ON TWO CONTINUOUS COLLOCATION METHODS FOR SOLVING GENERAL SECOND ORDER INITIAL VALUE PROBLEMS OF ORDINARY DIFFERENTIAL EQUATIONS

BY

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CERTIFICATION

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DEDICATION

This thesis is dedicated to God Almighty who made this work possible.

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ABSTRACT

In this thesis two zero stable Linear Multistep methods (LMM) with continuous coefficients for solving general second order initial value problems of ordinary differential equations which does not require that the equation be reduced to a system of first order equation are considered. The approach is based on collocation of the differential systems arising from the basis function at the grid points $x = x_{n+i}$, $o \le i \le k$ and interpolation of the approximate solution at the selected grid points $x = x_{n+i}$, for $1 \le i < 3$ and $2 \le i < 4$ for the step numbers k = 3 and 4 respectively.

Some predictors and their first derivatives are proposed to calculate y_{n+k} and y'_{n+k} for k = 3, 4. The use of Taylor series expansion is employed for the calculation of y_{n+i} for $i = 1,2$. Evaluation of each method and its predictors at $x = x_{n+k}$ gives particular discrete schemes as special cases of the methods and their predictors respectively. The new 4-step method was analysed and found to be consistent and zero stable, hence convergent.

The new method was tested on some general second order initial value problems of ordinary differential equations The result showed that the method converges as h decreases. The new results were compared with the exact and the earlier result of Awoyemi (1999), it was found that the new result improved over that of Awoyemi (1999).

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CHAPTER ONE

INTRODUCTION

1.1 PREAMBLE

The subject of differential equations constitutes a large and very important aspect of today's mathematics. Differential equations occur in connection with the numerous problems that are encountered in various branches of science and engineering. Such problems to mention a few include:

- Determination of the motion of a projectile, rocket, satellite or planet
- Determination of change or current in an electric circuit
- Conduction of heat in a rod or in a slab
- Determination of the vibrations of a wire or a membrane
- Rate of decomposition of a radioactive substance or the rate of growth of a population
- Studying of the reactions of chemicals, etc.

Though these problems exist by theory or principle it is the mathematical formulation of such problems that gives rise to differential equations. The underlying principle is in the fact that in all situations, objects involved obey certain laws involving various rates of change [see Ross 1989]. The resulting differential systems may be an ordinary or a partial differential equation. In many cases, it is found that the resulting equations does not posses closed form solutions. This implies that the dependent variable cannot be given specific expressions.

The general concept of a differential equation refers to any equation that involves an independent variable, a dependent variable and one or more of its differential coefficient. Any differential equation can be classified into type, order or degree. When the dependent variable *y* is a function of a single independent variable *x*, an ordinary differential equation (ODE) evolves; but when y is a function of two or more independent variable then, a partial differential equation (PDE) evolves. Depending upon given conditions, an ordinary differential equation (ODE) can be categorized into:

i. Initial Value

ii. Boundary value problems of ODE

The general solution of the m order ordinary differential equation contains m independent arbitrary constants. To determine the arbitrary constants in the general solution we prescribe m conditions at one point, called initial conditions. The differential equation together with the initial conditions is called the initial value problem (IVP)

Example $y'' + y' - 6y = 0$, $y(0) = 6$, $y'(0) = 2$. (I.I.I)

These conditions are called boundary conditions if they are prescribed at more than one point and then the differential equation together with the boundary conditions is known as boundary value problem (BVP)

Example $y^{11} + y = 0; y(0) = 0, y^{1}({}^{\pi}/_{2}) = -1$ (1.1.2)

1.2 FORMATION AND KINDS OF DIFFERENTIAL EQUATION

Ordinary differential equations can be stiff or non-stiff. Stiff ordinary differential systems arise frequently in the fields of chemical kinematics, nuclear reactors, control theory and electrical circuit theory. Generally speaking, whenever there is a quickly changing dynamics, there is stiffness.

Differential equation originates from mathematical formulation of a great variety of problems in science and engineering resulting in a variety of differential equations, such as first, second and higher order differential equations. For the first order we have orthogonal trajectories, oblique trajectories, rate of change of momentum, the falling body problem, rate of growth and decay problems, and mixture problem. In the second order category we have.

i. Free undamped motion

Consider the differential equation for the motion of the mass on the spring

$$mx^{11} + ax^{1} + kx = F(t). \tag{1.2.1}$$

If $a = 0$ the motion is said to be undamped. Thus for a free, undamped motion where

$a = 0$, $F(t) = 0$, equation (1.2.1) reduces to

$$mx^{11} + kx = 0 \qquad (1.2.2)$$

where: x is the displacement

m is the mass

k is the spring constant ($m, k > 0$),

ii. Free Damped Motion

Here $a \neq 0$ but $F(t) = 0$ for all t and equation (1.2.1) takes the form

$$mx^{11} + ax^{1} + kx = 0 \qquad (1.2.3)$$

where a is the damping coefficient ($a > 0$)

iii. Forced Motion

Here the mass is acted upon by a damping ($a > 0$) effect and also by a periodic externally impressed force F, defined as

$$F(t) = F_1 \text{ Cos } wt \text{ for all } t, \qquad (1.2.4)$$

where F_1 and w are constant. Then equation (1.2.1) becomes

$$mx^{11} + ax^{1} + kx = F_1 \text{ Cos } wt. \qquad (1.2.5)$$

iv. Electrical Circuit Problem

Applying kirchhoffs voltage law to a circuit we have

$$Li + Ri + \frac{1}{C}q = E \qquad (1.2.6)$$

Since $i = q$ is a popular circuit relationship, by any suitable elimination method we get

$$\left.\begin{array}{c} Lq^{II} + Rq^{I} + \frac{1}{C}q = E \\ \text{Or} \\ Li^{II} + Ri^{I} + \frac{1}{C}i = E \end{array}\right\} \qquad (1.2.7)$$

where

E is the electromotive force	(in volts, V)
R is the resistence	(in ohm Ω)
L is the inductor	(in henry H)
C is the capacitor	(in farad)
i is the current	(in ampere)
q is the charge	(in coulomb)

v. The torsion of a bar is described by the differential equation

$$\theta^{iv} + T\theta = 0 \qquad (1.2.8)$$

This equation (1.2.8) is an example of a fourth order ordinary differential equation.

1.3 INITIAL VALUE PROBLEMS OF ORDINARY DIFFERENTIAL EQUATION

An ordinary differential equation (ODE) is a relation between a function, its derivatives and the variable upon which they depend. An mth order differential system can be expressed in the form.

$$y^{(m)} = f(x, y, y^{1}, y^{11}, \quad \ldots\ldots y^{(m-1)}), \qquad (1.3.1)$$

The system as expressed in (1.3.1) is the canonical representation of

$$F(x, y, y^{1}, \ldots y^{(m-1)}, y^{(m)}) = 0 \qquad (1.3.2)$$

which is the most general form of ODEs.

As mentioned in section 1.1, a general solution to this mth order system contains m arbitrary constants, which are determined by prescribing m conditions at specified point. Often, mathematical formulation of physical phenomena leads to initial value problems of the kind.

$$y^{11} = f(x, y, y^{1}), y(x_o) = y_o, y^{1}(x_o) = \eta \qquad (1.3.3)$$

Rutishauser (1960) examined the direct solution of (1.3.1) and its equivalent first order IVP and concluded that the choice of approach depends on the particular problem at hand (see also Fatumla 1988).

1.4 PROPERTIES OF INITIAL VALUE PROBLEM

A major property of any IVP is that of the existence of a unique solution of the problem of interest. This is ensured by the theorem.

Theorem 1.4.1

Here it is assumed that f(x, y) satisfies the following conditions.

i. f(x, y) is a real function

ii. f(x, y) is defined and continuous in the strip $x \in (-\infty, \infty)$

iii. There exist a constant L such that for any $x \in [x_0, b]$ and for any y_1 and y_2

$$/f(x, y) - f(x, y_1)/ \leq L/y - y_1/ \qquad (1.4.1)$$

where L is called the Lipschitz constant and equation (1.3.4) Lipschitz condition. Then for any y_o the initial value problem (1.3.3) has a unique solution y(x) for $x \in [x_0, b]$.

For problems of the kind (1.3.3), their properties are very useful when seeking for its numerical solution as they determine whether or not (1.3.3) can have any solution. For instance, if "*f*" becomes infinite (undefined) at some point in the domain of integration, then the partial derivative of "*f*" does not exist or are unbounded when they exist.

Furthermore, the non-linearity of *f* makes it impossible to locate the points of singularities since there may be no clue regarding them in the problems. In particular, if *f* possesses a continuous derivative with *y* for all (x, y) in D, then, by the mean value theorem,

$$f(x, y) - f(x_1, y_1) = \frac{\partial f(x, \overline{y})}{\partial y} (y - y_1) \qquad (1.4.2)$$

where $\overline{y}$ is a point in the interior of the interval whose end points are y and y_1, and (x, y) and (x, y_1) are both in D. We therefore make bold to assume in this thesis, except otherwise stated that 'f' is Liptshitz continuous in the region of definition D of the x-y plane. If the partial derivatives of f w.r.t y are continuous and bounded in D, then the Liptschitz constant L of the system may be taken to be

$$L \quad = \sup_{(x, y) \in D} \left| \frac{\partial f(x, y)}{\partial y} \right| \qquad (1.4.3)$$

In addition if for $x \in [a, b]$ the eigenvalue of $\frac{df}{dy}$ given as λj, is such that

$$\left| Re \, \lambda j \right| \quad < \quad 1 \qquad (1.4.4)$$

and

$$\text{Max } |Re \, \lambda_j| > O, j = 1, 2, 3. \qquad (1.4.5)$$

then L>>O, and the system of initial value problem of ordinary differential equation with this property is a system with large Liptschitz constant. This imposes a severe restriction on the size of the step length of the method. This property is very crusial when the stability in stiff problems is being considered. Such stiff systems are referred to as "system with large Liptschitz constant, [see Lambert 1973]

1.5 BASIC CONCEPT AND PRINCIPLES

Our desire is to present a new numerical integration method for solving directly IVPs of second order ordinary differential equations of the kind (1.3.3). As such in this section we present the basic concepts and principles of this new method. This new method for solving (1.3.3) is based on linear multistep method (LMM) a mathematical concept discussed by Henrici (1962), Lambert (1973) and Awoyemi (1992, 1999).

1.5.1 STEP-LENGTH OR MESHSIZE

As mentioned in section 1.5 above we seek a direct solution of (1.3.3). We adopt the principle of discretization in which an approximation to an unknown $y = f(x)$ is sought on certain discrete points x_i; $i = 0, 1,$ ----

We can define the sequence of points (x_n) in the interval $[a, b]$, by

$$a = x_o < x_1 < x_2 \text{---} < x_n = b$$

where $h_i = x_{i+1} - x_i$, $i = O(1)n-1$

The parameter h_i is called the step length or mesh size and for this work h_i is constant. The point x_i; $i = O,(1)n$ are called grid or modal or mesh points. Each grid point is given in terms of the previous points by the relation.

$$x_{i+1} = x_i + h;\ i = O(1)n,\ x_o = a,\ x_n = x_o + nh. \qquad (1.5.1)$$

(see Kayode 2004, Adesanya 2005)

1.5.2 LINEAR MULTISTEP METHOD (LMM)

A linear multistep method (LMM) for (1.3.3) is a computational method for determining (y_n) which takes the form of a linear relationship between y_{n+j} and f_{n+j}, $j = O(1)k$ in the form

$$\sum_{j=o}^{k} \alpha_i \; y_{n+j} = h^m \sum_{j=o}^{k} \beta_j \, f_{n+j} \qquad (1.5.2)$$

in which α_i, B_j are constants. For this thesis $m = 2$;

$$f_{n+j} = f(x_{n+j}, y_{n+j}, y'_{n+j}, y''_{n+j}, \qquad \ldots\ldots\ldots\ldots y^{m-1});\; j = O(1)\,k$$

and h (>0) represents the method's steplength and is used as a constant. For the purpose of our work we also assume that $\alpha_o \neq \beta_o \neq 0$.

Equation (1.5.2) is said to be implicit if $\beta_k \neq O$ or otherwise it is explicit. It is possible to determine the current value of y_{n+k} directly for an explicit method from (1.5.2) in terms of

y_{n+j}, f_{n+j}, $J = O(1)k-1$

whose values have already been found. For the implicit method, we are required to generate a good initial estimate for

$y_{n+k}, y'_{n+k} \ldots\ldots\ldots y^{m-1}{}_{n+k}$. In the form

$f(x_{n+k}, y_{n+k}, y'_{n+k}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots y^{m-1}{}_{n+k})$

the previous values y_{n+j}, $j = 0(1)k-1$ are also required to be evaluated. Once these additional starting values are generated, these initial estimates for y_{n+j}, $j = O(1)k-1$ are then adopted for calculation in the form

$$y_{n+k} = \sum_{i=j=o}^{k} \alpha_i \; y_{n+j} + h^2 \sum_{j=o}^{k} \beta_j \, f_{n+j} \qquad 1.5.3$$

Equation (1.5.3) is called the corrector method and those methods that provide initial values for y_{n+k} are called the predictors for (1.5.3).

Ironically, according to Fatunla (1988), though implicit method *for* a step number k requires a substantially greater computational effort than explicit methods, but implicit methods are more accurate and have more favourable stability properties.

1.6 EXISTING METHODS

The numerical methods for the solution of the differential equation (1.3.3) whenever $x \in [x_0, b]$ are called multistep methods if the value of $y(x)$ at $x = x_{n+1}$ uses the values of the dependent variable and its derivative at more than one grid or mesh point. Attempting analytic solution for (1.3.3) has remained a problem. According to Ross (1989), a closed form solution is really a luxury in differential equations. The commonest methods used in solving, (1.3.3) in the views of Lambert (1973), Fatunla (1988), Awoyemi (1999) is to reduce the problem into first order and then develop appropriate numerical method to solve the resulting system. The major constraints inherent in this approach is in the need to develop separate computer subprograms needed to initialize the starting values for evaluating the functions arising from the system. The validity of such approaches depends to a large extent on the ability of the researcher to cope with the time and cost of getting a good result. This is a serious hindrance though it ironically doubles as an incentive for investigation into cheaper and faster approaches.

Great attention has been given by some eminent scholars to problems of the form (1.3.3) In their separate works, Henrici (1962) and Lambert (1973) both agreed on the concept of derivation of Linear multistep methods (LMM) in terms of power series involving finite difference operators for solving (1.3.3) Jain (1984) described the characteristics of LMM for solving (1.3.3) Nystrom (1925) and Fatunla (1988) considered step by step methods based on the classical Runge —Kutta methods. Gear (1971), Hairer (1979), Chawla and Sharma *(1985)*and Vander Houwen (1979), worked on explicit and implicit Runge —Kutta — Nystrom methods for the solution of initial value problems (IVP) of the form (1.3.3). Fatunla (1984, *1985*, and 1988) on his part developed a P stable one-leg LMM and a one-leg hybrid methods in which pade approximation was used as the basis function for solving (1.3.3). These methods overcome the "order barrier". This means that the attainable order *P* of an implicit P-stable method cannot exceed two ((Lambert (1973) and Dahlquist (1978)). Independent works by Chawla (1981) and cash (1981) showed that considering certain hybrid two-step methods, the barrier could be overcomed. This gave rise to their fourth and sixth order p-stable methods. In another work on sixth order p-stable symmetric multistep method for periodic IVPs of type (1.3.3) Jain et al (1984), observed that the cost of implementing the method of cash (1981) is very high due to the evaluation of many functions per iteration. Awoyemi (1992, 1999) worked on LMM by collocation in which the numerov's nethods for solving (1.3.3) was reaffirmed. Very recently, Janssen and Van Hentenryck (2003) developed one-leg multistep methods for IVPs in ODEs. This involves the usage of a nonlinear multistep formula to compute the olution at the next integration point "an evaluation point t".

There are many other contributors whose methods have many ymbols and function evaluation per iteration, resulting in complexity for ny serious practical application. According to Cong and Xuam (2003), the omputational efforts required for a numerical method is measured by the otal number of functions evaluations over the total number of integration eps and the efficiency of the method is reduced if the number of output oints becomes very large.

Regrettably little has been done in considering LMM with continuous oefficients as a means of solving (1.3.3). LMM with continuous oefficients directly solves initial and boundary value problems of general econd order ordinary differential equations of the form (1.3.1).

Consequently, we propose a collocation procedure, which leads to distinct continuous schemes and their derivatives for different values of step number K for solving (1.3.3) directly.

1.7 RESEARCH OBJECTIVES

The purpose of this study is to:

(i) adopt power series as a basis function in place of existing basis functions like pade approximation

(ii) develop two zero stable continuous collocation methods of high order to solve (1.3.3) directly without reducing it to a system of first order equations.

(iii) develop predictor for calculating Y_{n+k}

(iv) analyze the basic properties of the method with respect to zero stability, consistency, order, error constant and interval of stability

(v) develop a FORTRAN programme for the methods and implement it on a computer with sample problems.

.8 RESEARCH METHODOLOGY

The methods are based on the collocation of the second derivative of he approximate solution and the interpolation of the approximate solution at pecific grid points for $3 \leq k \leq 4$. The values of the unknown parameters btained by solving the set of generated system of linear equations involving the parameters of the method are substituted into the approximate solution and evaluated to yield the continous schemes in t. When **t =1** => $x = x_{n+j}$ **j** = 3, 4, these scheme produces the desired discrete schemes (methods).

The resulting discrete schemes are analyzed for accuracy, consistency, convergence and stability properties using Dalhguist (1978) stability theorems and boundary locus method in Lambert (1973) and Fatunla (1988). A FORTRAN programme was developed and implemented on a computer for the solution of some sample second order initial value problems of rdinary differential equations.

.9 COLLOCATION METHODS

Collocation method is a procedure for determining the numerical olution to given problems. It involves the determination of an approximate olution in a suitable set of functions called the basis functions. In this work, he basis function is taken to be power series of degree n. The approximate olution is required to satisfy the differential equation (1.3.3) and its initial onditions at certain points referred to as collocation points.

According to Awoyemi (1992), collocation methods dates as far back s to the 1930's when Antorovich (1934), first proposed spline collocation rocedure for the solution of partial differential equation.

Lanczos *(1956,* 1973) became the first proponents of the use of collocation by polynomials at the root of orthogonal polynomial rather than at the equidistant point. This type of collocation method is called orthogonal collocation. Fox and Parker (1968) also in their own contribution adopted Chebyshev orthogonal collocation for the solution of ordinary and partial differential equations as well as for integral equations.

1.10 EXPECTED CONTRIBUTION TO KNOWLEDGE

The study is expected to produce schemes of improved order of accuracy which reduces the amount of computational burden when compared with existing schemes. It is expected that our results (method) will be useful in constructing global error estimation

CHAPTER TWO

DERIVATION OF THE METHODS

2.1 INTRODUCTION

We propose an approximate solution to (1.3.3) in the form

$$y(x) = \sum_{i=0}^{2k+1} a_i x^i \qquad (2.1.1)$$

in which $\sum_{i=0}^{2k+1} x^i$ serves as the basis function of (2.1.1).

2.2 DERIVATION OF 2, 3 AND 4-STEP METHODS

2.2.1. General Overview

A typical line segment can be partitioned such that for k = i, i = 0(1)m we have an m+1 number of grid points. That is, a line segment such as

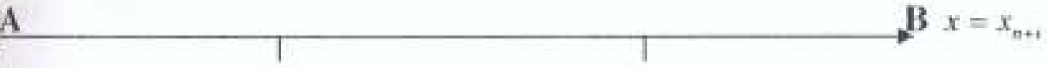

Fig 2.1

will have $x = x_{n+i}$ and $x = x_{n+m}$ as its start and end points respectively.

Thus for the two methods considered in this thesis a sketch of the grid points for the main methods and of their predictors are as follows.

1. For k = 3

(a) Main method

Fig 2.2

(b) Predictor 1

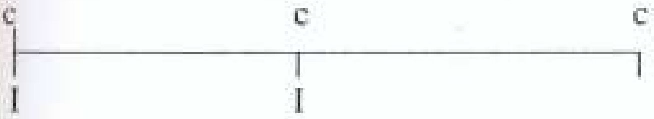

Fig 2.3

(c) Predictor 2: The use of Taylor series expansion is employed here to enable us evaluate predictor 2.

2. For K = 4 we have the following

(a) Main method for 4

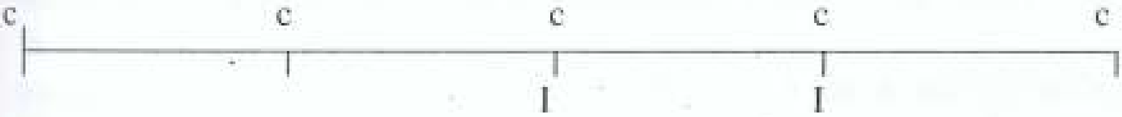

Fig 2.4

(b) Predictors 1

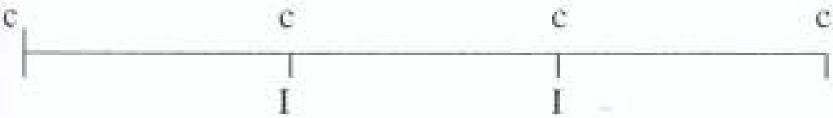

Fig 2. 5.

(c) Predictor 2

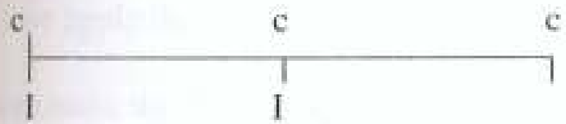

Fig 2.6

(d) Predictor 3: Taylor series expansion is used to evaluate this predictor.

The first and second derivatives of (2.1.1) are respectively

$$y'(x) = \sum_{i=0}^{2k+1} i \; a_i x^{i-1} \tag{2.2.1}$$

and

$$y''(x) = \sum_{i=1}^{2k+1} i(i-1)a_i x^{i-2} \tag{2.2.2}$$

From (1.3.3) and (2.2.2) we have that

$$\sum_{i=0}^{2k+1} i(i-1)a_i x^{i-2} = f\left(x, \sum_{i=0}^{2k+1} aix^i, \sum_{i=0}^{2k+1} ia_i x^{i-1}\right) \tag{2.2.3}$$

where the $a_i s$ are parameters to be determined. Thus we collocate (2.2.3) at the grid or mesh points, $x = x_{n+i}$, $o \le i \le k$ for $k = 2,3$ and 4 and interpolate (2.1.1) at $x = x_{n+i}$, for $0 \le i \le 2$, $1 \le i < 3$ and $2 \le i < 4$ respectively.

The resulting non-linear system of equations are

$$\sum_{i=0}^{2k+1} i(i-1)a_i x^{i-2} = f_{n+i} \quad o \le i \le k \tag{2.2.4}$$

$$\sum_{i=0}^{2k+1} a_i x^i = y_{n+i} \qquad 1, 2 \le i < 3, 4 \tag{2.2.5}$$

where $X_{n+i} = x_n + ih$ (2.2.6)

If we apply the Guasian elimination method to equations (2.2.4) and (2.2.5) we obtain the following ai values for k = 3.

$$a_5 = \frac{1}{120h^3}\{f_{n+3} - 3f_{n+2} + 3f_{n+1} - f_n\} \qquad (2.2.7)$$

$$a_4 = \frac{1}{24h^2}\{f_{n+2} - 2f_{n+1} + f_n\} - \frac{5}{3}a_5(x_{n+2} + x_{n+1} + x_n) \qquad (2.2.8)$$

$$a_3 = \frac{1}{6h}\{f_{n+1} - f_n\} - 2a_4(x_{n+1} + x_n) - \frac{10}{3}a_5(x^2_{n+1} + x_{n+1}x_n + x_n^2) \qquad (2.2.9)$$

$$a_2 = \frac{1}{2}f_n - 3a_3x_n - 6a_4x_n - 10a_5x_n^3 \qquad (2.2.10)$$

$$a_1 = \frac{1}{h}\{y_{n+2} - y_{n+1}\} - a_2(x_{n+2} + x_{n+1}) - a_3(x^2_{n+2} + x_{n+2}x_{n+1} + x^2_{n+1})$$

$$-a_4(x^3_{n+2} + x^2_{n+2}x_{n+1} + x_{n+2}x^2_{n+1} + x^3_{n+1})$$

$$-a_5(x^4_{n+2} + x^3_{n+2}x_{n+1} + x^2_{n+2}x^2_{n+1} + x_{n+2}x^3_{n+1} + x^4_{n+1}) \qquad (2.2.11)$$

$$a_0 = y_{n+1} - a_1x_{n+1} - a_2x^2_{n+1} - a_3x^3_{n-1} - a_4x^4_{n+1} - a_5x^5_{n+1} \qquad (2.2.12)$$

Also applying Guassian elimination method to (2.2.4) and (2.2.5) gives the following values for a_i s i=0 (1)6 when K = 4

$$a_6 = \frac{1}{720h^4}\{f_{n+4} - 4f_{n+3} + 6f_{n+2} - 4f_{n+1} + f_n\} \qquad (2.2.13)$$

$$a_5 = \frac{1}{120h^3}\{f_{n+3} - f_{n+2} + 3f_{n+1} - f_n\} - \frac{3}{2}a_6(x_{n+3} + x_{n+2} + x_{n+1} + x_n) \qquad (2.2.14)$$

$$a_4 = \frac{1}{24}h^2\{f_{n+2} - 2f_{n+1} + f_n\} - \frac{5}{3}a_5(x_{n+2} + x_{n+1} + x_n)$$

$$-\frac{5}{2}a_6(x^2_{n+2} + x_{n+2}x_{n+1} + x^2_{n+1} + x_{n+2}x_n + x_{n+1}x_n + x_n) \qquad (2.2.15)$$

$$a_3 = \frac{1}{6h}\{f_{n+1} - f_n\} - 2a_4(x_{n+1} + x_n) - \frac{10}{3}a_5(x_{n+1}^2 + x_{n+1}x_n + x_n^2)$$

$$-5a_6(x^3_{n+1} + x^2_{n+1}x_n + x_{n+1}x_n^2 + x_n^3) \qquad (2.2.16)$$

$$a_2 = \frac{1}{2}f_n - 3a_3x_n - 6a_4x_n^2 - 10a_5x_n^3 - 15a_6x_n^4 \qquad (2.2.17)$$

$$a_1 = \frac{1}{h}\{y_{n+3} - y_{n+2}\} - a_2(x_{n+3} + x_{n+2}) - a_3(x^2_{n+3} + x_{n+3}x_{n+2} + x^2_{n+2})$$

$$-a_4(x^3_{n+3} + x^2_{n+3}x_{n+2} + x_{n+3}x^2_{n+2} + x^3_{n+2}) - a_5(x^4_{n+3} + x^3_{n+3}x_{n+2} + x^2_{n+3}x^2_{n+2} + x_{n+3}x^3_{n+2} + x^4_{n+2})$$
$$-a_6(x^5_{n+3} + x^4_{n+3}x_{n+2} + x^3_{n+3}x^2_{n+2} + x^2_{n+3}x^3_{n+2} + x_{n+3}x^4_{n+2} + x^5_{n+2}) \qquad (2.2.18)$$

$$a_0 = y_{n+2} - a_1x_{n+2} - a_2x^2_{n+2} - a_3x^3_{n+2} - a_4x^4_{n+2} - a_5x^5_{n+2} - a_6x^6_{n+2} \qquad (2.2.19)$$

When the a_i values are substituted into (2.1.1) the results can be expressed in the following forms

$$y(x) = Y_n + \sum_{i=1}^{4} a_i(x^i - x_n^i) \qquad \text{for } k = 2 \qquad (2.2.20)$$

$$y(x) = y_{n+1} + \sum_{i=1}^{5} a_i(x^i - x_{n+1}^i) \qquad \text{for } k = 3 \qquad (2.2.21)$$

$$y(x) = y_{n+2} + \sum_{i=1}^{6} a_i(x^i - x_{n+2}^i) \qquad \text{for } k = 4 \qquad (2.2.22)$$

Equation (2.2.22) is evaluated at $x = x_{n+4}$ while (2.2.21) is at $x = x_{n+3}$.Note is taken in that $y(x) = \sum \alpha_i(x)y_{n+i} + \sum \beta_i(x)f_{n+i}$ (2.2.23)

where $f_{n+i} = f(x_{n+i}, y_{n+i}, y'_{n+i})$; and $\alpha_i(x)$, $\beta_i(x)$ are both functions of x.

Next we proceed to the actual establishment of the methods for the different K steps.

2.2.2 The 2-step Method

Here it was discovered that Awoyemi (1999) and Kayode (2004) had extensively worked on K = 2. Their work confirmed the famous Numerouv's formula.

2.2.3 The 3-step Method

However some of our predictors for K = 3 exhibited similar characteristic as the above mentioned 2-step methods.

The continuous scheme for k = 3 is

$$y(x) = y_{n+1} + \tfrac{1}{h}\{y_{n+2} - y_{n+1}\}[(x - x_{n+1})]$$

$$+ \tfrac{1}{2} f_n [(x-x_{n+2})^2 + h\,(x-x_{n+2})]$$

$$+ \tfrac{1}{6h}\{f_{n+1} - f_n\}[(x-x_{n+2})^3 + 6h\,(x-x_{n+2})^2 + 5h^2(x-x_{n+2})]$$

$$+ \tfrac{1}{24h^2}\{f_{n+2} - 2f_{n+1} + f_n\}[(x-x_{n+2})^4 + 6h(x-x_{n+2})^3 + 12h^2(x-x_{n+2})^2 + 7h^3(x-x_{n+2})$$

$$+ \tfrac{1}{360h^3}\{f_{n+3} - 3f_{n+2} + 3f_{n+1} - f_n\}[3(x-x_{n+2})^5 + 15h(x-x_{n+2})^4 + 20h^2(x-x_{n+2})^3$$

$$-8h^4(x-x_{n+2})] \qquad (2.2.24)$$

If we now let

$$t = \frac{x - x_{n+2}}{h}, \qquad (2.2.25)$$

$$\frac{dt}{dx}=\frac{1}{h} \qquad (2.2.26)$$

Then equation (2.2.24) becomes

$$y_{n+3}-(t+1)\,y_{n+2}+y_{n+1}=\tfrac{h^2}{360}\,[(-3t^5+10t^3-7t)f_n+(9t^5+15t^4-60t^3+66t)\,f_{n+1}$$

$$+(-9t^5-30t^4+30t^3+180t^2+129t)\,f_{n+2}+(3t^5+15t^4+20t^3-8t)f_{n+3}] \qquad (2.2.27)$$

The coefficients of (2.2.27) are put as follows:

$$\left.\begin{aligned}
&\alpha_0(t)=0\\
&\alpha_1(t)=t\\
&\alpha_2(t)=-(t+1)\\
&\alpha_3(t)=1\\
&\\
&\beta_0(t)=\frac{h^2}{360}\left(-3t^5+10t^3-7t\right)\\
&\beta_1(t)=\frac{h^2}{120}\left(3t^5+5t^4-20t^3+22t\right)\\
&\beta_2(t)=\frac{h^2}{120}\left(-3t^5-10t^4+10t^3+60t^2+43t\right)\\
&\beta_3(t)=\frac{h^2}{360}\left(3t^5+15t^4+20t^3-8t\right)
\end{aligned}\right\} \qquad (2.2.28)$$

The first derivatives of equation (2.2.28) are as follows:

$$\left.\begin{aligned}
\alpha_1^1(t) &= -\frac{1}{h} \\
\alpha_2^1(t) &= \frac{1}{h} \\
\beta_0^1(t) &= \frac{h}{360}\left(-15t^4 + 30t^2 - 7\right) \\
\beta_1^1(t) &= \frac{h}{120}\left(15t^4 - 20t^3 - 60t^2 + 22\right) \\
\beta_2^1(t) &= \frac{h}{120}\left(-15t^4 - 40t^3 + 30t^2 + 120t + 43\right) \\
\beta_3^1(t) &= \frac{h}{360}\left(15t^4 + 60t^3 + 60t^2 - 8\right)
\end{aligned}\right\} \qquad (2.2.29)$$

Evaluating (2.2.28) at t = 1 which implies x = x_{n+3}

gives

$$y_{n+3} - 2y_{n+2} + y_{n+1} = \frac{h^2}{12}\{f_{n+3} + 10f_{n+2} + f_{n+1}\} \qquad (2.2.30)$$

which again is the numerov's method.

The derivatives of (2.2.30) is

$$y'_{n+3} - \frac{(y_{n+2} - y_{n+1})}{h} = \frac{h}{360}\{127f'_{n+3} + 414f'_{n+2} - 9f'_{n+1} + 8f'_n\} \qquad (2.2.31)$$

where $f'_{n+i} = f'(x_{n+i}, y_{n+i}, y'_{n+i})$ is computed from

$$y''' = f'(x, y, y') = \frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} + y''\frac{\partial f}{\partial y'} \qquad (2.2.32)$$

The first derivative of the 2-step method is given as

$$y'_{n+2} = \frac{y_{n+1} - y_n}{h} + \frac{h}{24}\{9f'_{n+2} + 26f'_{n+1} + f'_n\} \qquad (2.2.33)$$

which is different from the first derivative (2.2.31) obtained for the 3-step method.

2.2.4 The 4-step Method

Substituting all the parameters and after some simplification we now have the following continuous method for k = 4.

$$\begin{aligned}
y(x_n) = {} & y_{n+2} + \tfrac{1}{h}\{y_{n+3} - y_{n+2}\}(x - x_{n+3}) + \tfrac{1}{2} fn[(x - x_{n+3})^2 + \\
& h(x - x_{n+3})] + \tfrac{1}{60}\{f_{n+1} - fn\}[(x - x_{n+3})^3 + 9h(x - x_{n+3})^2 + 8h^2(x - x_{n+3})] \\
& + \tfrac{1}{24} h^2\{f_{n+2} - 2f_{n+1} + fn\}[(x - x_{n+3})^4 + 10h(x - x_{n+3})^3 + 36h^2(x - x_{n+3}) + 27h^3(x - x_{n+3})] \\
& + \tfrac{1}{360} h^3\{f_{n+3} - 3f_{n+2} + 3f_{n+1} - fn\}[3(x - x_{n+3})^5 + 30h(x - x_{n+3})^4 + 110h^2(x - x_{n+3})^3] \\
& + 180h^3(x - x_{n+3})^2 + 97h^4(x - x_{n+3})] + \tfrac{1}{1440h^4}\{f_{n+4} - 4fn_{n+3} + 6f_{n+2} - 4f_{n+1} + f_n\} \\
& [2(x - x_{n+3})^6 + 18h(x - x_{n+3})^5 + 55h^2(x - x_{n+3})^4 + 60h^3(x - x_{n+3})^3 - 21h^5(x - x_{n+3})] \quad \ldots\ldots\ldots\ldots (2.2.34)
\end{aligned}$$

let t = $(x-x_{n+3})/h$,

then coefficient of (2.2.34) becomes

$$\left.\begin{aligned}
\alpha_0(t) &= 0 \\
\alpha_1(t) &= 0 \\
\alpha_2(t) &= t \\
\alpha_3(t) &= -(t-1) \\
\alpha_4(t) &= 1 \\
\\
\beta_0(t) &= \frac{h^2}{1440}\left(2t^6 + 6t^5 - 5t^4 - 20t^3 + 11t\right) \\
\beta_1(t) &= \frac{h^2}{360}\left(-2t^6 - 9t^5 + 5t^4 + 30t^3 - 18t\right) \\
\beta_2(t) &= \frac{h^2}{240}\left(2t^6 + 12t^5 + 5t^4 - 60t^3 + 55t\right) \\
\beta_3(t) &= \frac{h^2}{360}\left(-2t^6 - 15t^5 - 25t^4 + 50t^3 + 180t^2 + 118t\right) \\
\beta_4(t) &= \frac{h^2}{1440}\left(2t^6 + 18t^5 + 55t^4 + 60t^3 - 21t\right)
\end{aligned}\right\}(2.2.35)$$

Also, the first derivatives of equation (2.2.35) gives

$$\left.\begin{aligned}
\alpha_2^1(t) &= \frac{1}{h} \\
\alpha_3^1(t) &= -\frac{1}{h} \\
\alpha_4^1(t) &= 0 \\
\\
\beta_0^1(t) &= \frac{h}{1440}\left(12t^5 + 30t^4 - 20t^3 - 60t^2 + 11\right) \\
\beta_1^1(t) &= \frac{h}{360}\left(-12t^5 - 45t^4 + 20t^3 + 90t^2 - 18\right) \\
\beta_2^1(t) &= \frac{h}{240}\left(12t^5 + 60t^4 + 20t^3 - 180t^2 + 55\right) \\
\beta_3^1(t) &= \frac{h}{360}\left(-12t^5 - 75t^4 - 100t^3 + 150t^2 + 360t + 118\right) \\
\beta_4^1(t) &= \frac{h}{1440}\left(12t^5 + 90t^4 + 220t^3 + 180t^2 - 21\right)
\end{aligned}\right\} \quad (2.2.36)$$

Evaluating (2.2.35) and (2.2.36) at $t = 1$ which implies $x = x_{n+4}$ gives

$$y_{n+4} - 2y_{n+3} + y_{n+2} = \frac{h^2}{240}\left\{19f_{n+4} + 204f_{n+3} + 14f_{n+2} + 4f_{n+1} - f_n\right\} \qquad (2.2.37)$$

and

$$y'_{n+4} - \frac{\left(y_{n+3} - y_{n+2}\right)}{h} = \frac{h}{1440}\left\{481f'_{n+4} + 1764f'_{n+3} - 198f'_{n+2} + 140f'_{n+1} - 27f'_n\right\} \qquad (2.2.38)$$

2.3 DETERMINATION OF PREDICTORS FOR y_{n+3} and y_{n+4}

The procedure for the determination of the predictors is similar to that adopted for the main methods. Thus the discrete schemes arising from each continuous predictor for K=2, 3 and 4 are listed below:

Thus for K = 2

$$y_{n+2} = 2y_{n+1} - y_n + h^2 f_{n+1} \tag{2.3.1a}$$

then

$$y'_{n+2} = \frac{(y_{n+1} - y_n)}{h} + \left(\frac{3}{2}\right) hf'_{n+1} \tag{2.3.1b}$$

For K=3

$$y_{n+3} = 2y_{n+2} - y_{n+1} + \frac{h^2}{12}\{13f_{n+2} - 2f_{n+1} + f_n\} \tag{2.3.2a}$$

$$y'_{n+3} = (y_{n+2} - y_{n+1}) + \frac{h}{24}\{53f'_{n+2} - 26f'_{n+1} + 9f'_n\} \tag{2.3.2b}$$

Finally, for K=4

$$y_{n+4} = 2y_{n+3} - y_{n+2} + \frac{h^2}{12}\{14f_{n+3} - 5f_{n+2} + 4f_{n+1} - f_n\} \tag{2.3.3a}$$

and

$$y'_{n+4} = (y_{n+3} - y_{n+2}) + \frac{h}{360}\{922f'_{n+3} - 771f'_{n+2} + 516f'_{n+1} - 7f'_n\} \tag{2.3.3b}$$

We also express y_{n+i} and y'_{n+i} by Taylor series expansion for use in (2.2.28), (2.2.33), (2.2.35), (2.3.2) and (2.3.3) respectively as follows:

$$y_{n+i} = y(x_n + ih) = y(x_n) + ihy'(x_n) + \frac{(ih)^2}{2!}y''(x_n) + \frac{(ih)^3}{3!}y'''(x_n) + \ldots$$

$$= y(x_n) + ihZ_n + \frac{(ih)^2}{2!}f(x_n, y_n, z_n) +$$

$$\frac{(ih)^3}{3!}\left[\frac{\partial f}{\partial x}(x_n, y_n, z_n) + z_n\frac{\partial f}{\partial y}(x_n, y_n, z_n) + fn\frac{\partial f}{\partial z}(x_n, y_n z_n)\right] + - - - - (2.3.4a)$$

and the first derivative is

$$y'_{n+1} = y'(x_n + ih) = y'(x_n) + ihy''(x_n) + \frac{(ih)^2}{2!} y'''(x_n) + \ldots = z_n + ihf(x_n, y_n, z_n) +$$

$$\frac{(ih)^2}{2!} \left[\frac{\partial f}{\partial x}(x_n, y_n, z_n) + z_n \frac{\partial f}{\partial y}(x_n, y_n, z_n) + f_n \frac{\partial f}{\partial z}(x_n, y_n, z_n) \right] + \ldots \qquad (2.3.4b)$$

where $Z = y'$

CHAPTER THREE

ANALYSIS OF BASIC PROPERTIES OF THE METHODS

3.1 INTRODUCTION

The order of the method, consistency, zero stability, convergence and region of absolute stability measure the basic properties of the method.

3.2 ORDER & ERROR CONSTANT

The order and error term of our method is found by defining a linear operator L as $L[y(x);h]=\sum_{j=0}^{k}\left\{\alpha_j \; y(x_{n+j})-h^2\beta_j y(x_{n+j})\right\}$, k = 4 (3.2.1)

With $\propto_4=1$, $\propto_0$ and β_0 are both not zero, $y(x)$ is an arbitrary function which is continuously differentiable on the interval $[a,b]$, $y(x_{n+j})=y(x_n+jh)$

If $y(x)$ represents the true solution of (1.3.3) and we adopt Taylor series expansion of

$$y(x_{n+j}) \text{ and } y''(x_{n+j}),\ \text{j} = 0, (1)\, 4 \tag{3.2.2}$$

about $x=x_n$ we have $y(x_{n+k})=\sum_{r=0}^{q}\frac{(kh)^r y^r(x_n)}{r!}$; (3.2.3)

and

$$y''(x_{n+k})=\sum_{r=2}^{q}\frac{(kh)^{r-2} y^r(x_n)}{(r-2)!} \tag{3.2.4}$$

[See Ademiluyi and Kayode (2001)], then using (3.2.3) and (3.2.4) for $0\le k\le 4$ and collecting like terms, we obtain

$$L[y(x),h] = C_0 y(x_n) + C_1 h y'(x_n) + C_2 h^2 y''(x_n) + \ldots + C_r h^r y^r(x_n) + C_{r+1} h^{r+1} y^{r+1}(x_n) +$$
$$C_{r+2} h^{r+2} y^{r+2}(x_n) + C_{r+3} h^{r+3} y^{r+3}(x_n) + ---(3.2.5)$$

Expanding (3.2.5) and comparing coefficients we get the following C_i values

$$C_0 = 1 - 2 + 1 = 0$$

$$C_1 = 4 - 6 + 2 = 0$$

$$\begin{aligned}
C_2 &= \frac{16}{2} - \frac{18}{2} + \frac{4}{2} - \frac{19}{240} - \frac{204}{240} - \frac{14}{240} - \frac{4}{240} + \frac{1}{240} = 0 \\
C_3 &= \frac{64}{6} - \frac{54}{6} + \frac{8}{6} - \frac{19}{60} - \frac{204}{80} - \frac{14}{120} - \frac{4}{240} = 0 \\
C_4 &= \frac{256}{24} - \frac{162}{24} + \frac{16}{24} - \frac{152}{240} - \frac{918}{240} - \frac{28}{240} - \frac{2}{240} = 0 \\
C_5 &= \frac{1024}{120} - \frac{486}{120} + \frac{32}{120} - \frac{1216}{1440} - \frac{5508}{1440} - \frac{112}{1440} - \frac{4}{1440} = 0
\end{aligned} \qquad (3.2.6)$$

$$C_6 = \frac{4096}{720} - \frac{1458}{720} + \frac{64}{720} - \frac{4864}{5760} - \frac{16524}{5760} - \frac{224}{5760} - \frac{4}{5760} = 0$$

$$C_7 = \frac{16384}{5040} - \frac{4374}{5040} + \frac{128}{5040} - \frac{19456}{28800} - \frac{49572}{28800} - \frac{448}{28800} - \frac{4}{28800} = -\frac{1517}{362880} \approx -\frac{209}{50000} = -0.00418.$$

From the result $C_0 = C_1 = C_2 = C_3 = \ldots = C_6$ and $C_7 = C_{p+2} \neq 0$. This implies that the scheme (2.2.21) is of order five (5) and has error constant $C_{p+2} = -0.00418$. [See Lambert ([1973, 1991), Fatunla (1988)]. Using the same procedure as that used above, the order and error constant for the other schemes are as shown in table 2 below.

3.3 CONSISTENCY

According to Lambert (1973, 1991) and Awoyemi (2001) for any scheme to be consistent, it must satisfy the following conditions:

(i) order $p > 1$

(ii) $\sum_{j=0}^{k} \alpha_j = 0$

(iii) $\rho(r) = \rho'(r) = 0$ and $\rho''(r) = 2!\delta(1)$

Here ρ and δ are first and second characteristic polynomials respectively. [see equation (3.5.1a) and (3.5.1b)] The result is as summarized in table 2.

3.4 ZERO STABILITY

According to Lambert [1973, 1991], the schemes and their predictors are zero stable when no roots of their first characteristic polynomial $\rho(r)$ has modulus greater than one and every root of modulus one has multiplicity not greater than two. Summary of result are as found in table 2.

For instance for scheme (2.2.38), we see that

$r^4 - 2r^3 + r^2 = 0$

$\Rightarrow r^2(r^2 - 2r + 1) = 0;$

$\Rightarrow \quad r = 0, 1$ twice.

3.5 REGION OF ABSOLUTE STABILITY

To establish the region of absolute stability we apply the boundary locus method as in Lambert (1973) and Fatunla (1988). This method implies that

$h(r)=\dfrac{\rho(r)}{\delta(r)}$, where $r=e^{i\theta}=Cos\theta+iSin\theta$

Thus, for scheme (2.2.38) we have that

$$\rho(r)=r^4-2r^3+r^2 \qquad (3.5.1a)$$

and $\delta(r)=\dfrac{1}{240}\left\{19r^4+204r^3+14r^2+4r-1\right\}$ (3.5.1b)

But $h(\theta)=x(\theta)+iy(\theta)$ (3.5.1c)

so that since $r=e^{i\theta}$, then

$$h(r)=\frac{240\left\{e^{4i\theta}-2e^{3i\theta}+e^{2i\theta}\right\}}{19e^{4i\theta}+204e^{3i\theta}+14e^{2i\theta}+4e^{i\theta}-1} \qquad (3.5.2)$$

Recalling that $\cos^2\theta+\sin^2\theta=1$ and $\cos(x-y)=\cos x\cos y+\sin x\sin y$ enables us to rationalize and simplify considering only the real part to obtain

$$x(\theta)=\frac{-90000+83040Cos\theta+5760Cos2\theta+1440Cos3\theta-240Cos4\theta}{42190+14384Cos\theta+1320Cos2\theta-256Cos3\theta-38Cos4\theta} \qquad (3.5.3)$$

Considering the values of θ for $0\le\theta\le180^\circ$ at intervals of 30° we get

TABLE 1: REGION OF STABILITY FOR THE 4-STEP METHOD

θ	0°	30°	60°	90°	120°	150°	180°
$x(\theta)$	0	-0.273	-1.075	-2.351	-3.895	-5.341	-5.758

Hence the region of absolute stability for scheme (2.2.38) is (-5.758,0).

Equally, we determine the region of absolute stability of the other schemes and summarize them as in table 2

The consistency, order, error constant and interval of stability for each of the methods were obtained and presented as in table 2 below.

TABLE 2: SUMMARY OF THE PROPERTIES OF ALL DERIVED METHODS, THEIR FIRST DERIVATIVES AND THOSE OF THEIR PREDICTORS

Methods	Equation Number	Order	Error Constant	Interval of Stability	Consistent	Convergen
lumerov's ethod for =2	-	Four (4)	$\frac{1}{240}$	(-6, 0)	Yes	Yes
irst erivative of =2 method	2.3.33	Two (2)	$\frac{1}{12}$	(-4, 0)	Yes	Yes
lain method r k=3	2.2.30	Four (4)	$-\frac{1}{240}$	(-6, 0)	Yes	Yes
rst rivative of =3 method	2.2.31	Four (4)	$-\frac{1}{5}$	(-2.37, 0)	Yes	Yes
edictor of 3 method	2.3.2a	Three (3)	$\frac{1}{12}$	(-3, 0)	Yes	Yes
ain method r k=4	2.2.37	Five (5)	$\frac{1}{240}$	(-5.758, 0)	Yes	Yes
rst rivative of = 4 method	2.2.38	Five (5)	$-\frac{1}{7}$	(-1.75, 0)	Yes	Yes
edictor for 4 method	2.3.3a	Four (4)	$\frac{1}{27}$	(-1.018, 0)	Yes	Yes

CHAPTER FOUR

NUMERICAL EXPERIMENT

4.1 INTRODUCTION

We solve the following examples to illustrate our method (2.2.38) for step sizes, $h = \frac{.1}{30}, \frac{.1}{32}, \frac{.1}{40}, \frac{.1}{50}, \frac{.1}{60}$

A computer program in FORTRAN language is developed and used to solve each of our problem. Our main aim is to determine the accuracy of our methods as the step length h is decreasing thus in our tabulation, YEX is the exact solution, YC is the continuous solution and ER is the error. The proposed 4-step method (2.2.37) and its predictors were used to solve the problems. The problems solved are listed as follows:

Problem 1.

$$y'' - x(y')^2 = 0, \quad y(0) = 1, \quad y'(0) = \frac{1}{2} \tag{4.1.1}$$

Exact solution: $y(x) = 1 + \frac{1}{2}\ln\frac{(2+x)}{(2-x)}$ (4.1.2)

Problem 2.

$$y'' = \frac{2x^2+1}{x^6} - \frac{2y'}{x} - \frac{y}{x4}, y\left(\frac{2}{\pi}\right) = \left(\frac{\pi^2}{4}\right) - 1, y'\left(\frac{2}{\pi}\right) = \left(\frac{\pi^2}{4}\right)(2-\pi) \tag{4.1.3}$$

Exact solution: $y = 2Cos\left(\frac{1}{x}\right) - Sin\left(\frac{1}{x}\right) + \frac{1}{x^2}$ (4.1.4)

4.2 PROBLEMS: Using the 4-step method (2.2.37) for the numerical solution of problems (4.11) and (4.13) we obtain the following results as presented in Tables 3(a-e) and Tables 4(a-e).

4.2.1 Solution to problem 1

$$y'' - x(y')^2 = 0, \quad y(0) = 1, \quad y'(0) = \frac{1}{2}$$

Exact solution: $y(x) = 1 + \frac{1}{2}\ln\frac{(2+x)}{(2-x)}$

TABLE (3a): RESULT OF PROBLEM 1 FOR THE 4-STEP METHOD WHEN

$h = \frac{.1}{30}$

X	YEX	YC	ER
0.1	0.10567274D+01	0.10567274D+01	0.47722137D-09
0.2	0.11070740D+01	0.11070740D+01	0.33174534D-08
0.3	0.11579676D+01	0.11579676D+01	0.10864673D-07
0.4	0.12079464D+01	0.12079463D+01	0.25282426D-07
0.5	0.12607533D+01	0.12607533D+01	0.51079613D-07
0.6	0.13150232D+01	0.13150231D+01	0.92772738D-07
0.7	0.13730615D+01	0.13730613D+01	0.15989156D-06
0.8	0.14316109D+01	0.14316106D+01	0.25855782D-06
0.9	0.14830915D+01	0.14930911D+01	0.40607917D-06
1.0	0.15582350D+01	0.15582343D+01	0.62707607D-06

TABLE (3b): RESULT OF PROBLEM 1 FOR THE 4-STEP METHOD WHEN

$h = \frac{.1}{32}$

X	YEX	YC	ER
0.1	0.10547421D+01	0.10547421D+01	0.37752268D-09
0.2	0.11066526D+01	0.11066525D+01	0.28855511D-08
0.3	0.11575405D+01	0.11575405D+01	0.94796342D-08
0.4	0.12092515D+01	0.12092515D+01	0.22670897D-07
0.5	0.12620907D+01	0.12620907D+01	0.45645560D-07
0.6	0.13146787D+01	0.13146786D+01	0.81295102D-07
0.7	0.13707957D+01	0.13707956D+01	0.13783109D-06
0.8	0.14292418D+01	0.14292416D+01	0.22320044D-06
0.9	0.14905937D+01	0.14905933D+01	0.35085905D-06
1.0	0.15555758D+01	0.15555753D+01	0.54205096D-06

TABLE (3c): RESULT OF PROBLEM 1 FOR THE 4-STEP METHOD WHEN

$h = \frac{.1}{40}$

X	YEX	YC	ER
0.1	0.10538019D+01	0.10538019D+01	0.23182123D-09
0.2	0.11041247D+01	0.11041247D+01	0.17248980D-08
0.3	0.11549790D+01	0.11549790D+01	0.57872027D-08
0.4	0.12066419D+01	0.12066419D+01	0.13987652D-07
0.5	0.12594128D+01	0.12594128D+01	0.28338729D-07
0.6	0.13711532D+01	0.13711531D+01	0.52286327D-07
0.7	0.13711532D+01	0.13711531D+01	0.88603635D-07
0.8	0.14296156D+01	0.14296154D+01	0.14343561D-06
0.9	0.14909877D+01	0.14909875D+01	0.22542795D-06
1.0	0.15559952D+01	0.15559949D+01	0.34823549D-06

TABLE (3d) : RESULT OF PROBLEM 1 FOR THE 4-STEP METHOD WHEN

$h = \frac{.1}{50}$

X	YEX	YC	ER
0.1	0.10530497D+01	0.10530497D+01	0.14336687D-09
0.2	0.11033666D+01	0.11033666D+01	0.10835686D-08
0.3	0.11542109D+01	0.11542109D+01	0.36561716D-08
0.4	0.12058595D+01	0.12058595D+01	0.88624779D-08
0.5	0.12586154D+01	0.12586154D+01	0.17985877D-07
0.6	0.13128196D+01	0.13128196D+01	0.32760861D-07
0.7	0.13688667D+01	0.13688666D+01	0.55634408D-07
0.8	0.14272255D+01	0.14272254D+01	0.90190652D-07
0.9	0.14884684D+01	0.14884683D+01	0.14187211D-06
1.0	0.15533142D+01	0.15533140D+01	0.21926171D-06

TABLE (3e) : RESULT OF PROBLEM 1 FOR THE 4-STEP METHOD WHEN

h = $\frac{.1}{60}$

X	YEX	YC	ER
0.1	0.10533840D+01	0.10533840D+01	0.10201617D-09
0.2	0.11028612D+01	0.11028612D+01	0.74306206D-09
0.3	0.11536990D+01	0.11536990D+01	0.25169731D-08
0.4	0.12062072D+01	0.12062072D+01	0.16931171D-08
0.5	0.12589715D+01	0.12589715D+01	0.12554757D-07
0.6	0.13122691D+01	0.13122691D+01	0.22640161D-07
0.7	0.13682956D+01	0.13682956D+01	0.38466343D-07
0.8	0.14266287D+01	0.14266286D+01	0.62379371D-07
0.9	0.14878395D+01	0.14878394D+01	0.98144624D-07
1.0	0.15526451D+01	0.15526449D+01	0.15169818D-06

4.2.2 Solution to problem 2

$$y'' = \frac{2x^2+1}{x^6} - \frac{2y'}{x} - \frac{y}{x4}, y\left(\frac{2}{\pi}\right) = \left(\frac{\pi^2}{4}\right) - 1, y'\left(\frac{2}{\pi}\right) = \left(\frac{\pi^2}{4}\right)(2-\pi)$$

Exact solution: $y = 2Cos\left(\frac{1}{x}\right) - Sin\left(\frac{1}{x}\right) + \frac{1}{x^2}$

TABLE (4a) : RESULT OF PROBLEM 2 FOR THE 4-STEP METHOD WHEN

h = $\frac{.1}{30}$

X	YEX	YC	ER
0.7	0.12763557D+01	0.12763422D+01	0.13506846D-04
0.8	0.12292067D+01	0.12291681D+01	0.38553911D-04
0.9	0.12298703D+01	0.12298067D+01	0.63563530D-04
1.0	0.12517078D+01	0.12516222D+01	0.85603434D-04
1.1	0.12823974D+01	0.12822931D+01	0.10429324D-03
1.2	0.13149077D+01	0.13147882D+01	0.11949490D-03
1.3	0.13486748D+01	0.13485421D+01	0.13273122D-03
1.4	0.13812142D+01	0.13810704D+01	0.14389134D-03
1.5	0.14119093D+01	0.14117560D+01	0.15336418D-03
1.6	0.14405409D+01	0.14403795D+01	0.16146431D-03
1.7	0.14670928D+01	0.14669244D+01	0.16844227D-03
1.8	0.14916491D+01	0.14914746D+01	0.17449688D-03
1.9	0.15143401D+01	0.15141603D+01	0.17978624D-03
2.0	0.15353126D+01	0.15351282D+01	0.18443671D-03

TABLE (4b) : RESULT OF PROBLEM 2 FOR THE 4-STEP METHOD WHEN

$$h = \frac{.1}{32}$$

X	YEX	YC	ER
0.7	0.12798695D+01	0.12798585D+01	0.11008213D-04
0.8	0.12299131D+01	0.12298802D+01	0.32882775D-04
0.9	0.12293274D+01	0.12292725D+01	0.54924758D-04
1.0	0.12506211D+01	0.12505467D+01	0.74400653D-04
1.1	0.12811002D+01	0.12810093D+01	0.90930263D-04
1.2	0.13146951D+01	0.13145903D+01	0.10480127D-03
1.3	0.13484667D+01	0.13483502D+01	0.11644106D-03
1.4	0.13810162D+01	0.13808900D+01	0.12625488D-03
1.5	0.14117238D+01	0.14115892D+01	0.13458473D-03
1.6	0.14403685D+01	0.14402268D+01	0.14170727D-03
1.7	0.14669332D+01	0.14667853D+01	0.14784289D-03
1.8	0.14915016D+01	0.14913484D+01	0.15316645D-03
1.9	0.15142038D+01	0.15140459D+01	0.15871703D-93
2.0	0.15351966D+01	0.15359014D+01	0.16190577D-03

TABLE (4c) : RESULT OF PROBLEM 2 FOR THE 4-STEP METHOD WHEN

$$h = \frac{.1}{40}$$

X	YEX	YC	ER
0.7	0.12793039D+01	0.12792969D+01	0.69081583D-05
0.8	0.12297968D+01	0.12297759D+01	0.20891701D-04
0.9	0.12294110D+01	0.12293760D+01	0.34964373D-04
1.0	0.12507918D+01	0.12507444D+01	0.47395332D-04
1.1	0.12813047D+01	0.12812468D+01	0.57945367D-04
1.2	0.13149077D+01	0.13148409D+01	0.66799125D-04
1.3	0.13486748D+01	0.13486006D+01	0.74229264D-04
1.4	0.13812142D+01	0.13811337D+01	0.80494885D-04
1.5	0.14119093D+01	0.14118235D+01	0.85813508D-04
1.6	0.14405409D+01	0.14404506D+01	0.90361780D-04
1.7	0.14670928D+01	0.14669985D+01	0.94280248D-04
1.8	0.14916491D+01	0.14915515D+01	0.97680457D-04
1.9	0.15143402D+01	0.15142394D+01	0.10065112D-03
2.0	0.15353126D+01	0.15352094D+01	0.10326314D-03

TABLE (4d) : RESULT OF PROBLEM 2 FOR THE 4-STEP METHOD WHEN

$$h = \frac{.1}{50}$$

X	YEX	YC	ER
0.7	0.12829960D+01	0.12829921D+01	0.39274569D-05
1.8	0.12305732D+01	0.12305604D+01	0.12804892D-04
1.9	0.12288902D+01	0.12268684D+01	0.21827519D-04
1.0	0.12497052D+01	0.12496754D+01	0.29820979D-04
1.1	0.12799978D+01	0.12799612D+01	0.36611820D-04
1.2	0.13135473D+01	0.13135050D+01	0.42312515D-04
1.3	0.13473417D+01	0.13472946D+01	0.47096718D-04
1.4	0.13799456D+01	0.13798945D+01	0.51130344D-04
1.5	0.14107205D+01	0.10106660D+01	0.54553805D-04
1.6	0.14394359D+01	0.14393784D+01	0.57480824D-04
1.7	0.14660698D+01	0.14660098D+01	0.60002030D-04
1.8	0.14907037D+01	0.14906415D+01	0.62189359D-04
1.9	0.15134664D+01	0.15134023D+01	0.64100021D-04
2.0	0.15349090D+01	0.15348432D+01	0.65811235D-04

TABLE (4e) : RESULT OF PROBLEM 2 FOR THE 4-STEP METHOD WHEN

$$h = \frac{.1}{60}$$

X	YEX	YC	ER
0.7	0.12839460D+01	0.12839434D+01	0.25381671D-05
0.8	0.12307792D+01	0.12307705D+01	0.86843195D-05
0.9	0.12287653D+01	0.12287504D+01	0.14948758D-04
1.0	0.12494358D+01	0.12494153D+01	0.20503827D-04
1.1	0.12796718D+01	0.12796465D+01	0.25225042D-04
1.2	0.13132072D+01	0.13131781D+01	0.29189157D-04
1.3	0.13470081D+01	0.13469756D+01	0.32516350D-04
1.4	0.13796280D+01	0.13795927D+01	0.35321756D-04
1.5	0.14104228D+01	0.14103851D+01	0.37702914D-04
1.6	0.14391591D+01	0.14391194D+01	0.39738863D-04
1.7	0.14658136D+01	0.14657721D+01	0.41492605D-04
1.8	0.14904665D+01	0.14904238D+01	0.43014156D-04
1.9	0.15132476D+01	0.15132033D+01	0.44343297D-04
2.0	0.15343025D+01	0.15342570D+01	0.45511818D-04

4.3 COMPARISM OF RESULTS AS h IS DECREASING

Here we present for both problems 1 and 2 respectively, the effect of decreasing h.

TABLE 5: SUMMARY OF RESULTS FOR PROBLEM 1 IN RESPECT OF THE 4-STEP METHOD

X	$h = \frac{1}{30}$	$h = \frac{1}{32}$	$h = \frac{1}{40}$	$h = \frac{1}{50}$	$h = \frac{1}{60}$
0.1	4.77D – 10	3.78D - 10	2.32D – 10	1.43 D - 10	1.02 D - 10
0.2	3.32 D – 09	2.89 D - 09	1.72 D – 09	1.08 D - 09	7.43 D - 10
0.3	1.09 D – 09	9.48 D - 09	5.79 D – 09	3.66 D - 09	2.52 D - 09
0.4	2.53 D – 08	2.27 D - 08	1.40 D – 08	8.86 D - 09	6.19 D - 09
0.5	5.11 D – 08	4.56 D - 08	2.83 D – 08	1.80 D - 08	1.26 D - 08
0.6	9.28 D – 08	8.13 D - 08	5.23 D – 08	3.28 D - 08	2.26 D - 08
0.7	1.60 D – 07	1.38 D - 07	8.86 D – 08	5.56 D - 08	3.85 D - 08
0.8	2.59 D – 07	2.23 D - 07	1.43 D – 07	9.02 D - 08	6.24 D - 08
0.9	4.06 D – 07	3.51 D - 07	2.25 D – 07	1.42 D - 07	9.81 D - 08
1.0	6.27 D – 07	5.42 D - 07	3.48 D – 07	2.19 D - 07	1.52 D - 07

TABLE 6: SUMMARY OF RESULTS FOR PROBLEM 2 IN RESPECT OF THE 4-STEP METHOD

X	$h=\frac{1}{30}$	$h=\frac{1}{32}$	$h=\frac{1}{40}$	$h=\frac{1}{50}$	$h=\frac{1}{60}$
1.1	1.04 D - 04	9.09 D - 05	5.79 D – 05	3.66 D - 05	2.52 D - 05
1.2	1.19 D - 04	1.05 D - 04	6.68 D – 05	4.23 D - 05	2.92 D - 05
1.3	1.33 D - 04	1.16 D - 04	7.42 D – 05	4.71 D - 05	3.25 D - 05
1.4	1.44 D - 04	1.26 D - 04	8.05 D -05	5.11 D - 05	3.53 D - 05
1.5	1.53 D - 04	1.35 D - 04	8.58 D – 05	5.46 D - 05	3.77 D - 05
1.6	1.61 D - 04	1.42 D - 04	9.04 D – 05	5.75 D - 05	3.97 D - 05
1.7	1.68 D - 04	1.48 D - 04	9.43 D – 05	6.00 D - 05	4.15 D - 05
1.8	1.74 D - 04	1.53 D - 04	9.77 D – 05	6.22 D - 05	4.30 D - 05
1.9	1.80 D - 04	1.58 D - 04	1.01 D – 04	6. 41 D - 05	4.43 D - 05
2.0	1.84 D - 04	1.62 D - 04	1.03 D – 04	6.58 D - 05	4.55 D - 05

4.4 COMPARING NEW METHOD WITH EXISTING ONES (AWOYEMI 1999)

TABLE 7: Comparing result for k = 2, 3 and 4. (For Problem 2)

X	Awoyemi (1999) $k=2$	Awoyemi (1999) $k=3$	New method $k=4$
1.1	0.45866726 D – 06	0.45866726 D – 06	0.10930263 D – 06
1.3	0.24418032 D – 06	0.22418032 D – 06	0.11644106 D – 06
1.5	0.47942720 D – 06	0.47942720 D – 06	0.13458473 D – 06
1.7	0.16847207 D – 05	0.16847207 D – 05	0.14784289 D – 05
1.9	0.31506677 D – 05	0.31506677 D – 05	0.15781703 D – 05
2.0	0.38686420 D – 05	0.38686420 D – 05	0.16190577 D – 05

CHAPTER FIVE

GENERAL CONCLUSION

5.1 DISCUSSION OF RESULTS

A collocation technique which produces an order five (5) continuous methods has been described for the direct approximate solution of problem (1.3.3). Two test examples are used to compare the accuracy of the new method with previously proposed methods [Awoyemi (1999)].

In table 5 the effect of decreasing h using problem 1 is examined. It is found that the smaller the value of h the better the accuracy. This assertion was confirmed in table 6 using problem 2. In table 7 we compared the accuracy of our new method with those of Awoyemi (1999) for $k = 2$, $k = 3$, $k = 4$. It was discovered that our new method displayed better accuracy (See graph).

We note from table 7 that additional step number improves the accuracy of the new method. We discovered that for $k = 4$, a method which yields a discrete scheme with an improved order of accuracy is obtained. This suggests that better accuracy can be achieved if we limit ourselves to even step numbers.
[See Awoyemi 1999]

5.2 CONCLUSION

It is noted here that methods with continuous coefficients have a lot of advantages over their discrete counterparts. This is so since with continuous schemes one can generate as many values as desired especially between the last two grid points. Derivatives of continuous schemes to any possible order could be computed. This enables an n^{th} order $(n \geq 2)$ ordinary differential equation to be solved directly without reduction to first order systems. A direct consequence of this is that complicated computer programs are avoided.

According to Onumanyi et al (1999) the new method can be used to construct global error estimation. More importantly, this method could be used to develop automatic codes for problem (1.3.1) (see [11], chapter 11).

5.3 RECOMMENDATION

In this work, power series was adopted as basis function. Other basis function could be adopted to see whether accuracy will be improved. From tables 7 it can be deduced that even step numbers seems to give better accuracy than odd step numbers. Thus, it is recommended that higher even step numbers should be investigated and adopted when good accuracy is the focus. Equally, hybrid type method for step numbers 3 and 4 for second order odes should be investigated for comparison with the new proposed method.

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APPENDIX 1

```
NAME OF FILE: UDOH1. FOR
K=4
SOLTION OF GENERAL SECOND ORDER INITIAL VALUE PROBLEMS OF THE
FORM Y"-F(X, Y, Y')
IMPLICIT DOUBLE PRECISION (A-H, O-Z)
DIMENSION Y4C(16, 32 10), YEX (16, 32, 10), ERC (16, 32, 10), A4C (16, 32, 10)
1, TT ((16, 32, 10)
F (X, Y, Z) = 5D0*Z*Z/Y – 2. D0*Y
Y(X)=DSIN (X) *DSIN (X)
OPEN (6, FILE=' UDOH1. OUT')
N = 10
NSTEP=32
P1=3.141593D0
A=PI/6.D0
B=A+0.1D0
DIST=B-A
H=DIST/FLOAT (NSTEP)
C=A+H
DX=H/FLOAT (N)
D=2.D0
XN=A
YN=.25D0
ZN=(DSQRT (3. D0) / 2. D0
XN1=XN+H
XN2=XN+2.D0*H
XN3=XN+3.D0*H
XN4=XN+4.D0*H
WRITE HEADINGS
WRITE (6, 9)
9 FORMAT (4X, 26HPROBLEM: Y'=.5*Y' Y' / Y – 2y/)
WRITE (6, 11)
11 FORMAT (4X, 12H Y (P1/6) =.25/)
WRITE (6, 66)
66 FORMAT (4X, 19H Y' (PI/6) = SQRT (3) /2/)
WRITE (6,8)
8 FORMAT (4X,       'H=.1/32'/)
WRITE (6,7)
7 FORMAT (4X, 'EXACT SOLUTION:      Y=3.X.X.X-2X+X.X.(1+X.LNX'/)
WRITE (6,6)
6 FROMAT (4X,'SOLUTION USING PROSOED PREDICTORS AND THEIR')
 WRITE (6,12)
12 FOEMAT (4X, 'FIRST DERIVATIVES '/)
WRITE (6,5)
5 FORMAT (7X, 'X' , 15X, 'YEX' ,20X, 'YC' ,20X, 'ER' /)
```

```
CALCULATE PREDICTORS AND THEIR DERIVATIVES
K=0
DO 1 I=1, 16
DO 2 J=1,NSTEP
F0=F(XN, YN, ZN)
CALCULATE FP
PDFX =0 .D0
PDFY =- .5D0*ZN*ZN/(YN*YN)-2.D0
PDFZ = ZN/YN
YN1=YN+H*ZN+ (H*H/2.D0) F0 + (H*3/6.D0)* (PDFX+ZN*PDFY+F0*PDFZ)
ZN1 = ZN+H*F0+ (H*H/2.D0) * (PDFX+ZN*PDFY+FO*PDFZ)
F1=F(XN1, YN1, ZN1)
YN2=2.D0*YN1 – YN+H*H*F1
ZN2=(YN1 – YN) /H+ (H+ (H/6.D0) * (11.D0*F1 – 2.D0*F0)
F2=F (XN2, YNS, ZN2)
YN3=.D0*YN2 – YN1 + (H*H/12.D0) * (12.D0*F2 – 2.D0*F2 – 2.D0*F1+F0)
ZN3=(YN2-YN1)/H+ (H/24.D0) * (53.D0*F2 – 26.D0* F1+9.D0*F0)
F3 = F (XN3, YN3, ZN3)
K=K+1
IF (K.GE.2) THEN
YN4=YC
YPN4=ZC.
ELSE
YN4=2.D0*YN3-YN2+ (H*H/12.D0) * (14.D0*F3-5.D0*F2+4.D0*F1 – F0)
ZN4=(YN3 – YN2) /H+ (H/360.D0) * (922.D0*F1 – 771.D0*F2+516.D0*F1 –
7.D0*F0)ENDIF
F4=F (XN4, YN4, ZN4)
CALCULATE COEFFICIENT OF CONTINUOUS METHOD
DO 3 K = 1, 10
TT (I, J, K) = XN3+DX*FLOAT (K)
X=TT(I, J, K)
T=(X-XN3) /H
A2T=-T
A3T=T+1.D0)
D1=H*H/1440.D0
BOT=D1* (2.D0*T**6+6.D0*T**5-5.D0*T**4-20.D0*T**3+11.D0*DO*T)
B1T=D1*(-8.D0*T**6-36.D0*T**5+20 . D0*T**4+120.D0*T**3-72.D0*T)
B2T=D1* (12.D0*T**6+72.D0*T**5+30.D0*T**4 – 360.D0*T**3 + 330.D0*T)
B3T=D1*(-8.D0*T**6–60.D*T**5–100.D0*T**4+200.D0*T**.D0*T**720.D0*T**T+
1472.D0*T)
B4T=D1*(2.D0*T**6+18.D0*T**5+55.D0*T** 4+60.D0*T**3 – 21.D0*T)
AP2T=-1.D0/H
AP3T=1.D0/H
P=H/1440.DO
BPOT=P* (12.DO*T** 5+30.D0*T**4 – 20.D0*T**3 – 60.D0*T*T+11.D0)
BPIT=P* (-48.D0*T**5 – 180.D0*T**.4 + 80.D0*T88 3 + 360.D0*T*T-72.D0)
```

```
BP2T=P* (72.D0*T** 5 + 360.D0*T**4 + 120.D0*T** 3- 1080.D0*T*T+330.D0)
BP3T=P* (-48.D0*T**5-300.D0*T**4-400.D0*T**3+600.D0*T+1440.D0*T+
1472.D0)
BPT4T=P* (12.D0*T** 5+90.D0*220.D0*T**3+180.D0*T*T-21.D0)Z4C (I, J, K)
=AP2T*YN2 + AP3T*YN3+BP4T*4F+BP3T*F3+BP3T*F2+BP1T*F1+1BP0T*F0
CALCULATE EXACT SOLUTION
YEX (I, J, K) = Y (X)
ERC (I, J, K) = DABS (Y4C (I, J, K) – YEX (I, J, E))
IF (X.GE.C) THEN
YC=Y4C (I, J, K)
ZC=Z4C (I, J, K)
YE=YEX (I, J, K)
ER=ERC (I, J, K)
GO TO 3
ELSE
ENDIF
3.      CONTINUE
IF (C.GE.B) THEN
WRITE (6; 10) X, YE, YC, ER
10      FORMAT (5X, F5.1, 3X, 3D20.10)
GO TO 4
ELSE
C       CHANGE VARIABLES
C=C+H
XN=XN1
XN1=XN2
YN=YN1
ZN=ZN1
XN1=XN2
YN1-YN2
ZN1=ZN2
ZN2=ZN3
YN2=YN3
ZN2=ZN3
ZN3=ZN4
YN3=YN4
ZN3=ZN4
XN4=XN4+H
ENDIF
GO TO 2
4       IF (B. GE. D) GO TO 1
B=B+DIST
2       CONTINEU
1       CONTINUE
        SOP
        END
```

APPENDIX 2

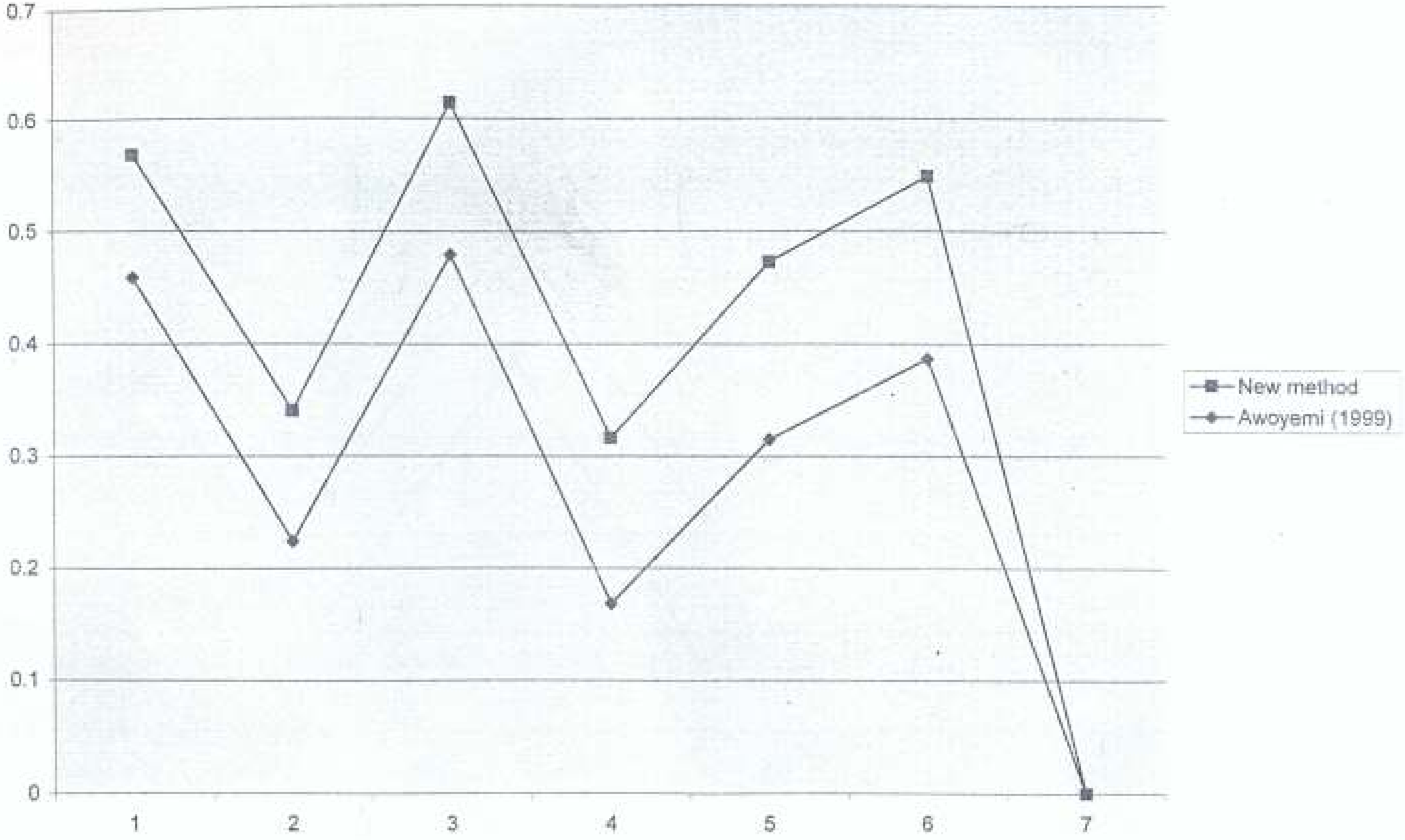
```
NAME OF FILE: UDOH1. FOR
K=4
SOLTION OF GENERAL SECOND ORDER INITIAL VALUE PROBLEMS
OF THE FORM Y''=F (X, Y, Y')
IMPLICIT DOUBLE PRECISION (A-H, O-Z)
DIMENSION Y4C (15, 32, 10), YEX (15, 32, 10), ERC (15, 32, 10), ZAC (15, 32, 10)
1, TT (15, 32, 10)
F(X, Y, Z)=(2.D0*X*X+1.D0) / X** 6 – 2. D0*Z/X-Y/X**4
Y(X)=2.D0*DCOS (1.DO/X) – DSIN (1.DO/X) + 1.DO/ (X*X)
OPEN (6, FILE= ‘UDOH2. OUT’)
N=10
NSTEP=32
PI=3.141593D0
A=2.D0/PI
DIST=B-A
H=DIST/FLOAT (NSTEP)
C-A+H
DX=H/FLOAT (N)
D=2.D0
XN=A
YN= (PI*PI) /4.D0 – 1.D0
ZN=(PI*PI/4.D0) * (2.D0-PI)
XN1=XN+H
XN2=XN+2.DO*H
XN3=XN+3.D0*H
XN4=XN+4.DO*H
WRITE HEADINGS
WRITE (6, 5)
5       FORMAT (7X, ‘Z’, 15X, ‘YEX’, 20X, ‘ER’ /)
CALCULATE PREDICTORS AND THEIR ERIVATIVES
K=0
DO 1 I = 1, 15
DO 2 J = 1, NSTEP
FO=F (XN, YN, ZN)
CALCULATE FP
PDFX=-8.DO / XN**D – 6.DO / XN**7+(2.DO/XN**2)* *ZN+(4.DO / XN **5) *YN
PDFY=-1.D0 / (XN**4)
PDFZ=-2.D0 / XN
YN1 = YN+H*ZN+ (H*H/2.D0) F0 +(H*3/6.D0)* (PDFX+ZN*PDFY+FO*PDFZ)
ZN1=ZN+H*F0 + (H*H/2.DO)* (PDFX+ZN*PDFY+F0*PDFZ)
F1=F (XN1, ZN1)
YN2=2.DO*YN1 – YN+H*H*F1
ZN2=(YN1 – YN) /H+ (H/6.D0) * (11.D0*F1 – 2.DO*F0)
```

```
F2=F (XN2, YN2, ZN2)
YN3=2.D0*YN –YN1+ (H*H/1.D0) * (13.D0*F2 – 2.DO*F1+F0
ZN3= (YN2 – YN1) /H+ (H/24.D0) * (53.D0*F2-26.DO*F1 + 9.D0*F0)
F3=F (XN3, YN3, ZN3)
K=K+1
IF (K.GE. 2) THEN
YN4 = YC
YPN4 =ZC
ELSE
YN4=2.D0*YN3 – YN2 + (H*H/12.D0) * (14.D0*F3-5.D0*F2+4.D0*F1-F0)
ZN4= (YN3-YN2)/H+ (H/360.D0) * (922.D0*F1-771.D0*F2+516.D0*F1 – 7.D0*F0)
ENDIF
F4=F (XN4, YN4, ZN4)
CALCULATE COEFICIENT OF CONTINUOUS METHOD
DO 3 K=1, 10
TT (I, J, K) = XN3 + DX*FLOAT (K)
X=TT (I, J, K)
T= (X-XN3) /H
A2T=-T
A3T=T+1.D0
D1=H*H/1440.D0
BOT=D1* (2.D0*T**6+6.D0*T**5 – 5.D0*T** 4-80 .D*T**3+60.D0*T*T+
111.D0*T)
BIT=-D1* (8.D0*T**6+36.D0*T** 5-20.D0*T**4 – 360.D0*T**3+240.D0*T*T
1+72.D0*T)
B2T=D1* (12.D0*T** 6 +72. D0*T** 5+ 30.D0*T** 4- 720.D0*T**3+360.D0*T*T+
1330. D0*T)
B3T=-D1* (8.D0*T**6+60.D0*T**5+ 100.D0*T**4 – 440.D0*T**3 – 480.D0*T*T-
1472. D0*T)
B4T=D1* (2.D0*T** 6+ 18.D0*T** 5 + 55. D0*T** 4 + 60. D0*T**.2 – 21. D0*T)
Y4C (I, J, K) =A2T*YNS+A3T*YN3+B4T*F4+B3T*F3+B2T*F2+B1T*F1+B0T*F0
AP2T=-1.D0/H
AP3T=1.D0/H
P=H/1440.D0
BP0T=P* (12.D0*T** 5 + 30.D0*T**4 – 20.D0*T**3240.D0*T*T+120.D0*T+
111.D0)
BPIT=-T* (48.D0*T**5+180.D0*T**4 –80. D0*T**3 – 1080. D0*T*T + 480.D0 +
172.D0)
BP2T=P*(72.D0*T**5+360.D0*T**4+120.D0.D0*T**3-2160.D0*T*T+720.D0*T+
1330.D0)
BP3T=-P*(48.D0*T**5+300.D0*T**4 +400.D0*T**3 – 1320. D0*T*T – 960.D0*T-
1472.D0)
BP4T=P* (12.D0*T** 5 +90.D0*T**4+220.D0*T**3+120.D0*T – 12.D0)
Z4C (I, J, K) =AP2T*YN2+AP3T*YN3+BP4T*F4+BP3T*F2+BP1T*F1+
1BP0T*F0
C        CALCULATE EXACT SOLUTION
```

```
        YEX (I, J, K) = Y(X)
        ER (I, J, K) = DABS (Y4C (I, J, K) – YEX (I, J, K))
        IF (X. GE. C) THEN
        YC=Y4C(I, J, K)
        ZC=Z4C (I, J, K)
        YE = YEX (I, J, K)
        ER=ERC (I, J, K)
        GO TO 3
        ELSE
        ENDIF
3       CONTINUE
        IF (C. GE. B) THEN
        WRITE (T, 10) X, YE, YC, ER
10      FORMAT (5X, F5.1, 3X, 3D20.8)
        GO TO 4
        ELSE
C       CHANGE VARIABLES
        C=C+H
        ZN=XN1
        EN1=XN2
        YN=YN1
        ZN=ZN1
        ZN1=XN2
        YN1=YN2
        ZN1=ZN2
        YN2=YN3
        ZN2=ZN3
        XN3=XN4
        YN3=YN4
        ZN3=ZN4
        XN4=XN4+H
        ENDIF
        GO TO 2
4       IF (B. GE. D) GO TO 1
        B=B+DIST
2       CONTINUE
        STOP
        END
```

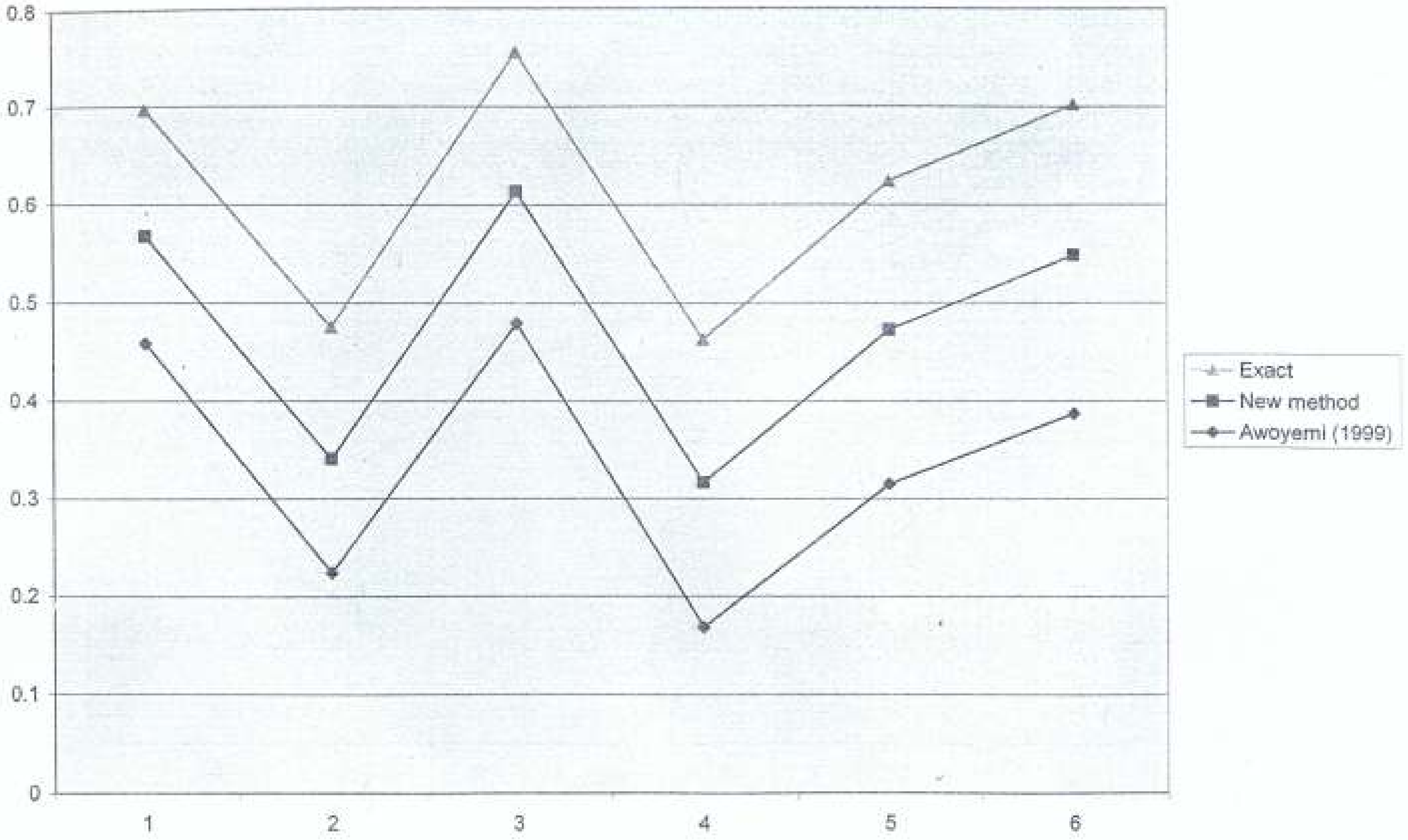
APPENDIX 3

Awoyemi (1999) Vs New Method



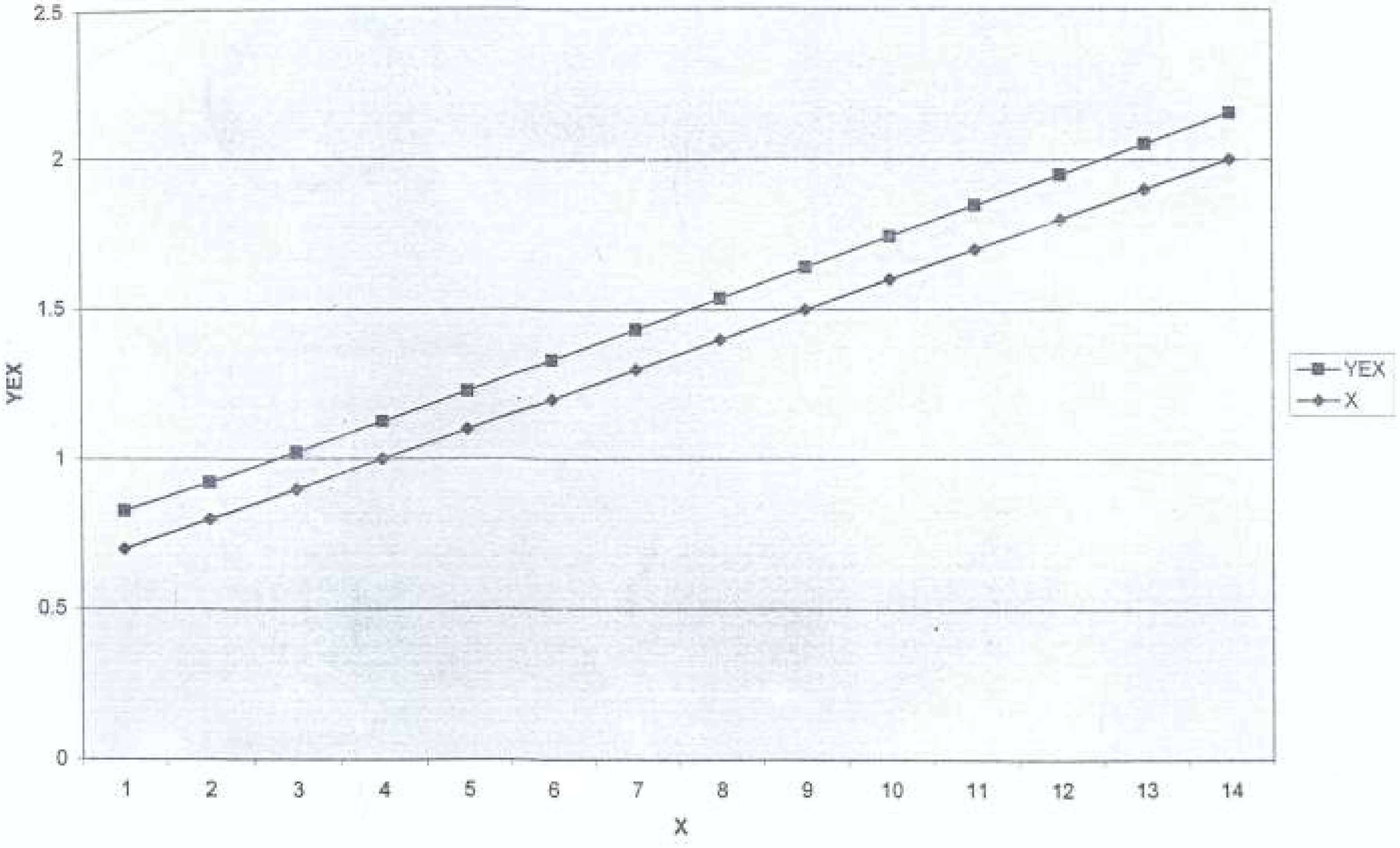
APPENDIX 4

Comparism of new method with the exact solution and Awoyemi (1999)

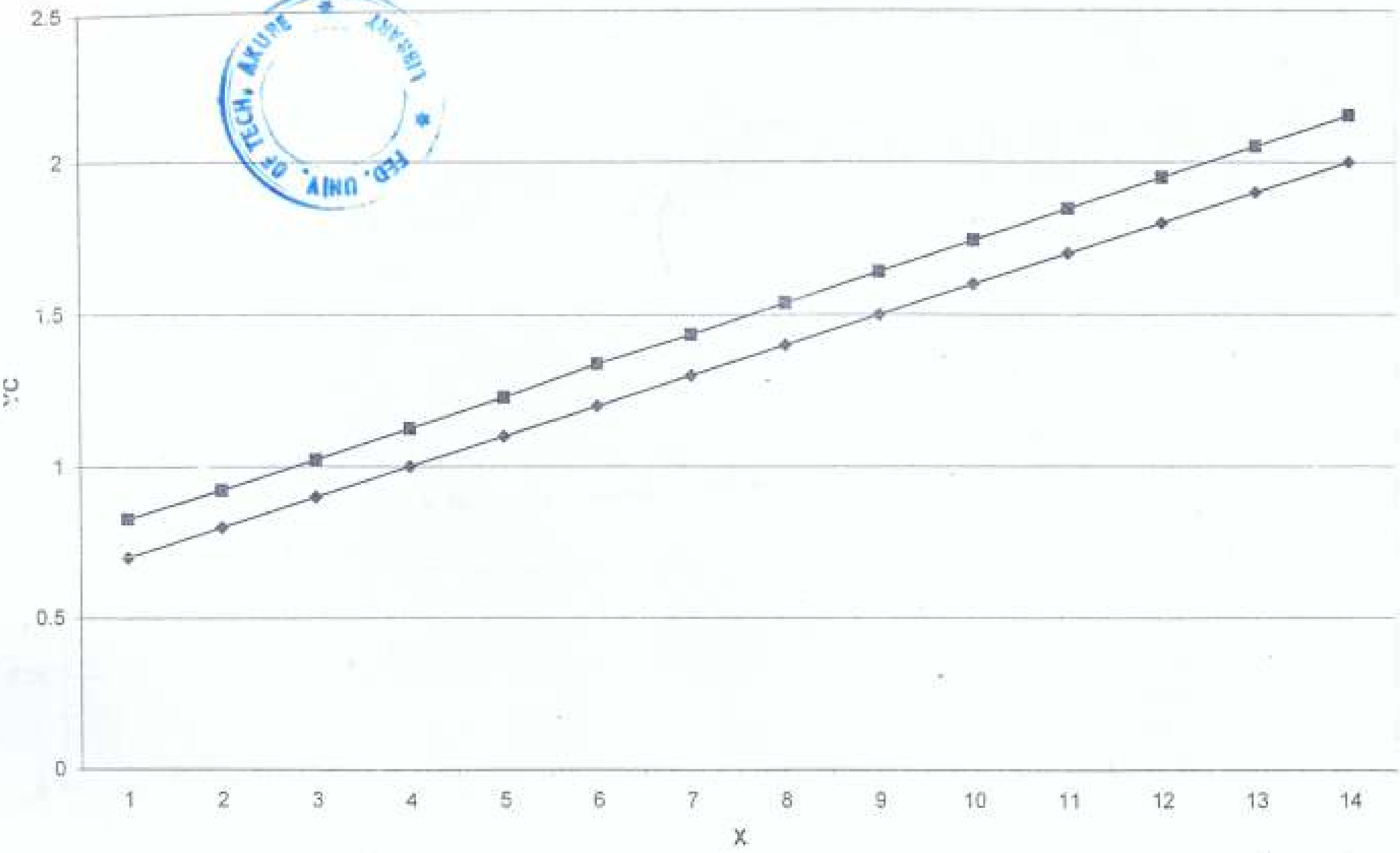


Appendix 5

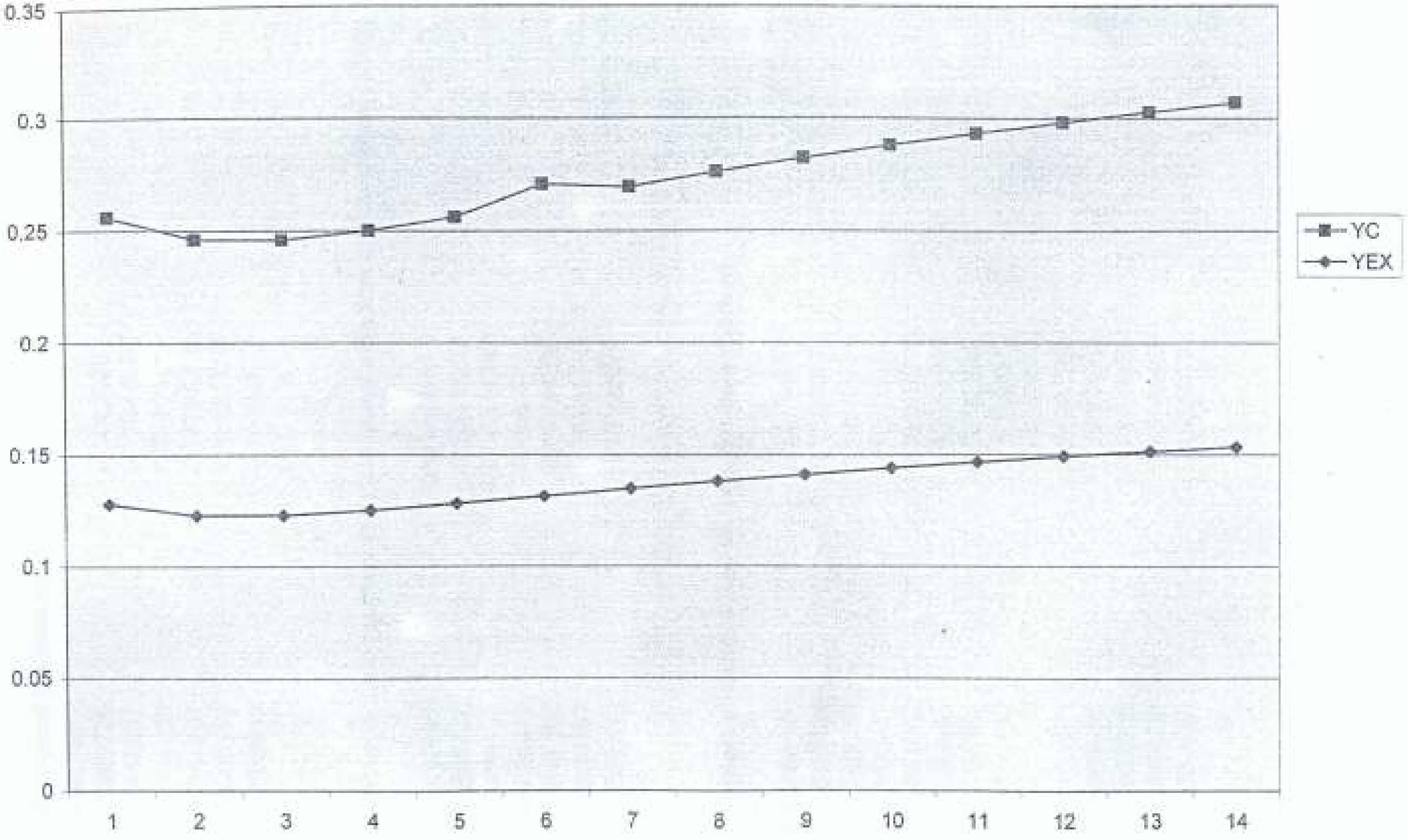
RESULT OF PROBLEM2 FOR THE 4-STEP METHOD WHEN h = 1/32



RESULT OF PROBLEM 2 FOR THE 4-STEP METHOD WHEN



Appendix 7


Graph of exact solution with the new method

APPENDIX 8

Awoyemi Vs New method

Awoyemi (1999)	New method
0.45866726	0.10930263
0.22418032	0.11644106
0.4794272	0.13458473
0.16847207	0.14784289
0.31506677	0.15781703
0.3868642	0.16190577

Comparism of new method with the exact solution and Awoyemi (1999)

Awoyemi (1999)	New method	Exact
0.45866726	0.10930263	0.12811002
0.22418032	0.11644106	0.13484667
0.4794272	0.13458473	0.14117238
0.16847207	0.14784289	0.14669332
0.31506677	0.15781703	0.15142038
0.3868642	0.16190577	0.15351966

Graph of exact solution with the new method

YEX	YC
0.12798695	0.12798585
0.12299131	0.12298802
0.12293274	0.12292725
0.12506211	0.12505467
0.12811002	0.12810093
0.13146951	0.13945903
0.13484667	0.13483502
0.13810162	0.138089
0.14117238	0.14115892
0.14403685	0.14402268
0.14669332	0.14667853
0.14915016	0.14913484
0.15142038	0.15140459
0.15351966	0.15359014

Result of problem 2 for the 4-step method when h = 1/32

X	YEX
0.7	0.12799
0.8	0.12299
0.9	0.12293
1	0.12506
1.1	0.12811
1.2	0.13147
1.3	0.13485
1.4	0.1381
1.5	0.14117
1.6	0.14404
1.7	0.14669
1.8	0.14915
1.9	0.15142
2	0.15352

Result of problem 2 for the 4-step method when h=1/32

X	YC
0.7	0.12799
0.8	0.12299
0.9	0.12293
1	0.12505
1.1	0.1281
1.2	0.13946
1.3	0.13484
1.4	0.13809
1.5	0.14116
1.6	0.14402
1.7	0.14668
1.8	0.14913
1.9	0.1514
2	0.15359