistic in an application is sufficiently low, which will happen if the value of the statistic is very

different from 0, one can reject the hypothesis that the residuals are normally distributed. But if the p value is reasonably high, which will happen if the value of the statistic is close to zero, we do not reject the normality assumption. If the sample size is small, one should not use the JB statistic (Gujarati [2004]).

There are so many other tests among which are Goodness-of-Fit Tests (e.g. Kolmogorov-Smirnov Test), Shapiro-Wilk's Test (W-statistic), D'Agostino Test (D-statistic), e.t.c.

In this work, histogram of residual was adopted for checking the normality property of residuals in equation (2.16).

2.4 MODEL SELECTION CRITERIA

2.4.1 Akaike Information Criterion

Akaike's information criterion, developed by Hirotugu Akaike under the name of "an information criterion" (AIC) and proposed in Akaike (1977), is a measure of the goodness of fit of an estimated statistical model. It is grounded in the concept of entropy, in effect offering a relative measure of the information lost when a given model is used to describe reality and can be said to describe the trade-off between bias and variance in model construction, or loosely speaking that of accuracy and complexity of the model.

It is a test between models - a tool for model selection. Given a data set, several competing models may be ranked according to their AIC, with the one having the lowest AIC being the best.

The AIC is

$$AIC = 2k + n(\ln RSS) \quad (2.08)$$

where k is the number of parameters in the statistical model, n is the number of observation and $RSS = \sum_{i=1}^T \hat{\epsilon}_i^2$

2.4.2 Bayesian Information Criterion

In statistics, the Bayesian information criterion (BIC) or Schwarz criterion is a criterion for model selection among a class of parametric models with different numbers of parameters. Choosing a model to optimize BIC is a form of regularization.

The BIC is an asymptotic result derived under the assumptions that the data distribution is in the exponential family. Let:

$$BIC = -2\ln(L) + k\ln(n) \quad (2.09)$$

where n = sample size

k = the number of free parameters to be estimated

L = the maximized value of the likelihood function for the estimated model

Under the assumption that the model errors or disturbances are normally distributed, this becomes

$$BIC = n \ln \left(\frac{RSS}{n} \right) + k \ln(n) \quad (2.10)$$

where RSS = residual sum of squares from the estimated model

Given any two estimated models, the model with the lower value of BIC is the one to be preferred. The BIC is an increasing function of RSS and an increasing function of k . That is, unexplained variation in the dependent variable and the number of explanatory variables increase the value of BIC. The BIC penalizes free parameters more strongly than does the Akaike information criterion. It is important to keep in mind that the BIC can be used to compare estimated models only when the numerical values of the dependent variable are identical for all estimates being compared.

2.4.3 Characteristics of the Bayesian Information Criterion

- (1) It is independent of the prior or the prior is "vague" (a constant).
- (2) It can measure the efficiency of the parameterized model in terms of predicting the data.
- (3) It penalizes the complexity of the model where complexity refers to the number of parameters in model.
- (4) It is exactly equal to the minimum description length criterion but with negative sign.
- (5) It can be used to choose the number of clusters according to the intrinsic complexity present in a particular dataset [Schwarz (1978)].
- (6) It is closely related to other penalized likelihood criteria such as RIC and the Akaike information criterion.

In this work, Normalized BIC was adopted as model selection criterion.

2.5 TEST OF SIGNIFICANCE

Test of Significance of the Estimates of Parameters of ARMA model: The t-Test

One of the methods of testing statistical hypotheses is the test-of-significance approach developed along independent lines by R. A. Fisher and jointly by Neyman and Pearson. Broadly speaking, a test of significance is a procedure by which sample results are used to verify the truth or falsity of a null hypothesis.

H_0 : estimates of the parameters are not significantly different from 0.

H_1 : estimates of the parameters are significantly different from 0.

The key idea behind test of significance is that a test statistic (estimator) and the sampling distribution of the statistic under the null hypothesis is used to decide whether to accept or reject H_0 on the basis of the value of the test statistic obtained from the data at hand.

The test statistic

$$t = \frac{\hat{\phi}_p - \phi_p}{SE(\hat{\phi}_p)} = \frac{(\hat{\phi}_p - \phi_p)\sqrt{\sum y_t^2}}{\hat{\sigma}} \quad (2.11)$$

follows the t distribution with $n - 2$ df. If the value of true ϕ_p is specified under the null hypothesis, the t value in equation (2.11) can readily be computed from the available sample. Since this test statistic follows the t distribution

$$Pr \left[-t_{\alpha/2} \leq \frac{\hat{\phi}_p - \phi_p}{SE(\hat{\phi}_p)} \leq t_{\alpha/2} \right] = 1 - \alpha \quad (2.12)$$

where $t_{\alpha/2}$ is the value of t (the critical t value) obtained from the t table for $(\alpha/2)$ level of significance and $n - 2$ df.

Rearranging, we obtain

$$Pr \left[\hat{\phi}_p - t_{\alpha/2} SE(\hat{\phi}_p) \leq \phi_p \leq \hat{\phi}_p + t_{\alpha/2} SE(\hat{\phi}_p) \right] = 1 - \alpha \quad (2.13)$$

which gives the interval in which $\hat{\phi}_p$ will fall with $1 - \alpha$ probability. In hypothesis testing, the $100(1 - \alpha)\%$ confidence interval established in equation(2.13) is known as the region of acceptance (of the null hypothesis) and the region(s) outside the confidence interval is (are) called the region(s) of rejection (of H_0) or the critical region(s). As noted previously, the confidence limits (that is, the endpoints of the confidence interval) are also called critical values.

2.6 AUTOREGRESSIVE INTEGRATED MOVING AVERAGE PROCESS

2.6.1 Autoregressive Model

The notation AR(p) refers to the autoregressive model of order p. The AR(p) model is written as

$$y_t = c + \sum_{i=1}^p \phi_i y_{t-i} + \varepsilon_t \quad (2.14)$$

where $\phi_1, \phi_2, \dots, \phi_p$ are the parameters of AR model, c is a constant term, ε_t is white noise and y_t is a process at date, t . The constant term is omitted by many authors for simplicity.

2.6.2 Moving Average Model

The notation MA (q) refers to the moving average model of order q, defined as

$$y_t = \mu + \varepsilon_t + \sum_{i=1}^q \theta_i \varepsilon_{t-i} \quad (2.15)$$

where the $\theta_1, \theta_2, \dots, \theta_q$ are the parameters of MA model, μ is the expectation of y_t (sometimes assumed to equal 0) and the $\varepsilon_t, \varepsilon_{t-1}, \dots, \varepsilon_{t-q}$ are white noise error terms.

2.6.3 Autoregressive Moving Average Model

One subset of ARMA models are the so-called *autoregressive*, or AR models while the other is *moving average* or MA models. The notation ARMA(p, q) refers to the model with p autoregressive terms and q moving average terms. This model contains the AR(p) and MA(q) models,

$$y_t = c + \sum_{i=1}^p \phi_i y_{t-i} + \varepsilon_t + \sum_{i=1}^q \theta_i \varepsilon_{t-i}$$

i.e.
$$y_t = c + \phi_1 y_{t-1} + \dots + \phi_p y_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \dots + \theta_q \varepsilon_{t-q} \quad (2.16)$$

The terms $\phi_1 y_{t-1}$ through $\phi_p y_{t-p}$ are the autoregressive portion of the filter. The terms ε_t through $\theta_q \varepsilon_{t-q}$ are a moving average of the white noise input process.

The need for estimating the parameters of an ARMA (p, q) process arises in many applications both in signal processing and in automatic control.

2.6.4 Specification in terms of lag operator

When the models are specified in terms of the lag operator L , the AR(p) model is defined as

$$\varepsilon_t = \left(1 - \sum_{i=1}^p \phi_i L^i\right) y_t = \phi(L) y_t \quad (2.17)$$

where $\phi(L)$ represents the polynomial

$$\phi(L) = 1 - \sum_{i=1}^p \phi_i L^i \quad (2.18)$$

The MA (q) model is defined as

$$y_t = \left(1 + \sum_{i=1}^q \theta_i L^i\right) \varepsilon_t = \theta(L) \varepsilon_t \quad (2.19)$$

where θ represents the polynomial

$$\theta(L) = 1 + \sum_{i=1}^q \theta_i L^i \quad (2.20)$$

Finally, the combined ARMA(p, q) model is defined as

$$\left(1 - \sum_{i=1}^p \phi_i L^i\right) y_t = \left(1 + \sum_{i=1}^q \theta_i L^i\right) \varepsilon_t \quad (2.21)$$

or more concisely,

$$\phi(L) y_t = \theta(L) \varepsilon_t \quad (2.22)$$

$$y_t = \psi(L)\varepsilon_t \quad (2.23)$$

where

$$\psi(L) = \frac{\theta(L)}{\phi(L)} = \frac{1 + \theta_1 L + \theta_2 L^2 + \dots + \theta_p L^p}{1 - \phi_1 L - \phi_2 L^2 - \dots - \phi_p L^p} \quad (2.24)$$

The ARMA process is stationary if

$$\sum_{j=1}^{\infty} |\psi_j| < \infty.$$

This happens if the series $\psi(Z)$ converges for every Z with $|Z| \leq 1$. Since $\psi(Z)$ is a rational function, the series converges for every Z with $|Z| \leq 1$ if the complex zeros of $\phi(Z)$ lie outside the unit circle. If we have a stationary process, then the expected value of y_t is also 0 since $y_t = \psi(L)\varepsilon_t$ and the expected values of ε_t are all 0.

A related issue is that of invertibility. Recall that we can write ε_t in terms of y_t and previous values of y_{t-h} . That is,

$$\varepsilon_t = \pi(L)y_t$$

where $\pi(L) = \frac{\phi(L)}{\theta(L)}$. That is $y_t = \varepsilon_t + \pi_1 y_{t-1} + \pi_2 y_{t-2} + \dots$

This inverted form of the process provides a very useful way of generating a random sequence according to our ARMA process. However, this infinite sum must be truncated in practice. If the π_j coefficients do not decay to zero, then it is not possible to approximate this infinite sum by truncating it.

We say that the process is invertible if

$$\sum_{j=1}^{\infty} |\pi_j| < \infty$$

Since $\pi(Z)$ is a rational function, the series is invertible if the complex zeros of $\theta(Z)$ lie outside of the unit circle.

2.6.5 Autoregressive Integrated Moving Average Model

This is a generalisation of ARMA model. It consists of Autoregressive model (AR), Integrated part (I; which is differencing term) and moving average model (MA). It is employed when time series is non stationary. p is the order of AR, d is the number of times a series is differenced to assume stationarity and q is the order of MA.

2.7 CAUSALITY OF ARMA PROCESSES

An ARMA (p, q) process is causal if the absolute value of the parameters of ARMA (p, q) model satisfy $|\phi_i| < 1$ for $i = 1, \dots, p$. If an ARMA process is causal, it is stationary. If ϕ and θ have no common factors, a (unique) stationary solution to $\phi(L)y_t = \theta(L)\varepsilon_t$ exists if and only if $|Z| = 1$ implies $\phi(Z) = 1 - \sum_{i=1}^p \phi_i Z^i \neq 0$. The ARMA (p, q) process is *causal* if and only if $|Z| \leq 1$ implies $\phi(Z) = 1 - \sum_{i=1}^p \phi_i Z^i \neq 0$. It is *invertible* if and only if $|Z| \leq 1$ implies $\theta(Z) = 1 + \sum_{i=1}^q \theta_i Z^i \neq 0$.

2.8 PRESENTATION OF SOME ARMA PROCESSES IN CAUSAL FORM

Here we present some ARMA process of various orders in causal form of the process in order to provide a useful way of generating a random sequence. That is, we show a linear process y_t as a linear combination of white noise variates ε_t .

ARMA (1, 1)

$$y_t = c + \phi y_{t-1} + \varepsilon_t + \theta \varepsilon_{t-1} \quad (2.25)$$

$$y_t - \phi y_{t-1} = c + \varepsilon_t + \theta \varepsilon_{t-1}$$

$$(1 - \phi L)y_t = c + \varepsilon_t + \theta \varepsilon_{t-1}$$

$$y_t = (1 - \phi L)^{-1}c + (1 - \phi L)^{-1}(\varepsilon_t + \theta \varepsilon_{t-1})$$

$$y_t = (1 + \phi L + \phi^2 L^2 + \dots)c + (1 + \phi L + \phi^2 L^2 + \dots)\varepsilon_t + \theta(1 + \phi L + \phi^2 L^2 + \dots)\varepsilon_{t-1}$$

$$y_t = (1 + \phi + \phi^2 + \dots)c + (\varepsilon_t + \phi \varepsilon_{t-1} + \phi^2 \varepsilon_{t-2} + \phi^3 \varepsilon_{t-3} + \dots) + \theta(\varepsilon_{t-1} + \phi \varepsilon_{t-2} + \phi^2 \varepsilon_{t-3} + \dots)$$

$$y_t = (1 - \phi)^{-1}c + \varepsilon_t + (\phi + \theta)\varepsilon_{t-1} + (\phi^2 + \theta\phi)\varepsilon_{t-2} + (\phi^3 + \theta\phi^2)\varepsilon_{t-3} + \dots$$

$$y_t = \frac{c}{(1 - \phi)} + \varepsilon_t + \sum_{i=1}^{\infty} (\phi^i + \theta\phi^{i-1})\varepsilon_{t-i}, \quad i = 1, 2, \dots \quad (2.26)$$

$$y_t = \frac{c}{(1 - \phi)} + \sum_{i=0}^{\infty} \psi_i \varepsilon_{t-i} \quad (2.27)$$



where

$$\psi_i = \begin{cases} 1, & \text{if } i = 0 \\ \phi^i + \theta\phi^{i-1}, & \text{if } i = 1, 2, 3, \dots \end{cases} \quad (2.28)$$

ARMA (1, 2)

$$y_t = c + \phi y_{t-1} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} \quad (2.29)$$

$$y_t - \phi y_{t-1} = c + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2}$$

$$(1 - \phi L)y_t = c + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2}$$

$$y_t = (1 - \phi L)^{-1}c + (1 - \phi L)^{-1}(\varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2})$$

$$y_t = (1 + \phi L + \phi^2 L^2 + \dots)c + (1 + \phi L + \phi^2 L^2 + \dots)(\varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2})$$

$$y_t = (1 + \phi L + \phi^2 L^2 + \dots)c + (1 + \phi L + \phi^2 L^2 + \dots)\varepsilon_t + \theta_1(1 + \phi L + \phi^2 L^2 + \dots)\varepsilon_{t-1} \\ + \theta_2(1 + \phi L + \phi^2 L^2 + \dots)\varepsilon_{t-2}$$

$$y_t = (1 + \phi L + \phi^2 L^2 + \dots)c + (\varepsilon_t + \phi \varepsilon_{t-1} + \phi^2 \varepsilon_{t-2} + \phi^3 \varepsilon_{t-3} + \dots) \\ + \theta_1(\varepsilon_{t-1} + \phi \varepsilon_{t-2} + \phi^2 \varepsilon_{t-3} + \phi^3 \varepsilon_{t-4} + \dots) + \theta_2(\varepsilon_{t-2} + \phi \varepsilon_{t-3} + \phi^2 \varepsilon_{t-4} \\ + \dots)$$

$$y_t = (1 - \phi)^{-1}c + \varepsilon_t + (\phi + \theta_1)\varepsilon_{t-1} + (\phi^2 + \phi\theta_1 + \theta_2)\varepsilon_{t-2} + (\phi^3 + \theta_1\phi^2 + \theta_2\phi)\varepsilon_{t-3} \\ + \dots$$

$$y_t = \frac{c}{(1 - \phi)} + \varepsilon_t + (\phi + \theta_1)\varepsilon_{t-1} + \sum_{i=2}^{\infty} (\phi^i + \theta_1\phi^{i-1} + \theta_2\phi^{i-2})\varepsilon_{t-i} \quad (2.30)$$

$$y_t = \frac{c}{(1 - \phi)} + \sum_{i=0}^{\infty} \psi_i \varepsilon_{t-i} \quad (2.31)$$

where

$$\psi_i = \begin{cases} 1, & \text{if } i = 0 \\ \phi + \theta_1, & \text{if } i = 1 \\ \phi^i + \theta_1\phi^{i-1} + \theta_2\phi^{i-2}, & \text{if } i = 2, 3, 4, \dots \end{cases} \quad (2.32)$$

ARMA (1, 3)

$$y_t = c + \phi y_{t-1} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \theta_3 \varepsilon_{t-3} \quad (2.33)$$

$$y_t - \phi y_{t-1} = c + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \theta_3 \varepsilon_{t-3}$$

$$(1 - \phi L)y_t = c + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \theta_3 \varepsilon_{t-3}$$

$$y_t = (1 - \phi L)^{-1}c + (1 - \phi L)^{-1}(\varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \theta_3 \varepsilon_{t-3})$$

$$y_t = (1 + \phi L + \phi^2 L^2 + \phi^3 L^3 + \phi^4 L^4 + \dots)c + (1 + \phi L + \phi^2 L^2 + \phi^3 L^3 + \phi^4 L^4 + \dots)(\varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \theta_3 \varepsilon_{t-3})$$

$$y_t = (1 + \phi L + \phi^2 L^2 + \phi^3 L^3 + \phi^4 L^4 + \dots)c + (1 + \phi L + \phi^2 L^2 + \phi^3 L^3 + \phi^4 L^4 + \dots)\varepsilon_t + \theta_1(1 + \phi L + \phi^2 L^2 + \phi^3 L^3 + \phi^4 L^4 + \dots)\varepsilon_{t-1} + \theta_2(1 + \phi L + \phi^2 L^2 + \phi^3 L^3 + \phi^4 L^4 + \dots)\varepsilon_{t-2} + \theta_3(1 + \phi L + \phi^2 L^2 + \phi^3 L^3 + \phi^4 L^4 + \dots)\varepsilon_{t-3}$$

$$y_t = (1 + \phi + \phi^2 + \phi^3 + \phi^4 + \dots)c + (\varepsilon_t + \phi \varepsilon_{t-1} + \phi^2 \varepsilon_{t-2} + \phi^3 \varepsilon_{t-3} + \dots) + \theta_1(\varepsilon_{t-1} + \phi \varepsilon_{t-2} + \phi^2 \varepsilon_{t-3} + \phi^3 \varepsilon_{t-4} + \dots) + \theta_2(\varepsilon_{t-2} + \phi \varepsilon_{t-3} + \phi^2 \varepsilon_{t-4} + \phi^3 \varepsilon_{t-5} + \dots) + \theta_3(\varepsilon_{t-3} + \phi \varepsilon_{t-4} + \phi^2 \varepsilon_{t-5} + \dots)$$

$$y_t = (1 - \phi)^{-1}c + \varepsilon_t + (\phi + \theta_1)\varepsilon_{t-1} + (\phi^2 + \phi\theta_1 + \theta_2)\varepsilon_{t-2} + (\phi^3 + \theta_1\phi^2 + \theta_2\phi + \theta_3)\varepsilon_{t-3} + (\phi^4 + \theta_1\phi^3 + \theta_2\phi^2 + \theta_3\phi)\varepsilon_{t-4} + \dots$$

$$y_t = \frac{c}{(1 - \phi)} + \varepsilon_t + (\phi + \theta_1)\varepsilon_{t-1} + (\phi^2 + \phi\theta_1 + \theta_2)\varepsilon_{t-2} + \sum_{i=3}^{\infty} (\phi^i + \theta_1\phi^{i-1} + \theta_2\phi^{i-2} + \theta_3\phi^{i-3})\varepsilon_{t-i} \quad (2.34)$$

$$y_t = \frac{c}{(1 - \phi)} + \sum_{i=0}^{\infty} \psi_i \varepsilon_{t-i} \quad (2.35)$$

where

$$\psi_i = \begin{cases} 1, & i = 0 \\ \phi + \theta_1, & i = 1 \\ \phi^2 + \phi\theta_1 + \theta_2, & i = 2 \\ \phi^i + \theta_1\phi^{i-1} + \theta_2\phi^{i-2} + \theta_3\phi^{i-3}, & i = 3, 4, \dots \end{cases} \quad (2.36)$$

ARMA (1, 4)

$$y_t = c + \phi y_{t-1} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \theta_3 \varepsilon_{t-3} + \theta_4 \varepsilon_{t-4} \quad (2.37)$$

$$y_t - \phi y_{t-1} = c + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \theta_3 \varepsilon_{t-3} + \theta_4 \varepsilon_{t-4}$$

$$(1 - \phi L)y_t = c + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \theta_3 \varepsilon_{t-3} + \theta_4 \varepsilon_{t-4}$$

$$y_t = (1 - \phi L)^{-1}c + (1 - \phi L)^{-1}(\varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \theta_3 \varepsilon_{t-3} + \theta_4 \varepsilon_{t-4})$$

$$y_t = (1 + \phi L + \phi^2 L^2 + \phi^3 L^3 + \phi^4 L^4 + \phi^5 L^5 + \dots)c + (1 + \phi L + \phi^2 L^2 + \phi^3 L^3 + \phi^4 L^4 + \phi^5 L^5 + \dots)(\varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \theta_3 \varepsilon_{t-3} + \theta_4 \varepsilon_{t-4})$$

$$\begin{aligned} y_t = & (1 + \phi L + \phi^2 L^2 + \phi^3 L^3 + \phi^4 L^4 + \dots)c \\ & + (1 + \phi L + \phi^2 L^2 + \phi^3 L^3 + \phi^4 L^4 + \phi^5 L^5 + \dots)\varepsilon_t \\ & + \theta_1 (1 + \phi L + \phi^2 L^2 + \phi^3 L^3 + \phi^4 L^4 + \phi^5 L^5 + \dots)\varepsilon_{t-1} \\ & + \theta_2 (1 + \phi L + \phi^2 L^2 + \phi^3 L^3 + \phi^4 L^4 + \phi^5 L^5 + \dots)\varepsilon_{t-2} \\ & + \theta_3 (1 + \phi L + \phi^2 L^2 + \phi^3 L^3 + \phi^4 L^4 + \phi^5 L^5 + \dots)\varepsilon_{t-3} + \theta_4 (1 + \phi L \\ & + \phi^2 L^2 + \phi^3 L^3 + \phi^4 L^4 + \phi^5 L^5 + \dots)\varepsilon_{t-4} \end{aligned}$$

$$\begin{aligned} y_t = & (1 + \phi + \phi^2 + \phi^3 + \phi^4 + \phi^5 + \dots)c \\ & + (\varepsilon_t + \phi \varepsilon_{t-1} + \phi^2 \varepsilon_{t-2} + \phi^3 \varepsilon_{t-3} + \phi^4 \varepsilon_{t-4} + \phi^5 \varepsilon_{t-5} + \dots) \\ & + \theta_1 (\varepsilon_{t-1} + \phi \varepsilon_{t-2} + \phi^2 \varepsilon_{t-3} + \phi^3 \varepsilon_{t-4} + \phi^4 \varepsilon_{t-5} + \dots) \\ & + \theta_2 (\varepsilon_{t-2} + \phi \varepsilon_{t-3} + \phi^2 \varepsilon_{t-4} + \phi^3 \varepsilon_{t-5} + \dots) \\ & + \theta_3 (\varepsilon_{t-3} + \phi \varepsilon_{t-4} + \phi^2 \varepsilon_{t-5} + \dots) + \theta_4 (\varepsilon_{t-4} + \phi \varepsilon_{t-5} + \phi^2 \varepsilon_{t-6} + \dots) \end{aligned}$$

$$\begin{aligned} y_t = & (1 - \phi)^{-1}c + \varepsilon_t + (\phi + \theta_1)\varepsilon_{t-1} + (\phi^2 + \phi\theta_1 + \theta_2)\varepsilon_{t-2} \\ & + (\phi^3 + \theta_1\phi^2 + \theta_2\phi + \theta_3)\varepsilon_{t-3} + (\phi^4 + \theta_1\phi^3 + \theta_2\phi^2 + \theta_3\phi + \theta_4)\varepsilon_{t-4} \\ & + (\phi^5 + \theta_1\phi^4 + \theta_2\phi^3 + \theta_3\phi^2 + \theta_4\phi)\varepsilon_{t-5} + \dots \end{aligned}$$

$$\begin{aligned} y_t = & \frac{c}{(1 - \phi)} + \varepsilon_t + (\phi + \theta_1)\varepsilon_{t-1} + (\phi^2 + \phi\theta_1 + \theta_2)\varepsilon_{t-2} + (\phi^3 + \theta_1\phi^2 + \theta_2\phi + \theta_3)\varepsilon_{t-3} \\ & + \sum_{i=4}^{\infty} (\phi^i + \theta_1\phi^{i-1} + \theta_2\phi^{i-2} + \theta_3\phi^{i-3} + \theta_4\phi^{i-4})\varepsilon_{t-i} \end{aligned} \quad (2.38)$$

$$y_t = \frac{c}{(1 - \phi)} + \sum_{i=0}^{\infty} \psi_i \varepsilon_{t-i} \quad (2.39)$$

where

$$\psi_i = \begin{cases} 1, & i = 0 \\ \phi + \theta_1, & i = 1 \\ \phi^2 + \phi\theta_1 + \theta_2, & i = 2 \\ \phi^3 + \theta_1\phi^2 + \theta_2\phi + \theta_3, & i = 3 \\ \phi^i + \theta_1\phi^{i-1} + \theta_2\phi^{i-2} + \theta_3\phi^{i-3} + \theta_4\phi^{i-4}, & i = 4, 5, \dots \end{cases} \quad (2.40)$$

ARMA (1, q)

$$y_t = c + \phi y_{t-1} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \theta_3 \varepsilon_{t-3} + \theta_4 \varepsilon_{t-4} + \dots + \theta_q \varepsilon_{t-q} \quad (2.41)$$

$$y_t - \phi y_{t-1} = c + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \theta_3 \varepsilon_{t-3} + \theta_4 \varepsilon_{t-4} + \dots + \theta_q \varepsilon_{t-q}$$

$$(1 - \phi L)y_t = c + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \theta_3 \varepsilon_{t-3} + \theta_4 \varepsilon_{t-4} + \dots + \theta_q \varepsilon_{t-q}$$

$$y_t = (1 - \phi L)^{-1} c + (1 - \phi L)^{-1} (\varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \theta_3 \varepsilon_{t-3} + \theta_4 \varepsilon_{t-4} + \dots + \theta_q \varepsilon_{t-q})$$

$$y_t = (1 + \phi L + \phi^2 L^2 + \phi^3 L^3 + \phi^4 L^4 + \phi^5 L^5 + \dots) c + (1 + \phi L + \phi^2 L^2 + \phi^3 L^3 + \phi^4 L^4 + \phi^5 L^5 + \dots) (\varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \theta_3 \varepsilon_{t-3} + \theta_4 \varepsilon_{t-4} + \dots + \theta_q \varepsilon_{t-q})$$

$$\begin{aligned} y_t = & (1 + \phi L + \phi^2 L^2 + \phi^3 L^3 + \phi^4 L^4 + \dots) c \\ & + (1 + \phi L + \phi^2 L^2 + \phi^3 L^3 + \phi^4 L^4 + \phi^5 L^5 + \dots) \varepsilon_t \\ & + \theta_1 (1 + \phi L + \phi^2 L^2 + \phi^3 L^3 + \phi^4 L^4 + \phi^5 L^5 + \dots) \varepsilon_{t-1} \\ & + \theta_2 (1 + \phi L + \phi^2 L^2 + \phi^3 L^3 + \phi^4 L^4 + \phi^5 L^5 + \dots) \varepsilon_{t-2} \\ & + \theta_3 (1 + \phi L + \phi^2 L^2 + \phi^3 L^3 + \phi^4 L^4 + \phi^5 L^5 + \dots) \varepsilon_{t-3} \\ & + \theta_4 (1 + \phi L + \phi^2 L^2 + \phi^3 L^3 + \phi^4 L^4 + \phi^5 L^5 + \dots) \varepsilon_{t-4} + \dots \\ & + \theta_q (1 + \phi L + \phi^2 L^2 + \phi^3 L^3 + \phi^4 L^4 + \phi^5 L^5 + \dots) \varepsilon_{t-q} \end{aligned}$$

$$\begin{aligned} y_t = & (1 + \phi + \phi^2 + \phi^3 + \phi^4 + \phi^5 + \dots) c \\ & + (\varepsilon_t + \phi \varepsilon_{t-1} + \phi^2 \varepsilon_{t-2} + \phi^3 \varepsilon_{t-3} + \phi^4 \varepsilon_{t-4} + \phi^5 \varepsilon_{t-5} + \dots) \\ & + \theta_1 (\varepsilon_{t-1} + \phi \varepsilon_{t-2} + \phi^2 \varepsilon_{t-3} + \phi^3 \varepsilon_{t-4} + \phi^4 \varepsilon_{t-5} + \dots) \\ & + \theta_2 (\varepsilon_{t-2} + \phi \varepsilon_{t-3} + \phi^2 \varepsilon_{t-4} + \phi^3 \varepsilon_{t-5} + \dots) \\ & + \theta_3 (\varepsilon_{t-3} + \phi \varepsilon_{t-4} + \phi^2 \varepsilon_{t-5} + \dots) + \theta_4 (\varepsilon_{t-4} + \phi \varepsilon_{t-5} + \phi^2 \varepsilon_{t-6} + \dots) + \dots \\ & + \theta_q (\varepsilon_{t-q} + \phi \varepsilon_{t-q-1} + \phi^2 \varepsilon_{t-q-2} + \dots) \end{aligned}$$

$$\begin{aligned} y_t = & (1 - \phi)^{-1} c + \varepsilon_t + (\phi + \theta_1) \varepsilon_{t-1} + (\phi^2 + \phi \theta_1 + \theta_2) \varepsilon_{t-2} \\ & + (\phi^3 + \theta_1 \phi^2 + \theta_2 \phi + \theta_3) \varepsilon_{t-3} + (\phi^4 + \theta_1 \phi^3 + \theta_2 \phi^2 + \theta_3 \phi + \theta_4) \varepsilon_{t-4} \\ & + \dots + (\phi^q + \theta_1 \phi^{q-1} + \theta_2 \phi^{q-2} + \theta_3 \phi^{q-3} + \theta_4 \phi^{q-4} + \dots + \theta_q) \varepsilon_{t-q} \\ & + (\phi^{q+1} + \theta_1 \phi^q + \theta_2 \phi^{q-1} + \theta_3 \phi^{q-2} + \theta_4 \phi^{q-3} + \dots + \theta_q \phi) \varepsilon_{t-q-1} \\ & + (\phi^{q+2} + \theta_1 \phi^{q+1} + \theta_2 \phi^q + \theta_3 \phi^{q-1} + \theta_4 \phi^{q-2} + \dots + \theta_q \phi^2) \varepsilon_{t-q-2} + \dots \end{aligned}$$



$$\begin{aligned}
y_t = & \frac{c}{(1-\phi)} + \varepsilon_t + (\phi + \theta_1)\varepsilon_{t-1} + (\phi^2 + \phi\theta_1 + \theta_2)\varepsilon_{t-2} + (\phi^3 + \theta_1\phi^2 + \theta_2\phi + \theta_3)\varepsilon_{t-3} \\
& + (\phi^4 + \theta_1\phi^3 + \theta_2\phi^2 + \theta_3\phi + \theta_4)\varepsilon_{t-4} + \dots \\
& + \sum_{i=q}^{\infty} (\phi^i + \theta_1\phi^{i-1} + \theta_2\phi^{i-2} + \theta_3\phi^{i-3} + \theta_4\phi^{i-4} + \dots + \theta_q\phi^{i-q})\varepsilon_{t-i} \quad (2.42)
\end{aligned}$$

$$y_t = \frac{c}{(1-\phi)} + \sum_{i=0}^{\infty} \psi_i \varepsilon_{t-i} \quad (2.43)$$

where

$$\psi_i = \begin{cases} 1, & i = 0 \\ \phi + \theta_1, & i = 1 \\ \phi^2 + \phi\theta_1 + \theta_2, & i = 2 \\ \phi^3 + \theta_1\phi^2 + \theta_2\phi + \theta_3, & i = 3 \\ \phi^4 + \theta_1\phi^3 + \theta_2\phi^2 + \theta_3\phi + \theta_4, & i = 4 \\ \phi^i + \theta_1\phi^{i-1} + \theta_2\phi^{i-2} + \theta_3\phi^{i-3} + \theta_4\phi^{i-4} + \dots + \theta_q\phi^{i-q}, & i = q, q+1, \dots \end{cases} \quad (2.44)$$

This establish linear process y_t as a linear combination of white noise variates ε_t .

2.9 INVERTIBILITY OF ARIMA PROCESSES

An ARIMA (p, d, q) process is said to be invertible if the series converges in mean to ε_t as $p \rightarrow \infty$. This happens when $\theta(Z) = 0$ lie outside the unit circle. An ARIMA (p, d, q) process is invertible if the absolute value of the parameters of ARIMA (p, d, q) model satisfy $|\theta_i| < 1$ for $i = 1, \dots, q$. The invertibility condition is satisfied if the absolute value of the parameters of ARMA (p, q) model satisfy $|\theta_i| < 1$ for $i = 1, \dots, q$. If the roots of the polynomial $\theta(L)$ lie outside the unit circle, the ARMA (p, q) process is invertible and thus forecast can be done in as much it satisfies the invertibility condition. An ARMA $(p, 0)$ is stationary if the roots of the polynomial $\phi(L)$ lie outside the unit circle. This is equivalent to saying that the coefficients of the model are in absolute value less than 1, i.e. $|\phi_i| < 1$ for $i = 1, \dots, p$.

Let y_t be an ARMA process defined by $\phi(L)y_t = \theta(L)\varepsilon_t$. If $\phi(z) \neq 0$ for all $|z| = 0$, then there are polynomials $\tilde{\phi}$ and $\tilde{\theta}$ and a white noise sequence ε_t such that y_t satisfies $\tilde{\phi}(L)y_t = \tilde{\theta}(L)\varepsilon_t$, and this is a causal, invertible ARMA process.

2.10 PRESENTATION OF SOME ARIMA PROCESS INVERTED FORM

Here we present some ARIMA processes of various values of p and d in inverted form.

ARIMA (1,0,1)

Recall from equation (2.25), $y_t = c + \phi y_{t-1} + \varepsilon_t + \theta \varepsilon_{t-1}$ gives

$$\varepsilon_t + \theta \varepsilon_{t-1} = -c + y_t + \phi y_{t-1}$$

$$(1 + \theta L)\varepsilon_t = -c + y_t - \phi y_{t-1}$$

$$\varepsilon_t = -(1 + \theta L)^{-1}c + (1 + \theta L)^{-1}(y_t - \phi y_{t-1})$$

$$\varepsilon_t = -(1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \dots)c + (1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \dots)y_t - \phi(1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \dots)y_{t-1}$$

$$\varepsilon_t = -(1 - \theta + \theta^2 - \theta^3 + \dots)c + (y_t - \theta y_{t-1} + \theta^2 y_{t-2} - \theta^3 y_{t-3} + \dots) - \phi(y_{t-1} - \theta y_{t-2} + \theta^2 y_{t-3} - \dots)$$

$$\varepsilon_t = -(1 + \theta)^{-1}c + y_t - (\theta + \phi)y_{t-1} + (\theta^2 + \phi\theta)y_{t-2} - (\theta^3 + \phi\theta^2)y_{t-3} + \dots$$

$$\varepsilon_t = -\frac{c}{(1 + \theta)} + y_t + \sum_{i=1}^{\infty} (-1)^i (\theta^i + \phi\theta^{i-1})y_{t-i} \quad (2.45)$$

$$\varepsilon_t = -\frac{c}{(1 + \theta)} + \sum_{i=0}^{\infty} \pi_i y_{t-i} \quad (2.46)$$

where

$$\pi_i = \begin{cases} 1, & i = 0 \\ (-1)^i [\phi\theta^{i-1} + \theta^i], & i = 1, 2, 3, \dots \end{cases} \quad (2.47)$$

Equation (2.46) holds if $\theta \neq -1$.

ARIMA (1, 1, 1)

$$\Delta y_t = c + \phi \Delta y_{t-1} + \varepsilon_t + \theta \varepsilon_{t-1} \quad (2.48)$$

$$y_t - y_{t-1} = c + \phi(y_{t-1} - y_{t-2}) + \varepsilon_t + \theta \varepsilon_{t-1}$$

$$\varepsilon_t + \theta \varepsilon_{t-1} = y_t - (1 + \phi)y_{t-1} + \phi y_{t-2} - c$$

$$(1 + \theta L)\varepsilon_t = y_t - (1 + \phi)y_{t-1} + \phi y_{t-2} - c$$

$$\varepsilon_t = -(1 + \theta L)^{-1}c + (1 + \theta L)^{-1}[y_t - (1 + \phi)y_{t-1} + \phi y_{t-2}]$$

$$\begin{aligned}\varepsilon_t = & -(1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots)c + (1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots)y_t \\ & - (1 + \phi)[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots]y_{t-1} \\ & + \phi[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots]y_{t-2}\end{aligned}$$

$$\begin{aligned}\varepsilon_t = & -(1 - \theta + \theta^2 - \theta^3 + \theta^4 - \dots)c + (y_t - \theta y_{t-1} + \theta^2 y_{t-2} - \theta^3 y_{t-3} + \theta^4 y_{t-4} - \dots) \\ & - (1 + \phi)(y_{t-1} - \theta y_{t-2} + \theta^2 y_{t-3} - \theta^3 y_{t-4} - \dots) \\ & + \phi(y_{t-2} - \theta y_{t-3} + \theta^2 y_{t-4} - \theta^3 y_{t-5} - \dots)\end{aligned}$$

$$\begin{aligned}\varepsilon_t = & -(1 + \theta)^{-1}c + y_t - (\theta + \phi + 1)y_{t-1} + (\theta^2 + \phi\theta + \theta + \phi)y_{t-2} \\ & - (\theta^3 + \phi\theta^2 + \theta^2 + \phi\theta)y_{t-3} + (\theta^4 + \phi\theta^3 + \theta^3 + \phi\theta^2)y_{t-4} - \dots\end{aligned}\quad (2.49)$$

$$\begin{aligned}\varepsilon_t = & -\frac{c}{(1 + \theta)} + y_t - (\theta + \phi + 1)y_{t-1} \\ & + \sum_{i=2}^{\infty} (-1)^i [(\theta + \phi)(\theta^{i-1} + \theta^{i-2})]y_{t-i}\end{aligned}\quad (2.50)$$

$$\varepsilon_t = -\frac{c}{(1 + \theta)} + \sum_{i=0}^{\infty} \pi_i y_{t-i}\quad (2.51)$$

where

$$\pi_i = \begin{cases} 1, & i = 0 \\ -(\theta + \phi + 1), & i = 1 \\ (-1)^i [(\theta + \phi)(\theta^{i-1} + \theta^{i-2})], & i = 2, 3, 4, \dots \end{cases}\quad (2.52)$$

Equation (2.51) holds if $\theta \neq -1$.

ARIMA (1, 2, 1)

$$\Delta^2 y_t = c + \phi \Delta^2 y_{t-1} + \varepsilon_t + \theta \varepsilon_{t-1}\quad (2.53)$$

$$\Delta(y_t - y_{t-1}) = c + \phi \Delta(y_{t-1} - y_{t-2}) + \varepsilon_t + \theta \varepsilon_{t-1}$$

$$y_t - y_{t-1} - y_{t-1} + y_{t-2} = c + \phi(y_{t-1} - y_{t-2} - y_{t-2} + y_{t-3}) + \varepsilon_t + \theta \varepsilon_{t-1}$$

$$y_t - 2y_{t-1} + y_{t-2} = c + \phi(y_{t-1} - 2y_{t-2} + y_{t-3}) + \varepsilon_t + \theta \varepsilon_{t-1}$$

$$\varepsilon_t + \theta \varepsilon_{t-1} = y_t - (2 + \phi)y_{t-1} + (1 + 2\phi)y_{t-2} - \phi y_{t-3} - c$$

$$(1 + \theta L)\varepsilon_t = y_t - (2 + \phi)y_{t-1} + (1 + 2\phi)y_{t-2} - \phi y_{t-3} - c$$

$$\varepsilon_t = -(1 + \theta L)^{-1}c + (1 + \theta L)^{-1}[y_t - (2 + \phi)y_{t-1} + (1 + 2\phi)y_{t-2} - \phi y_{t-3}]$$

$$\begin{aligned} \varepsilon_t = & -(1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots)c + (1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots)y_t \\ & - (2 + \phi)[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots]y_{t-1} \\ & + (1 + 2\phi)[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots]y_{t-2} \\ & - \phi[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots]y_{t-3} \end{aligned}$$

$$\begin{aligned} \varepsilon_t = & -(1 - \theta + \theta^2 - \theta^3 + \theta^4 - \dots)c + (y_t - \theta y_{t-1} + \theta^2 y_{t-2} - \theta^3 y_{t-3} + \theta^4 y_{t-4} - \dots) \\ & - (2 + \phi)(y_{t-1} - \theta y_{t-2} + \theta^2 y_{t-3} - \theta^3 y_{t-4} - \dots) \\ & + (1 + 2\phi)(y_{t-2} - \theta y_{t-3} + \theta^2 y_{t-4} - \theta^3 y_{t-5} - \dots) \\ & - \phi(y_{t-3} - \theta y_{t-4} + \theta^2 y_{t-5} - \theta^3 y_{t-6} - \dots) \end{aligned}$$

$$\begin{aligned} \varepsilon_t = & -(1 + \theta)^{-1}c + y_t - (\theta + \phi + 2)y_{t-1} + [\theta^2 + \theta(\phi + 2) + (2\phi + 1)]y_{t-2} \\ & - [\theta^3 + \theta^2(\phi + 2) + \theta(2\phi + 1) + \phi]y_{t-3} \\ & + [\theta^4 + \theta^3(\phi + 2) + \theta^2(2\phi + 1) + \theta\phi]y_{t-4} \\ & - [\theta^5 + \theta^4(\phi + 2) + \theta^3(2\phi + 1) + \theta^2\phi]y_{t-5} + \dots \end{aligned} \quad (2.54)$$

$$\begin{aligned} \varepsilon_t = & -\frac{c}{(1 + \theta)} + y_t - (\theta + \phi + 2)y_{t-1} + [\theta^2 + \theta(\phi + 2) + (2\phi + 1)]y_{t-2} \\ & + \sum_{i=3}^{\infty} (-1)^i [\theta^i + \theta^{i-1}(\phi + 2) + \theta^{i-2}(2\phi + 1) + \theta^{i-3}\phi]y_{t-i} \end{aligned} \quad (2.55)$$

$$\varepsilon_t = -\frac{c}{(1 + \theta)} + \sum_{i=0}^{\infty} \pi_i y_{t-i} \quad (2.56)$$

where

$$\pi_i = \begin{cases} 1, & i = 0 \\ -(\theta + \phi + 2), & i = 1 \\ \theta^2 + \theta(\phi + 2) + (2\phi + 1), & i = 2 \\ (-1)^i [\theta^i + \theta^{i-1}(\phi + 2) + \theta^{i-2}(2\phi + 1) + \theta^{i-3}\phi], & i = 3, 4, 5, \dots \end{cases} \quad (2.57)$$

Equation (2.56) holds if $\theta \neq -1$.

ARIMA (1, 3, 1)

$$\Delta^3 y_t = c + \phi \Delta^3 y_{t-1} + \varepsilon_t + \theta \varepsilon_{t-1} \quad (2.58)$$

$$\Delta(y_t - 2y_{t-1} + y_{t-2}) = c + \phi \Delta(y_{t-1} - 2y_{t-2} + y_{t-3}) + \varepsilon_t + \theta \varepsilon_{t-1}$$

$$\begin{aligned} y_t - y_{t-1} - 2(y_{t-1} - y_{t-2}) + (y_{t-2} - y_{t-3}) \\ = c + \phi[(y_{t-1} - y_{t-2}) - 2(y_{t-2} - y_{t-3}) + (y_{t-3} - y_{t-4})] + \varepsilon_t + \theta \varepsilon_{t-1} \end{aligned}$$

$$y_t - 3y_{t-1} + 3y_{t-2} - y_{t-3} = c + \phi(y_{t-1} - 3y_{t-2} + 3y_{t-3} - y_{t-4}) + \varepsilon_t + \theta \varepsilon_{t-1}$$

$$\varepsilon_t + \theta \varepsilon_{t-1} = y_t - (3 + \phi)y_{t-1} + (3 + 3\phi)y_{t-2} - (1 + 3\phi)y_{t-3} + \phi y_{t-4} - c$$

$$(1 + \theta L)\varepsilon_t = y_t - (3 + \phi)y_{t-1} + (3 + 3\phi)y_{t-2} - (1 + 3\phi)y_{t-3} + \phi y_{t-4} - c$$

$$\begin{aligned} \varepsilon_t = -(1 + \theta L)^{-1}c \\ + (1 + \theta L)^{-1}[y_t - (3 + \phi)y_{t-1} + (3 + 3\phi)y_{t-2} - (1 + 3\phi)y_{t-3} + \phi y_{t-4}] \end{aligned}$$

$$\begin{aligned} \varepsilon_t = -(1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots)c + (1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots)y_t \\ - (3 + \phi)[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots]y_{t-1} \\ + (3 + 3\phi)[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots]y_{t-2} \\ - (1 + 3\phi)[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots]y_{t-3} \\ + \phi[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots]y_{t-4} \end{aligned}$$

$$\begin{aligned} \varepsilon_t = -(1 - \theta + \theta^2 - \theta^3 + \theta^4 - \dots)c \\ + (y_t - \theta y_{t-1} + \theta^2 y_{t-2} - \theta^3 y_{t-3} + \theta^4 y_{t-4} - \theta^5 y_{t-5} + \theta^6 y_{t-6} - \dots) \\ - (3 + \phi)(y_{t-1} - \theta y_{t-2} + \theta^2 y_{t-3} - \theta^3 y_{t-4} + \theta^4 y_{t-5} - \theta^5 y_{t-6} + \dots) \\ + (3 + 3\phi)(y_{t-2} - \theta y_{t-3} + \theta^2 y_{t-4} - \theta^3 y_{t-5} + \theta^4 y_{t-6} - \dots) \\ - (1 + 3\phi)(y_{t-3} - \theta y_{t-4} + \theta^2 y_{t-5} - \theta^3 y_{t-6} + \theta^4 y_{t-7} - \dots) \\ + \phi(y_{t-4} - \theta y_{t-5} + \theta^2 y_{t-6} - \theta^3 y_{t-7} + \dots) \end{aligned}$$

$$\begin{aligned} \varepsilon_t = -(1 + \theta)^{-1}c + y_t - (\theta + \phi + 3)y_{t-1} + [\theta^2 + \theta(\phi + 3) + (3 + 3\phi)]y_{t-2} \\ - [\theta^3 + \theta^2(\phi + 3) + \theta(3 + 3\phi) + (1 + 3\phi)]y_{t-3} \\ + [\theta^4 + \theta^3(\phi + 3) + \theta^2(3 + 3\phi) + \theta(1 + 3\phi) + \phi]y_{t-4} \\ - [\theta^5 + \theta^4(\phi + 3) + \theta^3(3 + 3\phi) + \theta^2(1 + 3\phi) + \theta\phi]y_{t-5} \\ + [\theta^6 + \theta^5(\phi + 3) + \theta^4(3 + 3\phi) + \theta^3(1 + 3\phi) + \theta^2\phi]y_{t-6} - \dots \quad (2.59) \end{aligned}$$



$$\begin{aligned}
\varepsilon_t = & -\frac{c}{(1+\theta)} + y_t - (\theta + \phi + 3)y_{t-1} + [\theta^2 + \theta(\phi + 3) + (3 + 3\phi)]y_{t-2} \\
& - [\theta^3 + \theta^2(\phi + 3) + \theta(3 + 3\phi) + (1 + 3\phi)]y_{t-3} \\
& + \sum_{i=4}^{\infty} (-1)^i [\theta^i + \theta^{i-1}(\phi + 3) + \theta^{i-2}(3\phi + 3) + \theta^{i-3}(1 + 3\phi) \\
& + \theta^{i-4}\phi] y_{t-i}
\end{aligned} \tag{2.60}$$

$$\varepsilon_t = -\frac{c}{(1+\theta)} + \sum_{i=0}^{\infty} \pi_i y_{t-i} \tag{2.61}$$

where

$$\pi_i = \begin{cases} 1, & i = 0 \\ -(\theta + \phi + 3), & i = 1 \\ \theta^2 + \theta(\phi + 3) + (3 + 3\phi), & i = 2 \\ -[\theta^3 + \theta^2(\phi + 3) + \theta(3 + 3\phi) + (1 + 3\phi)], & i = 3 \\ (-1)^i [\theta^i + \theta^{i-1}(\phi + 3) + \theta^{i-2}(3\phi + 3) \\ + \theta^{i-3}(1 + 3\phi) + \theta^{i-4}\phi], & i = 4, 5, 6, \dots \end{cases} \tag{2.62}$$

Equation (2.61) holds if $\theta \neq -1$.

ARIMA (2, 0, 1)

$$y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \varepsilon_t + \theta \varepsilon_{t-1} \tag{2.63}$$

$$\varepsilon_t + \theta \varepsilon_{t-1} = -c + y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2}$$

$$(1 + \theta L) \varepsilon_t = -c + y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2}$$

$$\varepsilon_t = -(1 + \theta L)^{-1} c + (1 + \theta L)^{-1} (y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2})$$

$$\begin{aligned}
\varepsilon_t = & -(1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots) c + (1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots) (y_t \\
& - \phi_1 y_{t-1} - \phi_2 y_{t-2})
\end{aligned}$$

$$\begin{aligned}
\varepsilon_t = & -(1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots) c + (1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots) y_t \\
& - \phi_1 (1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots) y_{t-1} - \phi_2 (1 - \theta L + \theta^2 L^2 \\
& - \theta^3 L^3 + \theta^4 L^4 - \dots) y_{t-2}
\end{aligned}$$

$$\begin{aligned}\varepsilon_t = & -(1 - \theta + \theta^2 - \theta^3 + \dots)c + (y_t - \theta y_{t-1} + \theta^2 y_{t-2} - \theta^3 y_{t-3} + \theta^4 y_{t-4} - \dots) \\ & - \phi_1(y_{t-1} - \theta y_{t-2} + \theta^2 y_{t-3} - \theta^3 y_{t-4} + \dots) - \phi_2(y_{t-2} - \theta y_{t-3} + \theta^2 y_{t-4} \\ & - \dots)\end{aligned}$$

$$\begin{aligned}\varepsilon_t = & -(1 + \theta)^{-1}c + y_t - (\theta + \phi_1)y_{t-1} + (\theta^2 + \phi_1\theta - \phi_2)y_{t-2} \\ & - (\theta^3 + \phi_1\theta^2 - \phi_2\theta)y_{t-3} + (\theta^4 + \phi_1\theta^3 - \phi_2\theta^2)y_{t-4} - \dots\end{aligned}$$

$$\varepsilon_t = -\frac{c}{(1 + \theta)} + y_t - (\theta + \phi_1)y_{t-1} + \sum_{i=2}^{\infty} (-1)^i [\theta^i + \phi_1\theta^{i-1} - \phi_2\theta^{i-2}]y_{t-i} \quad (2.64)$$

$$\varepsilon_t = -\frac{c}{(1 + \theta)} + \sum_{i=0}^{\infty} \pi_i y_{t-i} \quad (2.65)$$

where

$$\pi_i = \begin{cases} 1, & i = 0 \\ -(\theta + \phi_1), & i = 1 \\ (-1)^i [\theta^i + \phi_1\theta^{i-1} - \phi_2\theta^{i-2}], & i = 2, 3, 4, \dots \end{cases} \quad (2.66)$$

Equation (2.65) holds if $\theta \neq -1$.

ARIMA (2, 1, 1)

$$\Delta y_t = c + \phi_1 \Delta y_{t-1} + \phi_2 \Delta y_{t-2} + \varepsilon_t + \theta \varepsilon_{t-1} \quad (2.67)$$

$$y_t - y_{t-1} = c + \phi_1(y_{t-1} - y_{t-2}) + \phi_2(y_{t-2} - y_{t-3}) + \varepsilon_t + \theta \varepsilon_{t-1}$$

$$\varepsilon_t + \theta \varepsilon_{t-1} = y_t - (1 + \phi_1)y_{t-1} + (\phi_1 - \phi_2)y_{t-2} + \phi_2 y_{t-3} - c$$

$$(1 + \theta L)\varepsilon_t = y_t - (1 + \phi_1)y_{t-1} + (\phi_1 - \phi_2)y_{t-2} + \phi_2 y_{t-3} - c$$

$$\varepsilon_t = -(1 + \theta L)^{-1}c + (1 + \theta L)^{-1}[y_t - (1 + \phi_1)y_{t-1} + (\phi_1 - \phi_2)y_{t-2} + \phi_2 y_{t-3}]$$

$$\begin{aligned}\varepsilon_t = & -(1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots)c + (1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots)y_t \\ & - (1 + \phi_1)[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots]y_{t-1} \\ & + (\phi_1 - \phi_2)[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots]y_{t-2} \\ & + \phi_2[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots]y_{t-3}\end{aligned}$$



$$\begin{aligned}\varepsilon_t = & -(1 - \theta + \theta^2 - \theta^3 + \theta^4 - \dots)c + (y_t - \theta y_{t-1} + \theta^2 y_{t-2} - \theta^3 y_{t-3} + \theta^4 y_{t-4} - \dots) \\ & - (1 + \phi_1)(y_{t-1} - \theta y_{t-2} + \theta^2 y_{t-3} - \theta^3 y_{t-4} + \dots) \\ & + (\phi_1 - \phi_2)(y_{t-2} - \theta y_{t-3} + \theta^2 y_{t-4} - \dots) \\ & + \phi_2(y_{t-3} - \theta y_{t-4} + \theta^2 y_{t-5} - \dots)\end{aligned}$$

$$\begin{aligned}\varepsilon_t = & -(1 + \theta)^{-1}c + y_t - (\theta + \phi_1 + 1)y_{t-1} + (\theta^2 + \theta(1 + \phi_1) + \phi_1 - \phi_2)y_{t-2} \\ & - [\theta^3 + \theta^2(1 + \phi_1) + \theta(\phi_1 - \phi_2) - \phi_2]y_{t-3} \\ & + [\theta^4 + \theta^3(1 + \phi_1) + \theta^2(\phi_1 - \phi_2) - \phi_2\theta]y_{t-4} - \dots\end{aligned}$$

$$\begin{aligned}\varepsilon_t = & -\frac{c}{(1 + \theta)} + y_t - (\theta + \phi_1 + 1)y_{t-1} + (\theta^2 + \phi_1\theta + \theta + \phi_1 - \phi_2)y_{t-2} \\ & + \sum_{i=3}^{\infty} (-1)^i [\theta^i + \theta^{i-1}(1 + \phi_1) + \theta^{i-2}(\phi_1 - \phi_2) - \phi_2\theta^{i-3}]y_{t-i}\end{aligned}\quad (2.68)$$

$$\varepsilon_t = -\frac{c}{(1 + \theta)} + \sum_{i=0}^{\infty} \pi_i y_{t-i}\quad (2.69)$$

where

$$\pi_i = \begin{cases} 1, & i = 0 \\ -(\theta + \phi_1 + 1), & i = 1 \\ \theta^2 + \phi_1\theta + \theta + \phi_1 - \phi_2, & i = 2 \\ (-1)^i [\theta^i + \theta^{i-1}(1 + \phi_1) + \theta^{i-2}(\phi_1 - \phi_2) - \phi_2\theta^{i-3}], & i = 3, 4, \dots \end{cases}\quad (2.70)$$

Equation (2.69) holds if $\theta \neq -1$.

ARIMA (2, 2, 1)

$$\Delta^2 y_t = c + \phi_1 \Delta^2 y_{t-1} + \phi_2 \Delta^2 y_{t-2} + \varepsilon_t + \theta \varepsilon_{t-1}\quad (2.71)$$

$$\Delta(y_t - y_{t-1}) = c + \phi_1 \Delta(y_{t-1} - y_{t-2}) + \phi_2 \Delta(y_{t-2} - y_{t-3}) + \varepsilon_t + \theta \varepsilon_{t-1}$$

$$\begin{aligned}y_t - y_{t-1} - y_{t-1} + y_{t-2} \\ = c + \phi_1(y_{t-1} - y_{t-2} - y_{t-2} + y_{t-3}) + \phi_2(y_{t-2} - y_{t-3} - y_{t-3} + y_{t-4}) + \varepsilon_t \\ + \theta \varepsilon_{t-1}\end{aligned}$$

$$\begin{aligned}y_t - 2y_{t-1} + y_{t-2} \\ = c + \phi_1(y_{t-1} - 2y_{t-2} + y_{t-3}) + \phi_2(y_{t-2} - 2y_{t-3} + y_{t-4}) + \varepsilon_t + \theta \varepsilon_{t-1}\end{aligned}$$

$$\varepsilon_t + \theta \varepsilon_{t-1} = y_t - (2 + \phi_1)y_{t-1} + (1 + 2\phi_1 - \phi_2)y_{t-2} - (\phi_1 - 2\phi_2)y_{t-3} - \phi_2 y_{t-4} - c$$

$$(1 + \theta L)\varepsilon_t = y_t - (2 + \phi_1)y_{t-1} + (1 + 2\phi_1 - \phi_2)y_{t-2} - (\phi_1 - 2\phi_2)y_{t-3} - \phi_2 y_{t-4} - c$$

$$\begin{aligned}\varepsilon_t = & -(1 + \theta L)^{-1}c + (1 + \theta L)^{-1}y_t - (2 + \phi_1)(1 + \theta L)^{-1}y_{t-1} \\ & + (1 + 2\phi_1 - \phi_2)(1 + \theta L)^{-1}y_{t-2} - (\phi_1 - 2\phi_2)(1 + \theta L)^{-1}y_{t-3} \\ & - \phi_2(1 + \theta L)^{-1}y_{t-4}\end{aligned}$$

$$\begin{aligned}\varepsilon_t = & -(1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots)c + (1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots)y_t \\ & - (2 + \phi_1)[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots]y_{t-1} \\ & + (1 + 2\phi_1 - \phi_2)[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots]y_{t-2} \\ & - (\phi_1 - 2\phi_2)[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots]y_{t-3} \\ & - \phi_2[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots]y_{t-4}\end{aligned}$$

$$\begin{aligned}\varepsilon_t = & -(1 - \theta + \theta^2 - \theta^3 + \theta^4 - \dots)c + (y_t - \theta y_{t-1} + \theta^2 y_{t-2} - \theta^3 y_{t-3} + \theta^4 y_{t-4} - \dots) \\ & - (2 + \phi_1)(y_{t-1} - \theta y_{t-2} + \theta^2 y_{t-3} - \theta^3 y_{t-4} - \dots) \\ & + (1 + 2\phi_1 - \phi_2)(y_{t-2} - \theta y_{t-3} + \theta^2 y_{t-4} - \theta^3 y_{t-5} - \dots) \\ & - (\phi_1 - 2\phi_2)(y_{t-3} - \theta y_{t-4} + \theta^2 y_{t-5} - \theta^3 y_{t-6} - \dots) \\ & - \phi_2(y_{t-4} - \theta y_{t-5} + \theta^2 y_{t-6} - \theta^3 y_{t-7} - \dots)\end{aligned}$$

$$\begin{aligned}\varepsilon_t = & -(1 + \theta)^{-1}c + y_t - (\theta + \phi_1 + 2)y_{t-1} + [\theta^2 + \theta(2 + \phi_1) + (1 + 2\phi_1 - \phi_2)]y_{t-2} \\ & + [\theta^3 + \theta^2(2 + \phi_1) + \theta(1 + 2\phi_1 - \phi_2) + (\phi_1 - 2\phi_2)]y_{t-3} \\ & + [\theta^4 + \theta^3(2 + \phi_1) + \theta^2(1 + 2\phi_1 - \phi_2) + \theta(\phi_1 - 2\phi_2) - \phi_2]y_{t-4} \\ & - [\theta^5 + \theta^4(2 + \phi_1) + \theta^3(1 + 2\phi_1 - \phi_2) + \theta^2(\phi_1 - 2\phi_2) - \theta\phi_2]y_{t-5} \\ & + \dots\end{aligned}\quad (2.72)$$

$$\begin{aligned}\varepsilon_t = & -\frac{c}{(1 + \theta)} + y_t - (\theta + \phi_1 + 2)y_{t-1} + [\theta^2 + \theta(2 + \phi_1) + (1 + 2\phi_1 - \phi_2)]y_{t-2} \\ & - [\theta^3 + \theta^2(2 + \phi_1) + \theta(1 + 2\phi_1 - \phi_2) + (\phi_1 - 2\phi_2)]y_{t-3} \\ & + \sum_{i=4}^{\infty} (-1)^i [\theta^i + \theta^{i-1}(\phi + 2) + \theta^{i-2}(1 + 2\phi_1 - \phi_2) + \theta^{i-3}(\phi_1 - 2\phi_2) \\ & - \theta^{i-4}\phi_2]y_{t-i}\end{aligned}\quad (2.73)$$

$$\varepsilon_t = -\frac{c}{(1 + \theta)} + \sum_{i=0}^{\infty} \pi_i y_{t-i}\quad (2.74)$$

where

$$\pi_i = \begin{cases} 1, & i = 0 \\ -(\theta + \phi_1 + 2), & i = 1 \\ \theta^2 + \theta(2 + \phi_1) + (1 + 2\phi_1 - \phi_2), & i = 2 \\ -[\theta^3 + \theta^2(2 + \phi_1) + \theta(1 + 2\phi_1 - \phi_2) + (\phi_1 - 2\phi_2)], & i = 3 \\ (-1)^i [\theta^i + \theta^{i-1}(\phi_1 + 2) + \theta^{i-2}(1 + 2\phi_1 - \phi_2) \\ \quad + \theta^{i-3}(\phi_1 - 2\phi_2) - \theta^{i-4}\phi_2], & i = 4, 5, 6, \dots \end{cases} \quad (2.75)$$

Equation (2.74) holds if $\theta \neq -1$.

ARIMA (2, 3, 1)

$$\Delta^3 y_t = c + \phi_1 \Delta^3 y_{t-1} + \phi_2 \Delta^3 y_{t-2} + \varepsilon_t + \theta \varepsilon_{t-1} \quad (2.76)$$

$$\begin{aligned} & \Delta(y_t - 2y_{t-1} + y_{t-2}) \\ &= c + \phi_1 \Delta(y_{t-1} - 2y_{t-2} + y_{t-3}) + \phi_2 \Delta(y_{t-2} - 2y_{t-3} + y_{t-4}) + \varepsilon_t + \theta \varepsilon_{t-1} \\ & y_t - y_{t-1} - 2(y_{t-1} - y_{t-2}) + (y_{t-2} - y_{t-3}) \\ &= c + \phi_1 [(y_{t-1} - y_{t-2}) - 2(y_{t-2} - y_{t-3}) + (y_{t-3} - y_{t-4})] \\ & \quad + \phi_2 [(y_{t-2} - y_{t-3}) - 2(y_{t-3} - y_{t-4}) + (y_{t-4} - y_{t-5})] + \varepsilon_t + \theta \varepsilon_{t-1} \end{aligned}$$

$$\begin{aligned} & y_t - 3y_{t-1} + 3y_{t-2} - y_{t-3} \\ &= c + \phi_1 (y_{t-1} - 3y_{t-2} + 3y_{t-3} - y_{t-4}) + \phi_2 (y_{t-2} - 3y_{t-3} + 3y_{t-4} - y_{t-5}) \\ & \quad + \varepsilon_t + \theta \varepsilon_{t-1} \end{aligned}$$

$$\begin{aligned} \varepsilon_t + \theta \varepsilon_{t-1} &= y_t - (3 + \phi_1)y_{t-1} + (3 + 3\phi_1 - \phi_2)y_{t-2} - (1 + 3\phi_1 - 3\phi_2)y_{t-3} \\ & \quad + (\phi_1 - 3\phi_2)y_{t-4} + \phi_2 y_{t-5} - c \end{aligned}$$

$$\begin{aligned} (1 + \theta L)\varepsilon_t &= y_t - (3 + \phi_1)y_{t-1} + (3 + 3\phi_1 - \phi_2)y_{t-2} - (1 + 3\phi_1 - 3\phi_2)y_{t-3} \\ & \quad + (\phi_1 - 3\phi_2)y_{t-4} + \phi_2 y_{t-5} - c \end{aligned}$$

$$\begin{aligned} \varepsilon_t &= -(1 + \theta L)^{-1}c \\ & \quad + (1 + \theta L)^{-1} [y_t - (3 + \phi_1)y_{t-1} + (3 + 3\phi_1 - \phi_2)y_{t-2} \\ & \quad - (1 + 3\phi_1 - 3\phi_2)y_{t-3} + (\phi_1 - 3\phi_2)y_{t-4} + \phi_2 y_{t-5}] \end{aligned}$$

$$\begin{aligned}
\varepsilon_t = & -(1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots)c + (1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots)y_t \\
& - (3 + \phi_1)[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots]y_{t-1} \\
& + (3 + 3\phi_1 - \phi_2)[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots]y_{t-2} \\
& - (1 + 3\phi_1 - 3\phi_2)[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots]y_{t-3} \\
& + (\phi_1 - 3\phi_2)[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots]y_{t-4} \\
& + \phi_2[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots]y_{t-5}
\end{aligned}$$

$$\begin{aligned}
\varepsilon_t = & -(1 - \theta + \theta^2 - \theta^3 + \theta^4 - \dots)c \\
& + (y_t - \theta y_{t-1} + \theta^2 y_{t-2} - \theta^3 y_{t-3} + \theta^4 y_{t-4} - \theta^5 y_{t-5} + \theta^6 y_{t-6} - \dots) \\
& - (3 + \phi_1)(y_{t-1} - \theta y_{t-2} + \theta^2 y_{t-3} - \theta^3 y_{t-4} + \theta^4 y_{t-5} - \theta^5 y_{t-6} + \dots) \\
& + (3 + 3\phi_1 - \phi_2)(y_{t-2} - \theta y_{t-3} + \theta^2 y_{t-4} - \theta^3 y_{t-5} + \theta^4 y_{t-6} - \dots) \\
& - (1 + 3\phi_1 - 3\phi_2)(y_{t-3} - \theta y_{t-4} + \theta^2 y_{t-5} - \theta^3 y_{t-6} + \theta^4 y_{t-7} - \dots) \\
& + (\phi_1 - 3\phi_2)(y_{t-4} - \theta y_{t-5} + \theta^2 y_{t-6} - \theta^3 y_{t-7} + \dots) \\
& + \phi_2(y_{t-5} - \theta y_{t-6} + \theta^2 y_{t-7} - \theta^3 y_{t-8} + \dots)
\end{aligned}$$

$$\begin{aligned}
\varepsilon_t = & -(1 + \theta)^{-1}c + y_t - (\theta + \phi_1 + 3)y_{t-1} + [\theta^2 + \theta(3 + \phi_1) + (3 + 3\phi_1 - \phi_2)]y_{t-2} \\
& - [\theta^3 + \theta^2(3 + \phi_1) + \theta(3 + 3\phi_1 - \phi_2) + (1 + 3\phi_1 - 3\phi_2)]y_{t-3} \\
& + [\theta^4 + \theta^3(3 + \phi_1) + \theta^2(3 + 3\phi_1 - \phi_2) + \theta(1 + 3\phi_1 - 3\phi_2) \\
& + (\phi_1 - 3\phi_2)]y_{t-4} \\
& - [\theta^5 + \theta^4(3 + \phi_1) + \theta^3(3 + 3\phi_1 - \phi_2) + \theta^2(1 + 3\phi_1 - 3\phi_2) \\
& + \theta(\phi_1 - 3\phi_2) - \phi_2]y_{t-5} \\
& + [\theta^6 + \theta^5(3 + \phi_1) + \theta^4(3 + 3\phi_1 - \phi_2) + \theta^3(1 + 3\phi_1 - 3\phi_2) \\
& + \theta^2(\phi_1 - 3\phi_2) - \theta\phi_2]y_{t-6} - \dots
\end{aligned} \tag{2.77}$$

$$\begin{aligned}
\varepsilon_t = & -\frac{c}{(1 + \theta)} + y_t - (\theta + \phi_1 + 3)y_{t-1} + [\theta^2 + \theta(3 + \phi_1) + (3 + 3\phi_1 - \phi_2)]y_{t-2} \\
& - [\theta^3 + \theta^2(3 + \phi_1) + \theta(3 + 3\phi_1 - \phi_2) + (1 + 3\phi_1 - 3\phi_2)]y_{t-3} \\
& + [\theta^4 + \theta^3(3 + \phi_1) + \theta^2(3 + 3\phi_1 - \phi_2) + \theta(1 + 3\phi_1 - 3\phi_2) \\
& + (\phi_1 - 3\phi_2)]y_{t-4} \\
& + \sum_{i=5}^{\infty} (-1)^i [\theta^i + \theta^{i-1}(\phi_1 + 3) + \theta^{i-2}(3 + 3\phi_1 - \phi_2) + \theta^{i-3}(1 + 3\phi_1 - 3\phi_2) \\
& + \theta^{i-4}(\phi_1 - 3\phi_2) - \theta^{i-5}\phi_2]y_{t-i}
\end{aligned} \tag{2.78}$$

$$\varepsilon_t = -\frac{c}{(1+\theta)} + \sum_{i=0}^{\infty} \pi_i y_{t-i} \quad (2.79)$$

where

$$\pi_i = \begin{cases} 1, & i=0 \\ -(\theta + \phi_1 + 3), & i=1 \\ \theta^2 + \theta(3 + \phi_1) + (3 + 3\phi_1 - \phi_2), & i=2 \\ -[\theta^3 + \theta^2(3 + \phi_1) + \theta(3 + 3\phi_1 - \phi_2) + (1 + 3\phi_1 - 3\phi_2)], & i=3 \\ \theta^4 + \theta^3(3 + \phi_1) + \theta^2(3 + 3\phi_1 - \phi_2) \\ \quad + \theta(1 + 3\phi_1 - 3\phi_2) + (\phi_1 - 3\phi_2), & i=4 \\ (-1)^i [\theta^i + \theta^{i-1}(\phi_1 + 3) + \theta^{i-2}(3 + 3\phi_1 - \phi_2) \\ \quad + \theta^{i-3}(1 + 3\phi_1 - 3\phi_2) + \theta^{i-4}(\phi_1 - 3\phi_2) - \theta^{i-5}\phi_2], & i=5, 6, 7, \dots \end{cases} \quad (2.80)$$

Equation (2.79) holds if $\theta \neq -1$.

ARIMA (3, 0, 1)

$$y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \phi_3 y_{t-3} + \varepsilon_t + \theta \varepsilon_{t-1} \quad (2.81)$$

$$\varepsilon_t + \theta \varepsilon_{t-1} = -c + y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2} - \phi_3 y_{t-3}$$

$$(1 + \theta L)\varepsilon_t = -c + y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2} - \phi_3 y_{t-3}$$

$$\varepsilon_t = -(1 + \theta L)^{-1} c + (1 + \theta L)^{-1} (y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2} - \phi_3 y_{t-3})$$

$$\varepsilon_t = -(1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots)c + (1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots)(y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2} - \phi_3 y_{t-3})$$

$$\begin{aligned} \varepsilon_t = & -(1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots)c + (1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots)y_t \\ & - \phi_1 (1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots)y_{t-1} \\ & - \phi_2 (1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots)y_{t-2} - \phi_3 (1 - \theta L + \theta^2 L^2 - \theta^3 L^3 \\ & + \theta^4 L^4 - \dots)y_{t-3} \end{aligned}$$

$$\begin{aligned} \varepsilon_t = & -(1 - \theta + \theta^2 - \theta^3 + \theta^4 - \dots)c \\ & + (y_t - \theta y_{t-1} + \theta^2 y_{t-2} - \theta^3 y_{t-3} + \theta^4 y_{t-4} - \theta^5 y_{t-5} + \dots) \\ & - \phi_1 (y_{t-1} - \theta y_{t-2} + \theta^2 y_{t-3} - \theta^3 y_{t-4} + \theta^4 y_{t-5} - \dots) \\ & - \phi_2 (y_{t-2} - \theta y_{t-3} + \theta^2 y_{t-4} - \theta^3 y_{t-5} + \dots) - \phi_3 (y_{t-3} - \theta y_{t-4} + \theta^2 y_{t-5} \\ & - \dots) \end{aligned}$$



$$\begin{aligned}\varepsilon_t = & -(1 + \theta)^{-1}c + y_t - (\theta + \phi_1)y_{t-1} + (\theta^2 + \phi_1\theta - \phi_2)y_{t-2} \\ & - (\theta^3 + \phi_1\theta^2 - \phi_2\theta + \phi_3)y_{t-3} + (\theta^4 + \phi_1\theta^3 - \phi_2\theta^2 + \phi_3\theta)y_{t-4} \\ & - (\theta^5 + \phi_1\theta^4 - \phi_2\theta^3 + \phi_3\theta^2)y_{t-5} + \dots\end{aligned}$$

$$\begin{aligned}\varepsilon_t = & -\frac{c}{(1 + \theta)} + y_t - (\theta + \phi_1)y_{t-1} + (\theta^2 + \phi_1\theta - \phi_2)y_{t-2} \\ & + \sum_{i=3}^{\infty} (-1)^i [\theta^i + \phi_1\theta^{i-1} - \phi_2\theta^{i-2} + \phi_3\theta^{i-3}]y_{t-i}\end{aligned}\quad (2.82)$$

$$\varepsilon_t = \frac{c}{(1 + \theta)} + \sum_{i=0}^{\infty} \pi_i y_{t-i}\quad (2.83)$$

where

$$\pi_i = \begin{cases} 1, & i = 0 \\ -(\theta + \phi_1), & i = 1 \\ \theta^2 + \phi_1\theta - \phi_2, & i = 2 \\ (-1)^i [\theta^i + \phi_1\theta^{i-1} - \phi_2\theta^{i-2} + \phi_3\theta^{i-3}], & i = 3, 4, \dots \end{cases}\quad (2.84)$$

Equation (2.83) holds if $\theta \neq -1$.

ARIMA (3, 1, 1)

$$\Delta y_t = c + \phi_1 \Delta y_{t-1} + \phi_2 \Delta y_{t-2} + \phi_3 \Delta y_{t-3} + \varepsilon_t + \theta \varepsilon_{t-1}\quad (2.85)$$

$$y_t - y_{t-1} = c + \phi_1 (y_{t-1} - y_{t-2}) + \phi_2 (y_{t-2} - y_{t-3}) + \phi_3 (y_{t-3} - y_{t-4}) + \varepsilon_t + \theta \varepsilon_{t-1}$$

$$\varepsilon_t + \theta \varepsilon_{t-1} = y_t - (1 + \phi_1)y_{t-1} + (\phi_1 - \phi_2)y_{t-2} + (\phi_2 - \phi_3)y_{t-3} + \phi_3 y_{t-4} - c$$

$$(1 + \theta L)\varepsilon_t = y_t - (1 + \phi_1)y_{t-1} + (\phi_1 - \phi_2)y_{t-2} + (\phi_2 - \phi_3)y_{t-3} + \phi_3 y_{t-4} - c$$

$$\begin{aligned}\varepsilon_t = & -(1 + \theta L)^{-1}c \\ & + (1 + \theta L)^{-1}[y_t - (1 + \phi_1)y_{t-1} + (\phi_1 - \phi_2)y_{t-2} + (\phi_2 - \phi_3)y_{t-3} \\ & + \phi_3 y_{t-4}]\end{aligned}$$

$$\begin{aligned}
\varepsilon_t = & -(1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \theta^5 L^5 + \dots)c \\
& + (1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \theta^5 L^5 + \dots)y_t \\
& - (1 + \phi_1)[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \theta^5 L^5 + \dots]y_{t-1} \\
& + (\phi_1 - \phi_2)[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \theta^5 L^5 + \dots]y_{t-2} \\
& + (\phi_2 - \phi_3)[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \theta^5 L^5 + \dots]y_{t-3} \\
& + \phi_3[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \theta^5 L^5 + \dots]y_{t-4}
\end{aligned}$$

$$\begin{aligned}
\varepsilon_t = & -(1 - \theta + \theta^2 - \theta^3 + \theta^4 - \theta^5 + \dots)c \\
& + (y_t - \theta y_{t-1} + \theta^2 y_{t-2} - \theta^3 y_{t-3} + \theta^4 y_{t-4} - \theta^5 y_{t-5} \dots) \\
& - (1 + \phi_1)(y_{t-1} - \theta y_{t-2} + \theta^2 y_{t-3} - \theta^3 y_{t-4} + \theta^4 y_{t-5} - \dots) \\
& + (\phi_1 - \phi_2)(y_{t-2} - \theta y_{t-3} + \theta^2 y_{t-4} - \theta^3 y_{t-5} + \dots) \\
& + (\phi_2 - \phi_3)(y_{t-3} - \theta y_{t-4} + \theta^2 y_{t-5} - \dots) \\
& + \phi_3(y_{t-4} - \theta y_{t-5} + \theta^2 y_{t-6} - \dots)
\end{aligned}$$

$$\begin{aligned}
\varepsilon_t = & -(1 + \theta)^{-1}c + y_t - (\theta + \phi_1 + 1)y_{t-1} + (\theta^2 + \phi_1\theta + \theta + \phi_1 - \phi_2)y_{t-2} \\
& - (\theta^3 + \phi_1\theta^2 + \theta^2 + \phi_1\theta - \phi_2\theta - \phi_2 + \phi_3)y_{t-3} \\
& + (\theta^4 + \phi_1\theta^3 + \theta^3 + \phi_1\theta^2 - \phi_2\theta^2 - \phi_2\theta + \phi_3\theta + \phi_3)y_{t-4} \\
& - (\theta^5 + \phi_1\theta^4 + \theta^4 + \phi_1\theta^3 - \phi_2\theta^3 - \phi_2\theta^2 + \phi_3\theta^2 + \phi_3\theta)y_{t-5} + \dots
\end{aligned}$$

$$\begin{aligned}
\varepsilon_t = & -(1 + \theta)^{-1}c + y_t - (\theta + \phi_1 + 1)y_{t-1} + (\theta^2 + \theta(1 + \phi_1) + \phi_1 - \phi_2)y_{t-2} \\
& - [\theta^3 + \theta^2(1 + \phi_1) + \theta(\phi_1 - \phi_2) - \phi_2 + \phi_3]y_{t-3} \\
& + [\theta^4 + \theta^3(1 + \phi_1) + \theta^2(\phi_1 - \phi_2) - \theta(\phi_2 - \phi_3) + \phi_3]y_{t-4} \\
& - [\theta^5 + \theta^4(1 + \phi_1) + \theta^3(\phi_1 - \phi_2) - \theta^2(\phi_2 - \phi_3) + \theta\phi_3]y_{t-5} + \dots
\end{aligned}$$

$$\begin{aligned}
\varepsilon_t = & -\frac{c}{(1 + \theta)} + y_t - (\theta + \phi_1 + 1)y_{t-1} + [\theta^2 + \theta(1 + \phi_1) + \phi_1 - \phi_2]y_{t-2} \\
& - [\theta^3 + \theta^2(1 + \phi_1) + \theta(\phi_1 - \phi_2) - \phi_2 + \phi_3]y_{t-3} \\
& + \sum_{i=4}^{\infty} (-1)^i [\theta^i + \theta^{i-1}(1 + \phi_1) + \theta^{i-2}(\phi_1 - \phi_2) - \theta^{i-3}(\phi_2 - \phi_3) \\
& + \theta^{i-4}\phi_3]y_{t-i}
\end{aligned} \tag{2.86}$$

$$\varepsilon_t = -\frac{c}{(1 + \theta)} + \sum_{i=0}^{\infty} \pi_i y_{t-i} \tag{2.87}$$

where

$$\pi_l = \begin{cases} 1, & i = 0 \\ -(\theta + \phi_1 + 1), & i = 1 \\ \theta^2 + \theta(1 + \phi_1) + \phi_1 - \phi_2, & i = 2 \\ -[\theta^3 + \theta^2(1 + \phi_1) + \theta(\phi_1 - \phi_2) - (\phi_2 - \phi_3)], & i = 3 \\ (-1)^i [\theta^i + \theta^{i-1}(1 + \phi_1) + \theta^{i-2}(\phi_1 - \phi_2) - \theta^{i-3}(\phi_2 - \phi_3) + \theta^{i-4}\phi_3], & i = 4, 5, 6, \dots \end{cases} \quad (2.88)$$

Equation (2.87) holds if $\theta \neq -1$.

ARIMA (3, 2, 1)

$$\Delta^2 y_t = c + \phi_1 \Delta^2 y_{t-1} + \phi_2 \Delta^2 y_{t-2} + \phi_3 \Delta^2 y_{t-3} + \varepsilon_t + \theta \varepsilon_{t-1} \quad (2.89)$$

$$\Delta(y_t - y_{t-1}) = c + \phi_1 \Delta(y_{t-1} - y_{t-2}) + \phi_2 \Delta(y_{t-2} - y_{t-3}) + \phi_3 \Delta(y_{t-3} - y_{t-4}) + \varepsilon_t + \theta \varepsilon_{t-1}$$

$$\begin{aligned} y_t - y_{t-1} - y_{t-1} + y_{t-2} \\ = c + \phi_1 (y_{t-1} - y_{t-2} - y_{t-2} + y_{t-3}) + \phi_2 (y_{t-2} - y_{t-3} - y_{t-3} + y_{t-4}) \\ + \phi_3 (y_{t-3} - y_{t-4} - y_{t-4} + y_{t-5}) + \varepsilon_t + \theta \varepsilon_{t-1} \end{aligned}$$

$$\begin{aligned} y_t - 2y_{t-1} + y_{t-2} \\ = c + \phi_1 (y_{t-1} - 2y_{t-2} + y_{t-3}) + \phi_2 (y_{t-2} - 2y_{t-3} + y_{t-4}) + \phi_3 (y_{t-3} - 2y_{t-4} + y_{t-5}) + \varepsilon_t + \theta \varepsilon_{t-1} \end{aligned}$$

$$\begin{aligned} \varepsilon_t + \theta \varepsilon_{t-1} = y_t - (2 + \phi_1)y_{t-1} + (1 + 2\phi_1 - \phi_2)y_{t-2} - (\phi_1 - 2\phi_2 + \phi_3)y_{t-3} \\ - (\phi_2 - 2\phi_3)y_{t-4} - \phi_3 y_{t-5} - c \end{aligned}$$

$$\begin{aligned} (1 + \theta L)\varepsilon_t = y_t - (2 + \phi_1)y_{t-1} + (1 + 2\phi_1 - \phi_2)y_{t-2} - (\phi_1 - 2\phi_2 + \phi_3)y_{t-3} \\ - (\phi_2 - 2\phi_3)y_{t-4} - \phi_3 y_{t-5} - c \end{aligned}$$

$$\begin{aligned} \varepsilon_t = -(1 + \theta L)^{-1}c + (1 + \theta L)^{-1}y_t - (2 + \phi_1)(1 + \theta L)^{-1}y_{t-1} \\ + (1 + 2\phi_1 - \phi_2)(1 + \theta L)^{-1}y_{t-2} - (\phi_1 - 2\phi_2 + \phi_3)(1 + \theta L)^{-1}y_{t-3} \\ - (\phi_2 - 2\phi_3)(1 + \theta L)^{-1}y_{t-4} - \phi_3(1 + \theta L)^{-1}y_{t-5} \end{aligned}$$

$$\begin{aligned}\varepsilon_t = & -(1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots)c + (1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots)y_t \\ & - (2 + \phi_1)[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots]y_{t-1} \\ & + (1 + 2\phi_1 - \phi_2)[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots]y_{t-2} \\ & - (\phi_1 - 2\phi_2 + \phi_3)[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots]y_{t-3} \\ & - (\phi_2 - 2\phi_3)[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots]y_{t-4} \\ & - \phi_3[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots]y_{t-5}\end{aligned}$$

$$\begin{aligned}\varepsilon_t = & -(1 - \theta + \theta^2 - \theta^3 + \theta^4 - \dots)c + (y_t - \theta y_{t-1} + \theta^2 y_{t-2} - \theta^3 y_{t-3} + \theta^4 y_{t-4} - \dots) \\ & - (2 + \phi_1)(y_{t-1} - \theta y_{t-2} + \theta^2 y_{t-3} - \theta^3 y_{t-4} - \dots) \\ & + (1 + 2\phi_1 - \phi_2)(y_{t-2} - \theta y_{t-3} + \theta^2 y_{t-4} - \theta^3 y_{t-5} - \dots) \\ & - (\phi_1 - 2\phi_2 + \phi_3)(y_{t-3} - \theta y_{t-4} + \theta^2 y_{t-5} - \theta^3 y_{t-6} - \dots) \\ & - (\phi_2 - 2\phi_3)(y_{t-4} - \theta y_{t-5} + \theta^2 y_{t-6} - \theta^3 y_{t-7} - \dots) \\ & - \phi_3(y_{t-5} - \theta y_{t-6} + \theta^2 y_{t-7} - \theta^3 y_{t-8} - \dots)\end{aligned}$$

$$\begin{aligned}\varepsilon_t = & -(1 + \theta)^{-1}c + y_t - (\theta + \phi_1 + 2)y_{t-1} + [\theta^2 + \theta(2 + \phi_1) + (1 + 2\phi_1 - \phi_2)]y_{t-2} \\ & - [\theta^3 + \theta^2(2 + \phi_1) + \theta(1 + 2\phi_1 - \phi_2) + (\phi_1 - 2\phi_2 + \phi_3)]y_{t-3} \\ & + [\theta^4 + \theta^3(2 + \phi_1) + \theta^2(1 + 2\phi_1 - \phi_2) + \theta(\phi_1 - 2\phi_2 + \phi_3) \\ & - (\phi_2 - 2\phi_3)]y_{t-4} \\ & - [\theta^5 + \theta^4(2 + \phi_1) + \theta^3(1 + 2\phi_1 - \phi_2) + \theta^2(\phi_1 - 2\phi_2 + \phi_3) \\ & - \theta(\phi_2 - 2\phi_3) + \phi_3]y_{t-5} \\ & + [\theta^6 + \theta^5(2 + \phi_1) + \theta^4(1 + 2\phi_1 - \phi_2) + \theta^3(\phi_1 - 2\phi_2 + \phi_3) \\ & - \theta^2(\phi_2 - 2\phi_3) + \theta\phi_3]y_{t-6} + \dots\end{aligned}\quad (2.90)$$

$$\begin{aligned}\varepsilon_t = & -\frac{c}{(1 + \theta)} + y_t - (\theta + \phi_1 + 2)y_{t-1} + [\theta^2 + \theta(2 + \phi_1) + (1 + 2\phi_1 - \phi_2)]y_{t-2} \\ & - [\theta^3 + \theta^2(2 + \phi_1) + \theta(1 + 2\phi_1 - \phi_2) + (\phi_1 - 2\phi_2 + \phi_3)]y_{t-3} \\ & + [\theta^4 + \theta^3(2 + \phi_1) + \theta^2(1 + 2\phi_1 - \phi_2) + \theta(\phi_1 - 2\phi_2 + \phi_3) \\ & - (\phi_2 - 2\phi_3)]y_{t-4} \\ & + \sum_{i=5}^{\infty} (-1)^i [\theta^i + \theta^{i-1}(\phi_1 + 2) + \theta^{i-2}(1 + 2\phi_1 - \phi_2) \\ & + \theta^{i-3}(\phi_1 - 2\phi_2 + \phi_3) - \theta^{i-4}(\phi_2 - 2\phi_3) + \theta^{i-5}\phi_3]y_{t-i}\end{aligned}\quad (2.91)$$

$$\varepsilon_t = -\frac{c}{(1 + \theta)} + \sum_{i=0}^{\infty} \pi_i y_{t-i}\quad (2.92)$$

where

$$\pi_i = \begin{cases} 1, & i = 0 \\ -(\theta + \phi_1 + 2), & i = 1 \\ \theta^2 + \theta(2 + \phi_1) + (1 + 2\phi_1 - \phi_2), & i = 2 \\ -[\theta^3 + \theta^2(2 + \phi_1) + \theta(1 + 2\phi_1 - \phi_2) + (\phi_1 - 2\phi_2 + \phi_3)], & i = 3 \\ \theta^4 + \theta^3(2 + \phi_1) + \theta^2(1 + 2\phi_1 - \phi_2) & \\ \quad + \theta(\phi_1 - 2\phi_2 + \phi_3) - (\phi_2 - 2\phi_3), & i = 4 \\ (-1)^i [\theta^i + \theta^{i-1}(\phi_1 + 2) + \theta^{i-2}(1 + 2\phi_1 - \phi_2) + \theta^{i-3}(\phi_1 - 2\phi_2 + \phi_3) & \\ \quad - \theta^{i-4}(\phi_2 - 2\phi_3) + \theta^{i-5}\phi_3], & i = 4, 5, 6, \dots \end{cases} \quad (2.93)$$

Equation (2.92) holds if $\theta \neq -1$.

ARIMA (3, 3, 1)

$$\Delta^3 y_t = c + \phi_1 \Delta^3 y_{t-1} + \phi_2 \Delta^3 y_{t-2} + \phi_3 \Delta^3 y_{t-3} + \varepsilon_t + \theta \varepsilon_{t-1} \quad (2.94)$$

$$\begin{aligned} & \Delta(y_t - 2y_{t-1} + y_{t-2}) \\ &= c + \phi_1 \Delta(y_{t-1} - 2y_{t-2} + y_{t-3}) + \phi_2 \Delta(y_{t-2} - 2y_{t-3} + y_{t-4}) \\ & \quad + \phi_3 \Delta(y_{t-3} - 2y_{t-4} + y_{t-5}) + \varepsilon_t + \theta \varepsilon_{t-1} \end{aligned}$$

$$\begin{aligned} & y_t - y_{t-1} - 2(y_{t-1} - y_{t-2}) + (y_{t-2} - y_{t-3}) \\ &= c + \phi_1 [(y_{t-1} - y_{t-2}) - 2(y_{t-2} - y_{t-3}) + (y_{t-3} - y_{t-4})] \\ & \quad + \phi_2 [(y_{t-2} - y_{t-3}) - 2(y_{t-3} - y_{t-4}) + (y_{t-4} - y_{t-5})] + \phi_3 [(y_{t-3} - y_{t-4}) \\ & \quad - 2(y_{t-4} - y_{t-5}) + (y_{t-5} - y_{t-6})] + \varepsilon_t + \theta \varepsilon_{t-1} \end{aligned}$$

$$\begin{aligned} & y_t - 3y_{t-1} + 3y_{t-2} - y_{t-3} \\ &= c + \phi_1 (y_{t-1} - 3y_{t-2} + 3y_{t-3} - y_{t-4}) + \phi_2 (y_{t-2} - 3y_{t-3} + 3y_{t-4} - y_{t-5}) \\ & \quad + \phi_3 (y_{t-3} - 3y_{t-4} + 3y_{t-5} - y_{t-6}) + \varepsilon_t + \theta \varepsilon_{t-1} \end{aligned}$$

$$\begin{aligned} \varepsilon_t + \theta \varepsilon_{t-1} &= y_t - (3 + \phi_1)y_{t-1} + (3 + 3\phi_1 - \phi_2)y_{t-2} - (1 + 3\phi_1 - 3\phi_2 + \phi_3)y_{t-3} \\ & \quad + (\phi_1 - 3\phi_2 + 3\phi_3)y_{t-4} + (\phi_2 - 3\phi_3)y_{t-5} + \phi_3 y_{t-6} - c \end{aligned}$$

$$\begin{aligned} (1 + \theta L)\varepsilon_t &= y_t - (3 + \phi_1)y_{t-1} + (3 + 3\phi_1 - \phi_2)y_{t-2} - (1 + 3\phi_1 - 3\phi_2 + \phi_3)y_{t-3} \\ & \quad + (\phi_1 - 3\phi_2 + 3\phi_3)y_{t-4} + (\phi_2 - 3\phi_3)y_{t-5} + \phi_3 y_{t-6} - c \end{aligned}$$

$$\varepsilon_t = -(1 + \theta L)^{-1} c$$

$$+ (1 + \theta L)^{-1} [y_t - (3 + \phi_1)y_{t-1} + (3 + 3\phi_1 - \phi_2)y_{t-2} - (1 + 3\phi_1 - 3\phi_2 + \phi_3)y_{t-3} + (\phi_1 - 3\phi_2 + 3\phi_3)y_{t-4} + (\phi_2 - 3\phi_3)y_{t-5} + \phi_3 y_{t-6}]$$

$$\begin{aligned} \varepsilon_t = & -(1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots)c + (1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots)y_t \\ & - (3 + \phi_1)[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots]y_{t-1} \\ & + (3 + 3\phi_1 - \phi_2)[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots]y_{t-2} \\ & - (1 + 3\phi_1 - 3\phi_2 + \phi_3)[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots]y_{t-3} \\ & + (\phi_1 - 3\phi_2 + 3\phi_3)[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots]y_{t-4} \\ & + (\phi_2 - 3\phi_3)[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots]y_{t-5} \\ & + \phi_3[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots]y_{t-6} \end{aligned}$$

$$\begin{aligned} \varepsilon_t = & -(1 - \theta + \theta^2 - \theta^3 + \theta^4 - \dots)c \\ & + (y_t - \theta y_{t-1} + \theta^2 y_{t-2} - \theta^3 y_{t-3} + \theta^4 y_{t-4} - \theta^5 y_{t-5} + \theta^6 y_{t-6} - \dots) \\ & - (3 + \phi_1)(y_{t-1} - \theta y_{t-2} + \theta^2 y_{t-3} - \theta^3 y_{t-4} + \theta^4 y_{t-5} - \theta^5 y_{t-6} + \dots) \\ & + (3 + 3\phi_1 - \phi_2)(y_{t-2} - \theta y_{t-3} + \theta^2 y_{t-4} - \theta^3 y_{t-5} + \theta^4 y_{t-6} - \dots) \\ & - (1 + 3\phi_1 - 3\phi_2 + \phi_3)(y_{t-3} - \theta y_{t-4} + \theta^2 y_{t-5} - \theta^3 y_{t-6} + \theta^4 y_{t-7} - \dots) \\ & + (\phi_1 - 3\phi_2 + 3\phi_3)(y_{t-4} - \theta y_{t-5} + \theta^2 y_{t-6} - \theta^3 y_{t-7} + \dots) \\ & + (\phi_2 - 3\phi_3)(y_{t-5} - \theta y_{t-6} + \theta^2 y_{t-7} - \theta^3 y_{t-8} + \dots) \\ & + \phi_3(y_{t-6} - \theta y_{t-7} + \theta^2 y_{t-8} - \theta^3 y_{t-9} + \dots) \end{aligned}$$

$$\begin{aligned} \varepsilon_t = & -(1 + \theta)^{-1} c + y_t - (\theta + \phi_1 + 3)y_{t-1} + [\theta^2 + \theta(3 + \phi_1) + (3 + 3\phi_1 - \phi_2)]y_{t-2} \\ & - [\theta^3 + \theta^2(3 + \phi_1) + \theta(3 + 3\phi_1 - \phi_2) + (1 + 3\phi_1 - 3\phi_2 + \phi_3)]y_{t-3} \\ & + [\theta^4 + \theta^3(3 + \phi_1) + \theta^2(3 + 3\phi_1 - \phi_2) + \theta(1 + 3\phi_1 - 3\phi_2 + \phi_3) \\ & + (\phi_1 - 3\phi_2 + 3\phi_3)]y_{t-4} \\ & - [\theta^5 + \theta^4(3 + \phi_1) + \theta^3(3 + 3\phi_1 - \phi_2) + \theta^2(1 + 3\phi_1 - 3\phi_2 + \phi_3) \\ & + \theta(\phi_1 - 3\phi_2 + 3\phi_3) - (\phi_2 - 3\phi_3)]y_{t-5} \\ & + [\theta^6 + \theta^5(3 + \phi_1) + \theta^4(3 + 3\phi_1 - \phi_2) + \theta^3(1 + 3\phi_1 - 3\phi_2 + \phi_3) \\ & + \theta^2(\phi_1 - 3\phi_2 + 3\phi_3) - \theta(\phi_2 - 3\phi_3) + \phi_3]y_{t-6} \\ & - [\theta^7 + \theta^6(3 + \phi_1) + \theta^5(3 + 3\phi_1 - \phi_2) + \theta^4(1 + 3\phi_1 - 3\phi_2 + \phi_3) \\ & + \theta^3(\phi_1 - 3\phi_2 + 3\phi_3) - \theta^2(\phi_2 - 3\phi_3) + \theta\phi_3]y_{t-7} + \dots \end{aligned} \quad (2.95)$$

$$\begin{aligned}
\varepsilon_t = & -\frac{c}{(1+\theta)} + y_t - (\theta + \phi_1 + 3)y_{t-1} + [\theta^2 + \theta(3 + \phi_1) + (3 + 3\phi_1 - \phi_2)]y_{t-2} \\
& - [\theta^3 + \theta^2(3 + \phi_1) + \theta(3 + 3\phi_1 - \phi_2) + (1 + 3\phi_1 - 3\phi_2 + \phi_3)]y_{t-3} \\
& + [\theta^4 + \theta^3(3 + \phi_1) + \theta^2(3 + 3\phi_1 - \phi_2) + \theta(1 + 3\phi_1 - 3\phi_2 + \phi_3) \\
& + (\phi_1 - 3\phi_2 + 3\phi_3)]y_{t-4} \\
& - [\theta^5 + \theta^4(3 + \phi_1) + \theta^3(3 + 3\phi_1 - \phi_2) + \theta^2(1 + 3\phi_1 - 3\phi_2 + \phi_3) \\
& + \theta(\phi_1 - 3\phi_2 + 3\phi_3) + (\phi_2 - 3\phi_3)]y_{t-5} \\
& + \sum_{i=6}^{\infty} (-1)^i [\theta^i + \theta^{i-1}(\phi_1 + 3) + \theta^{i-2}(3 + 3\phi_1 - \phi_2) \\
& + \theta^{i-3}(1 + 3\phi_1 - 3\phi_2 + \phi_3) + \theta^{i-4}(\phi_1 - 3\phi_2 + 3\phi_3) - \theta^{i-5}(\phi_2 - 3\phi_3) \\
& + \theta^{i-6}\phi_3]y_{t-i}
\end{aligned} \tag{2.96}$$

$$\varepsilon_t = -\frac{c}{(1+\theta)} + \sum_{i=0}^{\infty} \pi_i y_{t-i} \tag{2.97}$$

where

$$\pi_i = \begin{cases} 1, & i = 0 \\ -(\theta + \phi_1 + 3), & i = 1 \\ \theta^2 + \theta(3 + \phi_1) + (3 + 3\phi_1 - \phi_2), & i = 2 \\ -[\theta^3 + \theta^2(3 + \phi_1) + \theta(3 + 3\phi_1 - \phi_2) + (1 + 3\phi_1 - 3\phi_2)], & i = 3 \\ \theta^4 + \theta^3(3 + \phi_1) + \theta^2(3 + 3\phi_1 - \phi_2) + \theta(1 + 3\phi_1 - 3\phi_2 + \phi_3) \\ \quad + (\phi_1 - 3\phi_2 + 3\phi_3), & i = 4 \\ -[\theta^5 + \theta^4(3 + \phi_1) + \theta^3(3 + 3\phi_1 - \phi_2) + \theta^2(1 + 3\phi_1 - 3\phi_2 + \phi_3) \\ \quad + \theta(\phi_1 - 3\phi_2 + 3\phi_3) + (\phi_2 - 3\phi_3)], & i = 5 \\ (-1)^i [\theta^i + \theta^{i-1}(\phi_1 + 3) + \theta^{i-2}(3 + 3\phi_1 - \phi_2) \\ \quad + \theta^{i-3}(1 + 3\phi_1 - 3\phi_2 + \phi_3) + \theta^{i-4}(\phi_1 - 3\phi_2 + 3\phi_3) \\ \quad - \theta^{i-5}(\phi_2 - 3\phi_3) + \theta^{i-6}\phi_3], & i = 6, 7, 8, \dots \end{cases} \tag{2.98}$$

Equation (2.97) holds if $\theta \neq -1$.

ARIMA (4, 0, 1)

$$y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \phi_3 y_{t-3} + \phi_4 y_{t-4} + \varepsilon_t + \theta \varepsilon_{t-1} \tag{2.99}$$

$$(1 + \theta L)\varepsilon_t = -c + y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2} - \phi_3 y_{t-3} - \phi_4 y_{t-4}$$

$$\varepsilon_t = -(1 + \theta L)^{-1} c + (1 + \theta L)^{-1} (y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2} - \phi_3 y_{t-3} - \phi_4 y_{t-4})$$

$$\begin{aligned}\varepsilon_t = & -(1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots)c \\ & + [1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots][y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2} - \phi_3 y_{t-3} \\ & - \phi_4 y_{t-4}]\end{aligned}$$

$$\begin{aligned}\varepsilon_t = & -(1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \theta^5 L^5 + \dots)c \\ & + (1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \theta^5 L^5 + \dots)y_t \\ & - \phi_1 (1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \theta^5 L^5 + \dots)y_{t-1} \\ & - \phi_2 (1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \theta^5 L^5 + \dots)y_{t-2} - \phi_3 (1 - \theta L + \theta^2 L^2 \\ & - \theta^3 L^3 + \theta^4 L^4 - \theta^5 L^5 + \dots)y_{t-3} - \phi_4 (1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 \\ & - \theta^5 L^5 + \dots)y_{t-4}\end{aligned}$$

$$\begin{aligned}\varepsilon_t = & -(1 - \theta + \theta^2 - \theta^3 + \theta^4 - \theta^5 + \dots)c \\ & + (y_t - \theta y_{t-1} + \theta^2 y_{t-2} - \theta^3 y_{t-3} + \theta^4 y_{t-4} - \theta^5 y_{t-5} + \dots) \\ & - \phi_1 (y_{t-1} - \theta y_{t-2} + \theta^2 y_{t-3} - \theta^3 y_{t-4} + \theta^4 y_{t-5} - \dots) \\ & - \phi_2 (y_{t-2} - \theta y_{t-3} + \theta^2 y_{t-4} - \theta^3 y_{t-5} + \dots) - \phi_3 (y_{t-3} - \theta y_{t-4} + \theta^2 y_{t-5} \\ & - \dots) - \phi_4 (y_{t-4} - \theta y_{t-5} + \theta^2 y_{t-6} - \dots)\end{aligned}$$

$$\begin{aligned}\varepsilon_t = & -(1 + \theta)^{-1}c + y_t - (\theta + \phi_1)y_{t-1} + (\theta^2 + \phi_1\theta - \phi_2)y_{t-2} \\ & - (\theta^3 + \phi_1\theta^2 - \phi_2\theta + \phi_3)y_{t-3} + (\theta^4 + \phi_1\theta^3 - \phi_2\theta^2 + \phi_3\theta - \phi_4)y_{t-4} \\ & - (\theta^5 + \phi_1\theta^4 - \phi_2\theta^3 + \phi_3\theta^2 - \phi_4\theta)y_{t-5} + \dots\end{aligned}$$

$$\begin{aligned}\varepsilon_t = & -\frac{c}{(1 + \theta)} + y_t - (\theta + \phi_1)y_{t-1} + (\theta^2 + \phi_1\theta - \phi_2)y_{t-2} \\ & - (\theta^3 + \phi_1\theta^2 - \phi_2\theta + \phi_3)y_{t-3} \\ & + \sum_{i=4}^{\infty} (-1)^i [\theta^i + \phi_1\theta^{i-1} - \phi_2\theta^{i-2} + \phi_3\theta^{i-3} - \phi_4\theta^{i-4}]y_{t-i} \quad (2.100)\end{aligned}$$

$$\varepsilon_t = -\frac{c}{(1 + \theta)} + \sum_{i=0}^{\infty} \pi_i y_{t-i} \quad (2.101)$$

where

$$\pi_i = \begin{cases} 1, & i = 0 \\ -(\theta + \phi_1), & i = 1 \\ \theta^2 + \phi_1\theta - \phi_2, & i = 2 \\ -(\theta^3 + \phi_1\theta^2 - \phi_2\theta + \phi_3), & i = 3 \\ (-1)^i [\theta^i + \phi_1\theta^{i-1} - \phi_2\theta^{i-2} + \phi_3\theta^{i-3} - \phi_4\theta^{i-4}], & i = 4, 5, \dots \end{cases} \quad (2.102)$$

Equation(2.101) holds if $\theta \neq -1$.

ARIMA (4, 1, 1)

$$\Delta y_t = c + \phi_1 \Delta y_{t-1} + \phi_2 \Delta y_{t-2} + \phi_3 \Delta y_{t-3} + \phi_4 \Delta y_{t-4} + \varepsilon_t + \theta \varepsilon_{t-1} \quad (2.103)$$

$$y_t - y_{t-1} = c + \phi_1 (y_{t-1} - y_{t-2}) + \phi_2 (y_{t-2} - y_{t-3}) + \phi_3 (y_{t-3} - y_{t-4}) \\ + \phi_4 (y_{t-4} - y_{t-5}) + \varepsilon_t + \theta \varepsilon_{t-1}$$

$$\varepsilon_t + \theta \varepsilon_{t-1} = y_t - (1 + \phi_1)y_{t-1} + (\phi_1 - \phi_2)y_{t-2} + (\phi_2 - \phi_3)y_{t-3} + (\phi_3 - \phi_4)y_{t-4} \\ + \phi_4 y_{t-5} - c$$

$$(1 + \theta L)\varepsilon_t = y_t - (1 + \phi_1)y_{t-1} + (\phi_1 - \phi_2)y_{t-2} + (\phi_2 - \phi_3)y_{t-3} + (\phi_3 - \phi_4)y_{t-4} \\ + \phi_4 y_{t-5} - c$$

$$\varepsilon_t = -(1 + \theta L)^{-1}c \\ + (1 + \theta L)^{-1}[y_t - (1 + \phi_1)y_{t-1} + (\phi_1 - \phi_2)y_{t-2} + (\phi_2 - \phi_3)y_{t-3} \\ + (\phi_3 - \phi_4)y_{t-4} + \phi_4 y_{t-5}]$$

$$\varepsilon_t = -(1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \theta^5 L^5 + \dots)c \\ + (1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \theta^5 L^5 + \dots)y_t \\ - (1 + \phi_1)[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \theta^5 L^5 + \dots]y_{t-1} \\ + (\phi_1 - \phi_2)[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \theta^5 L^5 + \dots]y_{t-2} \\ + (\phi_2 - \phi_3)[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \theta^5 L^5 + \dots]y_{t-3} \\ + (\phi_3 - \phi_4)[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \theta^5 L^5 + \dots]y_{t-4} \\ + \phi_4[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \theta^5 L^5 + \dots]y_{t-5}$$

$$\varepsilon_t = -(1 - \theta + \theta^2 - \theta^3 + \theta^4 - \theta^5 + \dots)c \\ + (y_t - \theta y_{t-1} + \theta^2 y_{t-2} - \theta^3 y_{t-3} + \theta^4 y_{t-4} - \theta^5 y_{t-5} + \theta^6 y_{t-6} - \dots) \\ - (1 + \phi_1)(y_{t-1} - \theta y_{t-2} + \theta^2 y_{t-3} - \theta^3 y_{t-4} + \theta^4 y_{t-5} - \theta^5 y_{t-6} + \dots) \\ + (\phi_1 - \phi_2)(y_{t-2} - \theta y_{t-3} + \theta^2 y_{t-4} - \theta^3 y_{t-5} + \theta^4 y_{t-6} + \dots) \\ + (\phi_2 - \phi_3)(y_{t-3} - \theta y_{t-4} + \theta^2 y_{t-5} - \theta^3 y_{t-6} + \dots) \\ + (\phi_3 - \phi_4)(y_{t-4} - \theta y_{t-5} + \theta^2 y_{t-6} - \dots) \\ + \phi_4(y_{t-5} - \theta y_{t-6} + \theta^2 y_{t-7} - \dots)$$



$$\begin{aligned}
\varepsilon_t = & -(1+\theta)^{-1}c + y_t - (\theta + \phi_1 + 1)y_{t-1} + (\theta^2 + \phi_1\theta + \theta + \phi_1 - \phi_2)y_{t-2} \\
& - (\theta^3 + \phi_1\theta^2 + \theta^2 + \phi_1\theta - \phi_2\theta - \phi_2 + \phi_3)y_{t-3} \\
& + (\theta^4 + \phi_1\theta^3 + \theta^3 + \phi_1\theta^2 - \phi_2\theta^2 - \phi_2\theta + \phi_3\theta + \phi_3 - \phi_4)y_{t-4} \\
& - (\theta^5 + \phi_1\theta^4 + \theta^4 + \phi_1\theta^3 - \phi_2\theta^3 - \phi_2\theta^2 + \phi_3\theta^2 + \phi_3\theta - \phi_4\theta \\
& - \phi_4)y_{t-5} \\
& + (\theta^6 + \phi_1\theta^5 + \theta^5 + \phi_1\theta^4 - \phi_2\theta^4 - \phi_2\theta^3 + \phi_3\theta^3 + \phi_3\theta^2 - \phi_4\theta^2 \\
& - \phi_4\theta)y_{t-6} + \dots
\end{aligned}$$

$$\begin{aligned}
\varepsilon_t = & -(1+\theta)^{-1}c + y_t - (\theta + \phi_1 + 1)y_{t-1} + (\theta^2 + \theta(1 + \phi_1) + \phi_1 - \phi_2)y_{t-2} \\
& - [\theta^3 + \theta^2(1 + \phi_1) + \theta(\phi_1 - \phi_2) - \phi_2 + \phi_3]y_{t-3} \\
& + [\theta^4 + \theta^3(1 + \phi_1) + \theta^2(\phi_1 - \phi_2) - \theta(\phi_2 - \phi_3) + \phi_3 - \phi_4]y_{t-4} \\
& - [\theta^5 + \theta^4(1 + \phi_1) + \theta^3(\phi_1 - \phi_2) - \theta^2(\phi_2 - \phi_3) + \theta(\phi_3 - \phi_4) \\
& - \phi_4]y_{t-5} \\
& + [\theta^6 + \theta^5(1 + \phi_1) + \theta^4(\phi_1 - \phi_2) - \theta^3(\phi_2 - \phi_3) + \theta^2(\phi_3 - \phi_4) \\
& - \theta\phi_4]y_{t-6} + \dots
\end{aligned}$$

$$\begin{aligned}
\varepsilon_t = & -\frac{c}{(1+\theta)} + y_t - (\theta + \phi_1 + 1)y_{t-1} + [\theta^2 + \theta(1 + \phi_1) + \phi_1 - \phi_2]y_{t-2} \\
& - [\theta^3 + \theta^2(1 + \phi_1) + \theta(\phi_1 - \phi_2) - \phi_2 + \phi_3]y_{t-3} \\
& + [\theta^4 + \theta^3(1 + \phi_1) + \theta^2(\phi_1 - \phi_2) - \theta(\phi_2 - \phi_3) + \phi_3 - \phi_4]y_{t-4} \\
& + \sum_{i=5}^{\infty} (-1)^i [\theta^i + \theta^{i-1}(1 + \phi_1) + \theta^{i-2}(\phi_1 - \phi_2) - \theta^{i-3}(\phi_2 - \phi_3) \\
& + \theta^{i-4}(\phi_3 - \phi_4) - \theta^{i-5}\phi_4]y_{t-i}
\end{aligned} \tag{2.104}$$

$$\varepsilon_t = -\frac{c}{(1+\theta)} + \sum_{i=0}^{\infty} \pi_i y_{t-i} \tag{2.105}$$

where

$$\pi_i = \begin{cases} 1, & i = 0 \\ -(\theta + \phi_1 + 1), & i = 1 \\ \theta^2 + \phi_1\theta + \theta + \phi_1 - \phi_2, & i = 2 \\ -[\theta^3 + \theta^2(1 + \phi_1) + \theta(\phi_1 - \phi_2) - \phi_2 + \phi_3], & i = 3 \\ \theta^4 + \theta^3(1 + \phi_1) + \theta^2(\phi_1 - \phi_2) - \theta(\phi_2 - \phi_3) + \phi_3 - \phi_4, & i = 4 \\ (-1)^i [\theta^i + \theta^{i-1}(1 + \phi_1) + \theta^{i-2}(\phi_1 - \phi_2) - \theta^{i-3}(\phi_2 - \phi_3) \\ + \theta^{i-4}(\phi_3 - \phi_4) - \theta^{i-5}\phi_4], & i = 5, 6, \dots \end{cases} \tag{2.106}$$



Equation (2.105) holds if $\theta \neq -1$.

ARIMA (4, 2, 1)

$$\Delta^2 y_t = c + \phi_1 \Delta^2 y_{t-1} + \phi_2 \Delta^2 y_{t-2} + \phi_3 \Delta^2 y_{t-3} + \phi_4 \Delta^2 y_{t-4} + \varepsilon_t + \theta \varepsilon_{t-1} \quad (2.107)$$

$$\begin{aligned} \Delta(y_t - y_{t-1}) &= c + \phi_1 \Delta(y_{t-1} - y_{t-2}) + \phi_2 \Delta(y_{t-2} - y_{t-3}) + \phi_3 \Delta(y_{t-3} - y_{t-4}) \\ &\quad + \phi_4 \Delta(y_{t-4} - y_{t-5}) + \varepsilon_t + \theta \varepsilon_{t-1} \end{aligned}$$

$$\begin{aligned} y_t - y_{t-1} - y_{t-1} + y_{t-2} \\ &= c + \phi_1 (y_{t-1} - y_{t-2} - y_{t-2} + y_{t-3}) + \phi_2 (y_{t-2} - y_{t-3} - y_{t-3} + y_{t-4}) \\ &\quad + \phi_3 (y_{t-3} - y_{t-4} - y_{t-4} + y_{t-5}) + \phi_4 (y_{t-4} - y_{t-5} - y_{t-5} + y_{t-6}) + \varepsilon_t \\ &\quad + \theta \varepsilon_{t-1} \end{aligned}$$

$$\begin{aligned} y_t - 2y_{t-1} + y_{t-2} \\ &= c + \phi_1 (y_{t-1} - 2y_{t-2} + y_{t-3}) + \phi_2 (y_{t-2} - 2y_{t-3} + y_{t-4}) + \phi_3 (y_{t-3} \\ &\quad - 2y_{t-4} + y_{t-5}) + \phi_4 (y_{t-4} - 2y_{t-5} + y_{t-6}) + \varepsilon_t + \theta \varepsilon_{t-1} \end{aligned}$$

$$\begin{aligned} \varepsilon_t + \theta \varepsilon_{t-1} &= y_t - (2 + \phi_1)y_{t-1} + (1 + 2\phi_1 - \phi_2)y_{t-2} - (\phi_1 - 2\phi_2 + \phi_3)y_{t-3} \\ &\quad - (\phi_2 - 2\phi_3 + \phi_4)y_{t-4} - (\phi_3 - 2\phi_4)y_{t-5} - \phi_4 y_{t-6} - c \end{aligned}$$

$$\begin{aligned} (1 + \theta L)\varepsilon_t &= y_t - (2 + \phi_1)y_{t-1} + (1 + 2\phi_1 - \phi_2)y_{t-2} - (\phi_1 - 2\phi_2 + \phi_3)y_{t-3} \\ &\quad - (\phi_2 - 2\phi_3 + \phi_4)y_{t-4} - (\phi_3 - 2\phi_4)y_{t-5} - \phi_4 y_{t-6} - c \end{aligned}$$

$$\begin{aligned} \varepsilon_t &= -(1 + \theta L)^{-1}c + (1 + \theta L)^{-1}y_t - (2 + \phi_1)(1 + \theta L)^{-1}y_{t-1} \\ &\quad + (1 + 2\phi_1 - \phi_2)(1 + \theta L)^{-1}y_{t-2} - (\phi_1 - 2\phi_2 + \phi_3)(1 + \theta L)^{-1}y_{t-3} \\ &\quad - (\phi_2 - 2\phi_3 + \phi_4)(1 + \theta L)^{-1}y_{t-4} - (\phi_3 - 2\phi_4)(1 + \theta L)^{-1}y_{t-5} \\ &\quad - \phi_4(1 + \theta L)^{-1}y_{t-6} \end{aligned}$$

$$\begin{aligned} \varepsilon_t &= -(1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots)c + (1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots)y_t \\ &\quad - (2 + \phi_1)[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots]y_{t-1} \\ &\quad + (1 + 2\phi_1 - \phi_2)[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots]y_{t-2} \\ &\quad - (\phi_1 - 2\phi_2 + \phi_3)[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots]y_{t-3} \\ &\quad - (\phi_2 - 2\phi_3 + \phi_4)[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots]y_{t-4} \\ &\quad - (\phi_3 - 2\phi_4)[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots]y_{t-5} \\ &\quad - \phi_4[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots]y_{t-6} \end{aligned}$$

$$\begin{aligned}
\varepsilon_t = & -(1 - \theta + \theta^2 - \theta^3 + \theta^4 - \dots)c + (y_t - \theta y_{t-1} + \theta^2 y_{t-2} - \theta^3 y_{t-3} + \theta^4 y_{t-4} - \dots) \\
& - (2 + \phi_1)(y_{t-1} - \theta y_{t-2} + \theta^2 y_{t-3} - \theta^3 y_{t-4} \dots) \\
& + (1 + 2\phi_1 - \phi_2)(y_{t-2} - \theta y_{t-3} + \theta^2 y_{t-4} - \theta^3 y_{t-5} \dots) \\
& - (\phi_1 - 2\phi_2 + \phi_3)(y_{t-3} - \theta y_{t-4} + \theta^2 y_{t-5} - \theta^3 y_{t-6} \dots) \\
& - (\phi_2 - 2\phi_3 + \phi_4)(y_{t-4} - \theta y_{t-5} + \theta^2 y_{t-6} - \theta^3 y_{t-7} \dots) \\
& - (\phi_3 - 2\phi_4)(y_{t-5} - \theta y_{t-6} + \theta^2 y_{t-7} - \theta^3 y_{t-8} \dots) \\
& - \phi_4(y_{t-6} - \theta y_{t-7} + \theta^2 y_{t-8} - \theta^3 y_{t-9} \dots)
\end{aligned}$$

$$\begin{aligned}
\varepsilon_t = & -(1 + \theta)^{-1}c + y_t - (\theta + \phi_1 + 2)y_{t-1} + [\theta^2 + \theta(2 + \phi_1) + (1 + 2\phi_1 - \phi_2)]y_{t-2} \\
& - [\theta^3 + \theta^2(2 + \phi_1) + \theta(1 + 2\phi_1 - \phi_2) + (\phi_1 - 2\phi_2 + \phi_3)]y_{t-3} \\
& + [\theta^4 + \theta^3(2 + \phi_1) + \theta^2(1 + 2\phi_1 - \phi_2) + \theta(\phi_1 - 2\phi_2 + \phi_3) \\
& - (\phi_2 - 2\phi_3 + \phi_4)]y_{t-4} \\
& - [\theta^5 + \theta^4(2 + \phi_1) + \theta^3(1 + 2\phi_1 - \phi_2) + \theta^2(\phi_1 - 2\phi_2 + \phi_3) \\
& - \theta(\phi_2 - 2\phi_3 + \phi_4) + (\phi_3 - 2\phi_4)]y_{t-5} \\
& + [\theta^6 + \theta^5(2 + \phi_1) + \theta^4(1 + 2\phi_1 - \phi_2) + \theta^3(\phi_1 - 2\phi_2 + \phi_3) \\
& - \theta^2(\phi_2 - 2\phi_3 + \phi_4) + \theta(\phi_3 - 2\phi_4) - \phi_4]y_{t-6} \\
& - [\theta^7 + \theta^6(2 + \phi_1) + \theta^5(1 + 2\phi_1 - \phi_2) + \theta^4(\phi_1 - 2\phi_2 + \phi_3) \\
& - \theta^3(\phi_2 - 2\phi_3 + \phi_4) + \theta^2(\phi_3 - 2\phi_4) - \theta\phi_4]y_{t-7} + \dots \quad (2.108)
\end{aligned}$$

$$\begin{aligned}
\varepsilon_t = & -\frac{c}{(1 + \theta)} + y_t - (\theta + \phi_1 + 2)y_{t-1} + [\theta^2 + \theta(2 + \phi_1) + (1 + 2\phi_1 - \phi_2)]y_{t-2} \\
& - [\theta^3 + \theta^2(2 + \phi_1) + \theta(1 + 2\phi_1 - \phi_2) + (\phi_1 - 2\phi_2 + \phi_3)]y_{t-3} \\
& + [\theta^4 + \theta^3(2 + \phi_1) + \theta^2(1 + 2\phi_1 - \phi_2) + \theta(\phi_1 - 2\phi_2 + \phi_3) \\
& - (\phi_2 - 2\phi_3 + \phi_4)]y_{t-4} \\
& - [\theta^5 + \theta^4(2 + \phi_1) + \theta^3(1 + 2\phi_1 - \phi_2) + \theta^2(\phi_1 - 2\phi_2 + \phi_3) \\
& - \theta(\phi_2 - 2\phi_3 + \phi_4) + (\phi_3 - 2\phi_4)]y_{t-5} \\
& + \sum_{i=6}^{\infty} (-1)^i [\theta^i + \theta^{i-1}(\phi_1 + 2) + \theta^{i-2}(1 + 2\phi_1 - \phi_2) \\
& + \theta^{i-3}(\phi_1 - 2\phi_2 + \phi_3) - \theta^{i-4}(\phi_2 - 2\phi_3 + \phi_4) + \theta^{i-5}(\phi_3 - 2\phi_4) \\
& - \theta^{i-6}\phi_4]y_{t-i} \quad (2.109)
\end{aligned}$$

$$\varepsilon_t = -\frac{c}{(1 + \theta)} + \sum_{i=0}^{\infty} \pi_i y_{t-i} \quad (2.110)$$

where

$$\pi_i = \begin{cases} 1, & i = 0 \\ -(\theta + \phi_1 + 2), & i = 1 \\ \theta^2 + \theta(2 + \phi_1) + (1 + 2\phi_1 - \phi_2), & i = 2 \\ -[\theta^3 + \theta^2(2 + \phi_1) + \theta(1 + 2\phi_1 - \phi_2) + (\phi_1 - 2\phi_2 + \phi_3)], & i = 3 \\ \theta^4 + \theta^3(2 + \phi_1) + \theta^2(1 + 2\phi_1 - \phi_2) \\ \quad + \theta(\phi_1 - 2\phi_2 + \phi_3) - (\phi_2 - 2\phi_3), & i = 4 \\ -[\theta^5 + \theta^4(2 + \phi_1) + \theta^3(1 + 2\phi_1 - \phi_2) + \theta^2(\phi_1 - 2\phi_2 + \phi_3) \\ \quad - \theta(\phi_2 - 2\phi_3 + \phi_4) + (\phi_3 - 2\phi_4)], & i = 5 \\ (-1)^i [\theta^i + \theta^{i-1}(\phi_1 + 2) + \theta^{i-2}(1 + 2\phi_1 - \phi_2) + \theta^{i-3}(\phi_1 - 2\phi_2 + \phi_3) \\ \quad - \theta^{i-4}(\phi_2 - 2\phi_3 + \phi_4) + \theta^{i-5}(\phi_3 - 2\phi_4) + \theta^{i-6}\phi_4], & i = 6, 7, 8, \dots \end{cases} \quad (2.111)$$

Equation (2.110) holds if $\theta \neq -1$.

ARIMA (4, 3, 1)

$$\Delta^3 y_t = c + \phi_1 \Delta^3 y_{t-1} + \phi_2 \Delta^3 y_{t-2} + \phi_3 \Delta^3 y_{t-3} + \phi_4 \Delta^3 y_{t-4} + \varepsilon_t + \theta \varepsilon_{t-1} \quad (2.112)$$

$$\begin{aligned} & \Delta(y_t - 2y_{t-1} + y_{t-2}) \\ &= c + \phi_1 \Delta(y_{t-1} - 2y_{t-2} + y_{t-3}) + \phi_2 \Delta(y_{t-2} - 2y_{t-3} + y_{t-4}) \\ & \quad + \phi_3 \Delta(y_{t-3} - 2y_{t-4} + y_{t-5}) + \phi_4 \Delta(y_{t-4} - 2y_{t-5} + y_{t-6}) + \varepsilon_t + \theta \varepsilon_{t-1} \\ & y_t - y_{t-1} - 2(y_{t-1} - y_{t-2}) + (y_{t-2} - y_{t-3}) \\ &= c + \phi_1 [(y_{t-1} - y_{t-2}) - 2(y_{t-2} - y_{t-3}) + (y_{t-3} - y_{t-4})] \\ & \quad + \phi_2 [(y_{t-2} - y_{t-3}) - 2(y_{t-3} - y_{t-4}) + (y_{t-4} - y_{t-5})] \\ & \quad + \phi_3 [(y_{t-3} - y_{t-4}) - 2(y_{t-4} - y_{t-5}) + (y_{t-5} - y_{t-6})] + \phi_4 [(y_{t-4} - y_{t-5}) \\ & \quad - 2(y_{t-5} - y_{t-6}) + (y_{t-6} - y_{t-7})] + \varepsilon_t + \theta \varepsilon_{t-1} \\ & y_t - 3y_{t-1} + 3y_{t-2} - y_{t-3} \\ &= c + \phi_1 (y_{t-1} - 3y_{t-2} + 3y_{t-3} - y_{t-4}) + \phi_2 (y_{t-2} - 3y_{t-3} + 3y_{t-4} - y_{t-5}) \\ & \quad + \phi_3 (y_{t-3} - 3y_{t-4} + 3y_{t-5} - y_{t-6}) + \phi_4 (y_{t-4} - 3y_{t-5} + 3y_{t-6} - y_{t-7}) \\ & \quad + \varepsilon_t + \theta \varepsilon_{t-1} \end{aligned} \quad (2.113)$$

$$\begin{aligned} \varepsilon_t + \theta \varepsilon_{t-1} &= y_t - (3 + \phi_1)y_{t-1} + (3 + 3\phi_1 - \phi_2)y_{t-2} - (1 + 3\phi_1 - 3\phi_2 + \phi_3)y_{t-3} \\ & \quad + (\phi_1 - 3\phi_2 + 3\phi_3 - \phi_4)y_{t-4} + (\phi_2 - 3\phi_3 + 3\phi_4)y_{t-5} + (\phi_3 - 3\phi_4)y_{t-6} \\ & \quad + \phi_4 y_{t-7} - c \end{aligned}$$

$$\begin{aligned}
(1 + \theta L)\varepsilon_t &= y_t - (3 + \phi_1)y_{t-1} + (3 + 3\phi_1 - \phi_2)y_{t-2} - (1 + 3\phi_1 - 3\phi_2 + \phi_3)y_{t-3} \\
&\quad + (\phi_1 - 3\phi_2 + 3\phi_3 - \phi_4)y_{t-4} + (\phi_2 - 3\phi_3 + 3\phi_4)y_{t-5} + (\phi_1 - 3\phi_4)y_{t-6} \\
&\quad + \phi_4 y_{t-7} - c
\end{aligned} \tag{2.114}$$

$$\begin{aligned}
\varepsilon_t &= -(1 + \theta L)^{-1}c \\
&\quad + (1 + \theta L)^{-1}[y_t - (3 + \phi_1)y_{t-1} + (3 + 3\phi_1 - \phi_2)y_{t-2} \\
&\quad - (1 + 3\phi_1 - 3\phi_2 + \phi_3)y_{t-3} + (\phi_1 - 3\phi_2 + 3\phi_3 - \phi_4)y_{t-4} \\
&\quad + (\phi_2 - 3\phi_3 + 3\phi_4)y_{t-5} + (\phi_3 - 3\phi_4)y_{t-6} + \phi_4 y_{t-7}]
\end{aligned}$$

$$\begin{aligned}
\varepsilon_t &= -(1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots)c + (1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots)y_t \\
&\quad - (3 + \phi_1)[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots]y_{t-1} \\
&\quad + (3 + 3\phi_1 - \phi_2)[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots]y_{t-2} \\
&\quad - (1 + 3\phi_1 - 3\phi_2 + \phi_3)[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots]y_{t-3} \\
&\quad + (\phi_1 - 3\phi_2 + 3\phi_3 - \phi_4)[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots]y_{t-4} \\
&\quad + (\phi_2 - 3\phi_3 + 3\phi_4)[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots]y_{t-5} \\
&\quad + (\phi_3 - 3\phi_4)[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots]y_{t-6} \\
&\quad + \phi_4[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots]y_{t-7}
\end{aligned}$$

$$\begin{aligned}
\varepsilon_t &= -(1 - \theta + \theta^2 - \theta^3 + \theta^4 - \dots)c \\
&\quad + (y_t - \theta y_{t-1} + \theta^2 y_{t-2} - \theta^3 y_{t-3} + \theta^4 y_{t-4} - \theta^5 y_{t-5} + \theta^6 y_{t-6} - \dots) \\
&\quad - (3 + \phi_1)(y_{t-1} - \theta y_{t-2} + \theta^2 y_{t-3} - \theta^3 y_{t-4} + \theta^4 y_{t-5} - \theta^5 y_{t-6} + \dots) \\
&\quad + (3 + 3\phi_1 - \phi_2)(y_{t-2} - \theta y_{t-3} + \theta^2 y_{t-4} - \theta^3 y_{t-5} + \theta^4 y_{t-6} - \dots) \\
&\quad - (1 + 3\phi_1 - 3\phi_2 + \phi_3)(y_{t-3} - \theta y_{t-4} + \theta^2 y_{t-5} - \theta^3 y_{t-6} + \theta^4 y_{t-7} - \dots) \\
&\quad + (\phi_1 - 3\phi_2 + 3\phi_3 - \phi_4)(y_{t-4} - \theta y_{t-5} + \theta^2 y_{t-6} - \theta^3 y_{t-7} + \dots) \\
&\quad + (\phi_2 - 3\phi_3 + 3\phi_4)(y_{t-5} - \theta y_{t-6} + \theta^2 y_{t-7} - \theta^3 y_{t-8} + \dots) \\
&\quad + (\phi_3 - 3\phi_4)(y_{t-6} - \theta y_{t-7} + \theta^2 y_{t-8} - \theta^3 y_{t-9} + \dots) \\
&\quad + \phi_4(y_{t-7} - \theta y_{t-8} + \theta^2 y_{t-9} - \theta^3 y_{t-10} + \dots)
\end{aligned}$$



$$\begin{aligned}
\varepsilon_t = & -(1+\theta)^{-1}c + y_t - (\theta + \phi_1 + 3)y_{t-1} + [\theta^2 + \theta(3 + \phi_1) + (3 + 3\phi_1 - \phi_2)]y_{t-2} \\
& - [\theta^3 + \theta^2(3 + \phi_1) + \theta(3 + 3\phi_1 - \phi_2) + (1 + 3\phi_1 - 3\phi_2 + \phi_3)]y_{t-3} \\
& + [\theta^4 + \theta^3(3 + \phi_1) + \theta^2(3 + 3\phi_1 - \phi_2) + \theta(1 + 3\phi_1 - 3\phi_2 + \phi_3) \\
& + (\phi_1 - 3\phi_2 + 3\phi_3 - \phi_4)]y_{t-4} \\
& - [\theta^5 + \theta^4(3 + \phi_1) + \theta^3(3 + 3\phi_1 - \phi_2) + \theta^2(1 + 3\phi_1 - 3\phi_2 + \phi_3) \\
& + \theta(\phi_1 - 3\phi_2 + 3\phi_3 - \phi_4) - (\phi_2 - 3\phi_3 + 3\phi_4)]y_{t-5} \\
& + [\theta^6 + \theta^5(3 + \phi_1) + \theta^4(3 + 3\phi_1 - \phi_2) + \theta^3(1 + 3\phi_1 - 3\phi_2 + \phi_3) \\
& + \theta^2(\phi_1 - 3\phi_2 + 3\phi_3 - \phi_4) - \theta(\phi_2 - 3\phi_3 + 3\phi_4) + (\phi_3 - 3\phi_4)]y_{t-6} \\
& - [\theta^7 + \theta^6(3 + \phi_1) + \theta^5(3 + 3\phi_1 - \phi_2) + \theta^4(1 + 3\phi_1 - 3\phi_2 + \phi_3) \\
& + \theta^3(\phi_1 - 3\phi_2 + 3\phi_3 - \phi_4) - \theta^2(\phi_2 - 3\phi_3 + 3\phi_4) + \theta(\phi_3 - 3\phi_4) - \phi_4]y_{t-7} \\
& + [\theta^8 + \theta^7(3 + \phi_1) + \theta^6(3 + 3\phi_1 - \phi_2) + \theta^5(1 + 3\phi_1 - 3\phi_2 + \phi_3) \\
& + \theta^4(\phi_1 - 3\phi_2 + 3\phi_3 - \phi_4) - \theta^3(\phi_2 - 3\phi_3 + 3\phi_4) + \theta^2(\phi_3 - 3\phi_4) - \theta\phi_4]y_{t-8} \\
& - \dots
\end{aligned}$$

$$\begin{aligned}
\varepsilon_t = & -\frac{c}{(1+\theta)} + y_t - (\theta + \phi_1 + 3)y_{t-1} + [\theta^2 + \theta(3 + \phi_1) + (3 + 3\phi_1 - \phi_2)]y_{t-2} \\
& - [\theta^3 + \theta^2(3 + \phi_1) + \theta(3 + 3\phi_1 - \phi_2) + (1 + 3\phi_1 - 3\phi_2 + \phi_3)]y_{t-3} \\
& + [\theta^4 + \theta^3(3 + \phi_1) + \theta^2(3 + 3\phi_1 - \phi_2) + \theta(1 + 3\phi_1 - 3\phi_2 + \phi_3) \\
& + (\phi_1 - 3\phi_2 + 3\phi_3 - \phi_4)]y_{t-4} \\
& - [\theta^5 + \theta^4(3 + \phi_1) + \theta^3(3 + 3\phi_1 - \phi_2) + \theta^2(1 + 3\phi_1 - 3\phi_2 + \phi_3) \\
& + \theta(\phi_1 - 3\phi_2 + 3\phi_3 - \phi_4) - (\phi_2 - 3\phi_3 + 3\phi_4)]y_{t-5} \\
& + [\theta^6 + \theta^5(3 + \phi_1) + \theta^4(3 + 3\phi_1 - \phi_2) + \theta^3(1 + 3\phi_1 - 3\phi_2 + \phi_3) \\
& + \theta^2(\phi_1 - 3\phi_2 + 3\phi_3 - \phi_4) - \theta(\phi_2 - 3\phi_3 + 3\phi_4) + (\phi_3 - 3\phi_4)]y_{t-6} \\
& + \sum_{i=7}^{\infty} (-1)^i [\theta^i + \theta^{i-1}(\phi_1 + 3) + \theta^{i-2}(3 + 3\phi_1 - \phi_2) \\
& + \theta^{i-3}(1 + 3\phi_1 - 3\phi_2 + \phi_3) + \theta^{i-4}(\phi_1 - 3\phi_2 + 3\phi_3 - \phi_4) \\
& - \theta^{i-5}(\phi_2 - 3\phi_3 + 3\phi_4) + \theta^{i-6}(\phi_3 - 3\phi_4) - \theta^{i-7}\phi_4]y_{t-i}
\end{aligned} \tag{2.115}$$

$$\varepsilon_t = -\frac{c}{(1+\theta)} + \sum_{i=0}^{\infty} \pi_i y_{t-i} \tag{2.116}$$

where

$$\pi_i = \begin{cases} 1, & i = 0 \\ -(\theta + \phi_1 + 3), & i = 1 \\ \theta^2 + \theta(3 + \phi_1) + (3 + 3\phi_1 - \phi_2), & i = 2 \\ -[\theta^3 + \theta^2(3 + \phi_1) + \theta(3 + 3\phi_1 - \phi_2) + (1 + 3\phi_1 - 3\phi_2)], & i = 3 \\ \theta^4 + \theta^3(3 + \phi_1) + \theta^2(3 + 3\phi_1 - \phi_2) \\ \quad + \theta(1 + 3\phi_1 - 3\phi_2 + \phi_3) + (\phi_1 - 3\phi_2 + 3\phi_3), & i = 4 \\ -[\theta^5 + \theta^4(3 + \phi_1) + \theta^3(3 + 3\phi_1 - \phi_2) + \theta^2(1 + 3\phi_1 - 3\phi_2 + \phi_3) \\ \quad + \theta(\phi_1 - 3\phi_2 + 3\phi_3 - \phi_4) + (\phi_2 - 3\phi_3 + 3\phi_4)], & i = 5 \\ [\theta^6 + \theta^5(3 + \phi_1) + \theta^4(3 + 3\phi_1 - \phi_2) + \theta^3(1 + 3\phi_1 - 3\phi_2 + \phi_3) \\ \quad + \theta^2(\phi_1 - 3\phi_2 + 3\phi_3 - \phi_4) + \theta(\phi_2 - 3\phi_3 + 3\phi_4) + (\phi_3 - 3\phi_4)], & i = 6 \\ (-1)^i [\theta^i + \theta^{i-1}(\phi_1 + 3) + \theta^{i-2}(3 + 3\phi_1 - \phi_2) \\ \quad + \theta^{i-3}(1 + 3\phi_1 - 3\phi_2 + \phi_3) + \theta^{i-4}(\phi_1 - 3\phi_2 + 3\phi_3 - \phi_4) \\ \quad - \theta^{i-5}(\phi_2 - 3\phi_3 + 3\phi_4) + \theta^{i-6}(\phi_3 - 3\phi_4) - \theta^{i-7}\phi_4], & i = 7, 8, 9, \dots \end{cases} \quad (2.117)$$

Equation (2.116) holds if $\theta \neq -1$.

ARIMA (p, 0, 1)

$$y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \phi_3 y_{t-3} + \dots + \phi_p y_{t-p} + \varepsilon_t + \theta \varepsilon_{t-1} \quad (2.118)$$

$$\varepsilon_t + \theta \varepsilon_{t-1} = -c + y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2} - \phi_3 y_{t-3} - \phi_4 y_{t-4} - \dots - \phi_p y_{t-p}$$

$$(1 + \theta L) \varepsilon_t = -c + y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2} - \phi_3 y_{t-3} - \phi_4 y_{t-4} - \dots - \phi_p y_{t-p}$$

$$\varepsilon_t = -(1 + \theta L)^{-1} c + (1 + \theta L)^{-1} (y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2} - \phi_3 y_{t-3} - \phi_4 y_{t-4} - \dots - \phi_p y_{t-p})$$

$$\begin{aligned} \varepsilon_t = & -(1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \theta^5 L^5 + \dots) c \\ & + (1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \theta^5 L^5 + \dots) y_t \\ & - \phi_1 (1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \theta^5 L^5 + \dots) y_{t-1} \\ & - \phi_2 (1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \theta^5 L^5 + \dots) y_{t-2} \\ & - \phi_3 (1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \theta^5 L^5 + \dots) y_{t-3} - \dots - \phi_p (1 - \theta L \\ & + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \theta^5 L^5 + \dots) y_{t-p} \end{aligned}$$

$$\begin{aligned}
\varepsilon_t = & -(1 - \theta + \theta^2 - \theta^3 + \theta^4 - \theta^5 + \dots)c \\
& + (y_t - \theta y_{t-1} + \theta^2 y_{t-2} - \theta^3 y_{t-3} + \theta^4 y_{t-4} - \theta^5 y_{t-5} + \dots) \\
& - \phi_1 (y_{t-1} - \theta y_{t-2} + \theta^2 y_{t-3} - \theta^3 y_{t-4} + \theta^4 y_{t-5} - \dots) \\
& - \phi_2 (y_{t-2} - \theta y_{t-3} + \theta^2 y_{t-4} - \theta^3 y_{t-5} + \dots) \\
& - \phi_3 (y_{t-3} - \theta y_{t-4} + \theta^2 y_{t-5} - \dots) - \dots - \phi_p (y_{t-p} - \theta y_{t-p-1} + \theta^2 y_{t-p-2} \\
& - \dots)
\end{aligned}$$

$$\begin{aligned}
\varepsilon_t = & -(1 + \theta)^{-1}c + y_t - (\theta + \phi_1)y_{t-1} + (\theta^2 + \phi_1\theta - \phi_2)y_{t-2} \\
& - (\theta^3 + \phi_1\theta^2 - \phi_2\theta + \phi_3)y_{t-3} + (\theta^4 + \phi_1\theta^3 - \phi_2\theta^2 + \phi_3\theta - \phi_4)y_{t-4} \\
& + \dots + (-1)^p(\theta^p + \phi_1\theta^{p-1} - \phi_2\theta^{p-2} + \phi_3\theta^{p-3} + \dots + (-1)^{p+1}\phi_p)y_{t-p} \\
& + (-1)^{p+1}(\theta^{p+1} + \phi_1\theta^p - \phi_2\theta^{p-1} + \phi_3\theta^{p-2} + \dots + (-1)^{p+1}\phi_p\theta)y_{t-p-1} \\
& + \dots
\end{aligned}$$

$$\begin{aligned}
\varepsilon_t = & -\frac{c}{(1 + \theta)} + y_t - (\theta + \phi_1)y_{t-1} + (\theta^2 + \phi_1\theta - \phi_2)y_{t-2} \\
& - (\theta^3 + \phi_1\theta^2 - \phi_2\theta + \phi_3)y_{t-3} \\
& + (\theta^4 + \phi_1\theta^3 - \phi_2\theta^2 + \phi_3\theta - \phi_4)y_{t-4} - \dots + \sum_{i=p}^{\infty} (-1)^i [\theta^i + \phi_1\theta^{i-1} \\
& - \phi_2\theta^{i-2} + \phi_3\theta^{i-3} - \phi_4\theta^{i-4} + \dots + (-1)^{p+1}\phi_p\theta^{i-p}]y_{t-i} \quad (2.119)
\end{aligned}$$

$$\varepsilon_t = -\frac{c}{(1 + \theta)} + \sum_{i=0}^{\infty} \pi_i y_{t-i} \quad (2.120)$$

$$\pi_i = \begin{cases} 1, & i = 0 \\ -(\theta + \phi_1), & i = 1 \\ \theta^2 + \phi_1\theta - \phi_2, & i = 2 \\ -(\theta^3 + \phi_1\theta^2 - \phi_2\theta + \phi_3), & i = 3 \\ \theta^4 + \phi_1\theta^3 - \phi_2\theta^2 + \phi_3\theta - \phi_4, & i = 4 \\ (-1)^i [\theta^i + \phi_1\theta^{i-1} - \phi_2\theta^{i-2} + \phi_3\theta^{i-3} - \phi_4\theta^{i-4} + \\ \dots + (-1)^{p+1}\phi_p\theta^{i-p}], & i = p, p+1, \dots \end{cases} \quad (2.121)$$

Equation (2.120) holds if $\theta \neq -1$.

ARIMA (p, 1, 1)

$$\Delta y_t = c + \phi_1 \Delta y_{t-1} + \phi_2 \Delta y_{t-2} + \phi_3 \Delta y_{t-3} + \dots + \phi_p \Delta y_{t-p} + \varepsilon_t + \theta \varepsilon_{t-1} \quad (2.122)$$

$$y_t - y_{t-1} = c + \phi_1(y_{t-1} - y_{t-2}) + \phi_2(y_{t-2} - y_{t-3}) + \phi_3(y_{t-3} - y_{t-4}) + \dots \\ + \phi_p(y_{t-p} - y_{t-p-1}) + \varepsilon_t + \theta \varepsilon_{t-1}$$

$$\varepsilon_t + \theta \varepsilon_{t-1} = y_t - (1 + \phi_1)y_{t-1} + (\phi_1 - \phi_2)y_{t-2} + (\phi_2 - \phi_3)y_{t-3} + (\phi_3 - \phi_4)y_{t-4} + \dots \\ + (\phi_{p-1} - \phi_p)y_{t-p} + \phi_p y_{t-p-1} - c$$

$$(1 + \theta L)\varepsilon_t = y_t - (1 + \phi_1)y_{t-1} + (\phi_1 - \phi_2)y_{t-2} + (\phi_2 - \phi_3)y_{t-3} + (\phi_3 - \phi_4)y_{t-4} + \dots \\ + (\phi_{p-1} - \phi_p)y_{t-p} + \phi_p y_{t-p-1} - c$$

$$\varepsilon_t = -(1 + \theta L)^{-1}c \\ + (1 + \theta L)^{-1}[y_t - (1 + \phi_1)y_{t-1} + (\phi_1 - \phi_2)y_{t-2} + (\phi_2 - \phi_3)y_{t-3} \\ + (\phi_3 - \phi_4)y_{t-4} + \dots + (\phi_{p-1} - \phi_p)y_{t-p} + \phi_p y_{t-p-1}]$$

$$\varepsilon_t = -(1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \theta^5 L^5 + \dots)c \\ + (1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \theta^5 L^5 + \dots)y_t \\ - (1 + \phi_1)[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \theta^5 L^5 + \dots]y_{t-1} \\ + (\phi_1 - \phi_2)[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \theta^5 L^5 + \dots]y_{t-2} \\ + (\phi_2 - \phi_3)[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \theta^5 L^5 + \dots]y_{t-3} \\ + (\phi_3 - \phi_4)[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \theta^5 L^5 + \dots]y_{t-4} + \dots \\ + (\phi_{p-1} - \phi_p)[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \theta^5 L^5 + \dots]y_{t-p} \\ + \phi_p[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \theta^5 L^5 + \dots]y_{t-p-1}$$

$$\varepsilon_t = -(1 - \theta + \theta^2 - \theta^3 + \theta^4 - \theta^5 + \dots)c \\ + (y_t - \theta y_{t-1} + \theta^2 y_{t-2} - \theta^3 y_{t-3} + \theta^4 y_{t-4} - \theta^5 y_{t-5} + \theta^6 y_{t-6} - \dots) \\ - (1 + \phi_1)(y_{t-1} - \theta y_{t-2} + \theta^2 y_{t-3} - \theta^3 y_{t-4} + \theta^4 y_{t-5} - \theta^5 y_{t-6} + \dots) \\ + (\phi_1 - \phi_2)(y_{t-2} - \theta y_{t-3} + \theta^2 y_{t-4} - \theta^3 y_{t-5} + \theta^4 y_{t-6} + \dots) \\ + (\phi_2 - \phi_3)(y_{t-3} - \theta y_{t-4} + \theta^2 y_{t-5} - \theta^3 y_{t-6} + \dots) \\ + (\phi_3 - \phi_4)(y_{t-4} - \theta y_{t-5} + \theta^2 y_{t-6} - \dots) + \dots \\ + (\phi_{p-1} - \phi_p)(y_{t-p} - \theta y_{t-p-1} + \theta^2 y_{t-p-2} - \dots) \\ + \phi_p(y_{t-p-1} - \theta y_{t-p-2} + \theta^2 y_{t-p-3} - \dots)$$

$$\begin{aligned}
\varepsilon_t = & -(1+\theta)^{-1}c + y_t - (\theta + \phi_1 + 1)y_{t-1} + (\theta^2 + \phi_1\theta + \theta + \phi_1 - \phi_2)y_{t-2} \\
& - (\theta^3 + \phi_1\theta^2 + \theta^2 + \phi_1\theta - \phi_2\theta - \phi_2 + \phi_3)y_{t-3} \\
& + (\theta^4 + \phi_1\theta^3 + \theta^3 + \phi_1\theta^2 - \phi_2\theta^2 - \phi_2\theta + \phi_3\theta + \phi_3 - \phi_4)y_{t-4} \\
& - (\theta^5 + \phi_1\theta^4 + \theta^4 + \phi_1\theta^3 - \phi_2\theta^3 - \phi_2\theta^2 + \phi_3\theta^2 + \phi_3\theta - \phi_4\theta \\
& - \phi_4)y_{t-5} + \dots \\
& + (\theta^{p+1} + \phi_1\theta^p + \theta^p + \phi_1\theta^{p-1} - \phi_2\theta^{p-1} - \phi_2\theta^{p-2} + \phi_3\theta^{p-2} + \dots \\
& + \phi_{p-1}\theta^2 - \phi_{p-1}\theta^2 - \phi_p\theta)y_{t-p-1} + \dots
\end{aligned}$$

$$\begin{aligned}
\varepsilon_t = & -(1+\theta)^{-1}c + y_t - (\theta + \phi_1 + 1)y_{t-1} + (\theta^2 + \theta(1+\phi_1) + \phi_1 - \phi_2)y_{t-2} \\
& - [\theta^3 + \theta^2(1+\phi_1) + \theta(\phi_1 - \phi_2) - \phi_2 + \phi_3]y_{t-3} \\
& + [\theta^4 + \theta^3(1+\phi_1) + \theta^2(\phi_1 - \phi_2) - \theta(\phi_2 - \phi_3) + \phi_3 - \phi_4]y_{t-4} \\
& - [\theta^5 + \theta^4(1+\phi_1) + \theta^3(\phi_1 - \phi_2) - \theta^2(\phi_2 - \phi_3) + \theta(\phi_3 - \phi_4) \\
& - \phi_4]y_{t-5} + \dots \\
& + [\theta^{p+1} + \theta^p(1+\phi_1) + \theta^{p-1}(\phi_1 - \phi_2) - \theta^{p-2}(\phi_2 - \phi_3) + \theta^{p-3}(\phi_3 \\
& - \phi_4) + \dots + (-1)^{p+1}\theta\phi_p]y_{t-p} + \dots
\end{aligned}$$

$$\begin{aligned}
\varepsilon_t = & -\frac{c}{(1+\theta)} + y_t - (\theta + \phi_1 + 1)y_{t-1} + [\theta^2 + \theta(1+\phi_1) + \phi_1 - \phi_2]y_{t-2} \\
& - [\theta^3 + \theta^2(1+\phi_1) + \theta(\phi_1 - \phi_2) - \phi_2 + \phi_3]y_{t-3} \\
& + [\theta^4 + \theta^3(1+\phi_1) + \theta^2(\phi_1 - \phi_2) - \theta(\phi_2 - \phi_3) + \phi_3 - \phi_4]y_{t-4} + \dots \\
& + \sum_{i=p}^{\infty} (-1)^i [\theta^i + \theta^{i-1}(1+\phi_1) + \theta^{i-2}(\phi_1 - \phi_2) - \theta^{i-3}(\phi_2 - \phi_3) \\
& + \theta^{i-4}(\phi_3 - \phi_4) + \dots + (-1)^{p+1}\theta^{i-p}\phi_p]y_{t-i}
\end{aligned} \tag{2.123}$$

$$\varepsilon_t = -\frac{c}{(1+\theta)} + \sum_{i=0}^{\infty} \pi_i y_{t-i} \tag{2.124}$$

where

$$\pi_i = \begin{cases} 1, & i = 0 \\ -(\theta + \phi_1 + 1), & i = 1 \\ \theta^2 + \theta(1+\phi_1) + \phi_1 - \phi_2, & i = 2 \\ -[\theta^3 + \theta^2(1+\phi_1) + \theta(\phi_1 - \phi_2) - \phi_2 + \phi_3], & i = 3 \\ (-1)^i [\theta^i + \theta^{i-1}(1+\phi_1) + \theta^{i-2}(\phi_1 - \phi_2) - \theta^{i-3}(\phi_2 - \phi_3) \\ + \theta^{i-4}(\phi_3 - \phi_4) - \theta^{i-5}(\phi_4 - \phi_5) + \dots + (-1)^{p+1}\theta^{i-p-1}\phi_p], & i = p+1, \dots \end{cases} \tag{2.125}$$

Equation (2.124) holds if $\theta \neq -1$.

$$\Delta^2 y_t = c + \phi_1 \Delta^2 y_{t-1} + \phi_2 \Delta^2 y_{t-2} + \phi_3 \Delta^2 y_{t-3} + \dots + \phi_p \Delta^2 y_{t-p} + \varepsilon_t + \theta \varepsilon_{t-1} \quad (2.126)$$

$$\begin{aligned} \Delta(y_t - y_{t-1}) &= c + \phi_1 \Delta(y_{t-1} - y_{t-2}) + \phi_2 \Delta(y_{t-2} - y_{t-3}) + \phi_3 \Delta(y_{t-3} - y_{t-4}) + \dots \\ &\quad + \phi_p \Delta(y_{t-p} - y_{t-p-1}) + \varepsilon_t + \theta \varepsilon_{t-1} \end{aligned}$$

$$\begin{aligned} y_t - y_{t-1} - y_{t-1} + y_{t-2} \\ &= c + \phi_1 (y_{t-1} - y_{t-2} - y_{t-2} + y_{t-3}) + \phi_2 (y_{t-2} - y_{t-3} - y_{t-3} + y_{t-4}) \\ &\quad + \phi_3 (y_{t-3} - y_{t-4} - y_{t-4} + y_{t-5}) + \dots + \phi_p (y_{t-p} - y_{t-p-1} - y_{t-p-1} \\ &\quad + y_{t-p-2}) + \varepsilon_t + \theta \varepsilon_{t-1} \end{aligned}$$

$$\begin{aligned} y_t - 2y_{t-1} + y_{t-2} \\ &= c + \phi_1 (y_{t-1} - 2y_{t-2} + y_{t-3}) + \phi_2 (y_{t-2} - 2y_{t-3} + y_{t-4}) \\ &\quad + \phi_3 (y_{t-3} - 2y_{t-4} + y_{t-5}) + \dots + \phi_p (y_{t-p} - 2y_{t-p-1} + y_{t-p-2}) + \varepsilon_t \\ &\quad + \theta \varepsilon_{t-1} \end{aligned}$$

$$\begin{aligned} \varepsilon_t + \theta \varepsilon_{t-1} &= y_t - (2 + \phi_1)y_{t-1} + (1 + 2\phi_1 - \phi_2)y_{t-2} - (\phi_1 - 2\phi_2 + \phi_3)y_{t-3} \\ &\quad - (\phi_2 - 2\phi_3 + \phi_4)y_{t-4} - \dots - (\phi_{p-1} - 2\phi_p)y_{t-p-1} - \phi_p y_{t-p-2} - c \end{aligned}$$

$$\begin{aligned} (1 + \theta L)\varepsilon_t &= y_t - (2 + \phi_1)y_{t-1} + (1 + 2\phi_1 - \phi_2)y_{t-2} - (\phi_1 - 2\phi_2 + \phi_3)y_{t-3} \\ &\quad - (\phi_2 - 2\phi_3 + \phi_4)y_{t-4} - \dots - (\phi_{p-1} - 2\phi_p)y_{t-p-1} - \phi_p y_{t-p-2} - c \end{aligned}$$

$$\begin{aligned} \varepsilon_t &= -(1 + \theta L)^{-1}c + (1 + \theta L)^{-1}y_t - (2 + \phi_1)(1 + \theta L)^{-1}y_{t-1} \\ &\quad + (1 + 2\phi_1 - \phi_2)(1 + \theta L)^{-1}y_{t-2} - (\phi_1 - 2\phi_2 + \phi_3)(1 + \theta L)^{-1}y_{t-3} \\ &\quad - (\phi_2 - 2\phi_3 + \phi_4)(1 + \theta L)^{-1}y_{t-4} - (\phi_{p-1} - 2\phi_p)(1 + \theta L)^{-1}y_{t-p-1} \\ &\quad - \phi_p (1 + \theta L)^{-1}y_{t-p-2} \end{aligned}$$

$$\begin{aligned} \varepsilon_t &= -(1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots)c + (1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots)y_t \\ &\quad - (2 + \phi_1)[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots]y_{t-1} \\ &\quad + (1 + 2\phi_1 - \phi_2)[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots]y_{t-2} \\ &\quad - (\phi_1 - 2\phi_2 + \phi_3)[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots]y_{t-3} \\ &\quad - (\phi_2 - 2\phi_3 + \phi_4)[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots]y_{t-4} - \dots \\ &\quad - (\phi_{p-1} - 2\phi_p)[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots]y_{t-p-1} \\ &\quad - \phi_p [1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots]y_{t-p-2} \end{aligned}$$



$$\begin{aligned}
\varepsilon_t = & -(1 - \theta + \theta^2 - \theta^3 + \theta^4 - \dots)c + (y_t - \theta y_{t-1} + \theta^2 y_{t-2} - \theta^3 y_{t-3} + \theta^4 y_{t-4} - \dots) \\
& - (2 + \phi_1)(y_{t-1} - \theta y_{t-2} + \theta^2 y_{t-3} - \theta^3 y_{t-4} + \dots) \\
& + (1 + 2\phi_1 - \phi_2)(y_{t-2} - \theta y_{t-3} + \theta^2 y_{t-4} - \theta^3 y_{t-5} + \dots) \\
& - (\phi_1 - 2\phi_2 + \phi_3)(y_{t-3} - \theta y_{t-4} + \theta^2 y_{t-5} - \theta^3 y_{t-6} + \dots) \\
& - (\phi_2 - 2\phi_3 + \phi_4)(y_{t-4} - \theta y_{t-5} + \theta^2 y_{t-6} - \theta^3 y_{t-7} + \dots) - \dots \\
& - (\phi_{p-1} - 2\phi_p)(y_{t-p-1} - \theta y_{t-p-2} + \theta^2 y_{t-p-3} - \theta^3 y_{t-p-4} + \dots) \\
& - \phi_p(y_{t-p-2} - \theta y_{t-p-3} + \theta^2 y_{t-p-4} - \theta^3 y_{t-p-5} + \dots)
\end{aligned}$$

$$\begin{aligned}
\varepsilon_t = & -(1 + \theta)^{-1}c + y_t - (\theta + \phi_1 + 2)y_{t-1} + [\theta^2 + \theta(2 + \phi_1) + (1 + 2\phi_1 - \phi_2)]y_{t-2} \\
& - [\theta^3 + \theta^2(2 + \phi_1) + \theta(1 + 2\phi_1 - \phi_2) + (\phi_1 - 2\phi_2 + \phi_3)]y_{t-3} \\
& + [\theta^4 + \theta^3(2 + \phi_1) + \theta^2(1 + 2\phi_1 - \phi_2) + \theta(\phi_1 - 2\phi_2 + \phi_3) \\
& - (\phi_2 - 2\phi_3 + \phi_4)]y_{t-4} \\
& - [\theta^5 + \theta^4(2 + \phi_1) + \theta^3(1 + 2\phi_1 - \phi_2) + \theta^2(\phi_1 - 2\phi_2 + \phi_3) \\
& - \theta(\phi_2 - 2\phi_3 + \phi_4) + (\phi_3 - 2\phi_4 + \phi_5)]y_{t-5} \\
& + [\theta^6 + \theta^5(2 + \phi_1) + \theta^4(1 + 2\phi_1 - \phi_2) + \theta^3(\phi_1 - 2\phi_2 + \phi_3) \\
& - \theta^2(\phi_2 - 2\phi_3 + \phi_4) + \theta(\phi_3 - 2\phi_4 + \phi_5) - (\phi_4 - 2\phi_5 + \phi_6)]y_{t-6} - \dots \\
& + (-1)^{p+2}[\theta^{p+2} + \theta^{p+1}(2 + \phi_1) + \theta^p(1 + 2\phi_1 - \phi_2) \\
& + \theta^{p-1}(\phi_1 - 2\phi_2 + \phi_3) - \theta^{p-2}(\phi_2 - 2\phi_3 + \phi_4) + \dots \\
& + (-1)^p\theta(\phi_{p-1} - 2\phi_p) + (-1)^{p+1}\phi_p]y_{t-p-2} \\
& + (-1)^{p+3}[\theta^{p+3} + \theta^{p+2}(2 + \phi_1) + \theta^{p+1}(1 + 2\phi_1 - \phi_2) \\
& + \theta^p(\phi_1 - 2\phi_2 + \phi_3) - \theta^{p-1}(\phi_2 - 2\phi_3 + \phi_4) + \dots \\
& + (-1)^p\theta^2(\phi_{p-1} - 2\phi_p) + (-1)^{p+1}\theta\phi_p]y_{t-p-3} + \dots
\end{aligned} \tag{2.127}$$

$$\begin{aligned}
\varepsilon_t = & -\frac{c}{(1+\theta)} + y_t - (\theta + \phi_1 + 2)y_{t-1} + [\theta^2 + \theta(2 + \phi_1) + (1 + 2\phi_1 - \phi_2)]y_{t-2} \\
& - [\theta^3 + \theta^2(2 + \phi_1) + \theta(1 + 2\phi_1 - \phi_2) + (\phi_1 - 2\phi_2 + \phi_3)]y_{t-3} \\
& + [\theta^4 + \theta^3(2 + \phi_1) + \theta^2(1 + 2\phi_1 - \phi_2) + \theta(\phi_1 - 2\phi_2 + \phi_3) \\
& - (\phi_2 - 2\phi_3 + \phi_4)]y_{t-4} - \dots \\
& + (-1)^{p+1}[\theta^{p+1} + \theta^p(\phi_1 + 2) + \theta^{p-1}(1 + 2\phi_1 - \phi_2) \\
& + \theta^{p-2}(\phi_1 - 2\phi_2 + \phi_3) - \theta^{p-3}(\phi_2 - 2\phi_3 + \phi_4) + \dots \\
& + \theta(\phi_{p-2} - 2\phi_{p-1} + \phi_p) + (-1)^{p+1}(\phi_{p-1} - 2\phi_p)]y_{t-p-1} \\
& + \sum_{i=p+2}^{\infty} (-1)^i [\theta^i + \theta^{i-1}(\phi_1 + 2) + \theta^{i-2}(1 + 2\phi_1 - \phi_2) \\
& + \theta^{i-3}(\phi_1 - 2\phi_2 + \phi_3) - \theta^{i-4}(\phi_2 - 2\phi_3 + \phi_4) + \dots \\
& + (-1)^p \theta^{i-p-1}(\phi_{p-1} - 2\phi_p) + (-1)^{p+1} \theta^{i-p-2} \phi_p] y_{t-i} \quad (2.128)
\end{aligned}$$

$$\varepsilon_t = -\frac{c}{(1+\theta)} + \sum_{i=0}^{\infty} \pi_i y_{t-i} \quad (2.129)$$

where

$$\pi_i = \begin{cases} 1, & i = 0 \\ -(\theta + \phi_1 + 2), & i = 1 \\ \theta^2 + \theta(2 + \phi_1) + (1 + 2\phi_1 - \phi_2), & i = 2 \\ -[\theta^3 + \theta^2(2 + \phi_1) + \theta(1 + 2\phi_1 - \phi_2) + (\phi_1 - 2\phi_2 + \phi_3)], & i = 3 \\ \theta^4 + \theta^3(2 + \phi_1) + \theta^2(1 + 2\phi_1 - \phi_2) \\ \quad + \theta(\phi_1 - 2\phi_2 + \phi_3) - (\phi_2 - 2\phi_3 + \phi_4), & i = 4 \\ -[\theta^5 + \theta^4(2 + \phi_1) + \theta^3(1 + 2\phi_1 - \phi_2) + \theta^2(\phi_1 - 2\phi_2 + \phi_3) \\ \quad - \theta(\phi_2 - 2\phi_3 + \phi_4) + (\phi_3 - 2\phi_4 + \phi_5)], & i = 5 \\ (-1)^{p+1}[\theta^{p+1} + \theta^p(\phi_1 + 2) + \theta^{p-1}(1 + 2\phi_1 - \phi_2) \\ \quad + \theta^{p-2}(\phi_1 - 2\phi_2 + \phi_3) - \theta^{p-3}(\phi_2 - 2\phi_3 + \phi_4) + \dots \\ \quad + \theta(\phi_{p-2} - 2\phi_{p-1} + \phi_p) + (-1)^{p+1}(\phi_{p-1} - 2\phi_p)], & i = p + 1 \\ (-1)^i [\theta^i + \theta^{i-1}(\phi_1 + 2) + \theta^{i-2}(1 + 2\phi_1 - \phi_2) \\ \quad - \theta^{i-4}(\phi_2 - 2\phi_3 + \phi_4) + \dots + (-1)^p \theta^{i-p-1}(\phi_{p-1} - 2\phi_p) \\ \quad + (-1)^{p+1} \theta^{i-p-2} \phi_p], & i = p + 2, p + 3, \dots \end{cases} \quad (2.130)$$

Equation (2.129) holds if $\theta \neq -1$.

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$$\Delta^3 y_t = c + \phi_1 \Delta^3 y_{t-1} + \phi_2 \Delta^3 y_{t-2} + \phi_3 \Delta^3 y_{t-3} + \dots + \phi_p \Delta^3 y_{t-p} + \varepsilon_t + \theta \varepsilon_{t-1} \quad (2.131)$$

$$\begin{aligned}
& \Delta(y_t - 2y_{t-1} + y_{t-2}) \\
&= c + \phi_1 \Delta(y_{t-1} - 2y_{t-2} + y_{t-3}) + \phi_2 \Delta(y_{t-2} - 2y_{t-3} + y_{t-4}) \\
&+ \phi_3 \Delta(y_{t-3} - 2y_{t-4} + y_{t-5}) + \dots + \phi_p \Delta(y_{t-p} - 2y_{t-p-1} + y_{t-p-2}) + \varepsilon_t \\
&+ \theta \varepsilon_{t-1}
\end{aligned}$$

$$\begin{aligned}
y_t - y_{t-1} - 2(y_{t-1} - y_{t-2}) + (y_{t-2} - y_{t-3}) \\
&= c + \phi_1 [(y_{t-1} - y_{t-2}) - 2(y_{t-2} - y_{t-3}) + (y_{t-3} - y_{t-4})] \\
&+ \phi_2 [(y_{t-2} - y_{t-3}) - 2(y_{t-3} - y_{t-4}) + (y_{t-4} - y_{t-5})] \\
&+ \phi_3 [(y_{t-3} - y_{t-4}) - 2(y_{t-4} - y_{t-5}) + (y_{t-5} - y_{t-6})] + \dots \\
&+ \phi_p [(y_{t-p} - y_{t-p-1}) - 2(y_{t-p-1} - y_{t-p-2}) + (y_{t-p-2} - y_{t-p-3})] + \varepsilon_t \\
&+ \theta \varepsilon_{t-1}
\end{aligned}$$

$$\begin{aligned}
y_t - 3y_{t-1} + 3y_{t-2} - y_{t-3} \\
&= c + \phi_1 (y_{t-1} - 3y_{t-2} + 3y_{t-3} - y_{t-4}) \\
&+ \phi_2 (y_{t-2} - 3y_{t-3} + 3y_{t-4} - y_{t-5}) + \phi_3 (y_{t-3} - 3y_{t-4} + 3y_{t-5} - y_{t-6}) \\
&+ \dots + \phi_p (y_{t-p} - 3y_{t-p-1} + 3y_{t-p-2} - y_{t-p-3}) + \varepsilon_t + \theta \varepsilon_{t-1}
\end{aligned}$$

$$\begin{aligned}
\varepsilon_t + \theta \varepsilon_{t-1} &= y_t - (3 + \phi_1)y_{t-1} + (3 + 3\phi_1 - \phi_2)y_{t-2} - (1 + 3\phi_1 - 3\phi_2 + \phi_3)y_{t-3} \\
&+ (\phi_{p-3} - 3\phi_{p-2} + 3\phi_{p-1} - \phi_p)y_{t-p} + (\phi_{p-2} - 3\phi_{p-1} + 3\phi_p)y_{t-p-1} \\
&+ (\phi_{p-1} - 3\phi_p)y_{t-p-2} + \phi_p y_{t-p-3} - c
\end{aligned}$$

$$\begin{aligned}
(1 + \theta L)\varepsilon_t &= y_t - (3 + \phi_1)y_{t-1} + (3 + 3\phi_1 - \phi_2)y_{t-2} - (1 + 3\phi_1 - 3\phi_2 + \phi_3)y_{t-3} \\
&+ (\phi_1 - 3\phi_2 + 3\phi_3 - \phi_4)y_{t-4} - (\phi_2 - 3\phi_3 + 3\phi_4 - \phi_5)y_{t-5} + \dots \\
&+ (\phi_{p-3} - 3\phi_{p-2} + 3\phi_{p-1} - \phi_p)y_{t-p} + (\phi_{p-2} - 3\phi_{p-1} + 3\phi_p)y_{t-p-1} \\
&+ (\phi_{p-1} - 3\phi_p)y_{t-p-2} + \phi_p y_{t-p-3} - c
\end{aligned}$$

$$\begin{aligned}
\varepsilon_t &= -(1 + \theta L)^{-1}c \\
&+ (1 + \theta L)^{-1} [y_t - (3 + \phi_1)y_{t-1} + (3 + 3\phi_1 - \phi_2)y_{t-2} \\
&- (1 + 3\phi_1 - 3\phi_2 + \phi_3)y_{t-3} + (\phi_1 - 3\phi_2 + 3\phi_3 - \phi_4)y_{t-4} \\
&- (\phi_2 - 3\phi_3 + 3\phi_4 - \phi_5)y_{t-5} + \dots + (\phi_{p-3} - 3\phi_{p-2} + 3\phi_{p-1} - \phi_p)y_{t-p} \\
&+ (\phi_{p-2} - 3\phi_{p-1} + 3\phi_p)y_{t-p-1} + (\phi_{p-1} - 3\phi_p)y_{t-p-2} + \phi_p y_{t-p-3}]
\end{aligned}$$



$$\begin{aligned}
\varepsilon_t = & -(1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots)c + (1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots)y_t \\
& - (3 + \phi_1)[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots]y_{t-1} \\
& + (3 + 3\phi_1 - \phi_2)[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots]y_{t-2} \\
& - (1 + 3\phi_1 - 3\phi_2 + \phi_3)[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots]y_{t-3} \\
& + (\phi_1 - 3\phi_2 + 3\phi_3 - \phi_4)[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots]y_{t-4} \\
& - (\phi_2 - 3\phi_3 + 3\phi_4 - \phi_5)[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots]y_{t-5} + \dots \\
& + (\phi_{p-2} - 3\phi_{p-1} + 3\phi_p)[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots]y_{t-p-1} \\
& + (\phi_{p-1} - 3\phi_p)[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots]y_{t-p-2} \\
& + \phi_p[1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \theta^4 L^4 - \dots]y_{t-p-3}
\end{aligned}$$

$$\begin{aligned}
\varepsilon_t = & -(1 - \theta + \theta^2 - \theta^3 + \theta^4 - \dots)c \\
& + (y_t - \theta y_{t-1} + \theta^2 y_{t-2} - \theta^3 y_{t-3} + \theta^4 y_{t-4} - \theta^5 y_{t-5} + \theta^6 y_{t-6} - \dots) \\
& - (3 + \phi_1)(y_{t-1} - \theta y_{t-2} + \theta^2 y_{t-3} - \theta^3 y_{t-4} + \theta^4 y_{t-5} - \theta^5 y_{t-6} + \dots) \\
& + (3 + 3\phi_1 - \phi_2)(y_{t-2} - \theta y_{t-3} + \theta^2 y_{t-4} - \theta^3 y_{t-5} + \theta^4 y_{t-6} - \dots) \\
& - (1 + 3\phi_1 - 3\phi_2 + \phi_3)(y_{t-3} - \theta y_{t-4} + \theta^2 y_{t-5} - \theta^3 y_{t-6} + \theta^4 y_{t-7} - \dots) \\
& + (\phi_1 - 3\phi_2 + 3\phi_3 - \phi_4)(y_{t-4} - \theta y_{t-5} + \theta^2 y_{t-6} - \theta^3 y_{t-7} + \dots) + \dots \\
& + (-1)^{p+1}(\phi_{p-2} - 3\phi_{p-1} + 3\phi_p)(y_{t-p-1} - \theta y_{t-p-2} + \theta^2 y_{t-p-3} \\
& - \theta^3 y_{t-p-4} + \dots) \\
& + (-1)^{p+2}(\phi_{p-1} - 3\phi_p)(y_{t-p-2} - \theta y_{t-p-3} + \theta^2 y_{t-p-4} - \theta^3 y_{t-p-5} + \dots) \\
& + (-1)^{p+3}\phi_p(y_{t-p-3} - \theta y_{t-p-4} + \theta^2 y_{t-p-5} - \theta^3 y_{t-p-6} + \dots)
\end{aligned}$$

$$\begin{aligned}
\varepsilon_t = & -(1 + \theta)^{-1}c + y_t - (\theta + \phi_1 + 3)y_{t-1} + [\theta^2 + \theta(3 + \phi_1) + (3 + 3\phi_1 - \phi_2)]y_{t-2} \\
& - [\theta^3 + \theta^2(3 + \phi_1) + \theta(3 + 3\phi_1 - \phi_2) + (1 + 3\phi_1 - 3\phi_2 + \phi_3)]y_{t-3} \\
& + [\theta^4 + \theta^3(3 + \phi_1) + \theta^2(3 + 3\phi_1 - \phi_2) + \theta(1 + 3\phi_1 - 3\phi_2 + \phi_3) \\
& + (\phi_1 - 3\phi_2 + 3\phi_3 - \phi_4)]y_{t-4} - \dots \\
& + (-1)^p[\theta^{p+2} + \theta^{p+1}(3 + \phi_1) + \theta^p(3 + 3\phi_1 - \phi_2) \\
& + \theta^{p-1}(1 + 3\phi_1 - 3\phi_2 + \phi_3) + \theta^{p-2}(\phi_1 - 3\phi_2 + 3\phi_3 - \phi_4) + \dots \\
& + (-1)^{p+1}\theta(\phi_{p-2} - 3\phi_{p-1} + 3\phi_p) + (-1)^{p+2}(\phi_{p-1} - 3\phi_p)] \\
& + (-1)^{p+1}[\theta^{p+3} + \theta^{p+2}(3 + \phi_1) + \theta^{p+1}(3 + 3\phi_1 - \phi_2) \\
& + \theta^p(1 + 3\phi_1 - 3\phi_2 + \phi_3) + \theta^{p-1}(\phi_1 - 3\phi_2 + 3\phi_3 - \phi_4) + \dots \\
& + (-1)^{p+1}\theta^2(\phi_{p-2} - 3\phi_{p-1} + 3\phi_p) + (-1)^{p+2}\theta(\phi_{p-1} - 3\phi_p) \\
& + (-1)^{p+3}\phi_p]y_{t-p-3} + \dots
\end{aligned} \tag{2.132}$$

$$\begin{aligned}
\varepsilon_t = & -\frac{c}{(1+\theta)} + y_t - (\theta + \phi_1 + 3)y_{t-1} + [\theta^2 + \theta(3 + \phi_1) + (3 + 3\phi_1 - \phi_2)]y_{t-2} \\
& - [\theta^3 + \theta^2(3 + \phi_1) + \theta(3 + 3\phi_1 - \phi_2) + (1 + 3\phi_1 - 3\phi_2 + \phi_3)]y_{t-3} \\
& + [\theta^4 + \theta^3(3 + \phi_1) + \theta^2(3 + 3\phi_1 - \phi_2) + \theta(1 + 3\phi_1 - 3\phi_2 + \phi_3) \\
& + (\phi_1 - 3\phi_2 + 3\phi_3 - \phi_4)]y_{t-4} + \dots \\
& + (-1)^{p+2}[\theta^{p+2} + \theta^{p+1}(3 + \phi_1) + \theta^p(3 + 3\phi_1 - \phi_2) \\
& + \theta^{p-1}(1 + 3\phi_1 - 3\phi_2 + \phi_3) + \dots \\
& + (-1)^p\theta^2(\phi_{p-3} - 3\phi_{p-2} + 3\phi_{p-1} - \phi_p) \\
& + (-1)^{p+1}\theta(\phi_{p-2} - 3\phi_{p-1} + 3\phi_p) + (-1)^{p+2}(\phi_{p-1} - 3\phi_p)]y_{t-p-2} \\
& + \sum_{i=p+3}^{\infty} (-1)^i [\theta^i + \theta^{i-1}(\phi_1 + 3) + \theta^{i-2}(3 + 3\phi_1 - \phi_2) \\
& + \theta^{i-3}(1 + 3\phi_1 - 3\phi_2 + \phi_3) + \theta^{i-4}(\phi_1 - 3\phi_2 + 3\phi_3 - \phi_4) + \dots \\
& + (-1)^{p+1}\theta^{i-p-1}(\phi_{p-2} - 3\phi_{p-1} + 3\phi_p) + (-1)^{p+2}\theta^{i-p-2}(\phi_{p-1} - 3\phi_p) \\
& + (-1)^{p+3}\theta^{i-p-3}\phi_p]y_{t-i}
\end{aligned} \tag{2.133}$$

$$\varepsilon_t = -\frac{c}{(1+\theta)} + \sum_{i=0}^{\infty} \pi_i y_{t-i} \tag{2.134}$$

where

$$\pi_i = \begin{cases} 1, & i = 0 \\ -(\theta + \phi_1 + 3), & i = 1 \\ \theta^2 + \theta(3 + \phi_1) + (3 + 3\phi_1 - \phi_2), & i = 2 \\ -[\theta^3 + \theta^2(3 + \phi_1) + \theta(3 + 3\phi_1 - \phi_2) + (1 + 3\phi_1 - 3\phi_2 + \phi_3)], & i = 3 \\ \theta^4 + \theta^3(3 + \phi_1) + \theta^2(3 + 3\phi_1 - \phi_2) \\ \quad + \theta(1 + 3\phi_1 - 3\phi_2 + \phi_3) + (\phi_1 - 3\phi_2 + 3\phi_3 - \phi_4), & i = 4 \\ (-1)^{p+2}[\theta^{p+2} + \theta^{p+1}(3 + \phi_1) + \theta^p(3 + 3\phi_1 - \phi_2) + \\ \quad \dots + \theta(\phi_{p-2} - 3\phi_{p-1} + 3\phi_p) + (\phi_{p-1} - 3\phi_p)], & i = p + 2 \\ (-1)^i [\theta^i + \theta^{i-1}(\phi_1 + 3) + \theta^{i-2}(3 + 3\phi_1 - \phi_2) \\ \quad + \theta^{i-3}(1 + 3\phi_1 - 3\phi_2 + \phi_3) + \dots + \theta^{i-p-1}(\phi_{p-2} - 3\phi_{p-1} + 3\phi_p) \\ \quad + \theta^{i-p-2}(\phi_{p-1} - 3\phi_p) + \theta^{i-p-3}\phi_p], & i = p + 3, p + 4, \dots \end{cases} \tag{2.135}$$

Equation (2.134) holds if $\theta \neq -1$.

The expression for ARIMA (p, d, I) for various values of $p \geq 1$ holds if $\theta \neq -1$ for the case where constant term c exists but holds for all values of θ if there is no constant term.

2.11 BEHAVIOURAL PATTERN OF CAUSALITY PARAMETERS OF ARMA (1, q) MODEL

Given that $\theta_k > \theta_{k+1} > \theta_{k+2} > \dots > \theta_{k+q} > 0$ for various values of q , the causality parameter ψ_i of ARMA (1, q) is skewed to right for positive values of ϕ and sinusoidal for negative values of ϕ as shown in Figures 2.1 – 2.3 below. The absolute value of causality parameter $|\psi_i|$ rises as d increases for positive values of ϕ but smaller the value of d , the bigger the absolute value of causality parameter ψ_i for negative values of ϕ . Also, ψ_i increases as ϕ increases as shown in Figure 2.4 for positive values of ϕ .

i	Causality parameter ψ_i of some ARMA (p, q) process			
	ARMA (1, 1)	ARMA (1, 2)	ARMA (1, 3)	ARMA (1, 4)
0	1.00000	1.00000	1.00000	1.00000
1	1.30000	1.30000	1.30000	1.30000
2	0.78000	0.88000	0.88000	0.88000
3	0.46800	0.52800	0.57800	0.57800
4	0.28080	0.31680	0.34680	0.36680
5	0.16848	0.19008	0.20808	0.22008
6	0.10109	0.11405	0.12485	0.13205
7	0.06065	0.06843	0.07491	0.07923
8	0.03639	0.04106	0.04495	0.04754
9	0.02184	0.02463	0.02697	0.02852
10	0.01310	0.01478	0.01618	0.01711
11	0.00786	0.00887	0.00971	0.01027

Table 2.1: Table showing behaviour of ψ_i of ARIMA (1, q) for some i given that $\phi = 0.6$, $\theta_1 = 0.7$, $\theta_2 = 0.1$, $\theta_3 = 0.05$ and $\theta_4 = 0.02$.

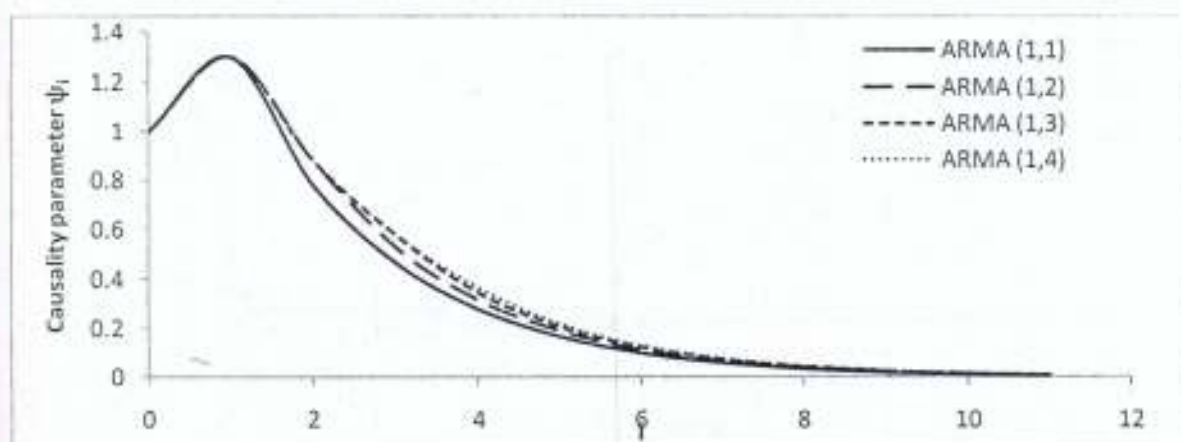


Figure 2.1: Figure showing behaviour of causality parameter ψ_i of ARMA (1, q) for some i given that $\phi = 0.6$, $\theta_1 = 0.7$, $\theta_2 = 0.1$, $\theta_3 = 0.05$ and $\theta_4 = 0.02$.

i	Causality parameter ψ_i of some ARMA (p, q) process			
	ARMA (1, 1)	ARMA (1, 2)	ARMA (1, 3)	ARMA (1, 4)
0	1.000000	1.000000	1.000000	1.000000
1	0.100000	0.100000	0.100000	0.100000
2	-0.060000	0.040000	0.040000	0.040000
3	0.036000	-0.024000	0.026000	0.026000
4	-0.021600	0.014400	-0.015600	0.004400
5	0.012960	-0.008640	0.009360	-0.002640
6	-0.007780	0.005184	-0.005620	0.001584
7	0.004666	-0.00311	0.003370	-0.000950
8	-0.002800	0.001866	-0.002020	0.000570
9	0.001680	-0.001120	0.001213	-0.000340
10	-0.001010	0.000672	-0.000730	0.000205

Table 2.2: Table showing behaviour of ψ_i of ARMA (l, q) for some i given that $\phi = -0.6$, $\theta_1 = 0.7$, $\theta_2 = 0.1$, $\theta_3 = 0.05$ and $\theta_4 = 0.02$.

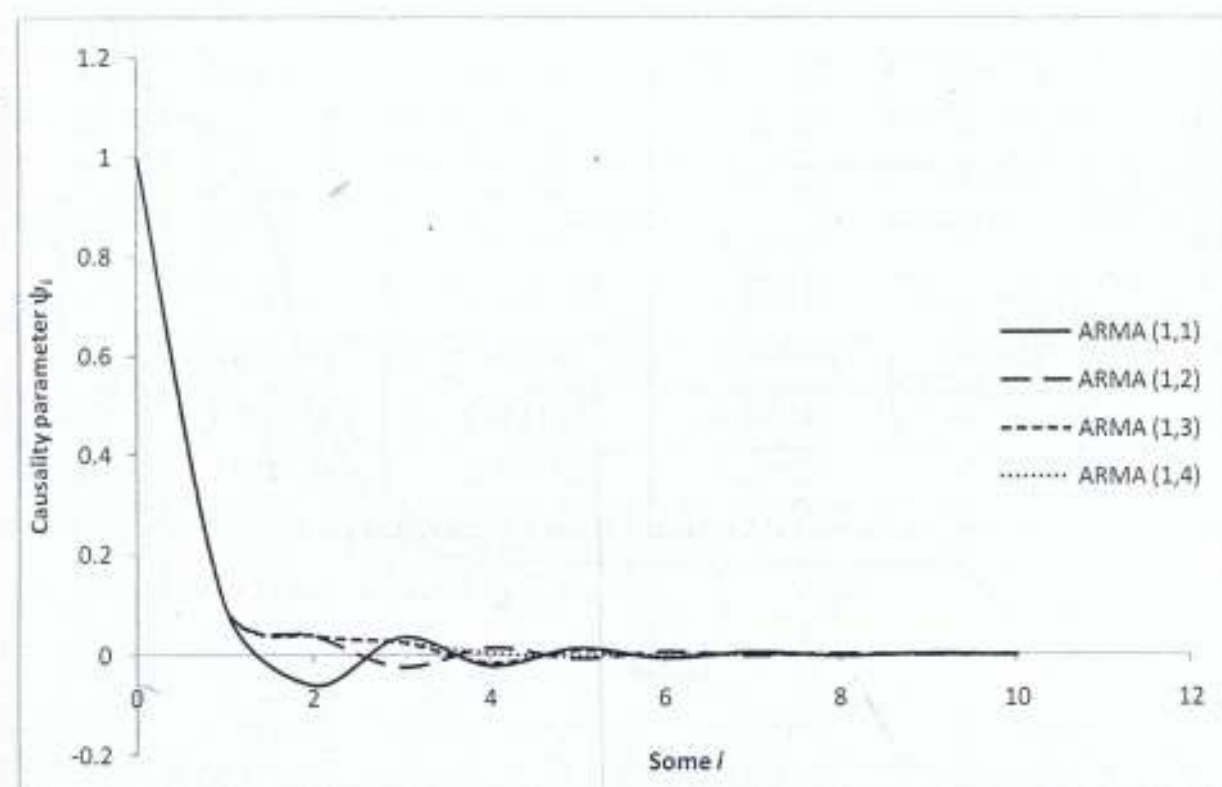


Figure 2.2: Figure showing behaviour of ψ_i of ARMA (l, q) for some i given that $\phi = -0.6$, $\theta_1 = 0.7$, $\theta_2 = 0.1$, $\theta_3 = 0.05$ and $\theta_4 = 0.02$.

i	Causality Parameter of some ARMA process			
	ARMA (1, 1)	ARMA (1, 2)	ARMA (1, 3)	ARMA (1, 4)
0	1.00000	1.00000	1.00000	1.00000
1	0.90000	0.90000	0.90000	0.90000
2	0.18000	0.28000	0.28000	0.28000
3	0.03600	0.05600	0.10600	0.10600
4	0.00720	0.01120	0.02120	0.04120
5	0.00144	0.00224	0.00424	0.00824
6	0.00029	0.00049	0.00085	0.00165
7	0.00006	0.00009	0.00017	0.00033
8	0.00001	0.00002	0.00003	0.00007
9	0.00000	0.00000	0.00001	0.00001
10	0.00000	0.00000	0.00000	0.00000

Table 2.3: Table showing behaviour of ψ_i of ARMA (l, q) for some i given that $\phi = 0.2$, $\theta_1 = 0.7$, $\theta_2 = 0.1$, $\theta_3 = 0.05$ and $\theta_4 = 0.02$.

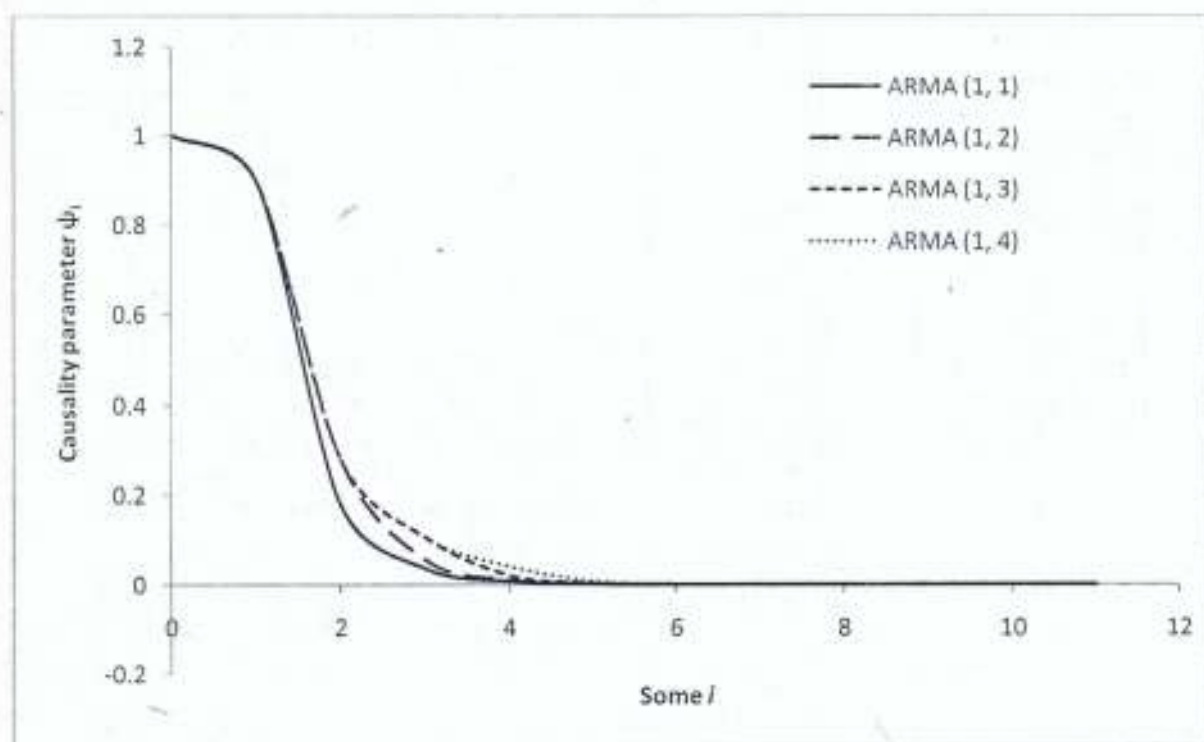


Figure 2.3: Figure showing behaviour of ψ_i of ARMA (l, q) for some i given that $\phi = 0.2$, $\theta_1 = 0.7$, $\theta_2 = 0.1$, $\theta_3 = 0.05$ and $\theta_4 = 0.02$.

i	Causality Parameter of some ARMA process		
	$\phi=0.2$	$\phi=0.4$	$\phi=0.6$
0	1.00000	1.00000	1.00000
1	0.90000	1.10000	1.30000
2	0.18000	0.44000	0.78000
3	0.03600	0.17600	0.46800
4	0.00720	0.07040	0.28080
5	0.00144	0.02816	0.16848
6	0.00029	0.01126	0.10109
7	0.00006	0.0045	0.06065
8	0.00001	0.0018	0.03639
9	0.00000	0.00072	0.02184
10	0.00000	0.00029	0.0131

Table 2.4: Table showing behaviour of ψ_i of ARMA $(1, q)$ for some i given that $\theta_1 = 0.7$, $\theta_2 = 0.1$, $\theta_3 = 0.05$ and $\theta_4 = 0.02$ for some real values of ϕ .

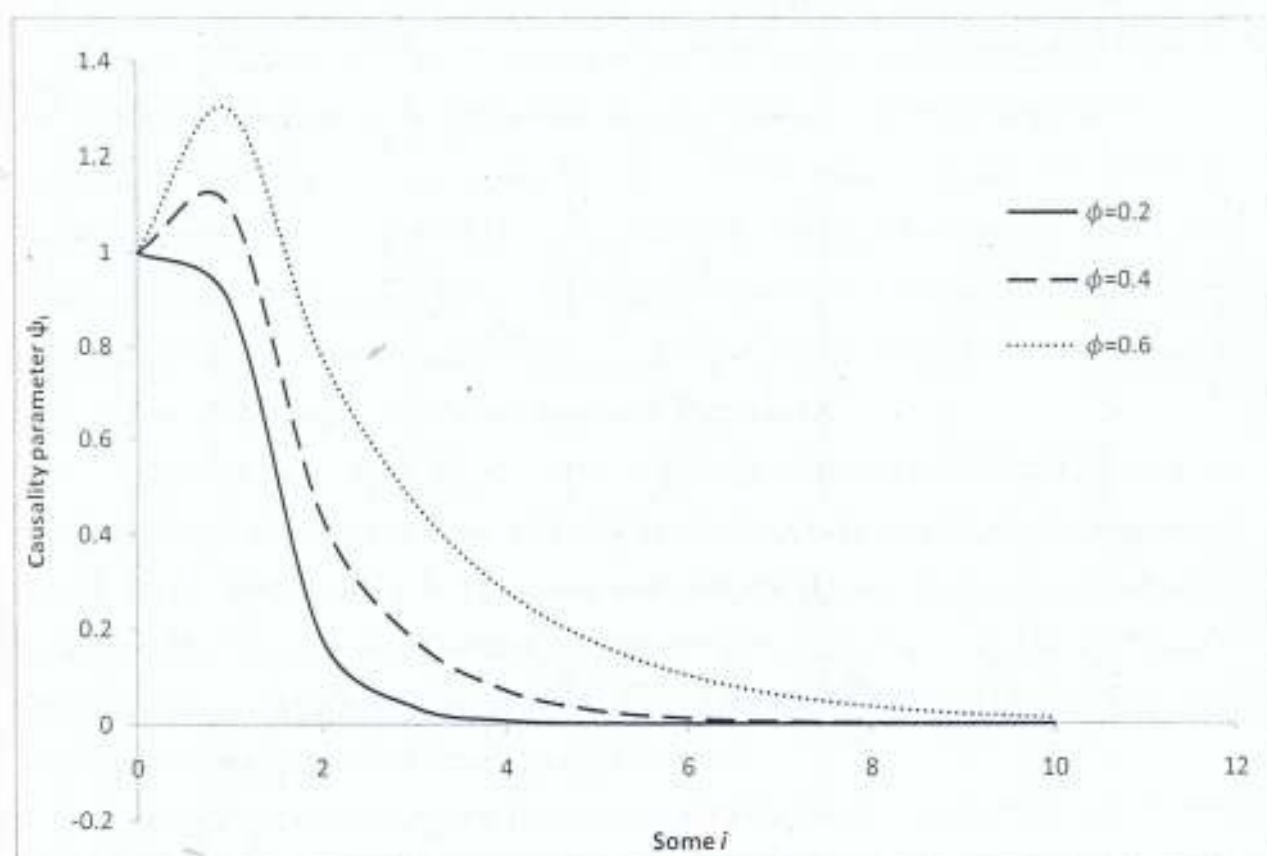


Figure 2.4: Figure showing behaviour of ψ_i of ARMA $(1, q)$ for some i given that $\theta_1 = 0.7$, $\theta_2 = 0.1$, $\theta_3 = 0.05$ and $\theta_4 = 0.02$ for some real values of ϕ .

2.12 BEHAVIOURAL PATTERN OF INVERTIBILITY PARAMETERS OF ARIMA ($p, d, 1$) MODEL

Here we consider the behaviour of π_i for some i for some orders of ϕ_i where $i = 1, 2, 3, \dots, p$.

(a) Given that $\phi_1 > \phi_2 > \phi_3 > \dots > \phi_p > 0$, we examine the behavioural pattern of π_i for both positive and negative values of θ and obtain the following result for $d = 0, 1, 2$ and 3 . Given that $\phi_1 > \phi_2 > \phi_3 > \dots > \phi_p > 0$ for various values of p , the invertibility parameter π_i of ARIMA ($p, d, 1$) are sinusoidal and the absolute value of the invertibility parameter, $|\pi_i|$ increases as d increases for positive values of θ . That is $|\pi_i|_{d=3} > |\pi_i|_{d=2} > |\pi_i|_{d=1} > |\pi_i|_{d=0}$ which implies that the lower the integer value of d , the faster $|\pi_i|$ converges to zero as shown in Table 2.5 and Figure 2.5 below. For negative integer values of θ , it is seen from Table 2.6 and Figure 2.6 that the invertibility parameter π_i of ARIMA ($p, d, 1$) model tends to oscillate as the value of d increases with $|\pi_i|_{d=k} < |\pi_i|_{d=k+2}$ for all even integer value $k > 0$ provided $i \geq 3$ and $|\pi_i|_{d=k} > |\pi_i|_{d=k+2}$ for all odd integer value $k > 0$. The invertibility parameter π_i of ARIMA ($p, 0, 1$) model is sinusoidal for $p = 1, 2, 3$ and 4 with $|\pi_i|_{p=4} > |\pi_i|_{p=3} > |\pi_i|_{p=2} > |\pi_i|_{p=1}$ for all positive values of θ as seen in Table 2.7 and Figure 2.7. For all negative values of θ , $|\pi_i|_{p=k} < |\pi_i|_{p=k+2}$ for odd integer value $k > 0$ and $|\pi_i|_{p=k} > |\pi_i|_{p=k+2}$ for even integer value $k > 0$ as seen in Figure 2.8. Similarly, invertibility parameter π_i of ARIMA ($p, 1, 1$) model is sinusoidal for $p = 1, 2, 3$ and 4 with $|\pi_i|_{p=4} > |\pi_i|_{p=3} > |\pi_i|_{p=2} > |\pi_i|_{p=1}$ for all positive values of θ as seen in Figure 2.7. For all negative values of θ , $|\pi_i|_{p=k} < |\pi_i|_{p=k+2}$ for odd integer value $k > 0$ and $|\pi_i|_{p=k} > |\pi_i|_{p=k+2}$ for even integer value $k > 0$ as seen in Figure 2.10.

(b) Given that $\phi_1 = \phi_2 = \phi_3 = \dots = \phi_p > 0$, we examine the behavioural pattern of π_i for positive of θ and obtain the following result for $d = 0, 1, 2$ and 3 , the invertibility parameter π_i of ARIMA ($p, d, 1$) are sinusoidal and the absolute value of the invertibility parameter, $|\pi_i|$ increases as d increases for positive values of θ . That is $|\pi_i|_{d=3} > |\pi_i|_{d=2} > |\pi_i|_{d=1} > |\pi_i|_{d=0}$ which implies that the lower the integer value of d , the faster $|\pi_i|$ converges to zero as shown in Figure 2.11 below.

(c) We examine the behavioural pattern of π_i for positive of θ and obtain the following result for $d = 0, 1, 2$ and 3 . Given that $0 < \phi_1 < \phi_2 < \phi_3 < \dots < \phi_p$ for various values of p , the invertibility parameter π_i of ARIMA ($4, d, 1$) are sinusoidal and the absolute value of the invertibility parameter, $|\pi_i|$ increases as d increases for positive values of θ . That is

$|\pi_i|_{d=3} > |\pi_i|_{d=2} > |\pi_i|_{d=1} > |\pi_i|_{d=0}$ which implies that the lower the integer value of d , the faster $|\pi_i|$ converges to zero as shown in Figure 2.12 below.

(d) We examine the behavioural pattern of π_i for different ϕ_p 's arrangement, it is seen in Figure 2.13 that $|\pi_i|$ is the greatest for decreasing order of ϕ_p (i.e. $\phi_1 > \phi_2 > \phi_3 > \dots > \phi_p > 0$), follow by for equal order of ϕ_p ($\phi_1 = \phi_2 = \phi_3 = \dots = \phi_p > 0$) and least for increasing order of ϕ_p ($0 < \phi_1 < \phi_2 < \phi_3 < \dots < \phi_p$).

i	Invertibility parameter π_i of some ARIMA $(p, d, 1)$ process			
	$d = 0$	$d = 1$	$d = 2$	$d = 3$
0	1.000000	1.000000	1.000000	1.000000
1	-0.900000	-1.900000	-2.900000	-3.900000
2	0.340000	1.240000	3.140000	6.040000
3	-0.304000	-0.644000	-1.884000	-5.024000
4	0.132400	0.436400	1.080400	2.964400
5	-0.104440	-0.236840	-1.321480	-1.803640
6	0.050164	0.154604	0.391444	1.094684
7	-0.036350	-0.086510	-0.241120	-0.663060
8	0.018684	0.055032	0.141545	0.400961
9	-0.01277	-0.031460	-0.086490	-0.242140
10	0.006883	0.019655	0.051112	0.140018

Table 2.5: Table showing behaviour of invertibility parameter π_i of ARIMA $(p, d, 1)$ for some i when $d = 0, 1, 2$ and 3, given that $\theta = 0.6$, $\phi_1 = 0.3$, $\phi_2 = 0.2$, $\phi_3 = 0.1$, $\phi_4 = 0.05$, $\phi_5 = 0.025$, $\phi_6 = 0.0125$, $\phi_7 = 0.00625$, $\phi_8 = 0.003125$, $\phi_9 = 0.0015625$ and $\phi_{10} = 0.00078125$.

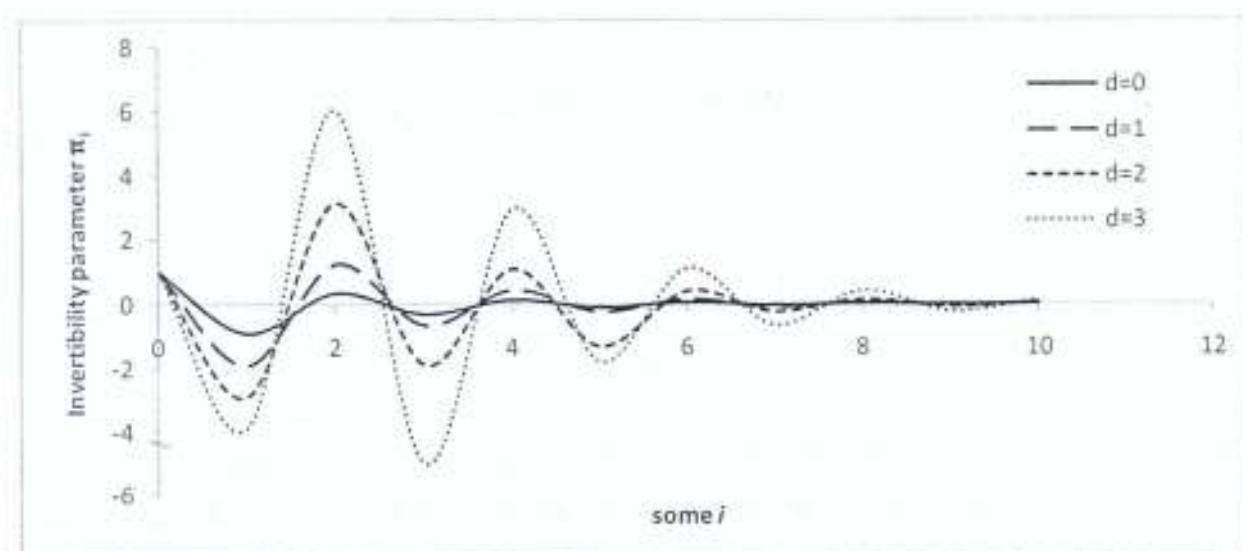


Figure 2.5: Graph showing behaviour of invertibility parameter π_i of ARIMA $(p, d, 1)$ for some i when $d = 0, 1, 2$ and 3, given that $\theta = 0.6$, $\phi_1 = 0.3$, $\phi_2 = 0.2$, $\phi_3 = 0.1$, $\phi_4 = 0.05$, $\phi_5 = 0.025$, $\phi_6 = 0.0125$, $\phi_7 = 0.00625$, $\phi_8 = 0.003125$, $\phi_9 = 0.0015625$ and $\phi_{10} = 0.00078125$.

i	Invertibility parameter π_i of some ARIMA ($p, d, 1$) process			
	$d = 0$	$d = 1$	$d = 2$	$d = 3$
0	1.00000	1.00000	1.00000	1.00000
1	0.30000	-0.70000	1.70000	-2.70000
2	-0.02000	-0.32000	0.38000	2.08000
3	-0.11200	-0.09200	0.22800	-0.15200
4	-0.11720	-0.00520	0.08680	-0.14120
5	-0.09532	0.02188	0.02710	-0.05972
6	-0.06969	0.025628	0.00375	-0.05333
7	-0.04807	0.021627	-0.00400	-0.02575
8	-0.03196	0.016101	-0.00553	-0.01983
9	-0.0207	0.011223	-0.00488	-0.01033
10	-0.00848	0.005895	-0.00371	-0.00729

Table 2.6: Table showing behaviour of invertibility parameter π_i of ARIMA ($p, d, 1$) for some i when $d = 0, 1, 2$ and 3 , given that $\theta = -0.6$, $\phi_1 = 0.3$, $\phi_2 = 0.2$, $\phi_3 = 0.1$, $\phi_4 = 0.05$, $\phi_5 = 0.025$, $\phi_6 = 0.0125$, $\phi_7 = 0.00625$, $\phi_8 = 0.003125$, $\phi_9 = 0.0015625$ and $\phi_{10} = 0.00078125$.

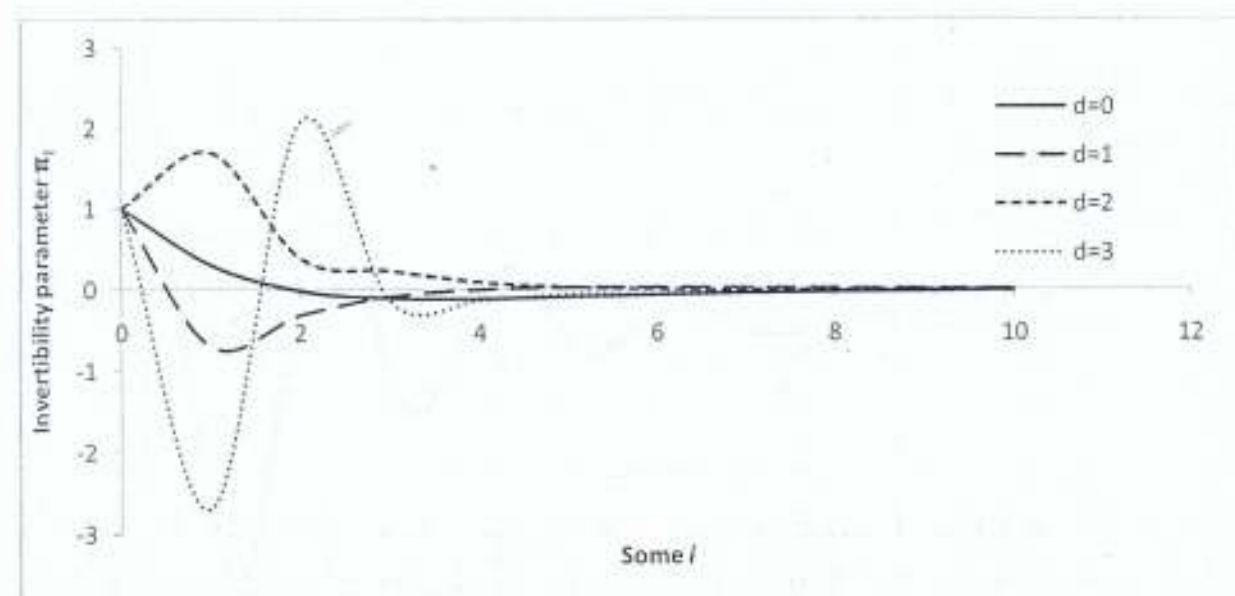


Figure 2.6: Graph showing behaviour of invertibility parameter π_i of ARIMA ($p, d, 1$) for some i when $d = 0, 1, 2$ and 3 , given that $\theta = -0.6$, $\phi_1 = 0.3$, $\phi_2 = 0.2$, $\phi_3 = 0.1$, $\phi_4 = 0.05$, $\phi_5 = 0.025$, $\phi_6 = 0.0125$, $\phi_7 = 0.00625$, $\phi_8 = 0.003125$, $\phi_9 = 0.0015625$ and $\phi_{10} = 0.00078125$.

i	Invertibility parameter π_i of some ARIMA ($p, d, 1$) process			
	ARIMA (1,0,1)	ARIMA (2,0,1)	ARIMA (3,0,1)	ARIMA (4,0,1)
0	1.00000	1.00000	1.00000	1.00000
1	-0.90000	-0.90000	-0.90000	-0.90000
2	0.54000	0.34000	0.34000	0.34000
3	-0.32400	-0.20400	-0.30400	-0.30400
4	0.19440	0.12240	0.18240	0.13240
5	-0.11664	-0.07344	-0.10944	-0.07944
6	0.06998	0.04406	0.06566	0.04766
7	-0.04199	-0.02644	-0.03940	-0.02860
8	0.02519	0.01586	0.02364	0.01716
9	-0.01512	-0.00952	-0.01418	-0.01030
10	0.00910	0.00571	0.00851	0.00618

Table 2.7: Table showing behaviour of invertibility parameter π_i of ARIMA (p, θ, I) for some i when $p = 1, 2, 3$ and 4 , given that $\theta = 0.6$, $\phi_1 = 0.3$, $\phi_2 = 0.2$, $\phi_3 = 0.1$ and $\phi_4 = 0.05$

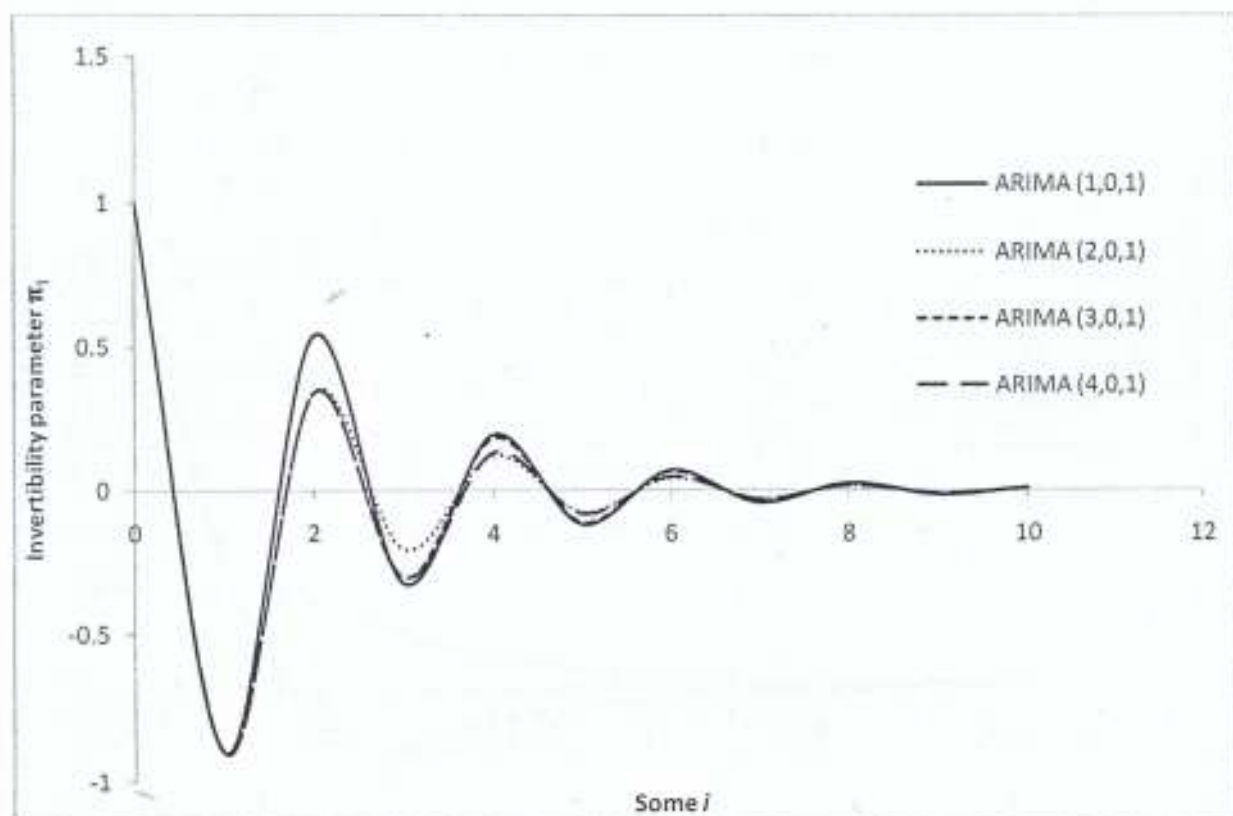


Figure 2.7: Graph showing behaviour of invertibility parameter π_i of ARIMA (p, θ, I) for some i when $p = 1, 2, 3$ and 4 , given that $\theta = 0.6$, $\phi_1 = 0.3$, $\phi_2 = 0.2$, $\phi_3 = 0.1$ and $\phi_4 = 0.05$.

i	Invertibility parameter π_i of some ARIMA ($p, d, 1$) process			
	ARIMA (1,0,1)	ARIMA (2,0,1)	ARIMA (3,0,1)	ARIMA (4,0,1)
0	1.00000	1.00000	1.00000	1.00000
1	0.30000	0.30000	0.30000	0.30000
2	0.18000	-0.02000	-0.02000	-0.02000
3	0.10800	-0.01200	-0.11200	-0.11200
4	0.06480	-0.00720	-0.06720	-0.11720
5	0.03888	-0.00432	-0.04032	-0.07032
6	0.02333	-0.00259	-0.02419	-0.04219
7	0.01400	-0.00156	-0.01452	-0.02532
8	0.00840	-0.00093	-0.00871	-0.01519
9	0.00504	-0.00056	-0.00523	-0.00911
10	0.00302	-0.00034	-0.00314	-0.00547

Table 2.8: Table showing behaviour of invertibility parameter π_i of ARIMA ($p, 0, 1$) for some i when $p = 1, 2, 3$ and 4, given that $\theta = -0.6$, $\phi_1 = 0.3$, $\phi_2 = 0.2$, $\phi_3 = 0.1$ and $\phi_4 = 0.05$.

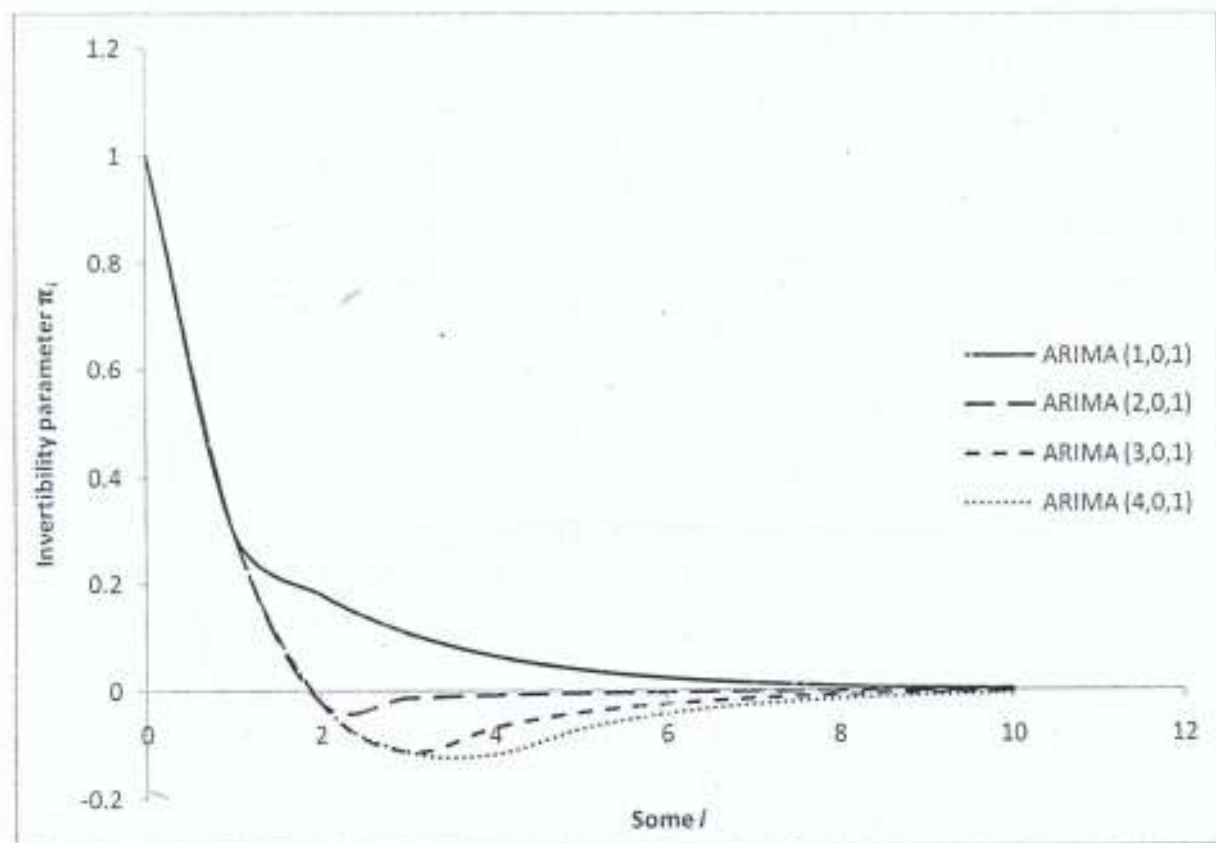


Figure 2.8: Graph showing behaviour of invertibility parameter π_i of ARIMA ($p, 0, 1$) for some i when $p = 1, 2, 3$ and 4, given that $\theta = -0.6$, $\phi_1 = 0.3$, $\phi_2 = 0.2$, $\phi_3 = 0.1$ and $\phi_4 = 0.05$.

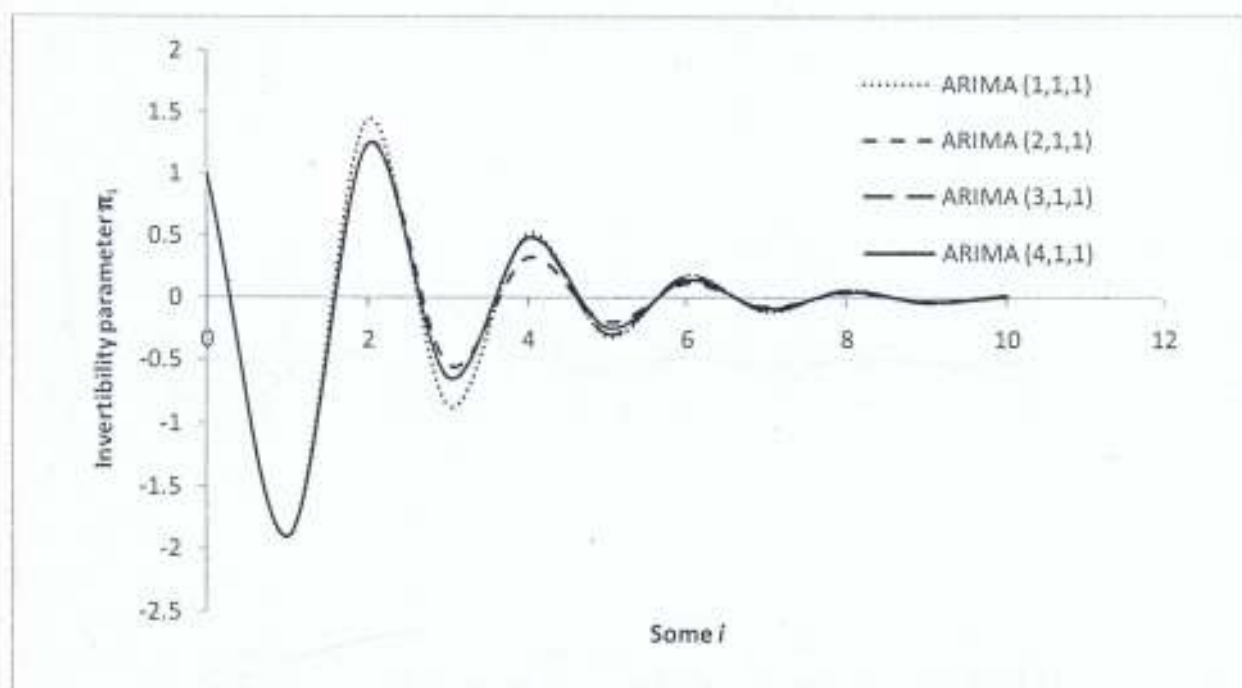


Figure 2.9: Graph showing behaviour of invertibility parameter π_i of ARIMA $(p, 1, 1)$ for some i when $p = 1, 2, 3$ and 4 , given that $\theta = 0.6$, $\phi_1 = 0.3$, $\phi_2 = 0.2$, $\phi_3 = 0.1$ and $\phi_4 = 0.05$.

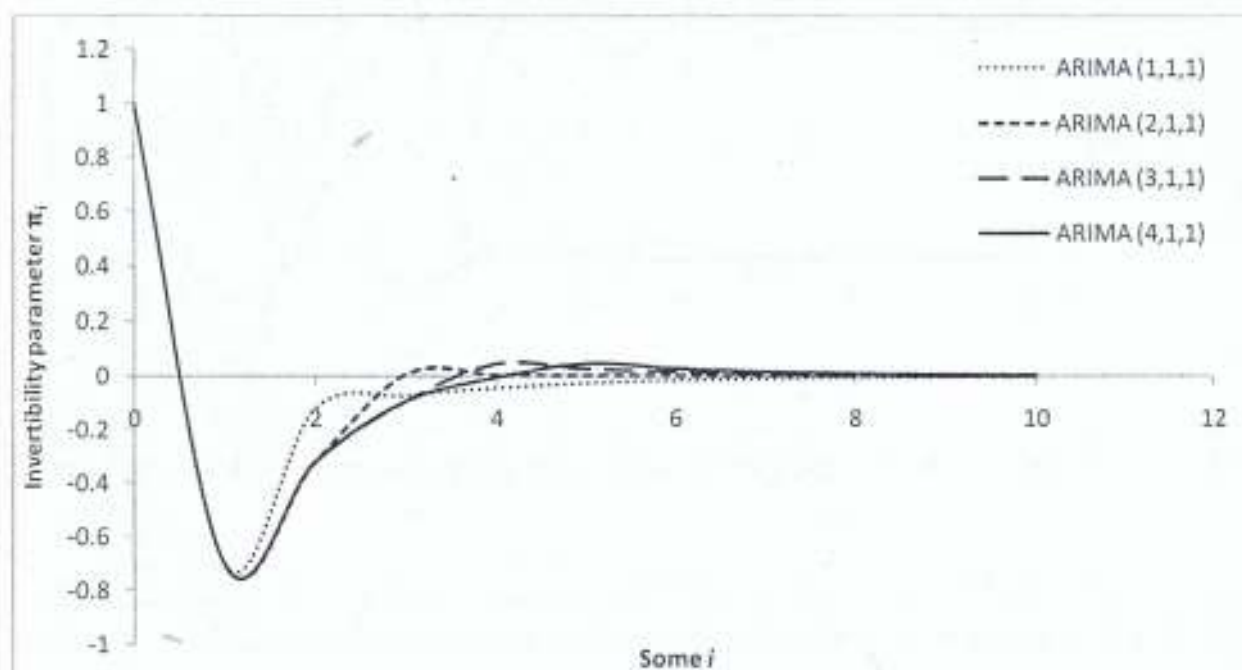


Figure 2.10: Graph showing behaviour of invertibility parameter π_i of ARIMA $(p, 1, 1)$ for some i when $p = 1, 2, 3$ and 4 , given that $\theta = -0.6$, $\phi_1 = 0.3$, $\phi_2 = 0.2$, $\phi_3 = 0.1$ and $\phi_4 = 0.05$.

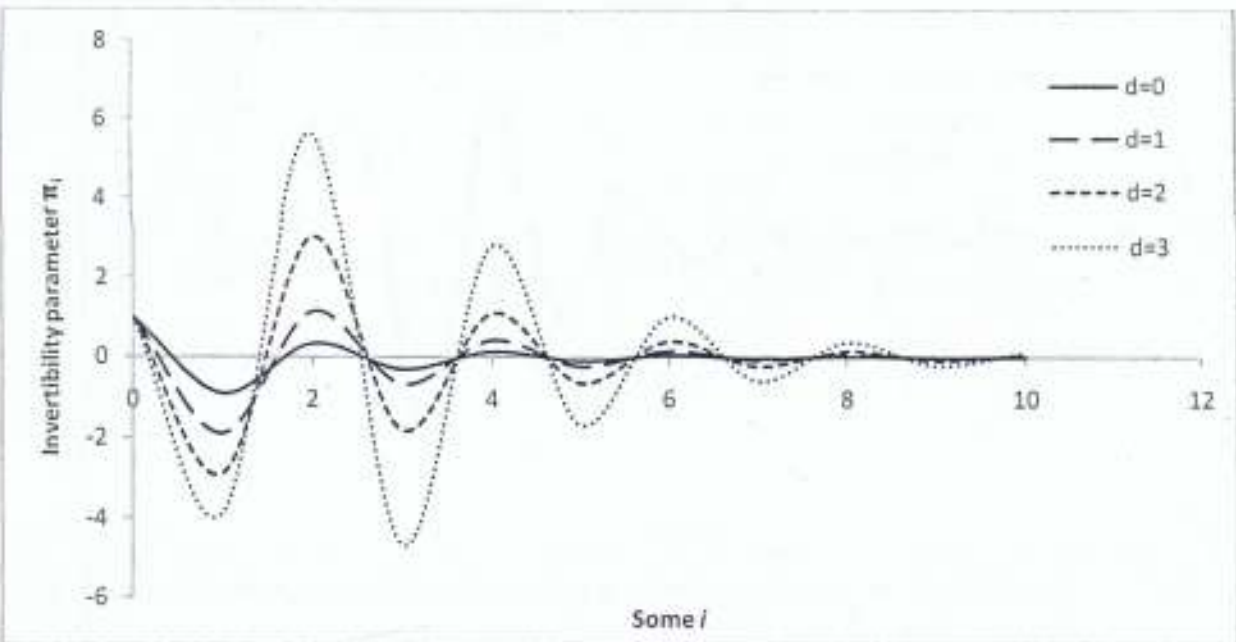


Figure 2.11: Graph showing behaviour of invertibility parameter π_i of ARIMA $(p, d, 1)$ for some i when $d = 0, 1, 2$ and 3 , given that $\theta = 0.6, \phi_1 = \phi_2 = \dots = \phi_{10} = 0.3$.

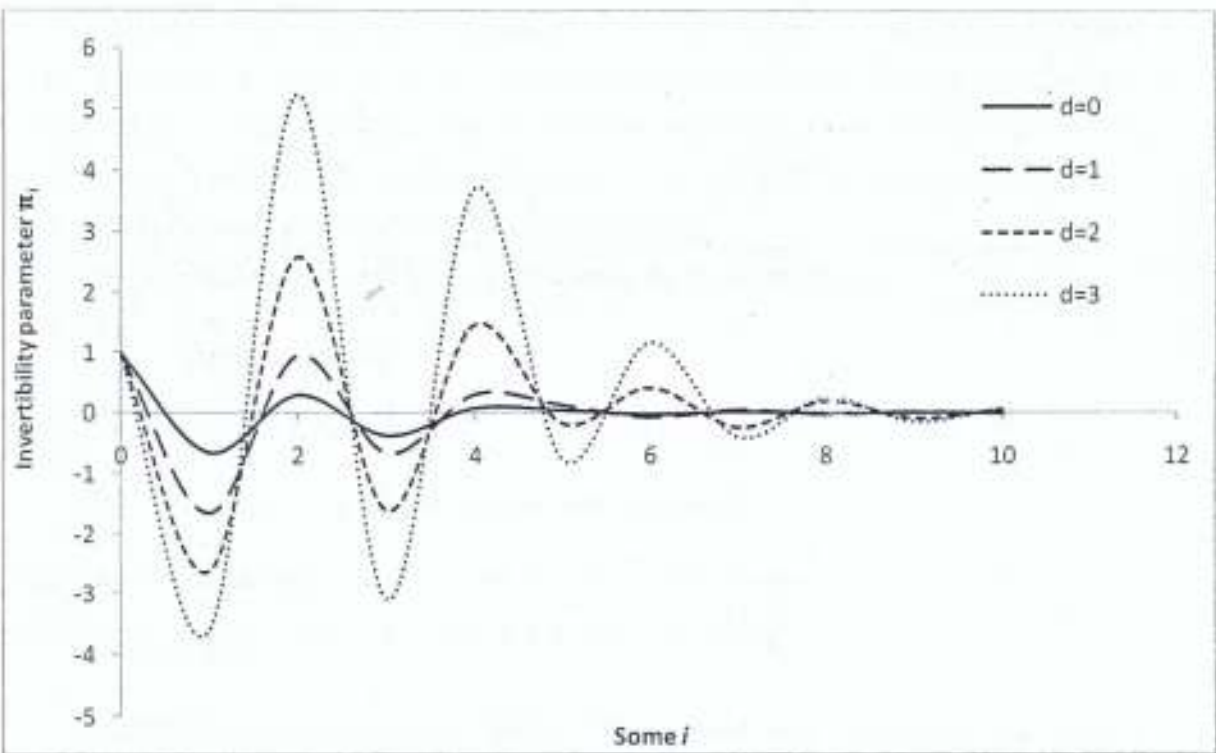


Figure 2.12: Graph showing behaviour of invertibility parameter π_i of ARIMA $(4, d, 1)$ for some i when $d = 0, 1, 2$ and 3 , given that $\theta = 0.6, \phi_1 = 0.05, \phi_2 = 0.1, \phi_3 = 0.2$ and $\phi_4 = 0.3$.

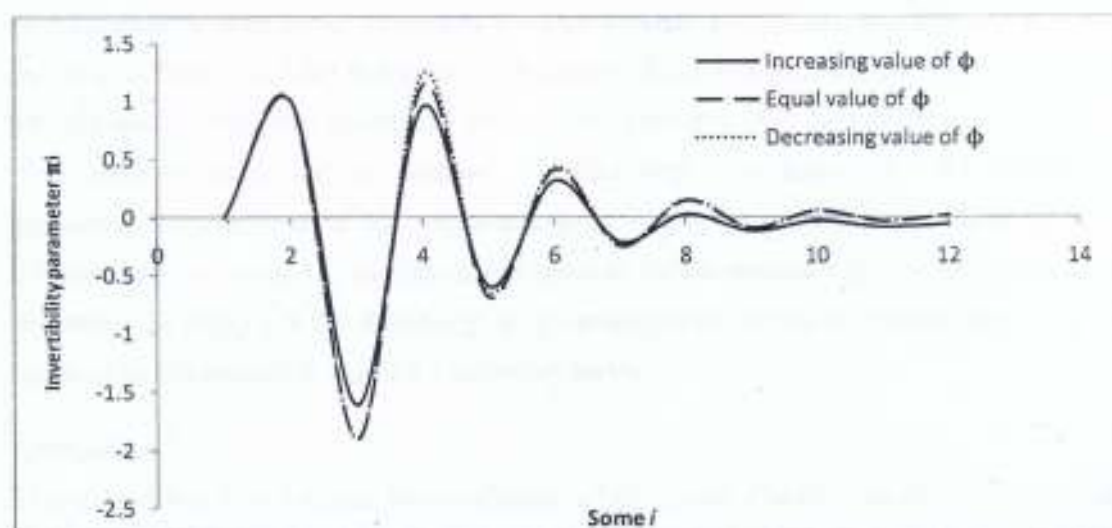


Figure 2.13: Graph showing behaviour of invertibility parameter π_i of ARIMA (p, d, l) for some i under different ϕ_p 's arrangement.

2.13 BOX-JENKINS METHODOLOGY

The objective of Box-Jenkins is to identify and estimate a statistical model which can be interpreted as having generated the sample data. If this estimated model is then to be used for forecasting, we must assume that the features of this model are constant through time, and particularly over future time periods. Thus the simple reason for requiring stationary data is that any model which is inferred from these data can itself be interpreted as stationary or stable, therefore providing valid basis for forecasting.

The Box-Jenkins approach typically comprises four parts, which are:

- (i) Identification of the model
- (ii) Estimation, usually OLS
- (iii) Diagnostic checking (mostly for autocorrelation)
- (iv) Forecasting

Identification

ARIMA modelling proceeds by a series of well-defined steps. The first step is to identify the model. Identification consists of specifying the appropriate structure (AR, MA or ARMA) and order of model. Identification is sometimes done by looking at plots of the Autocorrelation Function (ACF) and Partial Autocorrelation Function (PACF). Sometimes

identification is done by an automated iterative procedure - fitting many different possible model structures and orders and using a goodness-of-fit statistic to select the best model. Identification of the most appropriate model is the most important part of the process, where it becomes as much 'art' as 'science'. The first step is to determine if the variable is stationary, this can be done with the correlogram. If it is not stationary it needs to be first-differenced (it may need to be differenced again to induce stationarity). The next stage is to determine the p and q in the ARIMA (p, d, q) model (the d refers to how many times the data needs to be differenced to produce a stationary series)

Estimation

The second step is to estimate the coefficients of the model. Coefficients of AR models can be estimated by least-squares regression. Estimation of parameters of MA and ARMA models usually requires a more complicated iteration procedure (Chatfield 2004). In practice, estimation is fairly transparent to the user, as it is accomplished automatically by a computer program with little or no user interaction.

Diagnostic Check

The third step is to check the model. This step is also called diagnostic checking, or verification (Anderson 1977). Two important elements of checking are to ensure that the residuals of the model are random, and to ensure that the estimated parameters are statistically significant. Usually the fitting process is guided by the principle of parsimony, by which the best model is the simplest possible model – the model with the fewest parameters that adequately describes the data.

With this approach we test for autocorrelation, usually using the Q or Ljung-Box statistic. If there is evidence of autocorrelation, we need to go back to the identification stage and re-specify the model by adding more lags.



Here, parameters of ARIMA (p, d, q) were estimated for various values of p given that $q = 0$ for $d = 0, 1$. Two cases were considered, which are:

- (a) constant term $c = 0$
- (b) constant term $c \neq 0$

ARIMA (1, 0, 0)

$$y_t = c + \phi y_{t-1} + \varepsilon_t \quad (3.01)$$

Given that $c = 0$, we have

$$y_t = \phi y_{t-1} + \varepsilon_t \quad (3.02)$$

$$\varepsilon_t = y_t - \phi y_{t-1}$$

$$\varepsilon_t^2 = (y_t - \phi y_{t-1})^2$$

$$\sum_{t=1}^T \varepsilon_t^2 = \sum_{t=1}^T (y_t - \phi y_{t-1})^2$$

Let $S = \sum_{t=1}^T \varepsilon_t^2$, we have

$$S = \sum_{t=1}^T (y_t - \phi y_{t-1})^2$$

Differentiating S with respect to ϕ , we have

$$\frac{dS}{d\phi} = -2 \sum_{t=1}^T (y_t - \phi y_{t-1}) y_{t-1}$$

Setting $\frac{dS}{d\phi}$ to zero, we obtain

$$\sum_{t=1}^T y_t y_{t-1} - \phi \sum_{t=1}^T y_{t-1}^2 = 0$$

$$\hat{\phi} = \frac{\sum_{t=1}^T y_t y_{t-1}}{\sum_{t=1}^T y_{t-1}^2} \quad (3.03)$$

So we have

$$\hat{\phi} = \frac{\sum_{t=1}^T y_t y_{t-1}}{\sum_{t=1}^T y_{t-1}^2} \quad \text{and} \quad \text{Var}(\hat{\phi}) = \sigma_e^2 \sum_{t=1}^T y_{t-1}^2$$

When $c \neq 0$, equation (3.01) gives

$$\varepsilon_t = y_t - c - \phi y_{t-1} \quad (3.04)$$

$$S = \sum_{t=1}^T \varepsilon_t^2 = \sum_{t=1}^T (y_t - c - \phi y_{t-1})^2$$

where $S = \sum_{t=1}^T \varepsilon_t^2$.

Differentiating S with respect to c and ϕ , we have

$$\frac{dS}{dc} = -2 \sum_{t=1}^T (y_t - c - \phi y_{t-1})$$

and

$$\frac{dS}{d\phi} = -2 \sum_{t=1}^T (-c + y_t - \phi y_{t-1}) y_{t-1}$$

respectively.

Setting $\frac{dS}{d\phi}$ to zero, we obtain

$$nc + \phi \sum_{t=1}^T y_{t-1} = \sum_{t=1}^T y_t$$

$$c \sum_{t=1}^T y_{t-1} - \phi \sum_{t=1}^T y_{t-1}^2 = \sum_{t=1}^T y_t y_{t-1}$$

which can be written as



$$\begin{pmatrix} n & \sum_{t=1}^T y_{t-1} \\ \sum_{t=1}^T y_{t-1} & \sum_{t=1}^T y_{t-1}^2 \end{pmatrix} \begin{pmatrix} c \\ \phi \end{pmatrix} = \begin{pmatrix} \sum_{t=1}^T y_t \\ \sum_{t=1}^T y_t y_{t-1} \end{pmatrix}$$

Using Crammer's rule, we solve for c and ϕ

$$\hat{c} = \frac{\nabla_1}{\nabla} \text{ and } \hat{\phi} = \frac{\nabla_2}{\nabla} \quad (3.05)$$

where

$$\nabla = n \sum_{t=1}^T y_{t-1}^2 - \left[\sum_{t=1}^T y_{t-1} \right]^2$$

$$\nabla_1 = \left[\sum_{t=1}^T y_t \right] \left[\sum_{t=1}^T y_{t-1}^2 \right] - \left[\sum_{t=1}^T y_{t-1} \right] \left[\sum_{t=1}^T y_t y_{t-1} \right]$$

$$\nabla_2 = n \sum_{t=1}^T y_t y_{t-1} - \left[\sum_{t=1}^T y_t \right] \left[\sum_{t=1}^T y_{t-1} \right]$$

ARIMA (1, 1, 0)

$$\Delta y_t = c + \phi \Delta y_{t-1} + \varepsilon_t \quad (3.06)$$

When $c = 0$, we obtain

$$\Delta y_t = \phi \Delta y_{t-1} + \varepsilon_t \quad (3.07)$$

$$y_t - y_{t-1} = \phi(y_{t-1} - y_{t-2}) + \varepsilon_t$$

$$\varepsilon_t = y_t - (1 + \phi)y_{t-1} + \phi y_{t-2}$$

$$\varepsilon_t^2 = [y_t - (1 + \phi)y_{t-1} + \phi y_{t-2}]^2$$

$$\sum_{t=1}^T \varepsilon_t^2 = \sum_{t=1}^T [y_t - (1 + \phi)y_{t-1} + \phi y_{t-2}]^2$$

Let $S = \sum_{t=1}^T \varepsilon_t^2$, we have

$$S = \sum_{t=1}^T [y_t - (1 + \phi)y_{t-1} + \phi y_{t-2}]^2$$

Differentiating S with respect to ϕ , we have

$$\frac{dS}{d\phi} = 2 \sum_{t=1}^T [y_t - (1 + \phi)y_{t-1} + \phi y_{t-2}] [-y_{t-1} + y_{t-2}]$$

$$\frac{dS}{d\phi} = 2 \sum_{t=1}^T [-y_t y_{t-1} + y_t y_{t-2} + (1 + \phi)y_{t-1}^2 - (1 + \phi)y_{t-1}y_{t-2} - \phi y_{t-1}y_{t-2} + \phi y_{t-2}^2]$$

$$\begin{aligned} \frac{dS}{d\phi} = 2 \sum_{t=1}^T & [-y_t(y_{t-1} - y_{t-2}) + y_{t-1}(y_{t-1} - y_{t-2}) + \phi y_{t-1}(y_{t-1} - y_{t-2}) \\ & - \phi y_{t-2}(y_{t-1} - y_{t-2})] \end{aligned}$$

$$\frac{dS}{d\phi} = 2 \sum_{t=1}^T [-(y_t - y_{t-1})(y_{t-1} - y_{t-2}) + \phi(y_{t-1} - y_{t-2})^2]$$

Setting $\frac{dS}{d\phi}$ to zero, we obtain

$$\begin{aligned} - \sum_{t=1}^T (y_t - y_{t-1})(y_{t-1} - y_{t-2}) + \phi \sum_{t=1}^T (y_{t-1} - y_{t-2})^2 &= 0 \\ \hat{\phi} = \frac{\sum_{t=1}^T (y_t - y_{t-1})(y_{t-1} - y_{t-2})}{\sum_{t=1}^T (y_{t-1} - y_{t-2})^2} & \quad (3.08) \end{aligned}$$

When $c \neq 0$, equation (3.06) gives

$$\varepsilon_t = -c + \Delta y_t - \phi \Delta y_{t-1} + \phi y_{t-2} \quad (3.09)$$

$$S = \sum_{t=1}^T \varepsilon_t^2 = \sum_{t=1}^T (-c + \Delta y_t - \phi \Delta y_{t-1})^2$$

Differentiating S with respect to c and ϕ and setting $\frac{\partial S}{\partial c} = \frac{\partial S}{\partial \phi} = 0$, we obtain

$$\begin{pmatrix} n & \sum_{t=1}^T \Delta y_{t-1} \\ \sum_{t=1}^T \Delta y_{t-1} & \sum_{t=1}^T \Delta y_{t-1}^2 \end{pmatrix} \begin{pmatrix} c \\ \phi \end{pmatrix} = \begin{pmatrix} \sum_{t=1}^T \Delta y_t \\ \sum_{t=1}^T \Delta y_t \Delta y_{t-1} \end{pmatrix}$$

Using Crammer's rule, we solve for c and ϕ_2 thus

$$\hat{c} = \frac{\nabla_1}{\nabla} \text{ and } \hat{\phi} = \frac{\nabla_2}{\nabla} \quad (3.10)$$

where

$$\begin{aligned} \nabla &= n \sum_{t=1}^T \Delta y_{t-1}^2 - \left[\sum_{t=1}^T \Delta y_{t-1} \right]^2 \\ \nabla_1 &= \left[\sum_{t=1}^T \Delta y_t \right] \left[\sum_{t=1}^T \Delta y_{t-1}^2 \right] - \left[\sum_{t=1}^T \Delta y_{t-1} \right] \left[\sum_{t=1}^T \Delta y_t \Delta y_{t-1} \right] \\ \nabla_2 &= n \sum_{t=1}^T \Delta y_t \Delta y_{t-1} - \left[\sum_{t=1}^T \Delta y_t \right] \left[\sum_{t=1}^T \Delta y_{t-1} \right] \end{aligned}$$

ARIMA (2, 0, 0)

$$y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \varepsilon_t \quad (3.11)$$

Given that $c = 0$, we have

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \varepsilon_t \quad (3.12)$$

$$\varepsilon_t = y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2}$$

$$\varepsilon_t^2 = (y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2})^2$$

$$\sum_{t=1}^T \varepsilon_t^2 = \sum_{t=1}^T (y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2})^2$$

Let $S = \sum_{t=1}^T \varepsilon_t^2$, we have



$$S = \sum_{t=1}^T (y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2})^2$$

Differentiating S with respect to ϕ_1 and ϕ_2 , we have

$$\frac{\partial S}{\partial \phi_1} = -2 \sum_{t=1}^T (y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2}) y_{t-1}$$

$$\frac{\partial S}{\partial \phi_2} = -2 \sum_{t=1}^T (y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2}) y_{t-2}$$

Setting $\frac{\partial S}{\partial \phi_1} = \frac{\partial S}{\partial \phi_2} = 0$, we obtain

$$\phi_1 \sum_{t=1}^T y_{t-1}^2 + \phi_2 \sum_{t=1}^T y_{t-1} y_{t-2} = \sum_{t=1}^T y_t y_{t-1}$$

and

$$\phi_1 \sum_{t=1}^T y_{t-1} y_{t-2} + \phi_2 \sum_{t=1}^T y_{t-2}^2 = \sum_{t=1}^T y_t y_{t-2}$$

respectively.

$$\begin{pmatrix} \sum_{t=1}^T y_{t-1}^2 & \sum_{t=1}^T y_{t-1} y_{t-2} \\ \sum_{t=1}^T y_{t-1} y_{t-2} & \sum_{t=1}^T y_{t-2}^2 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \sum_{t=1}^T y_t y_{t-1} \\ \sum_{t=1}^T y_t y_{t-2} \end{pmatrix}$$

Using Cramer's rule, we obtain

$$\phi_1 = \frac{\nabla_1}{\nabla} \quad \text{and} \quad \phi_2 = \frac{\nabla_2}{\nabla} \quad (3.13)$$

where

$$\nabla = \begin{vmatrix} \sum_{t=1}^T y_{t-1}^2 & \sum_{t=1}^T y_{t-1}y_{t-2} \\ \sum_{t=1}^T y_{t-1}y_{t-2} & \sum_{t=1}^T y_{t-2}^2 \end{vmatrix}$$

$$\nabla = \sum_{t=1}^T y_{t-1}^2 \sum_{t=1}^T y_{t-2}^2 - \left(\sum_{t=1}^T y_{t-1}y_{t-2} \right)^2$$

$$\nabla_1 = \begin{vmatrix} \sum_{t=1}^T y_t y_{t-1} & \sum_{t=1}^T y_{t-1}y_{t-2} \\ \sum_{t=1}^T y_t y_{t-2} & \sum_{t=1}^T y_{t-2}^2 \end{vmatrix}$$

$$\nabla_1 = \sum_{t=1}^T y_t y_{t-1} \sum_{t=1}^T y_{t-2}^2 - \sum_{t=1}^T y_t y_{t-2} \sum_{t=1}^T y_{t-1}y_{t-2}$$

$$\nabla_2 = \begin{vmatrix} \sum_{t=1}^T y_{t-1}^2 & \sum_{t=1}^T y_t y_{t-1} \\ \sum_{t=1}^T y_{t-1}y_{t-2} & \sum_{t=1}^T y_t y_{t-2} \end{vmatrix}$$

$$\nabla_2 = \sum_{t=1}^T y_{t-1}^2 \sum_{t=1}^T y_t y_{t-2} - \sum_{t=1}^T y_t y_{t-1} \sum_{t=1}^T y_{t-1}y_{t-2}$$

So, we have

$$\left. \begin{aligned} \hat{\phi}_1 &= \frac{\sum_{t=1}^T y_t y_{t-1} \sum_{t=1}^T y_{t-2}^2 - \sum_{t=1}^T y_t y_{t-2} \sum_{t=1}^T y_{t-1}y_{t-2}}{\sum_{t=1}^T y_{t-1}^2 \sum_{t=1}^T y_{t-2}^2 - \left(\sum_{t=1}^T y_{t-1}y_{t-2} \right)^2} \\ \hat{\phi}_2 &= \frac{\sum_{t=1}^T y_{t-1}^2 \sum_{t=1}^T y_t y_{t-2} - \sum_{t=1}^T y_t y_{t-1} \sum_{t=1}^T y_{t-1}y_{t-2}}{\sum_{t=1}^T y_{t-1}^2 \sum_{t=1}^T y_{t-2}^2 - \left(\sum_{t=1}^T y_{t-1}y_{t-2} \right)^2} \end{aligned} \right\} \quad (3.14)$$

Given that $c \neq 0$, we have

$$\varepsilon_t = y_t - c - \phi_1 y_{t-1} - \phi_2 y_{t-2} \quad (3.15)$$

$$S = \sum_{t=1}^T \varepsilon_t^2 = \sum_{t=1}^T (y_t - c - \phi_1 y_{t-1} - \phi_2 y_{t-2})^2.$$

Differentiating S with respect to c , ϕ_1 and ϕ_2

$$\frac{\partial S}{\partial c} = -2 \sum_{t=1}^T (y_t - c - \phi_1 y_{t-1} - \phi_2 y_{t-2})$$

$$\frac{\partial S}{\partial \phi_1} = -2 \sum_{t=1}^T (y_t - c - \phi_1 y_{t-1} - \phi_2 y_{t-2}) y_{t-1}$$

$$\frac{\partial S}{\partial \phi_2} = -2 \sum_{t=1}^T (y_t - c - \phi_1 y_{t-1} - \phi_2 y_{t-2}) y_{t-2}$$

Setting $\frac{\partial S}{\partial c} = \frac{\partial S}{\partial \phi_1} = \frac{\partial S}{\partial \phi_2} = 0$, we obtain

$$nc + \phi_1 \sum_{t=1}^T y_{t-1} + \phi_2 \sum_{t=1}^T y_{t-2} = \sum_{t=1}^T y_t$$

$$c \sum_{t=1}^T y_{t-1} + \phi_1 \sum_{t=1}^T y_{t-1}^2 + \phi_2 \sum_{t=1}^T y_{t-1} y_{t-2} = \sum_{t=1}^T y_t y_{t-1}$$

and

$$c \sum_{t=1}^T y_{t-2} + \phi_1 \sum_{t=1}^T y_{t-1} y_{t-2} + \phi_2 \sum_{t=1}^T y_{t-2}^2 = \sum_{t=1}^T y_t y_{t-2}$$

$$\begin{pmatrix} n & \sum_{t=1}^T y_{t-1} & \sum_{t=1}^T y_{t-2} \\ \sum_{t=1}^T y_{t-1} & \sum_{t=1}^T y_{t-1}^2 & \sum_{t=1}^T y_{t-1} y_{t-2} \\ \sum_{t=1}^T y_{t-2} & \sum_{t=1}^T y_{t-1} y_{t-2} & \sum_{t=1}^T y_{t-2}^2 \end{pmatrix} \begin{pmatrix} c \\ \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \sum_{t=1}^T y_t \\ \sum_{t=1}^T y_t y_{t-1} \\ \sum_{t=1}^T y_t y_{t-2} \end{pmatrix}$$

Using Crammer's rule, we obtain

$$c = \frac{\nabla_1}{\nabla}, \phi_1 = \frac{\nabla_2}{\nabla} \text{ and } \phi_2 = \frac{\nabla_3}{\nabla} \quad (3.16)$$

$$\begin{aligned} \nabla &= \begin{vmatrix} n & \sum_{t=1}^T y_{t-1} & \sum_{t=1}^T y_{t-2} \\ \sum_{t=1}^T y_{t-1} & \sum_{t=1}^T y_{t-1}^2 & \sum_{t=1}^T y_{t-1}y_{t-2} \\ \sum_{t=1}^T y_{t-2} & \sum_{t=1}^T y_{t-1}y_{t-2} & \sum_{t=1}^T y_{t-2}^2 \end{vmatrix} \\ \nabla_1 &= \begin{vmatrix} \sum_{t=1}^T y_t & \sum_{t=1}^T y_{t-1} & \sum_{t=1}^T y_{t-2} \\ \sum_{t=1}^T y_t y_{t-1} & \sum_{t=1}^T y_{t-1}^2 & \sum_{t=1}^T y_{t-1}y_{t-2} \\ \sum_{t=1}^T y_t y_{t-2} & \sum_{t=1}^T y_{t-1}y_{t-2} & \sum_{t=1}^T y_{t-2}^2 \end{vmatrix} \\ \nabla_2 &= \begin{vmatrix} n & \sum_{t=1}^T y_t & \sum_{t=1}^T y_{t-2} \\ \sum_{t=1}^T y_{t-1} & \sum_{t=1}^T y_t y_{t-1} & \sum_{t=1}^T y_{t-1}y_{t-2} \\ \sum_{t=1}^T y_{t-2} & \sum_{t=1}^T y_t y_{t-2} & \sum_{t=1}^T y_{t-2}^2 \end{vmatrix} \\ \nabla_3 &= \begin{vmatrix} n & \sum_{t=1}^T y_{t-1} & \sum_{t=1}^T y_t \\ \sum_{t=1}^T y_{t-1} & \sum_{t=1}^T y_{t-1}^2 & \sum_{t=1}^T y_t y_{t-1} \\ \sum_{t=1}^T y_{t-2} & \sum_{t=1}^T y_{t-1}y_{t-2} & \sum_{t=1}^T y_t y_{t-2} \end{vmatrix} \end{aligned}$$

ARIMA (2, 1, 0)

$$\Delta y_t = c + \phi_1 \Delta y_{t-1} + \phi_2 \Delta y_{t-2} + \varepsilon_t \quad (3.17)$$

When $c = 0$, we have

$$\Delta y_t = \phi_1 \Delta y_{t-1} + \phi_2 \Delta y_{t-2} + \varepsilon_t \quad (3.18)$$

$$y_t - y_{t-1} = \phi_1(y_{t-1} - y_{t-2}) + \phi_2(y_{t-2} - y_{t-3}) + \varepsilon_t$$

$$\varepsilon_t = y_t - (1 + \phi_1)y_{t-1} + (\phi_1 - \phi_2)y_{t-2} + \phi_2 y_{t-3}$$

$$\varepsilon_t^2 = [y_t - (1 + \phi_1)y_{t-1} + (\phi_1 - \phi_2)y_{t-2} + \phi_2 y_{t-3}]^2$$

$$\varepsilon_t^2 = [y_t - y_{t-1} - \phi_1(y_{t-1} - y_{t-2}) - \phi_2(y_{t-2} - y_{t-3})]^2$$

$$\sum_{t=1}^T \varepsilon_t^2 = \sum_{t=1}^T [y_t - y_{t-1} - \phi_1(y_{t-1} - y_{t-2}) - \phi_2(y_{t-2} - y_{t-3})]^2$$

Let $S = \sum_{t=1}^T \varepsilon_t^2$, we have

$$S = \sum_{t=1}^T [y_t - y_{t-1} - \phi_1(y_{t-1} - y_{t-2}) - \phi_2(y_{t-2} - y_{t-3})]^2$$

Differentiating S with respect to ϕ_1 and ϕ_2 , we have

$$\begin{aligned} \frac{\partial S}{\partial \phi_1} &= -2 \sum_{t=1}^T [y_t - y_{t-1} - \phi_1(y_{t-1} - y_{t-2}) - \phi_2(y_{t-2} - y_{t-3})] (y_{t-1} - y_{t-2}) \\ &= -2 \sum_{t=1}^T [(y_t - y_{t-1})(y_{t-1} - y_{t-2}) - \phi_1(y_{t-1} - y_{t-2})^2 - \phi_2(y_{t-1} - y_{t-2})(y_{t-2} - y_{t-3})] \end{aligned}$$

and

$$\begin{aligned} \frac{\partial S}{\partial \phi_2} &= -2 \sum_{t=1}^T [y_t - y_{t-1} - \phi_1(y_{t-1} - y_{t-2}) - \phi_2(y_{t-2} - y_{t-3})] (y_{t-2} - y_{t-3}) \\ &= -2 \sum_{t=1}^T [(y_t - y_{t-1})(y_{t-2} - y_{t-3}) - \phi_1(y_{t-1} - y_{t-2})(y_{t-2} - y_{t-3}) - \phi_2(y_{t-2} - y_{t-3})^2] \end{aligned}$$

respectively.

Setting $\frac{\partial S}{\partial \phi_1} = \frac{\partial S}{\partial \phi_2} = 0$, we obtain

$$\phi_1 \sum_{t=1}^T (y_{t-1} - y_{t-2})^2 + \phi_2 \sum_{t=1}^T (y_{t-1} - y_{t-2})(y_{t-2} - y_{t-3}) = \sum_{t=1}^T (y_t - y_{t-1})(y_{t-1} - y_{t-2})$$

and

$$\phi_1 \sum_{t=1}^T (y_{t-1} - y_{t-2})(y_{t-2} - y_{t-3}) + \phi_2 \sum_{t=1}^T (y_{t-2} - y_{t-3})^2 = \sum_{t=1}^T (y_t - y_{t-1})(y_{t-2} - y_{t-3})$$

respectively. That is

$$\begin{pmatrix} \sum_{t=1}^T (y_{t-1} - y_{t-2})^2 & \sum_{t=1}^T (y_{t-1} - y_{t-2})(y_{t-2} - y_{t-3}) \\ \sum_{t=1}^T (y_{t-1} - y_{t-2})(y_{t-2} - y_{t-3}) & \sum_{t=1}^T (y_{t-2} - y_{t-3})^2 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \sum_{t=1}^T (y_t - y_{t-1})(y_{t-1} - y_{t-2}) \\ \sum_{t=1}^T (y_t - y_{t-1})(y_{t-2} - y_{t-3}) \end{pmatrix}$$

Using Crammer's rule, we obtain

$$\hat{\phi}_1 = \frac{\nabla_1}{\nabla} \text{ and } \hat{\phi}_2 = \frac{\nabla_2}{\nabla} \quad (3.18)$$

where

$$\nabla = \begin{vmatrix} \sum_{t=1}^T (y_{t-1} - y_{t-2})^2 & \sum_{t=1}^T (y_{t-1} - y_{t-2})(y_{t-2} - y_{t-3}) \\ \sum_{t=1}^T (y_{t-1} - y_{t-2})(y_{t-2} - y_{t-3}) & \sum_{t=1}^T (y_{t-2} - y_{t-3})^2 \end{vmatrix}$$

$$\nabla = \sum_{t=1}^T (y_{t-1} - y_{t-2})^2 \sum_{t=1}^T (y_{t-2} - y_{t-3})^2 - \left[\sum_{t=1}^T (y_{t-1} - y_{t-2})(y_{t-2} - y_{t-3}) \right]^2 \quad (3.19)$$

$$\begin{aligned} \nabla_1 &= \left| \begin{array}{cc} \sum_{t=1}^T (y_t - y_{t-1})(y_{t-1} - y_{t-2}) & \sum_{t=1}^T (y_{t-1} - y_{t-2})(y_{t-2} - y_{t-3}) \\ \sum_{t=1}^T (y_t - y_{t-1})(y_{t-2} - y_{t-3}) & \sum_{t=1}^T (y_{t-2} - y_{t-3})^2 \end{array} \right| \\ \nabla_1 &= \sum_{t=1}^T (y_t - y_{t-1})(y_{t-1} - y_{t-2}) \sum_{t=1}^T (y_{t-2} - y_{t-3})^2 \\ &\quad - \sum_{t=1}^T (y_t - y_{t-1})(y_{t-2} - y_{t-3}) \sum_{t=1}^T (y_{t-1} - y_{t-2})(y_{t-2} - y_{t-3}) \end{aligned} \quad (3.20)$$

$$\begin{aligned} \nabla_2 &= \left| \begin{array}{cc} \sum_{t=1}^T (y_{t-1} - y_{t-2})^2 & \sum_{t=1}^T (y_t - y_{t-1})(y_{t-1} - y_{t-2}) \\ \sum_{t=1}^T (y_{t-1} - y_{t-2})(y_{t-2} - y_{t-3}) & \sum_{t=1}^T (y_t - y_{t-1})(y_{t-2} - y_{t-3}) \end{array} \right| \\ \nabla_2 &= \sum_{t=1}^T (y_{t-1} - y_{t-2})^2 \sum_{t=1}^T (y_t - y_{t-1})(y_{t-2} - y_{t-3}) - \sum_{t=1}^T (y_t - y_{t-1})(y_{t-1} - y_{t-2}) \\ &\quad * \sum_{t=1}^T (y_{t-1} - y_{t-2})(y_{t-2} - y_{t-3}) \end{aligned} \quad (3.21)$$

ARIMA (P, 0, 0)

$$y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \phi_3 y_{t-3} + \dots + \phi_p y_{t-p} + \varepsilon_t \quad (3.22)$$

Given that $c = 0$, we have

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \phi_3 y_{t-3} + \dots + \phi_p y_{t-p} + \varepsilon_t \quad (3.23)$$

$$\varepsilon_t = y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2} - \phi_3 y_{t-3} - \dots - \phi_p y_{t-p}$$

$$\varepsilon_t^2 = (y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2} - \phi_3 y_{t-3} - \dots - \phi_p y_{t-p})^2$$

$$\sum_{t=1}^T \varepsilon_t^2 = \sum_{t=1}^T (y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2} - \phi_3 y_{t-3} - \dots - \phi_p y_{t-p})^2$$

Let $S = \sum_{t=1}^T \varepsilon_t^2$, we have

$$S = \sum_{t=1}^T (y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2} - \phi_3 y_{t-3} - \dots - \phi_p y_{t-p})^2$$

Differentiating S with respect to ϕ_1 and ϕ_2 , we have

$$\frac{\partial S}{\partial \phi_1} = -2 \sum_{t=1}^T (y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2} - \phi_3 y_{t-3} - \dots - \phi_p y_{t-p}) y_{t-1}$$

$$\frac{\partial S}{\partial \phi_2} = -2 \sum_{t=1}^T (y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2} - \phi_3 y_{t-3} - \dots - \phi_p y_{t-p}) y_{t-2}$$

$$\frac{\partial S}{\partial \phi_3} = -2 \sum_{t=1}^T (y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2} - \phi_3 y_{t-3} - \dots - \phi_p y_{t-p}) y_{t-3}$$

$$\vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots$$

$$\frac{\partial S}{\partial \phi_p} = -2 \sum_{t=1}^T (y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2} - \phi_3 y_{t-3} - \dots - \phi_p y_{t-p}) y_{t-p}$$

respectively.

Setting $\frac{\partial S}{\partial \phi_1} = \frac{\partial S}{\partial \phi_2} = \frac{\partial S}{\partial \phi_3} = \dots = \frac{\partial S}{\partial \phi_p} = 0$, we obtain

$$\phi_1 \sum_{t=1}^T y_{t-1}^2 + \phi_2 \sum_{t=1}^T y_{t-2} y_{t-1} + \phi_3 \sum_{t=1}^T y_{t-3} y_{t-1} + \dots + \phi_p \sum_{t=1}^T y_{t-p} y_{t-1} = \sum_{t=1}^T y_t y_{t-1}$$

$$\phi_1 \sum_{t=1}^T y_{t-1} y_{t-2} + \phi_2 \sum_{t=1}^T y_{t-2}^2 + \phi_3 \sum_{t=1}^T y_{t-3} y_{t-2} + \dots + \phi_p \sum_{t=1}^T y_{t-p} y_{t-2} = \sum_{t=1}^T y_t y_{t-2}$$

$$\phi_1 \sum_{t=1}^T y_{t-1} y_{t-3} + \phi_2 \sum_{t=1}^T y_{t-2} y_{t-3} + \phi_3 \sum_{t=1}^T y_{t-3}^2 + \dots + \phi_p \sum_{t=1}^T y_{t-p} y_{t-3} = \sum_{t=1}^T y_t y_{t-3}$$

$$\vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots$$

$$\phi_1 \sum_{t=1}^T y_{t-1} y_{t-p} + \phi_2 \sum_{t=1}^T y_{t-2} y_{t-p} + \phi_3 \sum_{t=1}^T y_{t-3} y_{t-p} + \dots + \phi_p \sum_{t=1}^T y_{t-p}^2 = \sum_{t=1}^T y_t y_{t-p}$$

respectively.

This can be written as $A\Psi = B$. That is

$$\begin{pmatrix} \sum_{t=1}^T y_{t-1}^2 & \sum_{t=1}^T y_{t-2}y_{t-1} & \sum_{t=1}^T y_{t-3}y_{t-1} & \cdots & \sum_{t=1}^T y_{t-p}y_{t-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \sum_{t=1}^T y_{t-1}y_{t-p} & \sum_{t=1}^T y_{t-2}y_{t-p} & \sum_{t=1}^T y_{t-3}y_{t-p} & \cdots & \sum_{t=1}^T y_{t-p}^2 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_p \end{pmatrix} = \begin{pmatrix} \sum_{t=1}^T y_t y_{t-1} \\ \vdots \\ \sum_{t=1}^T y_t y_{t-p} \end{pmatrix}$$

where A is $p \times p$ matrix and Ψ is a column matrix, that is $\Psi = (\phi_1 \ \phi_2 \ \phi_3 \ \dots \ \phi_p)'$. B is a column matrix, $\begin{pmatrix} \sum_{t=1}^T y_t y_{t-1} \\ \vdots \\ \sum_{t=1}^T y_t y_{t-p} \end{pmatrix}$. The expression for each parameter $\phi_i, i = 1, 2, \dots, p$ can thus be determined using Crammer's rule or Gauss-Schidel method.

But given that $c \neq 0$, we have

$$\varepsilon_t = y_t - c - \phi_1 y_{t-1} - \phi_2 y_{t-2} - \phi_3 y_{t-3} - \dots - \phi_p y_{t-p} \quad (3.24)$$

$$\varepsilon_t^2 = (y_t - c - \phi_1 y_{t-1} - \phi_2 y_{t-2} - \phi_3 y_{t-3} - \dots - \phi_p y_{t-p})^2$$

$$\sum_{t=1}^T \varepsilon_t^2 = \sum_{t=1}^T (y_t - c - \phi_1 y_{t-1} - \phi_2 y_{t-2} - \phi_3 y_{t-3} - \dots - \phi_p y_{t-p})^2$$

Let $S = \sum_{t=1}^T \varepsilon_t^2$, we have

$$S = \sum_{t=1}^T (y_t - c - \phi_1 y_{t-1} - \phi_2 y_{t-2} - \phi_3 y_{t-3} - \dots - \phi_p y_{t-p})^2$$

Differentiating S with respect to ϕ_1 and ϕ_2 , we have

$$\frac{\partial S}{\partial c} = -2 \sum_{t=1}^T (y_t - c - \phi_1 y_{t-1} - \phi_2 y_{t-2} - \phi_3 y_{t-3} - \dots - \phi_p y_{t-p})$$

$$\frac{\partial S}{\partial \phi_1} = -2 \sum_{t=1}^T (y_t - c - \phi_1 y_{t-1} - \phi_2 y_{t-2} - \phi_3 y_{t-3} - \dots - \phi_p y_{t-p}) y_{t-1}$$

$$\frac{\partial S}{\partial \phi_2} = -2 \sum_{t=1}^T (y_t - c - \phi_1 y_{t-1} - \phi_2 y_{t-2} - \phi_3 y_{t-3} - \dots - \phi_p y_{t-p}) y_{t-2}$$

$$\frac{\partial S}{\partial \phi_3} = -2 \sum_{t=1}^T (y_t - c - \phi_1 y_{t-1} - \phi_2 y_{t-2} - \phi_3 y_{t-3} - \dots - \phi_p y_{t-p}) y_{t-3}$$

$$\vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots$$

$$\frac{\partial S}{\partial \phi_p} = -2 \sum_{t=1}^T (y_t - c - \phi_1 y_{t-1} - \phi_2 y_{t-2} - \phi_3 y_{t-3} - \dots - \phi_p y_{t-p}) y_{t-p}$$

respectively.

Setting $\frac{\partial S}{\partial \phi_1} = \frac{\partial S}{\partial \phi_2} = \frac{\partial S}{\partial \phi_3} = \dots = \frac{\partial S}{\partial \phi_p} = 0$, we obtain

$$nc + \phi_1 \sum_{t=1}^T y_{t-1} + \phi_2 \sum_{t=1}^T y_{t-2} + \dots + \phi_p \sum_{t=1}^T y_{t-p} = \sum_{t=1}^T y_t$$

$$c \sum_{t=1}^T y_{t-1} + \phi_1 \sum_{t=1}^T y_{t-1}^2 + \phi_2 \sum_{t=1}^T y_{t-2} y_{t-1} + \dots + \phi_p \sum_{t=1}^T y_{t-p} y_{t-1} = \sum_{t=1}^T y_t y_{t-1}$$

$$c \sum_{t=1}^T y_{t-2} + \phi_1 \sum_{t=1}^T y_{t-1} y_{t-2} + \phi_2 \sum_{t=1}^T y_{t-2}^2 + \dots + \phi_p \sum_{t=1}^T y_{t-p} y_{t-2} = \sum_{t=1}^T y_t y_{t-2}$$

$$c \sum_{t=1}^T y_{t-3} + \phi_1 \sum_{t=1}^T y_{t-1} y_{t-3} + \phi_2 \sum_{t=1}^T y_{t-2} y_{t-3} + \dots + \phi_p \sum_{t=1}^T y_{t-p} y_{t-3} = \sum_{t=1}^T y_t y_{t-3}$$

$$\vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots$$

$$c \sum_{t=1}^T y_{t-p} + \phi_1 \sum_{t=1}^T y_{t-1} y_{t-p} + \phi_2 \sum_{t=1}^T y_{t-2} y_{t-p} + \dots + \phi_p \sum_{t=1}^T y_{t-p}^2 = \sum_{t=1}^T y_t y_{t-p}$$

respectively.

This can be written as $A\Psi = B$, where $\Psi = (c, \phi_1, \phi_2, \phi_3, \dots, \phi_p)'$.



$$\begin{pmatrix} \sum_{t=1}^T y_{t-1} y_{t-p} & \sum_{t=1}^T y_{t-2} y_{t-p} & \sum_{t=1}^T y_{t-3} y_{t-p} & \cdots & \sum_{t=1}^T y_{t-p}^2 \end{pmatrix} \begin{pmatrix} c \\ \phi_1 \\ \vdots \\ \phi_p \end{pmatrix} = \begin{pmatrix} \sum_{t=1}^T y_t \\ \sum_{t=1}^T y_t y_{t-1} \\ \vdots \\ \sum_{t=1}^T y_t y_{t-p} \end{pmatrix}$$

ARIMA (P, 1, 0)

$$\Delta y_t = c + \phi_1 \Delta y_{t-1} + \phi_2 \Delta y_{t-2} + \phi_3 \Delta y_{t-3} + \cdots + \phi_p \Delta y_{t-p} + \varepsilon_t \quad (3.25)$$

When $c = 0$, we have

$$\Delta y_t = \phi_1 \Delta y_{t-1} + \phi_2 \Delta y_{t-2} + \phi_3 \Delta y_{t-3} + \cdots + \phi_p \Delta y_{t-p} + \varepsilon_t \quad (3.25)$$

$$y_t - y_{t-1} = \phi_1 (y_{t-1} - y_{t-2}) + \phi_2 (y_{t-2} - y_{t-3}) + \cdots + \phi_p (y_{t-p} - y_{t-p-1}) + \varepsilon_t$$

$$\varepsilon_t = y_t - y_{t-1} - \phi_1 (y_{t-1} - y_{t-2}) - \phi_2 (y_{t-2} - y_{t-3}) - \cdots - \phi_p (y_{t-p} - y_{t-p-1})$$

$$\varepsilon_t^2 = [y_t - y_{t-1} - \phi_1 (y_{t-1} - y_{t-2}) - \phi_2 (y_{t-2} - y_{t-3}) - \cdots - \phi_p (y_{t-p} - y_{t-p-1})]^2$$

$$\sum_{t=1}^T \varepsilon_t^2 = \sum_{t=1}^T [y_t - y_{t-1} - \phi_1 (y_{t-1} - y_{t-2}) - \phi_2 (y_{t-2} - y_{t-3}) - \cdots - \phi_p (y_{t-p} - y_{t-p-1})]^2$$

Let $S = \sum_{t=1}^T \varepsilon_t^2$, we have

$$S = \sum_{t=1}^T [y_t - y_{t-1} - \phi_1 (y_{t-1} - y_{t-2}) - \phi_2 (y_{t-2} - y_{t-3}) - \cdots - \phi_p (y_{t-p} - y_{t-p-1})]^2$$

Differentiating S with respect to $\phi_1, \phi_2, \dots, \phi_p$, we have

$$\frac{\partial S}{\partial \phi_1} = -2 \sum_{t=1}^T [y_t - y_{t-1} - \phi_1 (y_{t-1} - y_{t-2}) - \phi_2 (y_{t-2} - y_{t-3}) - \cdots - \phi_p (y_{t-p} - y_{t-p-1})] \cdot (y_{t-1} - y_{t-2})$$



$$\frac{\partial S}{\partial \phi_2} = -2 \sum_{t=1}^T [y_t - y_{t-1} - \phi_1(y_{t-1} - y_{t-2}) - \phi_2(y_{t-2} - y_{t-3}) - \dots - \phi_p(y_{t-p} - y_{t-p-1})] \\ * (y_{t-2} - y_{t-3})$$

$$\frac{\partial S}{\partial \phi_3} = -2 \sum_{t=1}^T [y_t - y_{t-1} - \phi_1(y_{t-1} - y_{t-2}) - \phi_2(y_{t-2} - y_{t-3}) - \dots - \phi_p(y_{t-p} - y_{t-p-1})] \\ * (y_{t-3} - y_{t-4})$$

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$$\frac{\partial S}{\partial \phi_p} = -2 \sum_{t=1}^T [y_t - y_{t-1} - \phi_1(y_{t-1} - y_{t-2}) - \phi_2(y_{t-2} - y_{t-3}) - \dots - \phi_p(y_{t-p} - y_{t-p-1})] \\ * (y_{t-p} - y_{t-p-1})$$

Setting $\frac{\partial S}{\partial \phi_1} = \frac{\partial S}{\partial \phi_2} = \frac{\partial S}{\partial \phi_3} = \dots = \frac{\partial S}{\partial \phi_p} = 0$, we obtain

$$\phi_1 \sum_{t=1}^T (y_{t-1} - y_{t-2})^2 + \phi_2 \sum_{t=1}^T (y_{t-1} - y_{t-2})(y_{t-2} - y_{t-3}) \\ + \dots \phi_p \sum_{t=1}^T (y_{t-1} - y_{t-2})(y_{t-p} - y_{t-p-1}) = \sum_{t=1}^T (y_t - y_{t-1})(y_{t-1} - y_{t-2})$$

$$\phi_1 \sum_{t=1}^T (y_{t-1} - y_{t-2})(y_{t-2} - y_{t-3}) + \phi_2 \sum_{t=1}^T (y_{t-2} - y_{t-3})^2 \\ + \dots \phi_p \sum_{t=1}^T (y_{t-2} - y_{t-3})(y_{t-p} - y_{t-p-1}) = \sum_{t=1}^T (y_t - y_{t-1})(y_{t-2} - y_{t-3})$$

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$$\phi_1 \sum_{t=1}^T (y_{t-1} - y_{t-2})(y_{t-p} - y_{t-p-1}) + \phi_2 \sum_{t=1}^T (y_{t-2} - y_{t-3})(y_{t-p} - y_{t-p-1}) \\ + \dots \phi_p \sum_{t=1}^T (y_{t-p} - y_{t-p-1})^2 = \sum_{t=1}^T (y_t - y_{t-1})(y_{t-p} - y_{t-p-1})$$

respectively.

This can also be written as $A\Psi = B$. That is

$$\begin{aligned}
 & \begin{pmatrix} \sum_{t=1}^T (y_{t-1} - y_{t-2})^2 & \sum_{t=1}^T (y_{t-1} - y_{t-2})(y_{t-2} - y_{t-3}) & \cdots & \sum_{t=1}^T (y_{t-1} - y_{t-2})(y_{t-p} - y_{t-p-1}) \\ \vdots & \vdots & \vdots & \vdots \\ \sum_{t=1}^T (y_{t-1} - y_{t-2})(y_{t-p} - y_{t-p-1}) & \sum_{t=1}^T (y_{t-2} - y_{t-3})(y_{t-p} - y_{t-p-1}) & \cdots & \sum_{t=1}^T (y_{t-p} - y_{t-p-1})^2 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_p \end{pmatrix} \\
 &= \begin{pmatrix} \sum_{t=1}^T (y_t - y_{t-1})(y_{t-1} - y_{t-2}) \\ \vdots \\ \sum_{t=1}^T (y_t - y_{t-1})(y_{t-p} - y_{t-p-1}) \end{pmatrix} \\
 & \begin{pmatrix} \sum_{t=1}^T (\Delta y_{t-1})^2 & \sum_{t=1}^T (\Delta y_{t-1})(\Delta y_{t-2}) & \cdots & \sum_{t=1}^T (\Delta y_{t-1})(\Delta y_{t-p}) \\ \vdots & \vdots & \vdots & \vdots \\ \sum_{t=1}^T (\Delta y_{t-1})(\Delta y_{t-p}) & \sum_{t=1}^T (\Delta y_{t-2})(\Delta y_{t-p}) & \cdots & \sum_{t=1}^T (\Delta y_{t-p})^2 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_p \end{pmatrix} \\
 &= \begin{pmatrix} \sum_{t=1}^T (\Delta y_t)(\Delta y_{t-1}) \\ \vdots \\ \sum_{t=1}^T (\Delta y_t)(\Delta y_{t-p}) \end{pmatrix}
 \end{aligned}$$

When $c \neq 0$, we have

$$\varepsilon_t^2 = [-c + y_t - y_{t-1} - \phi_1(y_{t-1} - y_{t-2}) - \phi_2(y_{t-2} - y_{t-3}) - \cdots - \phi_p(y_{t-p} - y_{t-p-1})]^2$$

$$\sum_{t=1}^T \varepsilon_t^2 = \sum_{t=1}^T [y_t - y_{t-1} - \phi_1(y_{t-1} - y_{t-2}) - \phi_2(y_{t-2} - y_{t-3}) - \cdots - \phi_p(y_{t-p} - y_{t-p-1})]^2$$

Let $S = \sum_{t=1}^T \varepsilon_t^2$, we have

$$\begin{aligned}
 S = \sum_{t=1}^T & [-c + y_t - y_{t-1} - \phi_1(y_{t-1} - y_{t-2}) - \phi_2(y_{t-2} - y_{t-3}) - \cdots - \phi_p(y_{t-p} \\
 & - y_{t-p-1})]^2
 \end{aligned}$$



Differentiating S with respect to $c, \phi_1, \phi_2, \dots, \phi_p$, we have

$$\frac{\partial S}{\partial c} = -2 \sum_{t=1}^T [-c + y_t - y_{t-1} - \phi_1(y_{t-1} - y_{t-2}) - \phi_2(y_{t-2} - y_{t-3}) - \dots - \phi_p(y_{t-p} - y_{t-p-1})] * (y_{t-1} - y_{t-2})$$

$$\frac{\partial S}{\partial \phi_1} = -2 \sum_{t=1}^T [-c + y_t - y_{t-1} - \phi_1(y_{t-1} - y_{t-2}) - \phi_2(y_{t-2} - y_{t-3}) - \dots - \phi_p(y_{t-p} - y_{t-p-1})] * (y_{t-1} - y_{t-2})$$

$$\frac{\partial S}{\partial \phi_2} = -2 \sum_{t=1}^T [-c + y_t - y_{t-1} - \phi_1(y_{t-1} - y_{t-2}) - \phi_2(y_{t-2} - y_{t-3}) - \dots - \phi_p(y_{t-p} - y_{t-p-1})] * (y_{t-2} - y_{t-3})$$

$$\frac{\partial S}{\partial \phi_3} = -2 \sum_{t=1}^T [-c + y_t - y_{t-1} - \phi_1(y_{t-1} - y_{t-2}) - \phi_2(y_{t-2} - y_{t-3}) - \dots - \phi_p(y_{t-p} - y_{t-p-1})] * (y_{t-3} - y_{t-4})$$

$$\vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots$$

$$\frac{\partial S}{\partial \phi_p} = -2 \sum_{t=1}^T [-c + y_t - y_{t-1} - \phi_1(y_{t-1} - y_{t-2}) - \phi_2(y_{t-2} - y_{t-3}) - \dots - \phi_p(y_{t-p} - y_{t-p-1})] * (y_{t-p} - y_{t-p-1})$$

Setting $\frac{\partial S}{\partial c} = \frac{\partial S}{\partial \phi_1} = \frac{\partial S}{\partial \phi_2} = \frac{\partial S}{\partial \phi_3} = \dots = \frac{\partial S}{\partial \phi_p} = 0$, we obtain

$$c \sum_{t=1}^T (y_{t-1} - y_{t-2}) + \phi_1 \sum_{t=1}^T (y_{t-1} - y_{t-2})^2 + \phi_2 \sum_{t=1}^T (y_{t-1} - y_{t-2})(y_{t-2} - y_{t-3}) \\ + \dots + \phi_p \sum_{t=1}^T (y_{t-1} - y_{t-2})(y_{t-p} - y_{t-p-1}) = \sum_{t=1}^T (y_t - y_{t-1})(y_{t-1} - y_{t-2})$$

$$c \sum_{t=1}^T (y_{t-2} - y_{t-3}) + \phi_1 \sum_{t=1}^T (y_{t-1} - y_{t-2})(y_{t-2} - y_{t-3}) + \phi_2 \sum_{t=1}^T (y_{t-2} - y_{t-3})^2 \\ + \dots \phi_p \sum_{t=1}^T (y_{t-2} - y_{t-3})(y_{t-p} - y_{t-p-1}) = \sum_{t=1}^T (y_t - y_{t-1})(y_{t-2} - y_{t-3})$$

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$$c \sum_{t=1}^T (y_{t-p} - y_{t-p-1}) + \phi_1 \sum_{t=1}^T (y_{t-1} - y_{t-2})(y_{t-p} - y_{t-p-1}) \\ + \phi_2 \sum_{t=1}^T (y_{t-2} - y_{t-3})(y_{t-p} - y_{t-p-1}) + \dots \phi_p \sum_{t=1}^T (y_{t-p} - y_{t-p-1})^2 \\ = \sum_{t=1}^T (y_t - y_{t-1})(y_{t-p} - y_{t-p-1})$$

respectively.

This can also be written as $A\Psi = B$. That is

$$\begin{pmatrix} \sum_{t=1}^T (y_{t-1} - y_{t-2}) & \sum_{t=1}^T (y_{t-1} - y_{t-2})^2 & \dots & \sum_{t=1}^T (y_{t-1} - y_{t-2})(y_{t-p} - y_{t-p-1}) \\ \sum_{t=1}^T (y_{t-2} - y_{t-3}) & \sum_{t=1}^T (y_{t-1} - y_{t-2})(y_{t-2} - y_{t-3}) & \dots & \sum_{t=1}^T (y_{t-1} - y_{t-2})(y_{t-p} - y_{t-p-1}) \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{t=1}^T (y_{t-p} - y_{t-p-1}) & \sum_{t=1}^T (y_{t-1} - y_{t-2})(y_{t-p} - y_{t-p-1}) & \dots & \sum_{t=1}^T (y_{t-p} - y_{t-p-1})^2 \end{pmatrix} \begin{pmatrix} c \\ \phi_1 \\ \vdots \\ \phi_p \end{pmatrix} \\ = \begin{pmatrix} \sum_{t=1}^T (y_t - y_{t-1})(y_{t-1} - y_{t-2}) \\ \sum_{t=1}^T (y_t - y_{t-1})(y_{t-2} - y_{t-3}) \\ \vdots \\ \sum_{t=1}^T (y_t - y_{t-1})(y_{t-p} - y_{t-p-1}) \end{pmatrix}$$



$$\begin{pmatrix} \sum_{t=1}^T (\Delta y_{t-1})^2 & \sum_{t=1}^T (\Delta y_{t-1})(\Delta y_{t-2}) & \cdots & \sum_{t=1}^T (\Delta y_{t-1})(\Delta y_{t-p}) \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{t=1}^T (\Delta y_{t-1})(\Delta y_{t-p}) & \sum_{t=1}^T (\Delta y_{t-2})(\Delta y_{t-p}) & \cdots & \sum_{t=1}^T (\Delta y_{t-p})^2 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_p \end{pmatrix} \\
= \begin{pmatrix} \sum_{t=1}^T (\Delta y_t)(\Delta y_{t-1}) \\ \vdots \\ \sum_{t=1}^T (\Delta y_t)(\Delta y_{t-p}) \end{pmatrix}$$

For the estimate of parameters in ARIMA (p, 0, 0) and (p, 1, 0), it is deduced that every term y_{t-j} in ARIMA (p, 0, 0) is replaced by $y_{t-j} - y_{t-j-1}$ in ARIMA (p, 1, 0). Also, Ψ is p column matrix for $c = 0$ while Ψ is (p + 1) column matrix for $c \neq 0$.





4.1 DAILY MAXIMUM METEOROLOGICAL TEMPERATURE

The daily temperature data is used in this work. This data reports daily maximum meteorological temperature of Ondo, Nigeria ($07^{\circ}06'27''N$, $04^{\circ}50'19''E$) and Zaira, Nigeria ($11^{\circ}04'N$, $07^{\circ}42'E$) from January 1995 to November 2005.

Stationarity of the series is determined by the use of correlogram for describing autocorrelation function and partial autocorrelation function for the series. Appendix Figures 1 and 2 (in Appendices 5 and 6) show that the series is non-stationary for both Ondo, Nigeria ($07^{\circ}06'27''N$, $04^{\circ}50'19''E$) and Zaira, Nigeria ($11^{\circ}04'N$, $07^{\circ}42'E$) since autocorrelations for each of Ondo, Nigeria ($07^{\circ}06'27''N$, $04^{\circ}50'19''E$) and Zaira, Nigeria ($11^{\circ}04'N$, $07^{\circ}42'E$) start high and decline slowly, it is therefore differenced once (i.e. $d = 1$) to ensure stationarity. Appendix Figure 3 and 4 (in Appendices 7 and 8) give the correlograms for the series after differencing each once (stationary at $d = 1$). Also, the residual terms (white noise process or innovation series) are independently and identically distributed since the autocorrelation function at various lags hover around zero [Bobba et al (2006)], as can be seen in Appendix Figure 3 and 4 [Gujarati (2004)]. Similarly, Appendix Figures 5(a) and (b) (in Appendix 9) show that residuals for both Ondo, Nigeria ($07^{\circ}06'27''N$, $04^{\circ}50'19''E$) and Zaira, Nigeria ($11^{\circ}04'N$, $07^{\circ}42'E$) are normally distributed. Hence, ε_t is independently, identically and normally distributed for maximum daily temperature series of each of Ondo, Nigeria ($07^{\circ}06'27''N$, $04^{\circ}50'19''E$) and Zaira, Nigeria ($11^{\circ}04'N$, $07^{\circ}42'E$).

Using normalized BIC as the model selection criterion (that is, to test for goodness of fit) for various values of p and q as can be seen in Table 4.1 below, ARIMA ($1, 1, 1$) model has the least normalized BIC value ($= 0.915$ for Ondo, Nigeria ($07^{\circ}06'27''N$, $04^{\circ}50'19''E$) and 0.515 for Zaira, Nigeria ($11^{\circ}04'N$, $07^{\circ}42'E$)). Hence, ARIMA ($1, 1, 1$) model is considered the best model for the maximum daily temperature data of Ondo, Nigeria ($07^{\circ}06'27''N$, $04^{\circ}50'19''E$) and Zaira, Nigeria ($11^{\circ}04'N$, $07^{\circ}42'E$). To test for the adequacy of the model, Ljung-Box Statistic is used to test the randomness of residuals. From Table 4.2, Ljung – Box Statistic for the model is 17.770 for Ondo, Nigeria ($07^{\circ}06'27''N$, $04^{\circ}50'19''E$) with p -value 0.338 and 28.999 for Zaira, Nigeria ($11^{\circ}04'N$, $07^{\circ}42'E$) with $p - value = 0.024$ which establish that the model is adequate at 1% level of significance.

City	Normalized BIC Values for Model Fit							
	ARIMA (0,1,0)	ARIMA (0,1,1)	ARIMA (1,1,0)	ARIMA (1,1,1)	ARIMA (1,1,2)	ARIMA (2,1,0)	ARIMA (2,1,1)	ARIMA (2,1,2)
Ondo, Nigeria	1.354	0.917	1.144	0.915	0.917	1.049	0.917	0.917
Zaira, Nigeria	0.62	0.558	0.578	0.515	0.518	0.556	0.518	0.518

City	Normalized BIC Values for Model Fit							
	ARIMA (1,1,3)	ARIMA (3,1,1)	ARIMA (2,1,3)	ARIMA (3,1,2)	ARIMA (3,1,3)	ARIMA (1,1,4)	ARIMA (2,1,4)	ARIMA (3,1,4)
Ondo, Nigeria	0.919	0.919	0.920	0.920	0.922	0.920	0.922	0.922
Zaira, Nigeria	0.520	0.520	0.523	0.523	0.522	0.524	0.525	0.525

Table 4.1: Table showing various model fit with their Normalized BIC values for daily maximum temperature data for Ondo, Nigeria and Zaira, Nigeria.

Model Statistics						
City	Number of Predictors	Model Fit statistics	Ljung-Box Q(18)			Number of Outliers
		Stationary R-squared	Statistics	DF	Sig.	
Ondo, Nigeria	0	.359	17.770	16	.338	0
Zaira, Nigeria	0	.105	28.999	16	.024	0

Table 4.2: Table showing the result of test of adequacy of the model for daily maximum temperature data for Ondo, Nigeria and Zaira, Nigeria.

Table 4.3 below presents the estimates of parameters of the ARIMA (1, 1, 1) model for Ondo, Nigeria (07°06'27" N, 04°50'19" E) and Zaira, Nigeria (11°04' N, 07°42' E).

The model for Ondo, Nigeria (07°06'27" N, 04°50'19" E) is

$$\hat{y}_t = 0.00000001 + 0.087y_{t-1} + 0.836\varepsilon_{t-1}.$$

That is, $c = 0.00000001$, $\phi = 0.550$ and $\theta = 0.836$ as can be seen in Table 4.3. Also, in testing for significance of the estimates of parameters, Table 4.3 also gives t-statistic of the estimates of the parameters of the model. It is shown that c having $t = 0.052$ with p -value = 0.959 is not significantly different from zero. Hence, c is redundant.

		Ondo				Zaira			
		Estimate	SE	t	Sig.	Estimate	SE	t	Sig.
Maximum	Constant c	1.0E-8	0.005	0.052	0.959	1.0E-7	0.008	0.055	0.956
Temperature	AR ϕ	0.087	0.02	4.391	0.000	0.55	0.028	19.724	0.000
	Difference	1							
	MA θ	0.833	0.011	75.779	0.000	0.836	0.018	45.597	0.000

Table 4.3: Table showing estimates of parameters for maximum temperature data of Ondo, Nigeria and Zaira, Nigeria.

To improve the model result, we remove c being redundant. This removal has no effect on the estimates of other parameters and Ljung-Box value of the model, rather gives a smaller normalized BIC value ($= 0.912$). Hence, the optimal model for maximum temperature of Ondo, Nigeria ($07^{\circ}06'27''N$, $04^{\circ}50'19''E$) is $\hat{y}_t = 0.087y_{t-1} + 0.833\varepsilon_{t-1}$. Forecast is carried out for the subsequent days in month of December, 2005 for daily maximum meteorological temperature of Ondo, Nigeria ($07^{\circ}06'27''N$, $04^{\circ}50'19''E$), and compared with observed values. It can be seen in Appendix Table 10.

The invertibility behaviour of the optimal model for Ondo, Nigeria ($07^{\circ}06'27''N$, $04^{\circ}50'19''E$) is given as $\varepsilon_t = \sum_{i=0}^{\infty} \pi_i y_{t-i}$ since c is redundant, where

$$\pi_i = \begin{cases} 1, & i = 0 \\ -(1.920), & i = 1 \\ (-1)^i [0.920(0.833^{i-1} + 0.833^{i-2})], & i = 2, 3, 4, \dots \end{cases}$$

The model for Zaira, Nigeria ($11^{\circ}04'N$, $07^{\circ}42'E$) is

$$\hat{y}_t = 0.0000001 + 0.550y_{t-1} + 0.836\varepsilon_{t-1}.$$

That is, $c = 0.0000001$, $\phi = 0.550$ and $\theta = 0.836$ as can be seen in Table 4.3. Also, in testing for significance of the estimates of parameters, it is shown in Table 4.3 that c having $t = 0.055$ with p -value $= 0.956$ is not significantly different from zero. Hence, c is redundant. To improve the model result, we remove c being redundant. This removal has no effect on the estimates of other parameters and Ljung-Box value of the model, rather gives a smaller normalized BIC value ($= 0.512$). Hence, the optimal model for maximum temperature of Zaira, Nigeria ($11^{\circ}04'N$, $07^{\circ}42'E$) is $\hat{y}_t = 0.550y_{t-1} + 0.836\varepsilon_{t-1}$. Forecast is carried out for the subsequent days in month of December, 2005 for daily maximum

meteorological temperature of Zaira, Nigeria ($11^{\circ}04'N$, $07^{\circ}42'E$), and compared with observed values. It can be seen in Appendix Table 11.

The invertibility behaviour of the optimal model for Zaira, Nigeria ($11^{\circ}04'N$, $07^{\circ}42'E$) is given as $\varepsilon_t = \sum_{i=0}^{\infty} \pi_i y_{t-i}$ since c is redundant, where

$$\pi_i = \begin{cases} 1, & i = 0 \\ -(2.386), & i = 1 \\ (-1)^i [1.386(0.836^{i-1} + 0.836^{i-2})], & i = 2, 3, 4, \dots \end{cases}$$

4.2 DAILY RAIN AMOUNT

The rainfall data is used in this work. The data reports daily rain amount of Calabar, Nigeria ($07^{\circ}06'27''N$, $08^{\circ}20'E$) and Ondo, Nigeria ($07^{\circ}06'27''N$, $04^{\circ}50'19''E$) from January 1995 to December 2005. Appendix Figures 6 and 7 (in Appendices 14 and 15 respectively) give the correlogram for the series. The series are stationary and so, there is no need for differencing. Also, the residual terms are independently and identically distributed since the autocorrelation functions at various lags hover around zero for both Calabar, Nigeria ($07^{\circ}06'27''N$, $08^{\circ}20'E$) and Ondo, Nigeria ($07^{\circ}06'27''N$, $04^{\circ}50'19''E$) [Gujarati (2004)], as can be seen in Figures 6 and 7 and Appendix Tables 7 and 8 (in Appendices 12 and 13) respectively for Calabar, Nigeria ($07^{\circ}06'27''N$, $08^{\circ}20'E$) and Ondo, Nigeria ($07^{\circ}06'27''N$, $04^{\circ}50'19''E$) respectively. Similarly, Figures 16(a) and (b) (in Appendix 16) shows that residuals are normally distributed. Hence, ε_t is independently, identically and normally distributed for daily rainfall amount series of both Calabar, Nigeria ($07^{\circ}06'27''N$, $08^{\circ}20'E$) and Ondo, Nigeria ($07^{\circ}06'27''N$, $04^{\circ}50'19''E$). Using normalized BIC as the model selection criterion for various values of p and q , ARIMA (1, 0, 1) has the least normalized BIC value (= 5.700) for Calabar, Nigeria ($07^{\circ}06'27''N$, $08^{\circ}20'E$) but ARIMA (1, 0, 1), ARIMA (1, 0, 2) and ARIMA (2, 0, 1) models have the least normalized BIC value (= 4.773) for Ondo, Nigeria ($07^{\circ}06'27''N$, $04^{\circ}50'19''E$) as can be seen in Table 4.4. Using the principle of parsimony (the model with the fewest parameters that adequately describes the data), we consider ARIMA (1, 0, 1) model to be the most adequate for Ondo, Nigeria ($07^{\circ}06'27''N$, $04^{\circ}50'19''E$). Hence, ARIMA (1, 0, 1) model is considered the best model for the rain amount of Calabar, Nigeria ($07^{\circ}06'27''N$, $08^{\circ}20'E$) and Ondo, Nigeria ($07^{\circ}06'27''N$, $04^{\circ}50'19''E$). To test for the adequacy of the model, Ljung-Box Statistic is used to test the randomness of residuals. From Table 4.5, *Ljung - Box Statistic* for the model is 13.047 with p -value = 0.669 for Calabar, Nigeria ($07^{\circ}06'27''N$, $08^{\circ}20'E$) and

22.731 for Ondo, Nigeria ($07^{\circ}06'27''N$, $04^{\circ}50'19''E$) with $p - value = 0.121$ which establish that residuals are random at both 1% and 5% level of significance and hence the both models specifications are adequate for both stations.

City	Normalized BIC Values for Model Fit									
	(0,0,0)	(1,0,0)	(0,0,1)	(1,0,1)	(2,0,1)	(1,0,2)	(2,0,2)	(2,0,3)	(3,0,2)	(3,0,3)
Calabar	5.762	5.757	5.758	5.700	5.702	5.702	5.705	5.707	5.707	5.709
Ondo	4.819	4.817	4.818	4.773	4.773	4.773	4.776	4.778	4.778	4.780

Table 4.4: Table showing various model fit with their Normalized BIC values for daily rain amount data for Calabar, Nigeria and Ondo, Nigeria.

Model Statistics

City	Number of Predictors	Model Fit statistics	Ljung-Box Q(18)			Number of Outliers
		Stationary R-squared	Statistics	DF	Sig.	
Calabar	0	.065	13.047	16	.669	0
Ondo	0	.050	22.731	16	.121	0

Table 4.5: Table showing the result of test of adequacy of the model for rain amount of Calabar, Nigeria and Ondo, Nigeria.

ARIMA Model Parameters

		Calabar				Ondo			
		Estimate	SE	t	Sig.	Estimate	SE	t	Sig.
Rain Amount	Constant c	8.352	1.239	6.741	.000	4.439	0.541	8.207	0.000
	AR ϕ	0.988	0.004	227.867	.000	0.978	0.006	156.620	0.000
	MA θ	0.945	0.009	105.865	.000	0.931	0.011	84.586	0.000

Table 4.6: Table showing the estimates of parameters of the ARIMA(1, 0, 1) model for rain amount of Calabar, Nigeria and Ondo, Nigeria.

The model for Calabar, Nigeria ($07^{\circ}06'27''N$, $08^{\circ}20'E$) is

$$\hat{y}_t = 8.352 + 0.988y_{t-1} + 0.945\varepsilon_{t-1}$$

where $c = 8.352$, $\phi = 0.988$ and $\theta = 0.945$. The t- test of significance of parameters of the model shows that the three parameters c , ϕ and θ are significantly different from zero because $p - value < 0.01$ for each of c , ϕ and θ as can be seen in Table 4.6. Hence there is

no redundant variable in the model for rain amount for Calabar, Nigeria ($07^{\circ}06'27''N$, $08^{\circ}20'E$).

The causality behaviour of the optimal model for daily rain amount of Calabar, Nigeria ($07^{\circ}06'27''N$, $08^{\circ}20'E$) is given as $y_t = 696 + \sum_{i=0}^{\infty} \psi_i \varepsilon_{t-i}$ since $\frac{\varepsilon}{(1+\theta)} \neq 0$, where

$$\psi_i = \begin{cases} 1, & \text{if } i = 0 \\ 0.988^i + 0.945(0.988^{i-1}), & \text{if } i = 1, 2, 3, \dots \end{cases}$$

The invertibility behaviour of the optimal model for daily rain amount of Calabar, Nigeria ($07^{\circ}06'27''N$, $08^{\circ}20'E$) is given as $y_t = -4.294 + \sum_{i=0}^{\infty} \pi_i y_{t-i}$ since $\frac{\varepsilon}{(1+\theta)} \neq 0$, where

$$\pi_i = \begin{cases} 1, & i = 0 \\ (-1)^i [0.988(0.945^{i-1}) + 0.945^i], & i = 1, 2, 3, \dots \end{cases}$$

The model for Ondo, Nigeria ($07^{\circ}06'27''N$, $04^{\circ}50'19''E$) is

$$\hat{y}_t = 4.439 + 0.978y_{t-1} + 0.931\varepsilon_{t-1}$$

where $c = 4.439$, $\phi = 0.978$ and $\theta = 0.931$. The t- test of significance of parameters of the model shows that the three parameters c , ϕ and θ are significantly different from zero because $p - \text{value} < 0.01$ for each of c , ϕ and θ as can be seen in Table 4.6. Hence there is no redundant variable in the model for rain amount for Ondo, Nigeria.

The causality behaviour of the optimal model for daily rain amount of Ondo, Nigeria ($07^{\circ}06'27''N$, $04^{\circ}50'19''E$) is given as $y_t = 201.773 + \sum_{i=0}^{\infty} \psi_i \varepsilon_{t-i}$ since $\frac{\varepsilon}{(1-\phi)} \neq 0$, where

$$\psi_i = \begin{cases} 1, & \text{if } i = 0 \\ 0.978^i + 0.931(0.978^{i-1}), & \text{if } i = 1, 2, 3, \dots \end{cases}$$

The invertibility behaviour of the optimal model for daily rain amount of Ondo, Nigeria ($07^{\circ}06'27''N$, $04^{\circ}50'19''E$) is given as $y_t = -2.299 + \sum_{i=0}^{\infty} \pi_i y_{t-i}$ since $\frac{\varepsilon}{(1-\phi)} \neq 0$, where

$$\pi_i = \begin{cases} 1, & i = 0 \\ (-1)^i [0.978(0.931^{i-1}) + 0.931^i], & i = 1, 2, 3, \dots \end{cases}$$

4.3 DAILY CLOUD

The cloud data is also used in this work. This data reports daily cloud of Warri, Nigeria ($05^{\circ}30'N$, $05^{\circ}41'E$) and Benin City, Nigeria ($07^{\circ}06'27''N$, $05^{\circ}31'E$) from January 1995

to December 2005. Appendix Figures 9 and 10 (in Appendices 19 and 20 respectively) give the correlogram for the series. The series are stationary and so, there is no need for differencing. Also, the residual terms are independently and identically distributed for Warri, Nigeria ($05^{\circ}30'N$, $05^{\circ}41'E$) and Benin City, Nigeria ($07^{\circ}06'27''N$, $05^{\circ}31'E$) since the autocorrelation functions at various lags hover around zero [Gujarati (2004)], as can be seen in Appendix Tables 9 and 10 (in Appendices 17 and 18) and Appendix Figures 9 and 10 (in Appendices 19 and 20) for Warri, Nigeria ($05^{\circ}30'N$, $05^{\circ}41'E$) and Benin City, Nigeria ($07^{\circ}06'27''N$, $05^{\circ}31'E$) respectively. Similarly, Appendix Figures 11(a) and (b) (in Appendix 21) shows that residuals are normally distributed. Hence, ε_t is independently, identically and normally distributed.

Using normalized BIC as the model selection criterion for various values of p and q . From Table 4.7, it is shown that ARIMA ($I, 0, I$) model has the least normalized BIC value ($= -1.249$) for Warri, Nigeria ($05^{\circ}30'N$, $05^{\circ}41'E$) and ($= -3.686$) for Benin City, Nigeria ($07^{\circ}06'27''N$, $05^{\circ}31'E$). Hence, ARIMA ($I, 0, I$) is considered the best model for the cloud of Warri, Nigeria ($05^{\circ}30'N$, $05^{\circ}41'E$) and Benin City, Nigeria ($07^{\circ}06'27''N$, $05^{\circ}31'E$). To test for the adequacy of the model, Ljung-Box Statistic is used to test the randomness of residuals. From Table 4.8 below, Ljung – Box Statistic for the model is 14.152 with $p - value = 0.587$ for Warri, Nigeria ($05^{\circ}30'N$, $05^{\circ}41'E$) and 23.419 for Benin City, Nigeria ($07^{\circ}06'27''N$, $05^{\circ}31'E$) with $p - value = 0.103$ which establish that residuals are random at both 1% and 5% level of significance and hence the both models specifications are adequate for both stations.

The model for Warri, Nigeria ($05^{\circ}30'N$, $05^{\circ}41'E$) is

$$\hat{y}_t = 6.971 + 0.966y_{t-1} + 0.927\varepsilon_{t-1}$$

that is, $c = 6.971$, $\phi = 0.966$ and $\theta = 0.927$.

City	Normalized BIC Values for Model Fit									
	(0,0,0)	(1,0,0)	(0,0,1)	(1,0,1)	(2,0,1)	(1,0,2)	(2,0,2)	(2,0,3)	(3,0,2)	(3,0,3)
Warri	-1.231	-1.230	-1.230	-1.249	-1.247	-1.247	-1.245	-1.242	-1.242	-1.24
Benin City	-3.681	-3.681	-3.680	-3.686	-3.684	-3.684	-3.683	-3.681	-3.681	-3.67

Table 4.7: Table showing various model fit with their Normalized BIC values for daily rain amount data for Warri, Nigeria and Benin City, Nigeria.

Model Statistics

	Number of Predictors	Model Fit statistics	Ljung-Box Q(18)			Number of Outliers
		Stationary R-squared	Statistics	DF	Sig.	
Warri	0	.022	14.152	16	.587	0
Benin City	0	-0.001	23.419	16	.103	0

Table 4.8: Table showing the result of test of adequacy of model for cloud of Warri, Nigeria and Benin City, Nigeria.

ARIMA Model Parameters

		Warri				Benin City			
		Estimate	SE	t	Sig.	Estimate	SE	t	Sig.
Cloud	Constant c	6.971	0.018	378.949	.000	7.002	0.396	17.681	0.000
	AR ϕ	0.966	0.011	85.996	.000	1.000	0.004	265.515	0.000
	MA θ	0.927	0.016	56.916	.000	0.962	0.006	153.488	0.000

Table 4.9: Table showing the estimates of parameters of the ARIMA(1, 0, 1) model for rain amount of Warri, Nigeria and Benin City, Nigeria.

The t-test of significance of parameters of the model shows that the three parameters c , ϕ and θ are significantly different from zero because p -value < 0.01 for each of c , ϕ and θ as can be seen in Table 4.9.

The causality behaviour of the optimal model for daily cloud amount of Warri, Nigeria ($05^{\circ}30'N$, $05^{\circ}41'E$) is given as $y_t = 205.029 + \sum_{i=0}^{\infty} \psi_i \varepsilon_{t-i}$ since $\frac{c}{(1-\phi)} \neq 0$, where

$$\psi_i = \begin{cases} 1, & \text{if } i = 0 \\ 0.966^i + 0.927(0.966^{i-1}), & \text{if } i = 1, 2, 3, \dots \end{cases}$$

The model for Benin City, Nigeria ($07^{\circ}06'27''N$, $05^{\circ}31'E$) is

$$\hat{y}_t = 7.002 + y_{t-1} + 0.962\varepsilon_{t-1}$$

where $c = 7.002$, $\phi = 1.000$ and $\theta = 0.962$. The t-test of significance of parameters of the model shows that the three parameters c , ϕ and θ are significantly different from zero because p -value < 0.01 for each of c , ϕ and θ as can be seen in Table 4.9.



5.1 SUMMARY

Autoregressive Integrated Moving Average (ARIMA) model of various orders was presented with a view of identifying optimal model. ARIMA model of various orders was presented in causal and inverted forms, invertibility and causality conditions were investigated. In presenting ARMA model in causal form, we establish a process y_t as a linear combination of residual terms, ε_t . Equation (2.43) holds if $\phi \neq 1$ but holds for all values of ϕ if there is no constant term, c (i.e. $c = 0$). We presented ARIMA (p, d, q) model in inverted form for $q = 1$ and various values of p and d and examined the behaviour of both causality and invertibility parameters. Parameters of ARIMA (p, d, q) model were evaluated for some values of p and d at $q = 0$ given that $c = 0$ and $c \neq 0$ using Ordinary Least Square method and Crammer's rule. The ARIMA model of various orders p, d , and q were subjected to Normalized Bayesian Information criterion as model selection criterion to obtain optimal models for some daily average meteorological information in some stations in Nigeria from year 1995 to year 2005. Parameters of optimal models were analysed and t-test of significance was used to check for redundancy. Forecast was carried out for the subsequent days in the month of December, 2005 for daily maximum meteorological temperature of Ondo, Nigeria ($07^{\circ}06'27''N, 04^{\circ}50'19''E$) and Zaira, Nigeria ($11^{\circ}04'N, 07^{\circ}42'E$) and compared with observed values.

5.2 CONCLUSION

Autoregressive Integrated Moving Average (ARIMA) model of various orders was presented with a view of identifying optimal model. ARIMA model of various orders was presented in causal and inverted forms, invertibility and causality conditions were investigated. It was deduced that behaviour of causality parameter ψ_t and invertibility parameter π_t depend on positive and negative values of autoregressive parameter ϕ and moving average parameter θ respectively. Causality parameter ψ_t is skewed to right and sinusoidal for positive and negative values of ϕ as shown in Figures 2.1, 2.3 and 2.2 respectively. Absolute value of causality parameter ψ_t of ARMA $(1, q)$ model increases as the value of q increases for

positive values of ϕ . Also, ψ_i increases as ϕ increases as shown in Figure 2.4 for positive values of ϕ .

For various sequential arrangement of $\phi_1, \phi_2, \phi_3, \dots, \phi_p > 0$ for all integer values of p , it was deduced that the invertibility parameter π_i of ARIMA $(p, d, 1)$ for various integer values of d are sinusoidal, the absolute value of the invertibility parameter, $|\pi_i|$ increases as d increases for positive values of θ and the lower the integer value of d , the faster $|\pi_i|$ converges to zero as shown in Figure 2.5, 2.11, 2.12 above. For negative integer values of θ , it was shown in Figure 2.6 that the invertibility parameter π_i of ARIMA $(p, d, 1)$ model oscillates as the value of d increases with $|\pi_i|_{d=k} < |\pi_i|_{d=k+2}$ for all even integer value $k > 0$ provided $i \geq 3$ and $|\pi_i|_{d=k} > |\pi_i|_{d=k+2}$ for all odd integer value $k > 0$. Similarly, the invertibility parameter π_i of ARIMA $(p, d, 1)$ model was found to be sinusoidal for $p = 1, 2, 3$ and 4 and $d = 1$ and 2 with $|\pi_i|_{p=4} > |\pi_i|_{p=3} > |\pi_i|_{p=2} > |\pi_i|_{p=1}$ for all positive values of θ as seen in Figures 2.7 and 2.9 above. For all negative values of θ , $|\pi_i|_{p=k} < |\pi_i|_{p=k+2}$ for odd integer value $k > 0$ and $|\pi_i|_{p=k} > |\pi_i|_{p=k+2}$ for even integer value $k > 0$ as seen in Figures 2.8 and 2.10.

Furthermore, parameters of ARIMA (p, d, q) model were evaluated for some values of p and d at $q = 0$ given that $c = 0$ and $c \neq 0$ using Ordinary Least Square method and Crammer's rule. It was deduced that every term y_{t-j} in ARIMA $(p, 0, 0)$ is replaced by $y_{t-j} - y_{t-j-1}$ in ARIMA $(p, 1, 0)$. Also, Ψ is p column matrix for $c = 0$ while Ψ is $(p + 1)$ column matrix for $c \neq 0$.

In this study, stochastic models were developed which retained the identified trends and any persistence in each of the time series of daily maximum meteorological temperature, daily rain amount and daily cloud amount due to climate change. The developed models were parsimonious and of good statistical fit. ARIMA (p, d, q) was formulated for daily average meteorological outcomes in some stations in Nigeria from 1995 to year 2005. Normalized Bayesian Information Criterion was used for measuring performance of the models. ARIMA $(1, 1, 1)$ was adequate for modelling daily maximum meteorological temperature of Ondo, Nigeria ($07^{\circ}06'27''N$, $04^{\circ}50'19''E$) and Zaira, Nigeria ($11^{\circ}04'N$, $07^{\circ}42'E$), parameters of the model were estimated and redundant variable was removed. Forecast was carried out for the month of December, 2005 and compared with observed values, and was reported.

For modelling daily rain amount of Calabar, Nigeria ($07^{\circ}06'27''N$, $08^{\circ}20'E$) and Ondo, Nigeria ($07^{\circ}06'27''N$, $04^{\circ}50'19''E$), ARIMA (1, 0, 1) was found adequate, parameters of the model were estimated. In the model for both stations, there were no redundant variables as the estimates of all the parameters were significantly different from zero. Similarly, ARIMA (1, 0, 1) was adequate for modelling daily cloud of Warri, Nigeria ($05^{\circ}30'N$, $05^{\circ}41'E$) and Benin City, Nigeria ($07^{\circ}06'27''N$, $05^{\circ}31'E$), parameters of the model were estimated. In the model for both stations, there are no redundant variables as the estimates of all the parameters are significantly different from zero.

5.3 CONTRIBUTION TO KNOWLEDGE

This research has contributed the following to the existing knowledge by making available

- (1) an ARIMA model of various orders in causal and inverted form.
- (2) behavioural pattern of causality parameter of ARMA models and invertibility parameter of ARIMA models of various orders.
- (3) optimal autoregressive integrated moving average models for predicting various meteorological outcomes in some stations in Nigeria.



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APPENDIX 1

Autocorrelations

Series: Daily maximum temperature

Lag	Autocorrelation	Std. Error(a)	Box-Ljung Statistic		
	Value	df	Sig.(b)	Value	df
1	.753	.016	2230.383	1	.000
2	.722	.016	4283.249	2	.000
3	.718	.016	6313.255	3	.000
4	.715	.016	8329.577	4	.000
5	.709	.016	10309.904	5	.000
6	.698	.016	12231.969	6	.000
7	.697	.016	14148.960	7	.000
8	.695	.016	16058.029	8	.000
9	.689	.016	17933.121	9	.000
10	.679	.016	19756.821	10	.000
11	.684	.016	21612.632	11	.000
12	.666	.016	23372.363	12	.000
13	.663	.016	25117.462	13	.000
14	.655	.016	26821.430	14	.000
15	.655	.016	28527.636	15	.000
16	.651	.016	30213.498	16	.000
17	.637	.016	31830.251	17	.000
18	.636	.016	33440.469	18	.000
19	.618	.016	34964.574	19	.000
20	.625	.016	36521.542	20	.000
21	.612	.016	38017.907	21	.000
22	.607	.016	39488.791	22	.000
23	.595	.016	40905.716	23	.000
24	.594	.016	42316.807	24	.000

a. The underlying process assumed is independence (white noise).

b. Based on the asymptotic chi-square approximation.

Partial Autocorrelations

Series: Daily maximum temperature

Lag	Partial Autocorrelation	Std. Error
1	.753	.013
2	.358	.016
3	.263	.016
4	.202	.016
5	.150	.016
6	.104	.016
7	.101	.016
8	.090	.016
9	.066	.016
10	.041	.016
11	.072	.016
12	.006	.016
13	.025	.016
14	.009	.016
15	.028	.016
16	.021	.016
17	-.009	.016
18	.011	.016
19	-.033	.016
20	.023	.016
21	-.012	.016
22	-.004	.016
23	-.020	.016
24	.002	.016

Appendix Table 1: Correlogram before difference for daily maximum meteorological temperature of Ondo.



APPENDIX 2

Autocorrelations

Series: Maximum Temperature

Lag	Autocorrelation	Std. Error ^a	Box-Ljung Statistic		
			Value	df	Sig. ^b
1	-.439	.016	755.439	1	.000
2	-.053	.016	766.689	2	.000
3	-.004	.016	766.741	3	.000
4	.009	.016	767.045	4	.000
5	.008	.016	767.322	5	.000
6	-.021	.016	769.040	6	.000
7	.002	.016	769.055	7	.000
8	.009	.016	769.407	8	.000
9	.007	.016	769.580	9	.000
10	-.031	.016	773.375	10	.000
11	.049	.016	782.753	11	.000
12	-.032	.016	786.734	12	.000
13	.011	.016	787.252	13	.000
14	-.018	.016	788.486	14	.000
15	.009	.016	788.820	15	.000
16	.020	.016	790.339	16	.000
17	-.025	.016	792.804	17	.000
18	.032	.016	796.754	18	.000
19	-.047	.016	805.736	19	.000
20	.038	.016	811.565	20	.000
21	-.015	.016	812.438	21	.000
22	.013	.016	813.093	22	.000
23	.021	.016	814.866	23	.000
24	.006	.016	815.031	24	.000

a. The underlying process assumed is independence (white noise).

b. Based on the asymptotic chi-square approximation.

Partial Autocorrelations

Series: Maximum Temperature

Lag	Partial Autocorrelation	Std. Error
1	-.439	.016
2	-.304	.016
3	-.229	.016
4	-.169	.016
5	-.118	.016
6	-.115	.016
7	-.101	.016
8	-.077	.016
9	-.051	.016
10	-.080	.016
11	-.013	.016
12	-.032	.016
13	-.016	.016
14	-.035	.016
15	-.028	.016
16	.003	.016
17	-.016	.016
18	.027	.016
19	-.028	.016
20	.007	.016
21	-.002	.016
22	.015	.016
23	-.008	.016
24	-.009	.016

Appendix Table 2: Correlogram after difference for daily maximum meteorological temperature of Ondo.



APPENDIX 3

Autocorrelations

Series: Temperature

Lag	Autocorrelation	Std. Error(a)	Box-Ljung Statistic		
	Value	df	Sig.(b)	Value	df
1	.895	.018	2569.670	1	.000
2	.834	.018	4803.104	2	.000
3	.795	.018	6836.502	3	.000
4	.764	.018	8715.576	4	.000
5	.736	.018	10459.570	5	.000
6	.719	.018	12122.363	6	.000
7	.703	.018	13714.678	7	.000
8	.686	.018	15231.461	8	.000
9	.672	.018	16688.581	9	.000
10	.663	.018	18105.213	10	.000
11	.659	.018	19506.257	11	.000
12	.648	.018	20860.246	12	.000
13	.637	.018	22168.611	13	.000
14	.628	.018	23443.144	14	.000
15	.613	.018	24657.825	15	.000
16	.590	.018	25781.171	16	.000
17	.574	.018	26844.639	17	.000
18	.562	.018	27866.003	18	.000
19	.551	.018	28848.091	19	.000
20	.537	.018	29781.902	20	.000
21	.528	.018	30682.290	21	.000
22	.517	.018	31546.163	22	.000
23	.501	.018	32358.173	23	.000
24	.488	.018	33130.335	24	.000

a. The underlying process assumed is independence (white noise).

b. Based on the asymptotic chi-square approximation.

Partial Autocorrelations

Series: Temperature

Lag	Partial Autocorrelation	Std. Error
1	.895	.018
2	.167	.018
3	.126	.018
4	.076	.018
5	.047	.018
6	.077	.018
7	.051	.018
8	.028	.018
9	.038	.018
10	.046	.018
11	.064	.018
12	.006	.018
13	.015	.018
14	.026	.018
15	-.016	.018
16	-.050	.018
17	.003	.018
18	.013	.018
19	.010	.018
20	-.011	.018
21	.010	.018
22	-.003	.018
23	-.025	.018
24	-.001	.018

Appendix Table 3: Correlogram before differencing for daily maximum meteorological temperature of Zaira, Nigeria.

APPENDIX 4

Autocorrelations

Partial Autocorrelations

Series: Temperature

Series: Temperature

Lag	Autocorrelation	Std. Error(a)	Box-Ljung Statistic		
	Value	df	Sig. (b)	Value	df
1	-.210	.018	141.660	1	.000
2	-.105	.018	177.325	2	.000
3	-.036	.018	181.451	3	.000
4	-.015	.018	182.153	4	.000
5	-.052	.018	191.011	5	.000
6	-.010	.018	191.309	6	.000
7	.003	.018	191.341	7	.000
8	-.017	.018	192.292	8	.000
9	-.018	.018	193.386	9	.000
10	-.025	.018	195.459	10	.000
11	.034	.018	199.259	11	.000
12	.000	.018	199.259	12	.000
13	-.009	.018	199.498	13	.000
14	.034	.018	203.309	14	.000
15	.041	.018	208.684	15	.000
16	-.036	.018	212.780	16	.000
17	-.019	.018	213.995	17	.000
18	-.002	.018	214.005	18	.000
19	.011	.018	214.428	19	.000
20	-.019	.018	215.564	20	.000
21	.003	.018	215.602	21	.000
22	.021	.018	217.042	22	.000
23	-.018	.018	218.053	23	.000
24	.009	.018	218.302	24	.000

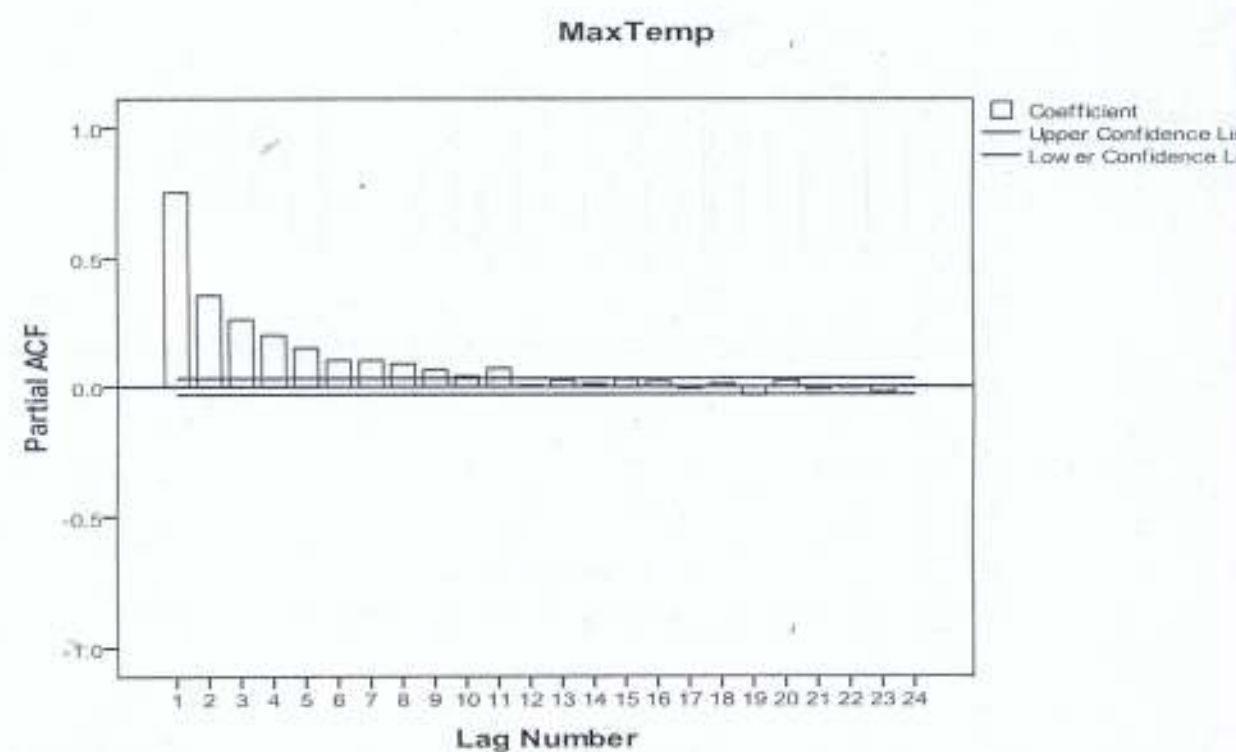
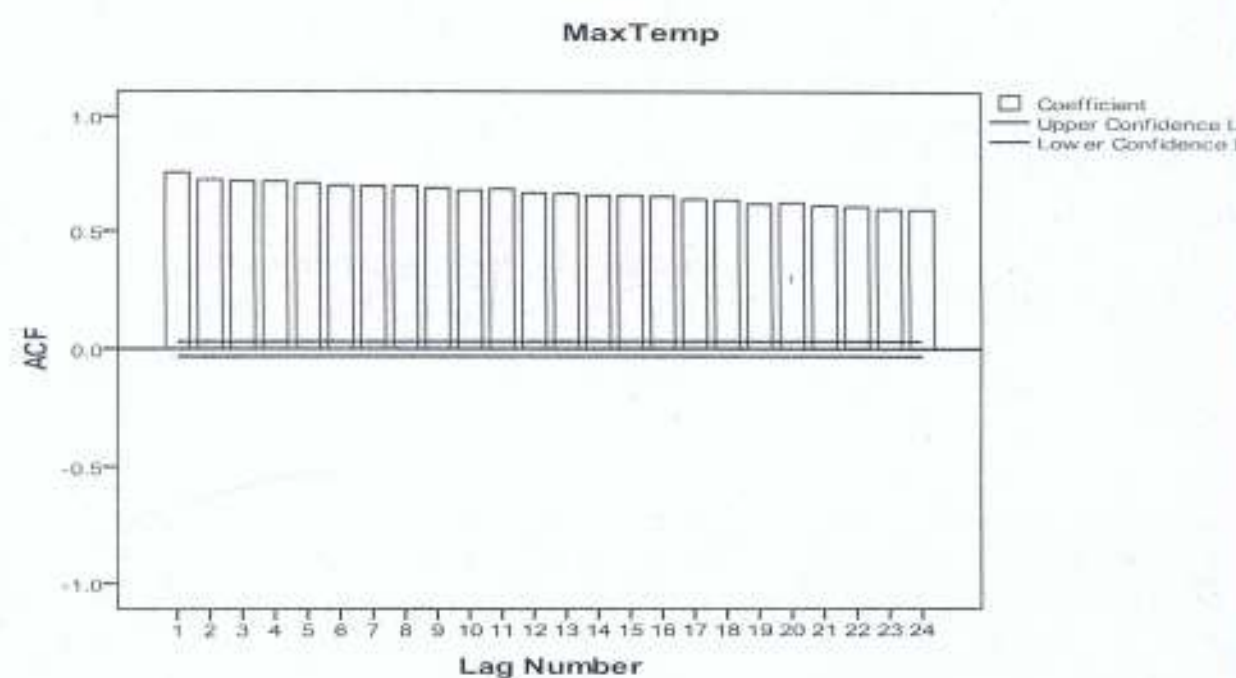
Lag	Partial Autocorrelation	Std. Error
1	-.210	.018
2	-.156	.018
3	-.101	.018
4	-.070	.018
5	-.099	.018
6	-.069	.018
7	-.048	.018
8	-.057	.018
9	-.061	.018
10	-.076	.018
11	-.019	.018
12	-.027	.018
13	-.033	.018
14	.011	.018
15	.043	.018
16	-.011	.018
17	-.018	.018
18	-.015	.018
19	.005	.018
20	-.016	.018
21	-.005	.018
22	.016	.018
23	-.010	.018
24	.011	.018

a. The underlying process assumed is independence (white noise).

b. Based on the asymptotic chi-square approximation.

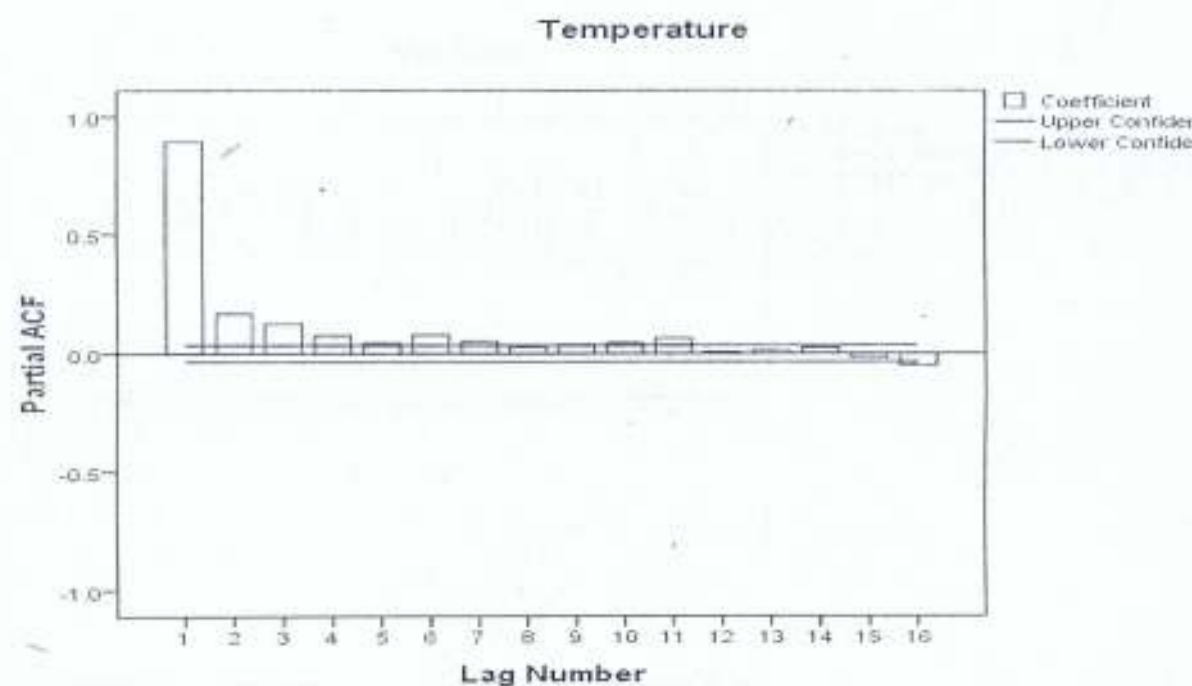
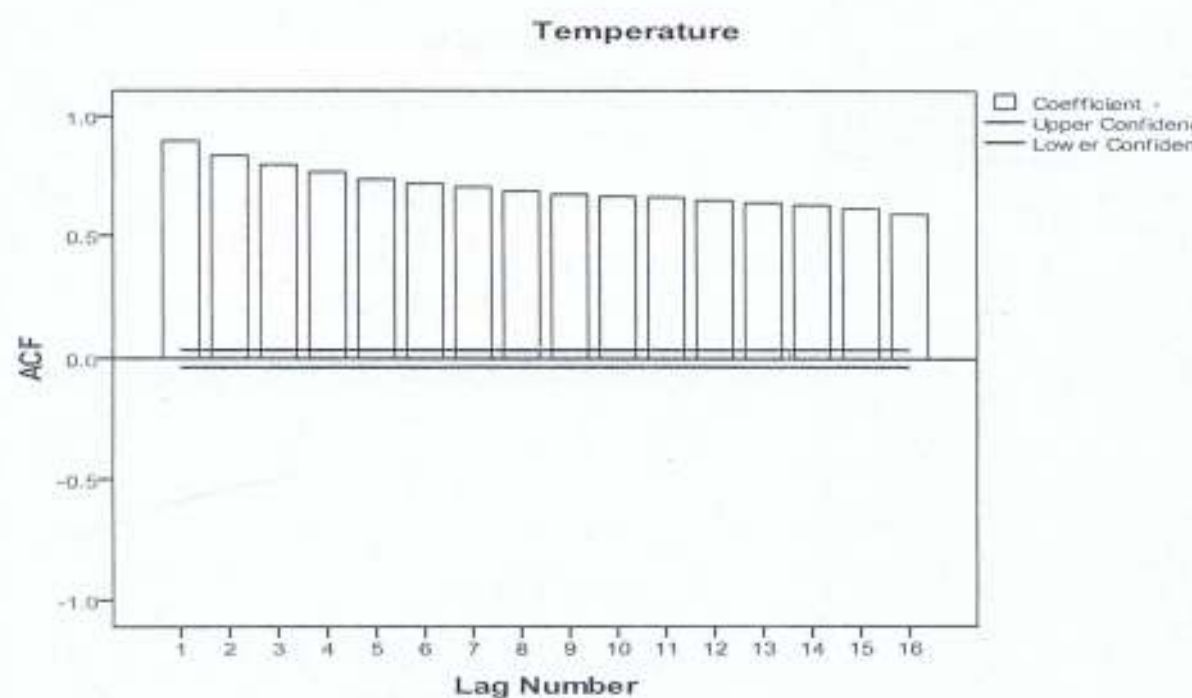
Appendix Table 4: Correlogram after differencing for daily maximum meteorological temperature of Zaira, Nigeria.

APPENDIX 5



Appendix Figure 1: Correlogram before difference for daily maximum temperature of Ondo, Nigeria.

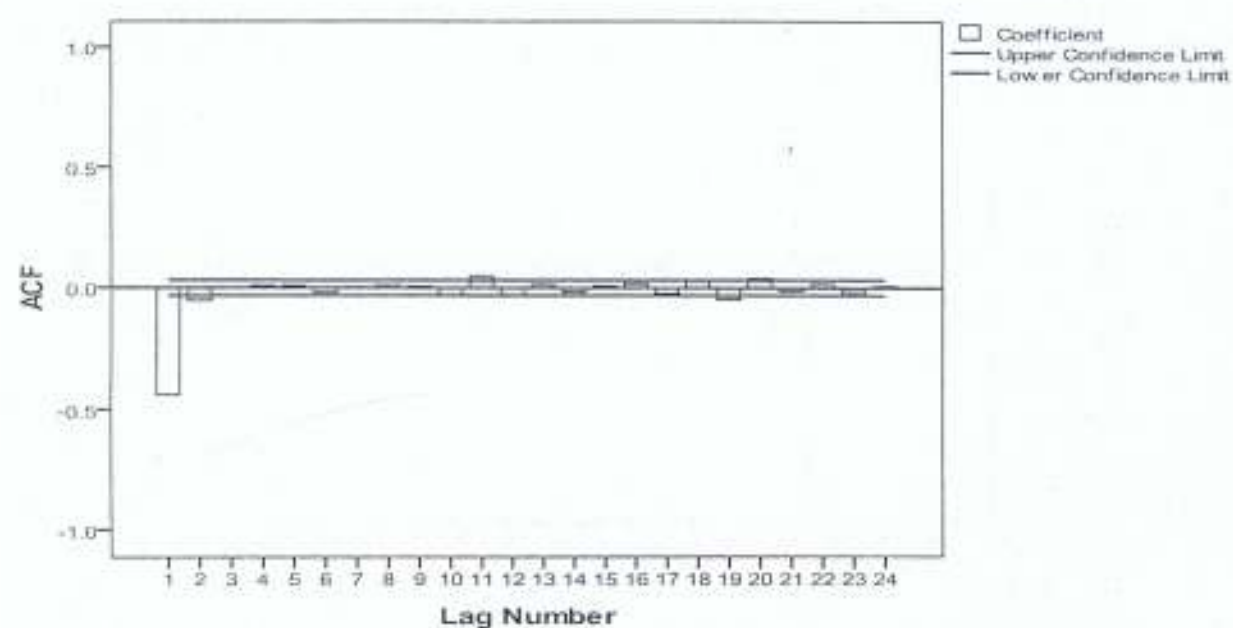
APPENDIX 6



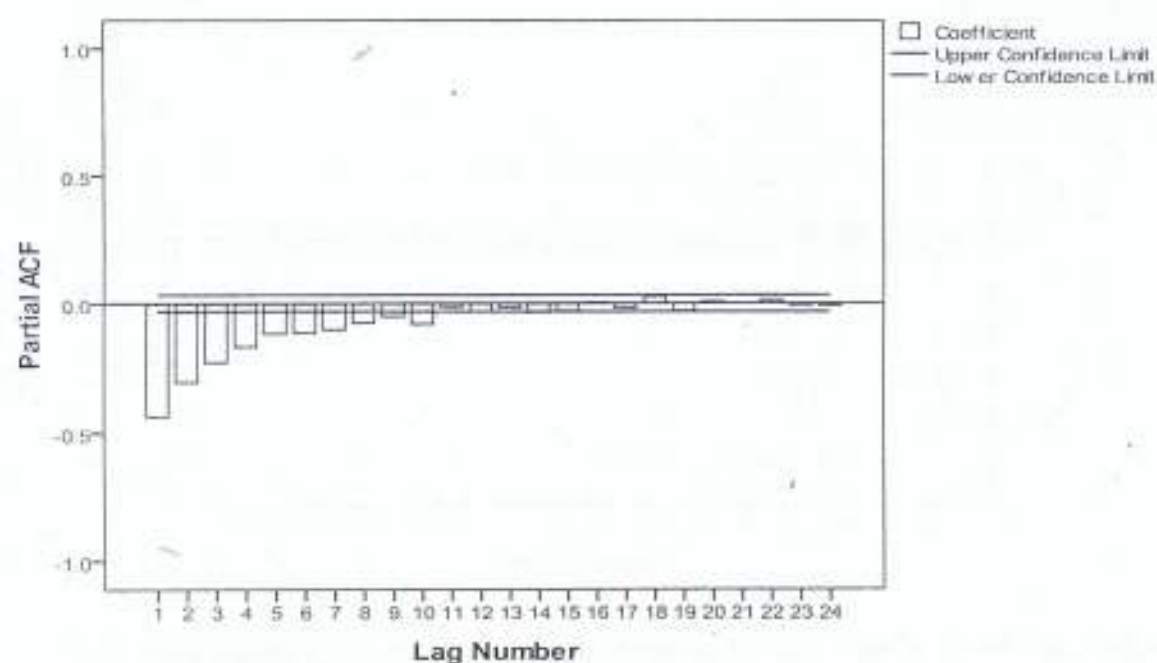
Appendix Figure 2: Correlogram before difference for daily maximum meteorological temperature of Zaira, Nigeria.

APPENDIX 7

MaxTemp

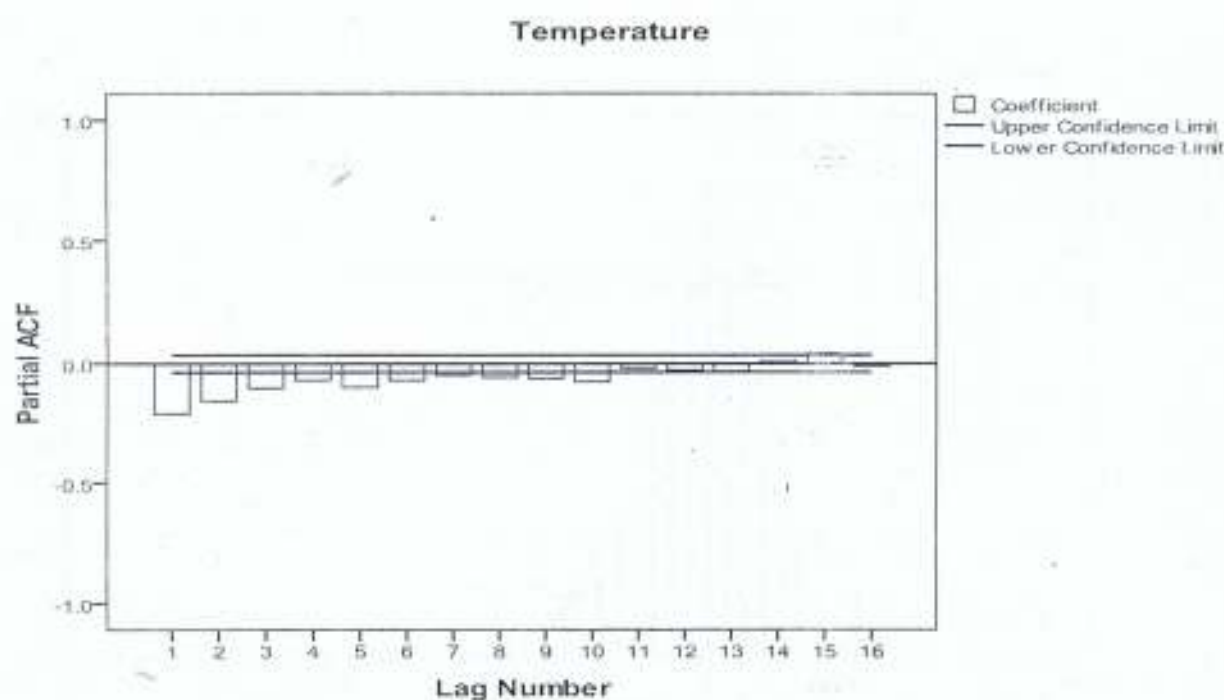
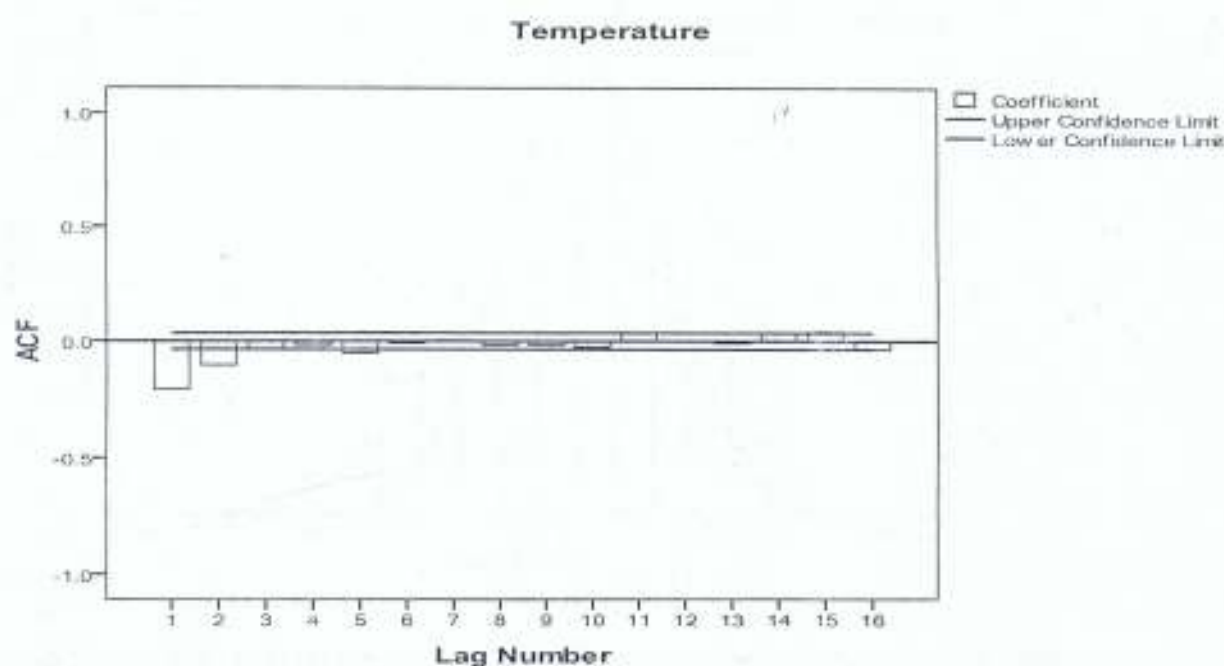


MaxTemp



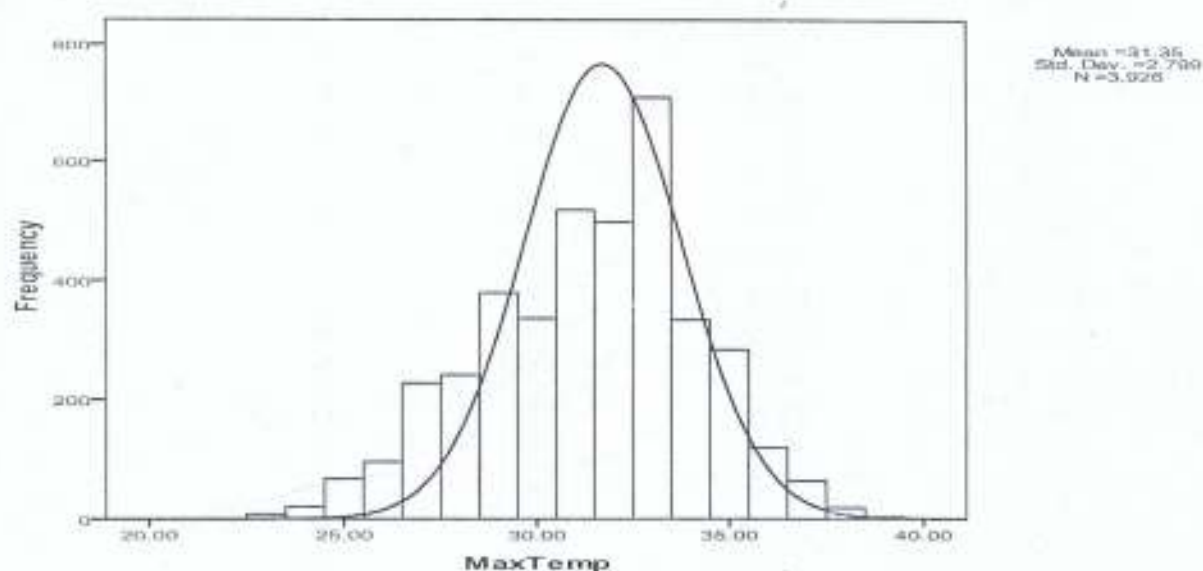
Appendix Figure 3: Correlogram after difference for daily maximum meteorological temperature of Ondo, Nigeria.

APPENDIX 8

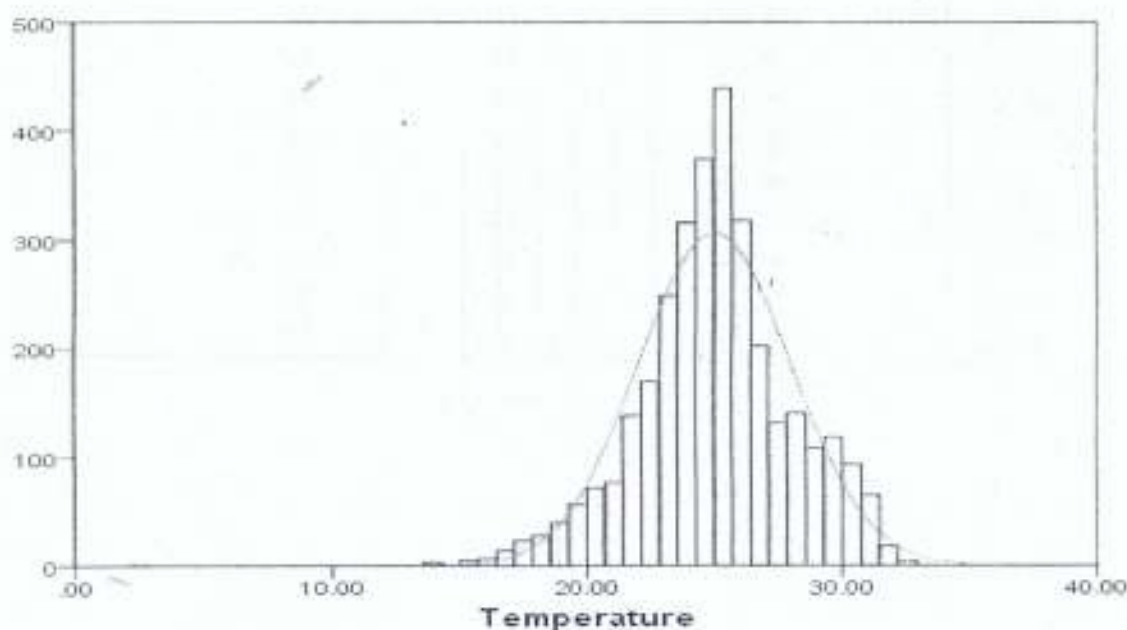


Appendix Figure 4: Correlogram after difference for daily maximum meteorological temperature of Zaira, Nigeria.

APPENDIX 9



Appendix Figure 5(a): Histogram of residuals for daily maximum meteorological temperature of Ondo, Nigeria.



Appendix Figure 5(b): Histogram of residuals for daily maximum meteorological temperature of Zaira, Nigeria.

APPENDIX 10

Month	Days	Observed	Forecast	Residual
Dec-05	1	34	33.8	0.20
	2	34	33.79	0.21
	3	35	33.79	1.21
	4	34	33.79	0.21
	5	34	33.79	0.21
	6	31	33.79	-2.79
	7	33	33.79	-0.79
	8	34	33.79	0.21
	9	34	33.79	0.21
	10	35	33.79	1.21
	11	32	33.79	-1.79
	12	32	33.79	-1.79
	13	33	33.79	-0.79
	14	33	33.79	-0.79
	15	33	33.79	-0.79
	16	33	33.79	-0.79
	17	34	33.79	0.21
	18	33	33.79	-0.79
	19	33	33.79	-0.79
	20	33	33.79	-0.79
	21	33	33.79	-0.79
	22	31	33.79	-2.79
	23	33	33.79	-0.79
	24	31	33.79	-2.79
	25	32	33.79	-1.79
	26	33	33.79	-0.79
	27	32	33.79	-1.79
	28	33	33.79	-0.79
	29	33	33.79	-0.79
	30	33	33.79	-0.79
	31	33	33.79	-0.79

Appendix Table 5: Comparison of Ondo, Nigeria daily maximum meteorological temperature data based on the optimal model and compared with observed values for December, 2005.

APPENDIX 11

Month	Days	Observed	Forecast	Residual
Dec-05	1	23.7	23.00626	0.693738
	2	24.1	22.95471	1.145295
	3	23.5	22.92635	0.573651
	4	22.2	22.91075	-0.710750
	5	22.5	22.90218	-0.402180
	6	22.1	22.89746	-0.797460
	7	22.7	22.89486	-0.194860
	8	23.4	22.89344	0.506564
	9	23.8	22.89265	0.907349
	10	24.4	22.89222	1.507781
	11	24.1	22.89198	1.208018
	12	23.8	22.89185	0.908149
	13	23.3	22.89178	0.408221
	14	22.8	22.89174	-0.091740
	15	23.4	22.89172	0.508282
	16	23.1	22.89171	0.208294
	17	23.1	22.89170	0.208300
	18	21.7	22.89170	-1.191700
	19	22.0	22.89169	-0.891690
	20	21.2	22.89169	-1.691690
	21	21.4	22.89169	-1.491690
	22	21.3	22.89169	-1.591690
	23	22.1	22.89169	-0.791690
	24	21.7	22.89169	-1.191690
	25	22.2	22.89169	-0.691690
	26	21.6	22.89169	-1.291690
	27	22.9	22.89169	0.008308
	28	24.5	22.89169	1.608308
	29	22.8	22.89169	-0.091690
	30	21.5	22.89169	-1.391690
	31	22.3	22.89169	-0.591690

Appendix Table 6: Comparison of Zaira, Nigeria daily maximum meteorological temperature data based on the optimal model and compared with observed values for December, 2005.

APPENDIX 12

Autocorrelations

Series: RainAmount

Lag	Autocorrelation	Std. Error(a)	Box-Ljung Statistic		
	Value	df	Sig (b)	Value	df
1	.087	.017	26.739	1	.000
2	.079	.017	48.722	2	.000
3	.110	.017	90.954	3	.000
4	.107	.017	131.194	4	.000
5	.103	.017	168.479	5	.000
6	.106	.017	208.006	6	.000
7	.073	.017	226.649	7	.000
8	.101	.017	262.449	8	.000
9	.097	.017	295.810	9	.000
10	.088	.017	323.214	10	.000
11	.076	.017	343.555	11	.000
12	.075	.017	363.360	12	.000
13	.097	.017	396.390	13	.000
14	.085	.017	422.128	14	.000
15	.100	.017	457.729	15	.000
16	.094	.017	489.292	16	.000
17	.071	.017	507.170	17	.000
18	.066	.017	522.726	18	.000
19	.091	.017	552.321	19	.000
20	.077	.017	573.682	20	.000
21	.073	.017	592.624	21	.000
22	.091	.017	622.297	22	.000
23	.078	.017	644.025	23	.000
24	.101	.017	680.817	24	.000

a. The underlying process assumed is independence (white noise).

b. Based on the asymptotic chi-square approximation.

Partial Autocorrelations

Series: RainAmount

Lag	Partial Autocorrelation	Std. Error
1	.087	.017
2	.072	.017
3	.098	.017
4	.087	.017
5	.077	.017
6	.074	.017
7	.034	.017
8	.063	.017
9	.054	.017
10	.044	.017
11	.029	.017
12	.027	.017
13	.049	.017
14	.035	.017
15	.052	.017
16	.042	.017
17	.016	.017
18	.010	.017
19	.036	.017
20	.022	.017
21	.017	.017
22	.037	.017
23	.021	.017
24	.046	.017

Appendix Table 7: Correlogram for daily rain amount of Calabar, Nigeria.

APPENDIX 13

Autocorrelations

Series: Rain amount

Lag	Autocorrelation	Std. Error(a)	Box-Ljung Statistic		
	Value	df	Sig.(b)	Value	df
1	.061	.016	14.451	1	.000
2	.087	.016	43.913	2	.000
3	.101	.016	84.195	3	.000
4	.094	.016	119.375	4	.000
5	.113	.016	170.116	5	.000
6	.090	.016	202.297	6	.000
7	.054	.016	213.806	7	.000
8	.094	.016	248.729	8	.000
9	.084	.016	276.589	9	.000
10	.082	.016	303.395	10	.000
11	.048	.016	312.665	11	.000
12	.076	.016	335.396	12	.000
13	.079	.016	359.982	13	.000
14	.087	.016	390.190	14	.000
15	.063	.016	405.903	15	.000
16	.079	.016	430.533	16	.000
17	.075	.016	453.122	17	.000
18	.048	.016	462.316	18	.000
19	.057	.016	475.240	19	.000
20	.068	.016	493.531	20	.000
21	.074	.016	515.397	21	.000
22	.047	.016	524.105	22	.000
23	.051	.016	534.655	23	.000
24	.082	.016	561.615	24	.000

- a. The underlying process assumed is independence (white noise).
 b. Based on the asymptotic chi-square approximation.

Partial Autocorrelations

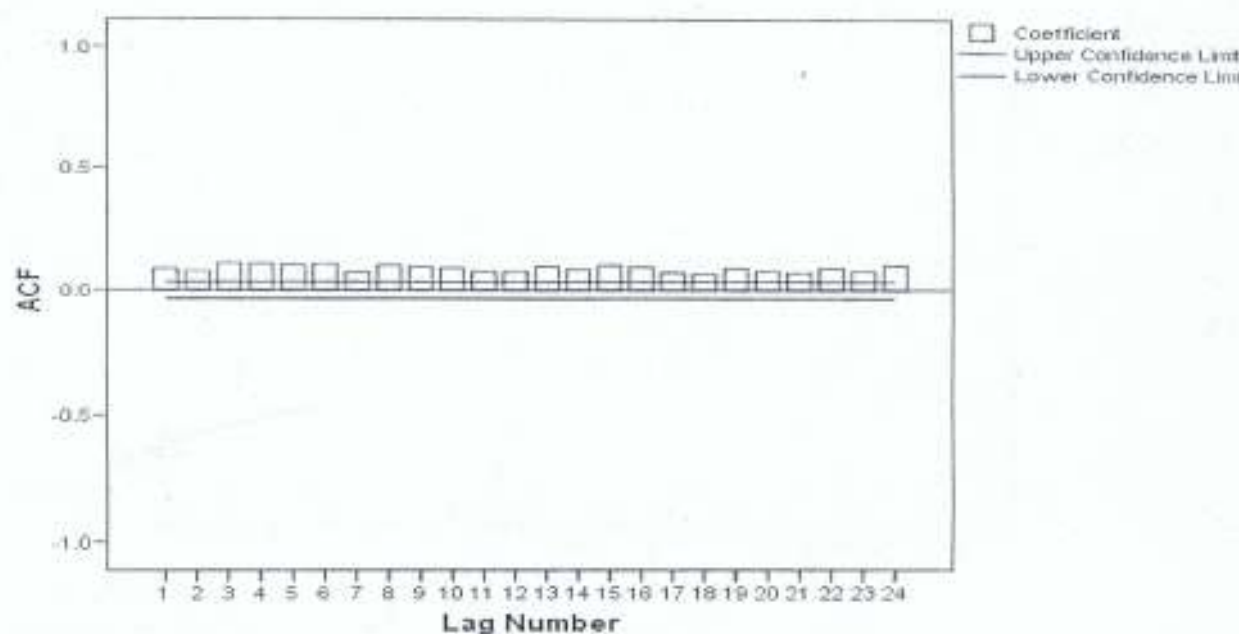
Series: Rain amount

Lag	Partial Autocorrelation	Std. Error
1	.061	.016
2	.083	.016
3	.092	.016
4	.079	.016
5	.092	.016
6	.062	.016
7	.019	.016
8	.059	.016
9	.048	.016
10	.044	.016
11	.007	.016
12	.037	.016
13	.039	.016
14	.047	.016
15	.021	.016
16	.037	.016
17	.031	.016
18	-.001	.016
19	.011	.016
20	.024	.016
21	.032	.016
22	.000	.016
23	.008	.016
24	.041	.016

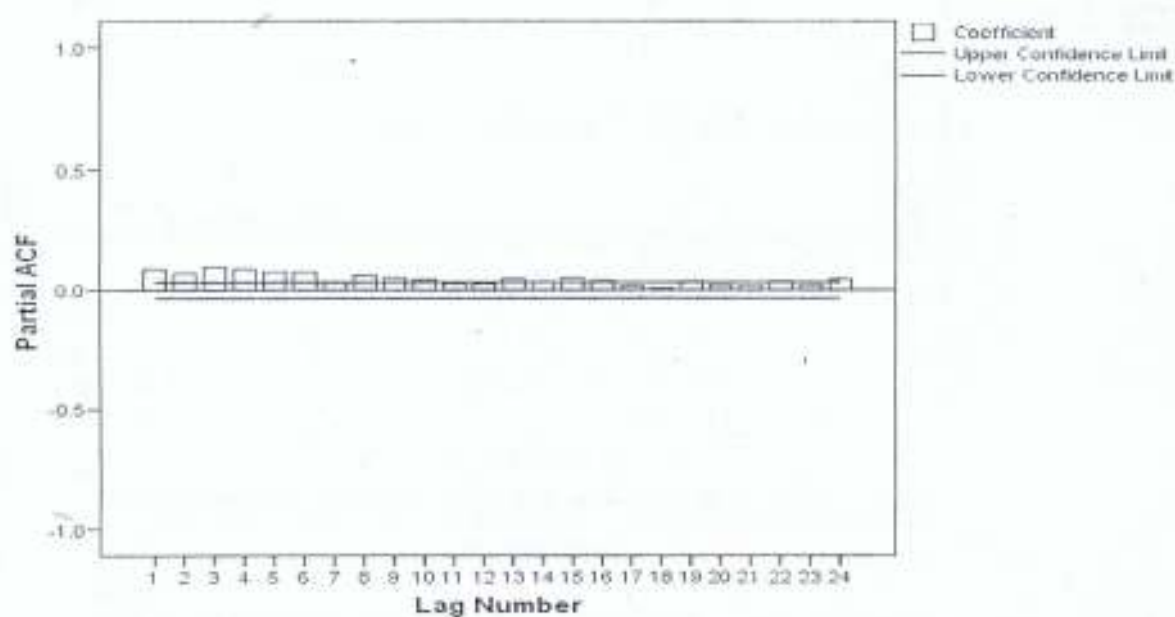
Appendix Table 8: Correlogram for daily rain amount in Ondo.



RainAmount



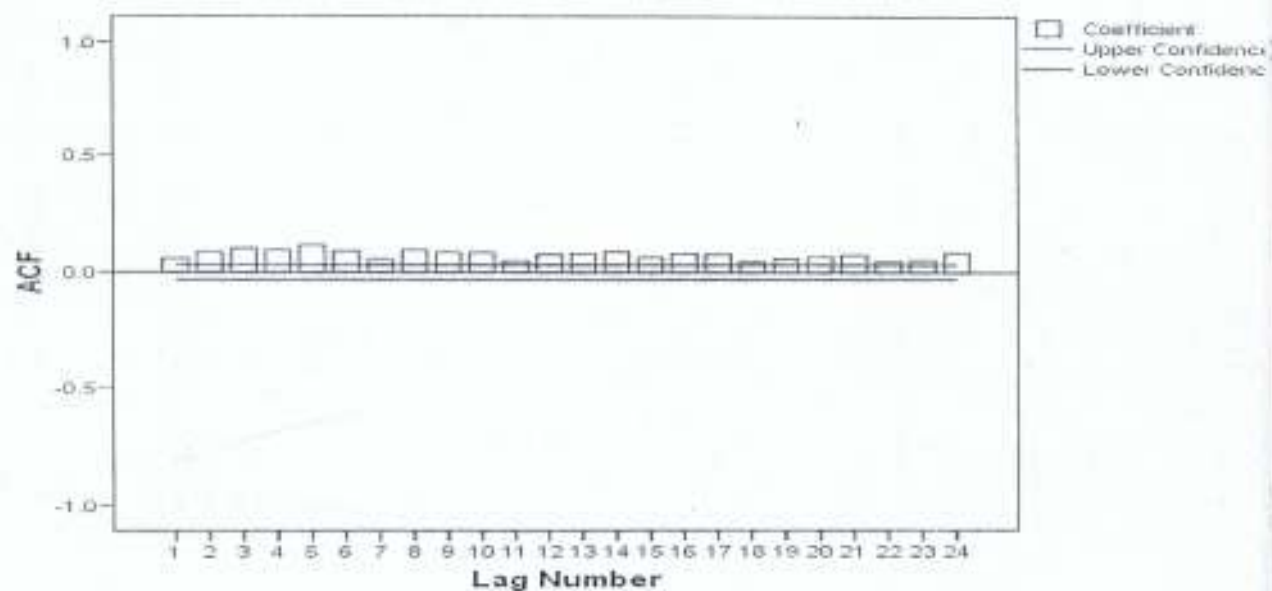
RainAmount



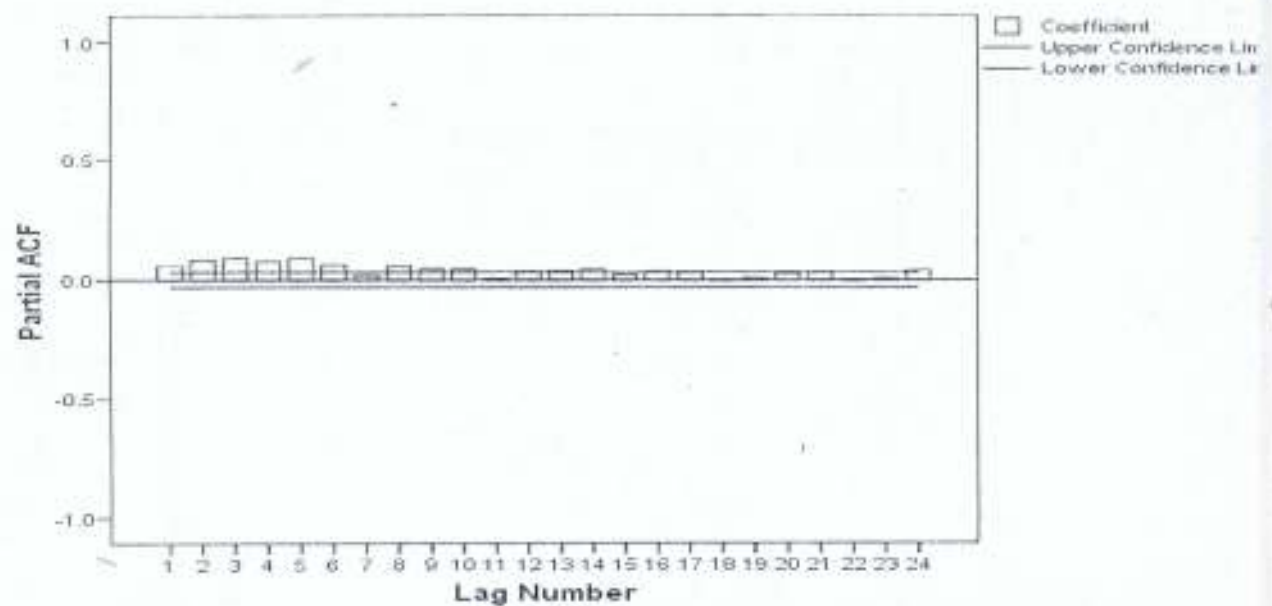
Appendix Figure 6: Correlogram for daily rain amount of Calabar, Nigeria.

APPENDIX 15

Rainamount

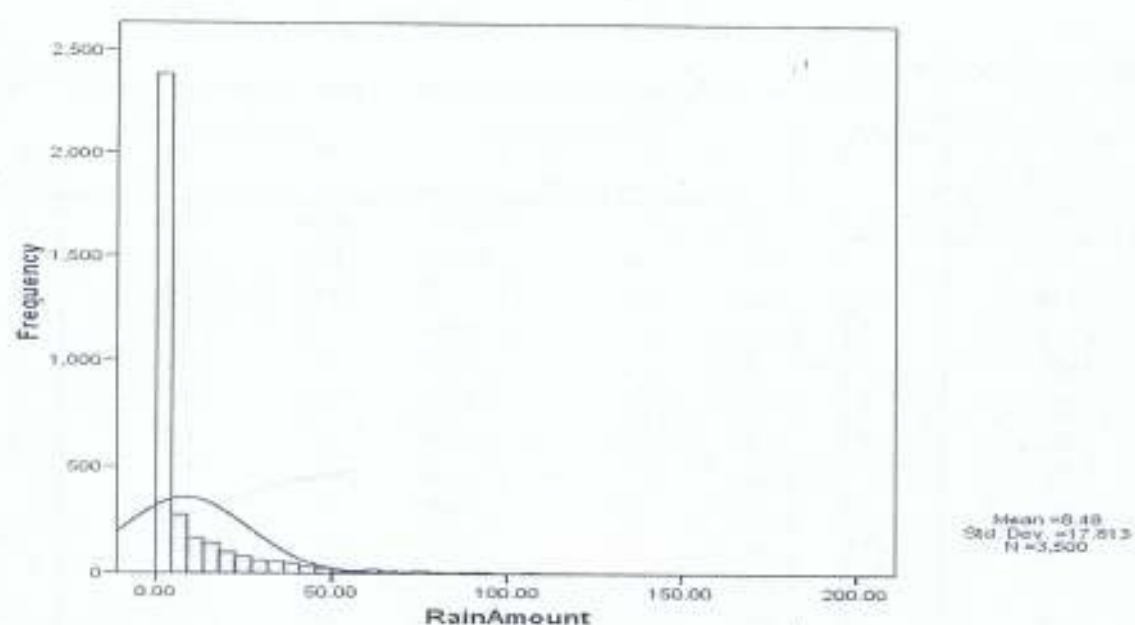


Rainamount

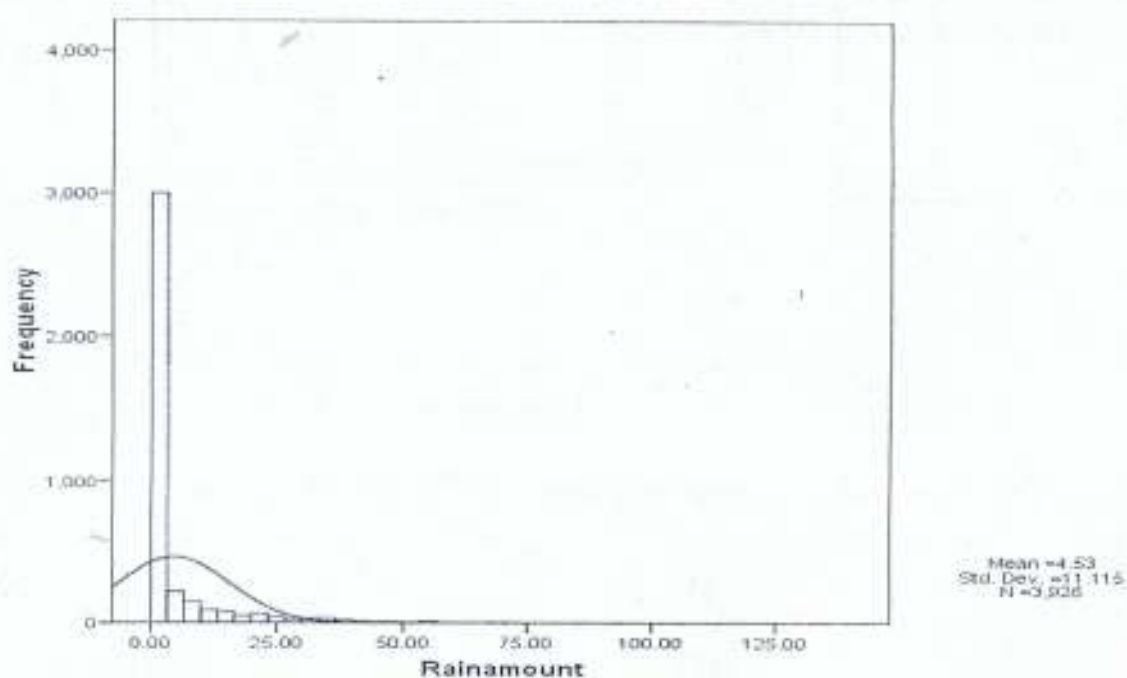


Appendix Figure 7: Correlogram for daily rain amount of Ondo, Nigeria.

APPENDIX 16



Appendix Figure 8(a): Histogram of residuals for daily rain amount of Calabar, Nigeria.



Appendix Figure 8(b): Histogram of residuals for daily rain amount of Ondo, Nigeria.

APPENDIX 17

Autocorrelations

Series: Cloud

Lag	Autocorrelation	Std. Error(a)	Box-Ljung Statistic		
	Value	df	Sig.(b)	Value	df
1	.038	.016	5.606	1	.018
2	.066	.016	22.320	2	.000
3	.043	.016	29.311	3	.000
4	.051	.016	39.196	4	.000
5	.064	.016	55.013	5	.000
6	.067	.016	72.187	6	.000
7	.039	.016	78.059	7	.000
8	.062	.016	93.016	8	.000
9	.058	.016	105.894	9	.000
10	.059	.016	119.176	10	.000
11	.053	.016	129.777	11	.000
12	.044	.016	137.071	12	.000
13	.026	.016	139.602	13	.000
14	.015	.016	140.456	14	.000
15	.027	.016	143.185	15	.000
16	.017	.016	144.267	16	.000
17	.033	.016	148.434	17	.000
18	.046	.016	156.705	18	.000
19	.046	.016	164.987	19	.000
20	.034	.016	169.387	20	.000
21	.058	.016	182.349	21	.000
22	.027	.016	185.081	22	.000
23	.018	.016	186.284	23	.000
24	.006	.016	186.423	24	.000

a. The underlying process assumed is independence (white noise).

b. Based on the asymptotic chi-square approximation.

Partial Autocorrelations

Series: Cloud

Lag	Partial Autocorrelation	Std. Error
1	.038	.016
2	.065	.016
3	.038	.016
4	.044	.016
5	.056	.016
6	.056	.016
7	.025	.016
8	.048	.016
9	.043	.016
10	.040	.016
11	.033	.016
12	.023	.016
13	.004	.016
14	-.007	.016
15	.007	.016
16	-.003	.016
17	.015	.016
18	.030	.016
19	.030	.016
20	.016	.016
21	.041	.016
22	.009	.016
23	-.003	.016
24	-.012	.016

Appendix Table 9: Correlogram for daily cloud of Warri, Nigeria.

APPENDIX 18

Autocorrelations

Partial Autocorrelations

Series: Cloud

Series: Cloud

Lag	Autocorrelation	Std. Error(a)	Box-Ljung Statistic		
	Value	df	Sig.(b)	Value	df
1	.039	.016	6.246	1	.012
2	.039	.016	12.493	2	.002
3	.030	.016	15.997	3	.001
4	.030	.016	19.501	4	.001
5	.020	.016	21.048	5	.001
6	.010	.016	21.427	6	.002
7	.030	.016	24.933	7	.001
8	.049	.016	34.729	8	.000
9	.020	.016	36.278	9	.000
10	.030	.016	39.787	10	.000
11	.010	.016	40.167	11	.000
12	.059	.016	54.306	12	.000
13	.000	.016	54.306	13	.000
14	.049	.016	64.117	14	.000
15	.030	.016	67.630	15	.000
16	.030	.016	71.145	16	.000
17	.010	.016	71.525	17	.000
18	-.010	.016	71.937	18	.000
19	.020	.016	73.489	19	.000
20	.039	.016	79.765	20	.000
21	.000	.016	79.765	21	.000
22	.030	.016	83.284	22	.000
23	.020	.016	84.839	23	.000
24	.020	.016	86.393	24	.000

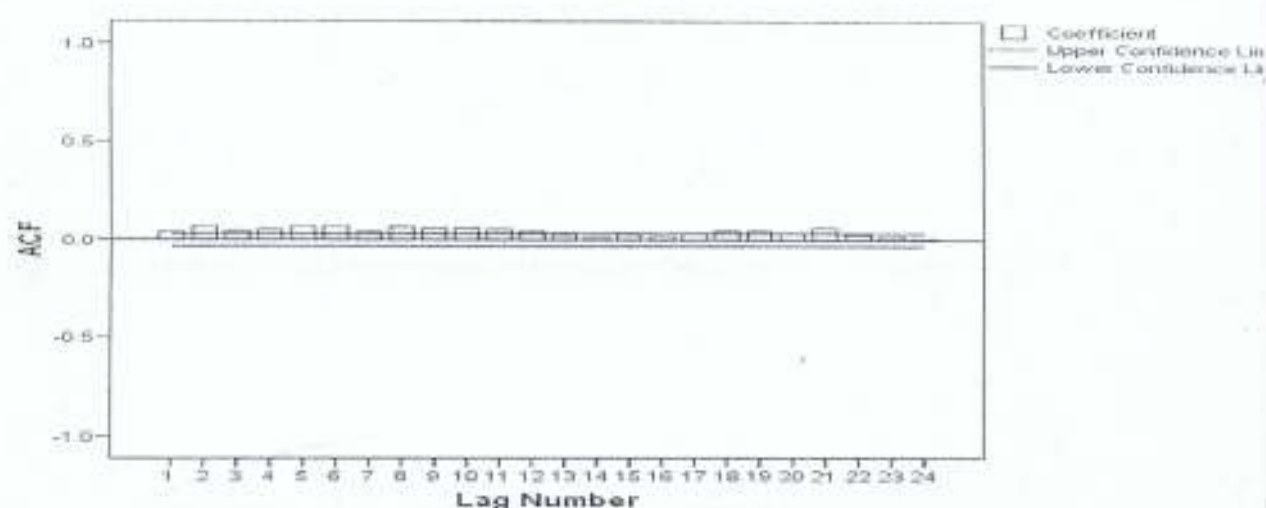
Lag	Partial Autocorrelation	Std. Error
1	.039	.016
2	.038	.016
3	.027	.016
4	.026	.016
5	.016	.016
6	.006	.016
7	.028	.016
8	.045	.016
9	.013	.016
10	.023	.016
11	.003	.016
12	.053	.016
13	-.008	.016
14	.043	.016
15	.021	.016
16	.019	.016
17	.000	.016
18	-.019	.016
19	.013	.016
20	.032	.016
21	-.008	.016
22	.020	.016
23	.012	.016
24	.007	.016

a. The underlying process assumed is independence (white noise).

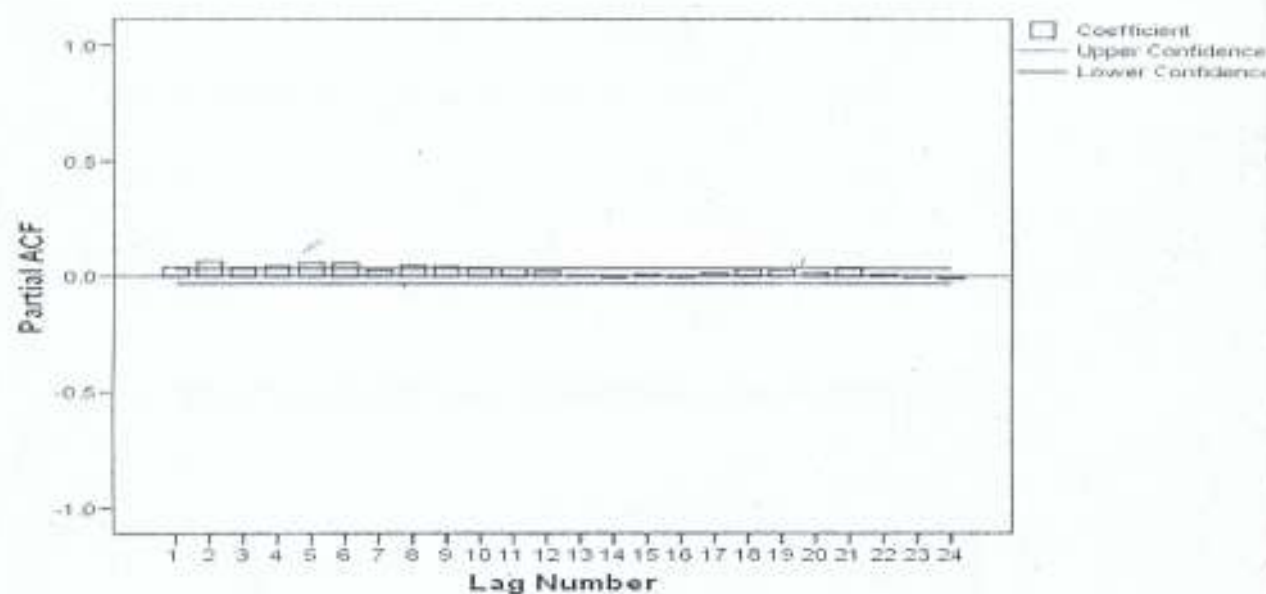
b. Based on the asymptotic chi-square approximation.

APPENDIX 19

Cloud

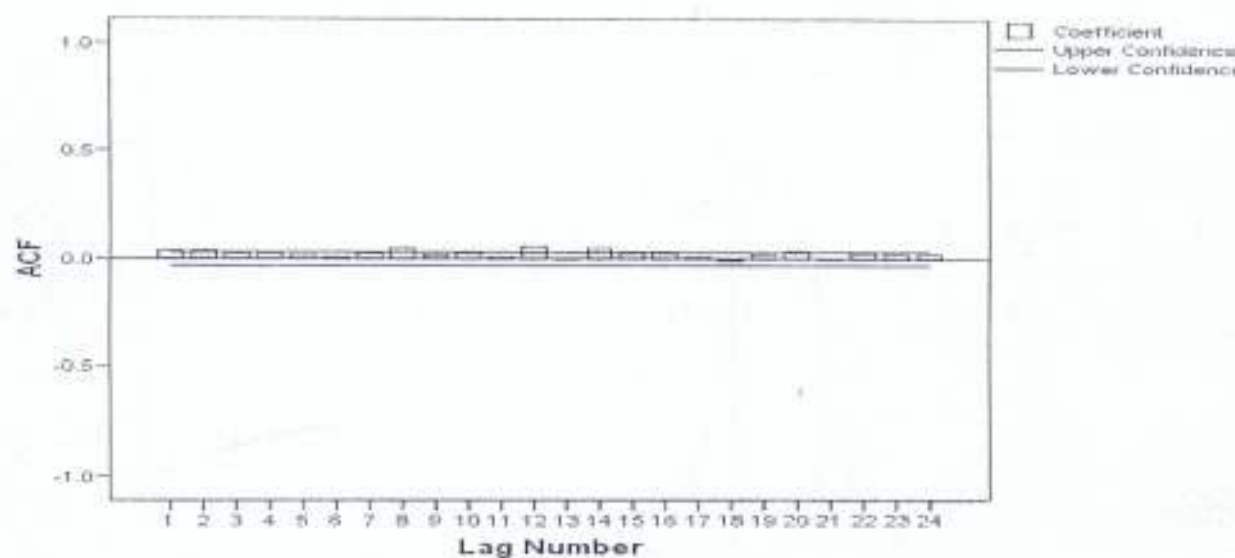


Cloud

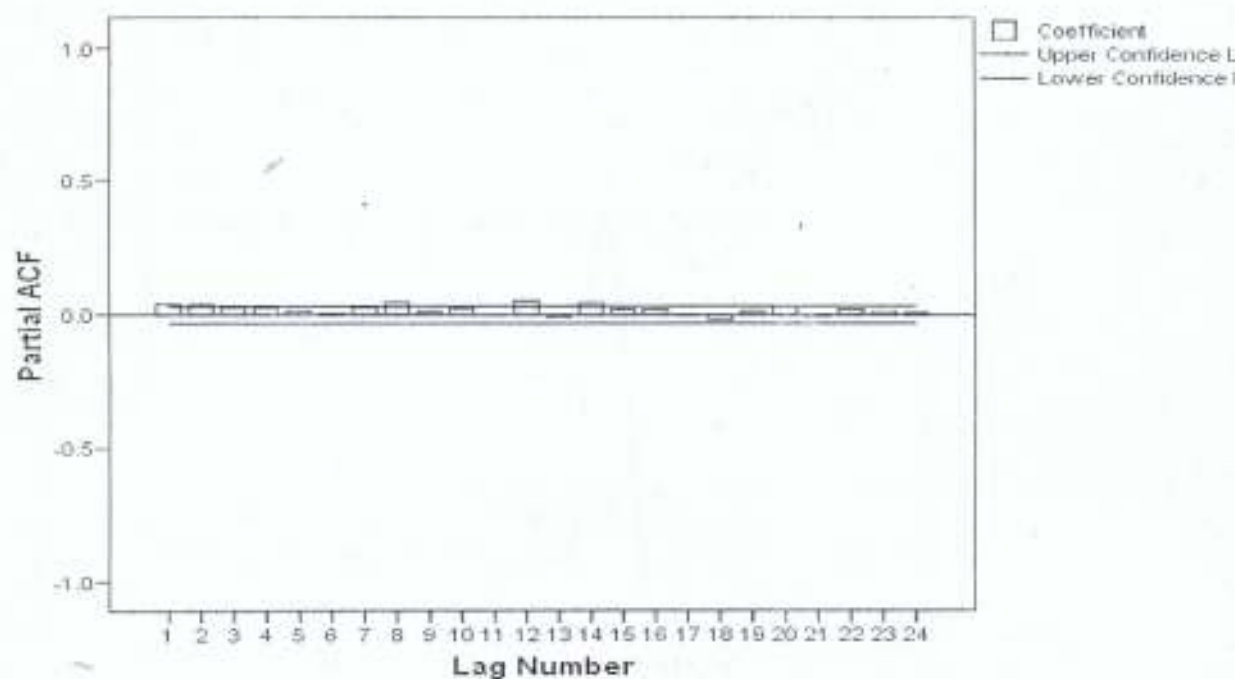


Appendix Figure 9: Correlogram for daily cloud of Warri, Nigeria.

Cloud

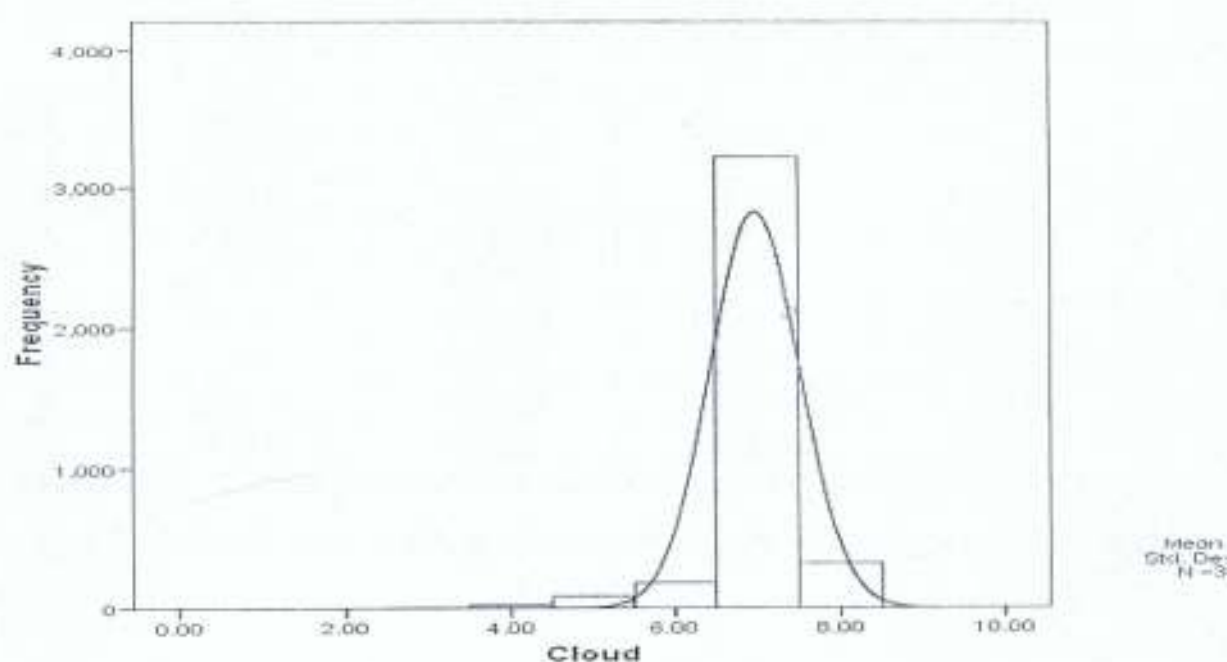


Cloud

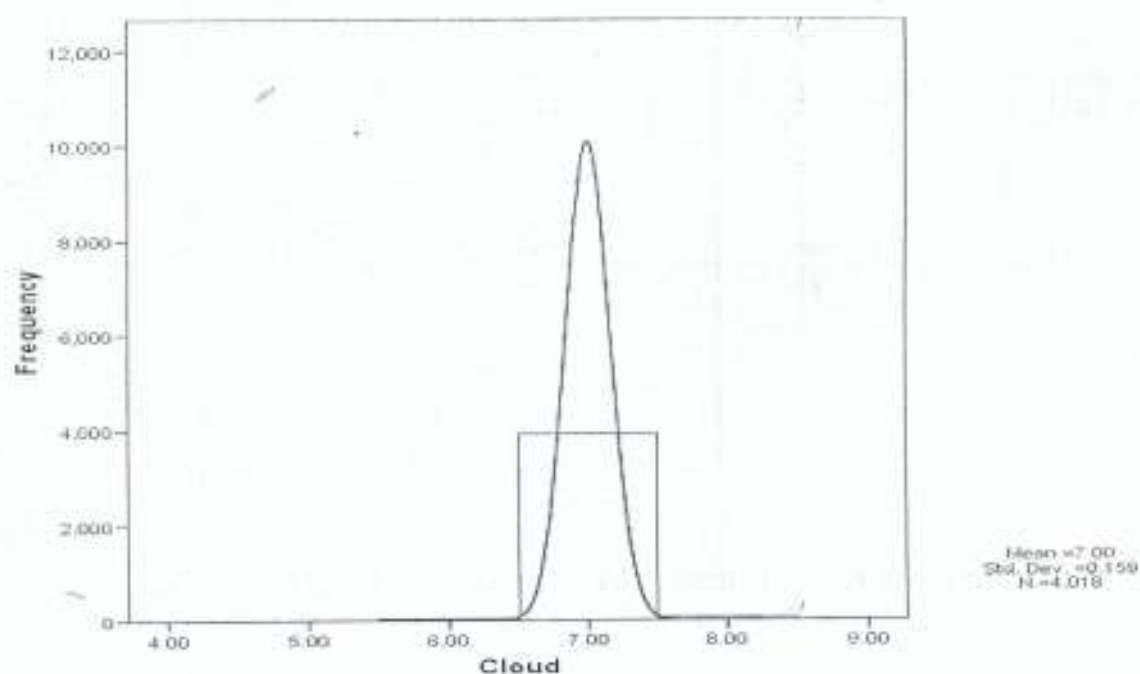


Appendix Figure 10: Correlogram for daily cloud of Benin, Nigeria.

APPENDIX 21



Appendix Figure 11(a): Histogram of residuals for daily cloud of Warri, Nigeria.



Appendix Figure 11(b): Histogram of residuals for daily cloud of Benin, Nigeria.

APPENDIX 22

i	Invertibility parameter π_i of some ARIMA ($p, d, 1$) process			
	ARIMA (1,1,1)	ARIMA (2,1,1)	ARIMA (3,1,1)	ARIMA (4,1,1)
0	1.00000	1.00000	1.00000	1.00000
1	-1.90000	-1.90000	-1.90000	-1.90000
2	1.44000	1.24000	1.24000	1.24000
3	-0.86400	-0.54400	-0.64400	-0.64400
4	0.51840	0.32640	0.48640	0.48640
5	-0.31104	-0.19584	-0.29184	-0.24184
6	0.18662	0.11750	0.17510	0.14510
7	-0.11197	-0.07050	-0.10506	-0.08706
8	0.06718	0.04230	0.06304	0.05224
9	-0.04031	-0.02538	-0.03782	-0.03134
10	0.02419	0.01523	0.02269	0.01276

Appendix Table 11: Table showing behaviour of invertibility parameter π_i of ARIMA ($p, 1, 1$) for some i when $p = 1, 2, 3$ and 4, given that $\theta = 0.6$, $\phi_1 = 0.3$, $\phi_2 = 0.2$, $\phi_3 = 0.1$ and $\phi_4 = 0.05$.

i	Invertibility parameter π_i of some ARIMA ($p, d, 1$) process			
	ARIMA (1,1,1)	ARIMA (2,1,1)	ARIMA (3,1,1)	ARIMA (4,1,1)
0	1.00000	1.00000	1.00000	1.00000
1	-0.70000	-0.70000	-0.70000	-0.70000
2	-0.12000	-0.32000	-0.32000	-0.32000
3	-0.07200	0.00800	-0.09200	-0.09200
4	-0.04320	0.00480	0.04480	-0.00520
5	-0.02592	0.00288	0.02688	0.04688
6	-0.01555	0.00173	0.01613	0.02813
7	-0.00933	0.00104	0.00968	0.01688
8	-0.00560	0.00062	0.00581	0.01013
9	-0.00336	0.00037	0.00348	0.00608
10	-0.00202	0.00022	0.00209	0.00365

Appendix Table 12: Table showing behaviour of invertibility parameter π_i of ARIMA ($p, 1, 1$) for some i when $p = 1, 2, 3$ and 4, given that $\theta = -0.6$, $\phi_1 = 0.3$, $\phi_2 = 0.2$, $\phi_3 = 0.1$ and $\phi_4 = 0.05$.

i	Invertibility parameter π_i of some ARIMA ($p, d, 1$) process			
	$d = 0$	$d = 1$	$d = 2$	$d = 3$
0	1.000000	1.000000	1.000000	1.000000
1	-0.900000	-1.900000	-2.900000	-3.900000
2	0.340000	1.140000	3.040000	5.640000
3	-0.304000	-0.684000	-1.824000	-4.684000
4	0.132400	0.410400	1.094400	2.810400
5	-0.104440	-0.246240	-0.656640	-1.686240
6	0.050164	0.147744	0.393984	1.011744
7	-0.036350	-0.088650	-0.236390	-0.607050
8	0.018684	0.053188	0.141834	0.364228
9	-0.012770	-0.031910	-0.085100	-0.218540
10	0.006883	0.019148	0.051060	0.131122

Appendix Table 13: Table showing behaviour of invertibility parameter π_i of ARIMA ($p, d, 1$) for some i when $d = 0, 1, 2$ and 3 , given that $\theta = 0.6$, $\phi_1 = \phi_2 = \dots = \phi_{10} = 0.3$.

i	Invertibility parameter π_i of some ARIMA ($p, d, 1$) process			
	$d = 0$	$d = 1$	$d = 2$	$d = 3$
0	1.000000	1.000000	1.000000	1.000000
1	-0.650000	-1.650000	-2.625000	-3.625000
2	0.290000	0.940000	2.590000	5.240000
3	-0.374000	-0.664000	-1.604000	-3.094000
4	0.075600	0.298400	1.462400	3.706400
5	0.045360	0.120960	-0.177440	-0.773840
6	-0.027220	-0.072580	0.406464	1.164304
7	0.016330	0.043546	-0.243880	-0.398580
8	-0.00980	-0.026130	0.180887	0.239149
9	0.005879	0.015676	-0.087800	-0.143490
10	-0.003530	-0.009410	0.052678	0.080047

Appendix Table 14: Table showing behaviour of invertibility parameter π_i of ARIMA ($4, d, 1$) for some i when $d = 0, 1, 2$ and 3 , given that $\theta = 0.6$, $\phi_1 = 0.05$, $\phi_2 = 0.1$, $\phi_3 = 0.2$ and $\phi_4 = 0.3$.

APPENDIX 24

i	Invertibility parameter π_i of some ARIMA ($p, d, 1$) process		
	Decreasing values of ϕ_p 's	Equal values of ϕ_p 's	Increasing values of ϕ_p 's
0	1.000000	1.000000	1.000000
1	-1.610000	-1.900000	-1.900000
2	0.956000	1.140000	1.240000
3	-0.593600	-0.684000	-0.644000
4	0.316160	0.410400	0.436400
5	-0.209700	-0.246240	-0.236840
6	0.025818	0.147744	0.154604
7	-0.115490	-0.088650	-0.086510
8	-0.030710	0.053188	0.055032
9	-0.081580	-0.031910	-0.031460
10	-0.057100	0.019148	0.019655

Appendix Table 15: Table showing behaviour of invertibility parameter π_i of ARIMA ($p, d, 1$) for some i under different ϕ_p 's arrangement.