ONE-STEP COLLOCATION HYBRID METHOD FOR SOLVING FIST ORDER ORDINARY DIFFERENTIAL EQUATIONS

BY

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CERTIFICATION

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DEDICATION

This project is dedicated to the glory of God, our LORD, in the name of our Lord Jesus Christ as it is written: Isaiah 42: 8, who by His mercy, loving-kindness and grace saw me through all the TRIBULATIONS during the duration of the course.

I am saying, thank thee God-Almighty my everlasting father, in Jesus Name. Amen

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ABSTRACT

In this thesis, two one-step collocation hybrid methods for treating first order ordinary differential equation were developed. They were obtained based on continuous collocation method. The resulting methods were evaluated at some points to obtain some discrete schemes. Stability properties analysis were done, and it showed that the methods converges (as $h \rightarrow 0$ and $n \rightarrow \infty$). Numerical computations were done on some sample problems on a micro-computer. The numerical results obtained demonstrate the efficiency of the method over existing methods.

CHAPTER ONE

INTRODUCTION

1.1 ORDINARY DIFFERENTIAL EQUATIONS

Numerical problems are encountered in the various branches of human activities such as science, engineering, management and technology. The mathematical formulation of these problems often leads to differential equations.

In mathematics, any equation which connects the derivatives of differentiable function of one independent variable with respect to itself is called ordinary differential equation (ODE).

The general form of an ordinary differential equation is:

$$F(x, y, y^{I}, y^{II}, \ldots\ldots\ldots\ldots y^{(n)}) = o \quad \ldots\ldots(1.1)$$

Where y is the dependent variable, x is the independent variable and n is the highest order of its derivatives.

This highest order derivative is also the order of the equation and its degree is the power to which the highest derivative is raised after rationalization.

If no product of the dependent variable y(x) with itself or any of the derivatives occurs, the equation is said to be linear, otherwise, it is non-linear.

A differential equation together with initial conditions as stated in ***(1.2)*** below, is called initial value problem (IVP).

That is, an n^{th} order initial value problem is of the form:

$$\left.\begin{aligned} &f(x, y^{1}, y^{11}, \ldots\ldots y^{(n)}) = 0 \\ &y^{(i)}(x_0) = \alpha_i, i=(1)\, n-1 \end{aligned}\right\} \qquad \ldots\ldots (1.2)$$

In general any equation of type (1.2) can be reduced to vector equation of the form:

$$y^{1} = f(x, y), \qquad y_i(x_n) = \alpha_i \qquad \ldots\ldots (1.3)$$

Where

$\alpha_i = (\alpha_o, \alpha_1\ \alpha_2 \ldots\ldots \alpha_{n-1})^T$

$f = (f_1, f_2, f_3 \ldots\ldots, fn)^T$

$y = (y_1, y_2, y_3 \ldots\ldots y_n)^T$ *and*

$$y_i = y^{(i-1)}, (i = 1, 2, 3, \ldots\ldots\ n-1)$$

The subject of differential equations constitutes a large and very important branch of mathematics. From early days till the present day, the subject has been an area of great theoretical research and practical applications.

In attempting to solve this, it will be assumed that $f(x,y)$ satisfies the following conditions.

(i) $f(x,y)$ is a real value vector function

(ii) $f(x,y)$ is defined and continuous in the region D of x,y-plane defined by:

$$D = \{(x,y)/ a \le x \le b, -\infty < y < \infty\} \qquad (1.4)$$

and contains initial point (x_0, y_0)

(iii) There exist a real constant L such that for any $x \in [a,b]$ and numbers y_1 and y_2 in D

$$/ f(x,y_1) - f(x,y_2) / \le L / y_1 - y_2 / \qquad (1.5)$$

Where L is the Lipschitz constant of order 1.

Thus, for any $Y_0 \in D$, the initial value problem (1.3) satisfying (i) – (iii) has a unique solution y (x) for $X \in [a,b]$.

If condition (i) and (ii) are satisfied and the partial derivatives f_x, f_y, are continuous and bounded in D, then the Lipschitz constant L of the system may be taken as

$$L = // \frac{\delta f}{\delta y} // \qquad (1.6)$$

Systems of initial value problem ***(1.3)*** in ordinary differential equations can be classified into stiff if its eigen values λj's are widely separated and non-stiff otherwise. We will soon observe that stiffness plays an important role in the development of the numerical methods for solution of ordinary differential equations.

In this thesis, we shall be concerned with initial value problems (IVPS) in ordinary differential equations (ODES) of the form;

$y^1 = f(x,y),\ y(x_0) = y_0$ over $[\ x_0,\ x_n]$..(1.7)

[With y satisfying additional initial or boundary conditions as in ***(1.2)***]

There are several existing algorithms designed to solve ***(1.7)***. These include;

(a) Ruge-Kuta methods, Euler's Methods, and Taylor's series expansion as discussed in Lamber (1973) and in Kreyszig (1979).

(b) Implicit BDM (Backward Difference Methods) by Gear (1971).

(c) Hybrid methods by Graff and Stetter (1964), Butcher (1965), Gear (1965).

(d) Hybrid methods by Ademiluyi (1987).

(e) Collocation methods by Awoyemi (1992, 2002).

(f) Numerical methods for differential equation and application by Butcher (1997).

(g) One-step methods of integration by Ademiluyi (2002).

(h) Single-step stable Implicit Runge-Kutta Method by Yakubu (2003).

Similarly, several researchers had carried out a lot of research on stiff equations. These include: Lambert (1973), Fatunla (1982), Ademiluyi (1985, 1987, 1992, 2002). Ademiluyi et al (2002), Gear (1971) and Onumanyi et al (2001), just to mention but only a few.

1.2 NATURE OF ORDINARY DIFFERENTIAL EQUATIONS

For easy clarification, the nature of ordinary differential equation *(1.7)*, can be looked into from the problems that can arise in connection with the equation as follows:

(i) f(x,y) is a real value function.

(ii) f(x,y) contains discontinuities in the form of finite jumps in the components of f itself or some derivatives of f.

(iii) f(x,y) is non-linear.

(iv) Eigen value λ of the Jacobian $J = \delta f/\delta y$ of f is large (i.e stiff problems).

(v) Lower order discontinuous derivatives in the solution.

Furthermore, since stiff ordinary differential equation is widely studied, we elaborate a little further on stiffness. Stiffness as it is used in the context of ordinary differential equations is a concept describing the nature of certain subset of ordinary differential equations whose solutions contains components with fast and slow responses. The fast responding components are called transient while the slow responding components are generally smooth and steady.

The stiff systems were first encountered by Hirchfelder (1952) in the study of the motion of the masses-spring system of varying stiffness from where the problem derives its name.

The class of stiff and non-stiff first order ordinary differential equations will be considered in this thesis.

1.3 PROBLEMS ASSOCIATED WITH ORDINARY DIFFERENTIAL EQUATIONS

Some of the basic problem facing the numerical solution of ordinary differential equations are:

(a) Error Analysis Problems.

(b) Order of Accuracy and error term of the method.

(c) Stability Properties.

(d) Consistency properties

(e) Convergent properties.

(f) Evaluation of methods criterials.

(g) Computational cost.

(h) Programming ease.

These points would be discussed in a broader sense in the preceding chapters and sections of this thesis as necessary.

However, some emphasized points would be briefly discussed as follows:

In numerical schemes errors are generated when they are adopted for approximation of solutions of ordinary differential equations. The magnitude of these errors determines the degree of accuracy of the schemes and its effect can be great. It can make the solution unstable.

How then do we manage error(s) becomes basic question of study. And qualitatively speaking, constituency controls the magnitude of the local truncation error committed at each stage of the calculation, while zero-stability controls the manner in which this error is propagated as the calculation proceeds. Both are essential if convergence is to be achieved. How do we choose a suitable value for the step length h, which is linked with how accurate is the Numerical solution we have obtained, constitute the major problem in the application of linear multistep methods for solving ordinary differential equations. And problem, if method is implicit will lead us to Predictor-Corrector pair methods.

Besides, instability problems are common in numerical solution of stiff ordinary differential equations. Desirable numerical methods for stiff O.D.ES are required to have infinite rather than finite region of absolute stability. In some cases (not general) stability criteria require that the numerical schemes must be implicit as proposed by Dalquist (1963), because of rigorous iterative process and high computations.

Other requirements include the necessity for the numerical schemes to be either A-Stable, Stiffly Stable, A(α)-Stable or A(0) –Stable. Other related properties will be discussed in chapter four.

1.4 METHOD OF SOLUTION

As earlier stated, many life situations ultimately come to numerical results, hence methods of obtaining numerical results from given data is desirable, which is numerical analysis.

As we need practical answers to given problems, which numerical answers have been giving, because in many cases the answers obtained from the theoretical methods may be almost useless for numerical purposes. Typical examples are the method of integrating factor, method of exact differential equation, method of variation parameters, Cramer's rule for solving systems of linear algebraic equations in term of determinants and any similar method to mention a few. In most cases the theoretical methods merely give the existence of a solution but give no indication of how to obtain it and what happened at several points. And the advent and use of automatic computer has influenced the use and the important of numerical methods of solution(s). This help to analyze differential equations at the desired points. It also help in creation of new methods, modification of existing methods in making them more effective, give room for theoretical/practical analyzation of algorithms for the standard computational process and pointing out those algorithms which are satisfactory in various situation. It allows proper analysis of errors by trying to eliminate arithmetic traps of all kinds.

Hence, numerical solution is preferable to analytical solution. There are several algorithms, but one-step method is chosen because it computes the solution at X_{K+1} on using information only from the very preceding mesh point X_K, thus being in contrast with the multistep methods which require information from several previous mesh points. The computational effort/step is generally higher in the case of one-step methods than for the Linear multistep methods of the same accuracy.

The major advantage of the one-step methods consists in the direct possibility of using non-equal step sizes. As for disadvantage, these come from the increased complexity of the calculations to derive one-step algorithms as well as from the fact that these algorithms are longer and less "authentic" than the multiplestep algorithms. Hybrid methods are preferred to other methods because of its remarkable small error constants.

The difficult nature of the solution process for ordinary differential equations has made researchers generate a lot of interest in numerical methods using one-step and hybrid algorithms.

The existing methods include:

(a). CONVENTIONAL RUNGE-KUTTA SCHEME

Since the methods proposed are Runge-kutta Like, we wish to discuss briefly on Runge-kutta schemes.

A Runge-Kutta scheme is one of the oldest numerical methods for solution of ordinary differential equation (ODES).

These schemes were proposed by Kutta (1901) and later improved by Runge (1915).

An S-stage Runge-Kutta scheme is defined as:

$$Y_{n+1} = y_n + \sum_{i=1}^{s} W_i K_i \quad \text{.......................................(1.3.1)}$$

Where $K_i = hf\,(X_n + a_i h,\ y_n + \sum_{j=1}^{i} b_{ij} K_j)$

$a_i = \sum_{j=1}^{i} b_{ij}$ and $\sum_{i=1}^{s} W_i = 1$

The numerical values of the unknown coefficients a_i, W_i, b_{ij} are normally obtained from set of non-linear equations generated by Taylor series expansion of K_i about points X_n for i = 1 (1)s and comparing the final expansion from (1.3.1) with the Taylor series expansion of Y_{n+1} about X_n in power of h.

These schemes are often divided into three classes, namely.

(i) Explicit: $B = \{b_{ij}\} = 0$ for $j \geq i$

(ii) Semi-implicit: $B = \{b_{ij}\} = 0$ for $j > i$.

(iii)Implicit: $b_{ij} \neq 0$, for at least $j \geq i$.

Some popular Runge-Kutta scheme are:

(i) the implicit Euler scheme

$y_{n+1} = y_n + hk_1$ ....(1.3.2)

where $k_1 = f(x_n+h, y_n+hk_1)$

(ii) two stages of trapezoidal scheme of order three

$y_{n+1} = y_n + \frac{h}{2}(k_1+k_2)$ ....(1.3.3)

where

$k_1 = f(x_n + h, y_n + k_1)$

$k_2 = f(x_n+h, y_n + k_2)$

(iii) two stage implicit R-K scheme of order four and defined by Harmmer and Hollingsworth (1955)

$y_{n+1} = y_n + \frac{h}{2}[k_1 + k_2]$ ....(1.3.4)

where

$k_1 = f[\ x_n + (\ \frac{1}{2} - \frac{\sqrt{3}}{6}\ )h, y_n + \frac{1}{4}k_1 + \left(\frac{1}{4} - \frac{\sqrt{3}}{6}\right)k_2$

$k_2 = f\left(x_n + \left[\frac{1}{2} + \frac{\sqrt{3}}{6}\right]h, y_n + \left[\frac{1}{4} + \frac{\sqrt{3}}{6}\right]hk + \frac{1}{4}hk_2 \right)$

(iv) Three-stage implicit Runge-Kutta scheme defined by Butcher (1964).

$$y_{n+1} = y_n + \frac{h}{18}\left[5k_1 + 8k_2 + 5k_3\right] \qquad (1.3.5)$$

where

$$k_1 = f\left[\left(x_n + \left(\frac{1}{2} - \frac{\sqrt{15}}{10}\right)h, y_n + \frac{5}{36}hk_1 + \left(\frac{2}{9} - \frac{\sqrt{15}}{15}\right)k_2 + \left(\frac{5}{36} - \frac{\sqrt{15}}{30}\right)hk_3\ \right)\right]$$

$$k_2 = f\left[\left(x_n + \frac{1}{2}h, y_n + \left(\frac{5}{3} - \frac{\sqrt{15}}{34}\right)hk_1 + \frac{5}{3}hk_2 + \left(\frac{2}{9} + \frac{\sqrt{15}}{34}\right)k_3\right)\right]$$

$$k_3 = f\left[\left(x_n + \left(\frac{1}{2} + \frac{\sqrt{15}}{10}\right)h, y_n + \left(\frac{5}{36} - \frac{\sqrt{15}}{30}\right)hk_1 + \left(\frac{2}{9} + \frac{\sqrt{15}}{18}\right)hk_2 + \frac{5}{36}hk_3\right)\right]$$

A few typical examples of current works done on these methods are:

(i) Towards efficient Runge-kutta methods for stiff systems by Butcher et al (1990).

(ii) A new type of singly-implicit Runge-Kutta method by Butcher et al (2000).

(iii) Estimates of variable step size Runge-Kutta methods for sectorial evolution equations with non-smooth data by Garay et al (2002).

(iv) Mono-implicit Runge-Kutta formulae for the numerical solution of second order nonlinear two-point boundary value problems by Cash et al (2002).

(v) A four stage index 2-diagonally implicit Runge-Kutta method by Cameron et al (2002).

(vi) Single-step stable implicit Runge-Kutta method based on Lobatto points for ordinary differential equations by Yakubu (2003).

b. **LINEAR MULTISTEP METHODS (LMM)**: these methods are among the most-popular numerical methods today for solving a first order system of ordinary differential equations of the form ***(1.7)***.

The LMM of step K is given by:

$$\sum_{j=0}^{k} \alpha_j \; y_{n+j} = h \sum_{j=0}^{k} \; \beta_j f_{n+j} \quad \text{......} (1.3.6)$$

Where α_j, β_j are constants and $\alpha_k \neq 0$, and that not both α_0 and β_0 are zero, but $\alpha_k = 1$, and Y_{n+j} is an approximation to the theoretical solution $y(X_{n+j})$.

Note: If $\beta_k = 0$, we have an explicit method but if $\beta_k \neq 0$ the method is said to be implicit.

Some methods have been found where continuous solution can be obtain through collocation.

Among researchers who have worked on this area are:

Adeniyi (1991), Lie and Norsett (1989), Awoyemi (1992)

ONUMANYI et al (1994), ONUMANYI et al (1999)

c. **BACKWARD DIFFERENTIATION FORMULA (B.D.F)**: This method has undergone various modifications for giving accurate solution to ordinary differential equation. The general form is given as:

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h\beta_k f_{n+k} \quad \text{......(1.3.7)}$$

Where α_j, β_j are constants and y_{n+j} is $(n+j)^{th}$ approximation to the solution. It was proposed by Gear (1969). Among researchers who have worked on this are: Gear (1971), Gear by Hind marsh (1974) Byne and Hind marsh (1975), Ndam (1998), ONUMANYI et al (2001).

(d) **SECOND DERIVATIVE FORMULA (S.D.F):** This is given as:

$$y_{n+k} - y_{n+k-1} = h\sum_{j=0}^{k} \beta_j f_{n+j} + h^2 \beta_{n+k} f_{n+k} \quad \text{......(1.3.8)}$$

It was proposed by Enright (1972), improved upon by Enright (1974), Ademiluyi) (1987).

(e) **HYBRID METHODS**: Hybrid methods were also incorporated in use in the 1960s. Between the period of 1964-1965, linear multistep formulae which incorporate a function evaluation at an off grid point emerged. Such formulae simultaneously proposed by Graff and Stetter (1964), Butcher (1965) and others were classified as "Hybrid" by Gear (1965). The Hybrid methods share with Runge-Kutta methods the property of utilizing data at points other than the step points ($x_n = a + nh$). A K-step hybrid formula is defined as

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h\sum_{j=0}^{k} \beta_j f_{n+j} + h\beta_v f_{n+v} \quad \text{......(1.3.9)}$$

Where $\alpha_k = 1$, $V \in$ (Rational numbers)

Many eminent scholars have given some attention at various times to solution of problems of types ***(1.7)*** by Hybrid or/collocation. These include:

(i) A collocation method for boundary value problems by Russel et al (1972).

(ii) One-step collocation, uniform super convergence prediction correction methods. Local error estimates by Zennaro (1985).

(iii) Some new collocation formulae for the continuous numerical solutions of initial value problems by OLADELE (1991).

(iv) Hybrid second derivative methods by Ademiluyi (1987)

(v) Conditioning collocation by BLAIR (1988)

(vi) Collocation methods by Awoyemi (1992)

(vii) Towards uniformly accurate continuous finite difference approximations ODEs by Sirisena et al (2001).

(viii) New continuous implicit Runge Kutta method for stiff ordinary differential equations by Yakubu (2002).

(ix) A recent work on single step stable implicit Runge-Kutta method based on collocation by Yakubu (2003)

In this thesis therefore, we would compliment on Ruge-Kutta of the improvement made by Butcher (1997) and Yakubu (2003) by collocation hybrid method.

1.5 MOTIVATION

The large variety of application areas and general acceptability of numerical methods for solving ordinary differential equations to almost all area of human endeavours motivated the research work.

1.6 AIMS AND OBJECTIVES

The aims and objective of this work are to:

(i) Derive a class of one-step methods with continuous coefficients by collocating the differential system at selected off grid points.

(ii) Analyze the consistency, order, convergence and stability of the methods.

(iii) Determine the interval of absolute stability of the method.

(iv) Develop computer programs for the implementation of the methods on computer.

(v) Implement the programs with specific sample problems on a micro- computer with a view to establishing its applicability and suitability.

1.7 RESEARCH METHODOLOGY

To accomplish the above aims and the objectives, we went into some literature review, adopted continuous collocation method to derive our continuous method and we used Pascal triangle and some algebraic manipulations to obtain our simplified continuous method. Evaluation was done at some points (as shown in chapter 3) to obtain our schemes.

The analysis of error, consistency, convergence and stability properties were carried out using Dalhquist stability theorems, and the boundary locus method of Lambert.

The algorithm was coded in Fortran programming language and implemented on a micro-computer to confirm the workability and accuracy of the new schemes with some sample problems.

1.8 ORGANIZATION OF WORK

The remaining chapters of this thesis are organized as described below:

In chapter two, the relevant general principles to the proposed method of one step schemes were discussed.

Chapter three discusses the development of the new proposed schemes.

Their order, consistency, convergence, stability, error and region of absolute stability properties were discussed in chapter four.

While chapter five, considers the implementation of the method on a micro-computer using some sample problems. Finally, chapter six summarizes the whole thesis and makes some appropriate recommendations.

CHAPTER TWO

PRELIMINARY CONCEPTS AND PRINCIPLES

2.1 PRINCIPLE OF ONE-STEP SCHEMES

Since the proposed schemes are based on one-step, it is necessary to discuss some of its principles and concepts.

The general one step scheme for solution of the differential equation of type ***(1.6)*** is the method in which the approximation of y_{n+1} to the solution at point X_{n+1} can be generated from the knowledge of y_n at x_n, and where h is also known. Generally, one-step schemes are written in the form:

$Y_{n+1} = y_n + h\phi(x_n, y_n, h)$..(2.1)

Where $\phi(x_n, y_n, h)$ is a function of the arguments x, y. h and in addition depends on f as in equation ***(1.7)***. This function $\phi(x_n, y_n, h)$ is called the increment function.

2.1.1 THE FAMILIES OF ONE-STEP SCHEMES INCLUDES:

(a) EULER SCHEME:

It is of the form:

$Y_{n+1} = y_n + hf(x_n, y_n)$ .... (2.2)

(b) TAYLOR'S SERIES METHOD:

It is of the form:

$Y_{n+1} = y_n + hf(x_n, y_n) + \frac{h^2}{2} f^1(x_1, y_x) +$.. (2.3)

(c) RUNGE – KUTTA FORMULA.

$$y_{n+1}=y_n+\sum_{j=1}^{k} w_j k_j \qquad (2.4)$$

Where

$$k_i=hf(x_n+a_i h, y_n+\sum_{j=1}^{i} b_{ij} k_j) \qquad (2.5)$$

With the constants

$$C_i=\sum_{j=1}^{i} b_{ij} \qquad (2.6)$$

And

$$\sum_{j=1}^{i} w_j=1 \qquad (2.7)$$

Where k – is the stage of the method.

2.2 THE GENERAL SCHEMES IN CONSIDERATION.

Since in this thesis our focus is on one-step collocation Hybrid schemes, we will limit our discussion to linear multistep method and Hybrid methods.

2.2.1 LINEAR MULTISTEP METHODS (LMM)

Considering ***(1.7)***, we define x_n by:

$X_n = a + nh$, $n = 0,1,2$……… the parameter h, which will always be regarded as constant, except where otherwise indicated, is called the step-length. An essential property of the majority of computational methods for the solution of ***(1.7)*** is that of dicretization: that is, we seek an approximate solution, not on the continuous interval a ≤ x ≤, but on the discrete point set $X_n / n = 0, 1, \ldots, n = \frac{(b-a)}{h} = \{x_n / x_n = x_{n-1} + nh, n = (1)\infty$

Let y_n be an approximation to the theoretical solution at x_n, that is, to $y(x_n)$, and let $f_n \equiv f\, f(x_n, y_n)$, if a computational method for determining the sequence $\{y_n\}$ takes the form of a linear relationship between y_{n+j}, f_{n+j}

j = 0, 1, 2 .... K, we call it a linear multistep method of step number K, or a linear K- step method. The general linear multistep method may thus be written as:

$$\sum_{j=0}^{K} \alpha_j y_{n+j} = h \sum_{j=0}^{K} \beta_j f_{n+j} \quad \text{......} (2.8)$$

Equation ***(2.8)*** is explicit if $\beta_K = 0$, and implicit if $\beta_K \neq 0$ where $\alpha_j \, and \, B_j$ are constants. We assume that $X_k \neq 0$ and that not both $\alpha_0 \, and \, \beta_0$ are zero. Since ***(2.8)*** can be multiplied on both sides by the same constant without altering the relationship, the coefficients $\alpha_j \, and \, B_j$ are arbitrary to the extent of a constant multiplier. We remove this arbitrariness by assuming throughout that $\alpha_k = 1$ (see Lambert, 1973, page, 11)

Thus the problem of determine the solution y(x) of the non-linear initial value problem ***(1.7)*** is replaced by that of finding a sequence $\{y_n\}$ which satisfies the differential equation ***(2.8)***.

We need to apply a set of starting values, y_0, y_1 ...... y_{K-1}, to do this. (NOTE: In the case of one-step method, only one such value, y_0 is needed and we normally choose $y_0 = y$).

2.2.2 HYBRID METHODS

The Hybrid methods are modified linear multistep formulae which incorporate a function evaluation at an off-grid point as earlier discussed in chapter one section ***(1.3)***. Hybrid methods retain linear multistep characteristics and it share with Runge-kutta methods the property of utilizing at points other than the step points (grid points) $\{x_n/x_n = a + nh\}$.

A K-step hybrid formula is defined as:

$$\sum_{j=0}^{k}\alpha_j y_{n+j} = h\sum_{j=0}^{k}\beta_j f_{n+j} + h\beta_v f_{n+v} \qquad (2.9)$$

Where $\alpha_k = +1$, α_0, β_0 are not both zero, $V \in$ {Rational numbers}

and $f_{n+v} = f(x_{n+v}, y_{n+v})$. In order to implement the formula, even when $\beta_K = 0$ (explicit), a special predictor to estimate y_{n+v} is necessary. Thus a hybrid formula, need a helper formula to get it start.

For this thesis we would use Taylor series method as our predictor.

2.2.3 TAYLOR SERIES EXPANSIONS

According to Taylor expansion of $y(x_n + h)$ about x_n, is of the form;

$$y(x_n+h)=y(x_n)+hy'(x_n)+\frac{h^2}{2i}y''(x_n)+.........$$

Where

$$y^{(q)}(x_n)=\frac{d^q y}{dx^q}\Big|_{x=x_n}, q=1,2,...(2.10)$$

2.3 CONVERGENCE OF THE LINEAR MULTISTEP METHODS AND THE HYBRID METHODS.

Convergence expresses the property that by using a sufficiently small step an accurate computation of the numerical solution of a differential equation can be made arbitrarily close to the true solution.

Furthermore, according to Dalhquist (1962) a linear multistep method is convergent if:

(i) It is consistent

(ii) It is zero-stable.

Definition 2.1: A linear multistep method ***(2.8)*** or ***(2.9)*** is said to converge if for all initial value problems ***(1.7)*** subject to the condition of existence and uniqueness of solution, we have that:

$$\lim_{\substack{h\to 0\\ nh=x-a}} y_n = y(x_n)$$

Holds for all $x \in [a, b]$, and for all solution $\{y_n\}$ of the difference equation ***(2.8)*** or ***(2.9)*** satisfying starting condition

$y_\mu = \alpha_\mu$ (h) for which $\lim_{h \to 0} \alpha_\mu (h) = \alpha$,

$\mu = 0, 1, \ldots\ldots\ldots\ldots\ldots\ldots k-1$, see Lambert (1973).

2.4 ORDER, ERROR CONSTANT AND TRUNCATION ERROR

Considering the linear multistep method;

$$\sum_{j=0}^{k} \alpha_j y_{n+j} - h \sum_{j=0}^{k} \beta_j f_{n+j} = 0 \ldots\ldots\ldots\ldots (2.11)$$

and its associated linear difference operator L defined as

$$L[y(x); h] = \sum_{j=0}^{k} \left[\alpha_j (x + jh) - h\beta_j y^1 (x + jh)\right] \ldots\ldots\ldots\ldots (2.12)$$

Where y(x) is an arbitrary function which is continuously differentiable on the interval [a, b]. Expanding the function y(x+jh) and its derivative y^1 (x + jh) by Taylor series method about x, and collecting terms in powers of h, ***(2.12)*** gives,

$$L[y(x), h] = C_0 y(x) + C_1 hy^1 (x) + \ldots\ldots\ldots + C_q h^q y^{(q)} (x) + \ldots\ldots\ldots\ldots (2.13)$$

Where the Cq's are constants.

Definition 2.2

The difference operator ***(2.12)*** and its associated linear multistep method ***(2.11)*** are said to be order p if in ***(2.13)***,

$C_0 = C_1 = C_2 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots C_p = 0,\ C_{p+1} \neq 0$

Where

$C_0 = \alpha_0 + \alpha_1 + \alpha_2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \alpha_k.$

$C_1 = \alpha_1 + 2\alpha_2 + \ldots.. + K\alpha_k - (\beta_0 + \beta_1 + \beta_2 + \ldots\ldots\ldots\ldots + \beta_K$

$$C_q = \frac{1}{q!}\left[\left(\alpha_1 + 2^q\alpha_2 + \ldots + K^q\alpha_k\right)\right] - \frac{1}{(q-1)!}\left[\left(\beta_1 + 2^{q-1}\beta_2 + \ldots + K^{q-1}\beta_K\right)\right] \ldots\ldots\ldots (2.14)$$

q = 2, 3, …….. See Lambert (1973)

In the case of the hybrid method ***(2.9)***

$$L\left[y(x);h\right] = D_0\, y(x+th) + D_1\, h y'(x+th) + \ldots\ldots\ldots + D_q\, h^q y^{(q)}(x+th) + \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.15)$$

Where y (x + th) can be expanded in Taylor series about x.

$$y^{(q)}(x+th) = y^q(x) + th\, y^{q+1}(x) + \ldots\ldots\ldots\ldots + \frac{(th)^s}{S!} y^{(q+s)}(x) + \ldots\ldots\ldots$$

q = 0, 1, 2………………………

Where $y^{(0)}(x) = y(x)$. Making the substitution and equating term by term, we have.

$C_0 = D_0$

$C_1 = D_1 + tD_0$

$$C_2 = D_2 + tD_1 + \frac{t^2}{2!}D_0$$

$$\vdots$$

$$C_p = D_p + tD_{p-1} + \dots\dots + \frac{t^p}{p!}D_0 \quad \dots\dots (2.16)$$

$$C_{p+1} = D_{p+1} + tDp + \dots\dots + \frac{t^{p+1}}{(P+1)!}D_0$$

$$C_{p+2} = D_{p+2} + tD_{p+1} + \dots\dots + \frac{t^{p+2}}{(P+2)!}D_0$$

It follow that $C_0 = C_1 = C_2 = \dots\dots = C_p = 0$, 1f and only 1f

$D_0 = D_1 = D_2 = \dots\dots = D_p = 0$. and if this is true then

$D_{p+1} = C_{p+1}$, $D_{p+2} = C_{p+2} - tC_{p+1}$.

And hence, the formulae giving the constants D_q in terms of the coefficient α_j and β_j are:

$$D_0 = \alpha_0 + \alpha_1 + \alpha_2 + \dots\dots + \alpha_k$$
$$D_1 = t\alpha_0 + (1-t)\alpha_1 + (2-t)\alpha_2 + \dots\dots + (k-t)\alpha_k - (\beta_0 + \beta_1 + \dots\dots + \beta_K)$$
$$\vdots$$
$$Dq = \frac{1}{q!}\left[(-t)^q \alpha_0 + (1-t)^q \alpha_1 + \dots\dots + (k-t)^q \alpha_k\right]$$
$$-\frac{1}{(q-1)!}\left[(-t)^{q-1}\beta_0 + (1-t)^{q-1}\beta_1 + \dots\dots + (k-\lambda)^{q-1}\beta_K + (1-t)^{q-1}\beta_r\right] \dots\dots (2.17)$$

Thus is equivalent to ***(2.14)*** if $t = 0$ and $\beta_v = 0$. This formula is employed in finding the order and the error constant of the hybrid methods.

Thus the error is got from the difference of the theoretical solution and the approximation solution as follows.

Assume that:

$y_{n+j} = y(x_{n+j}),\ j = 0, 1, \ldots\ldots k-1$

From ***(2.12)***

$$\sum_{j=0}^{k} \alpha_j y\ (x_n + jh) = h\sum_{j=0}^{k} \beta_n y'(x_n + jh) + L[y(x), h]$$
$$= h\sum_{j=0}^{k} \beta_n f(x_n + jh, y(x_n + jh)) + L[y(x), h] \qquad (2.18)$$

Since, in this context, y(x) is taken to be exact solution of ***(1.7)***. The value for y_{n+k} given by ***(2.8)*** satisfies

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h\sum_{j=0}^{k} \beta_j f(x_{n+j}, y_{n+j})$$

Subtracting and using the localizing assumption stated above, gives

$$y(x_{n+k}) - y_{n+k} = h\beta_k [f(x_{n+k}, y(x_{n+k})) - f(x_{n+k}, y_{n+k})] + L[y(x_n, h)]$$

By the mean value theorem

$$f(x_{n+k}, y(x_{n+k})) - f(x_{n+k}, y_{n+k}) = [y(x_{n+k}) - y_{n+k}]\frac{\partial f(x_{n+k}, P_{n+k})}{\delta y}$$

Where P_{n+k} is an interior point of the interval whose end points are y_{n+k} and $y(x_{n+k})$: Hence

$$y(x_{n+k}) - y_{n+k} - [y(x_{n+k}) - y_{n+k}]h\beta_k \frac{\partial f(_{m+k}, P_{n+k})}{\delta y} = L[y(x_n), h] = T_{n+k} \qquad (2.19)$$

Thus for an explicit method the local truncation error is the difference between the theoretical solution and the solution given by the linear multistep method under the above localizing assumption. For an implicit method, the local truncation error is (approximately) proportional to the difference, between the two solutions.

If we make the further assumption that the theoretical solution y(x) has continuous derivations of sufficiently high order, then for both explicit and implicit method, it can be deduced from ***(2.19)*** that

$y(x_{n+k}) - y_{n+k} = C_{p+1} h^{p+1} y^{(p+1)} (x_n) + 0 (h^{p+2})$ ....(2.20)

Where p is the order of the method, Cp+1 is the error constant and

$C_{p+1} h^{p+1} y^{(p+1)}(x_n) +$.......... is the so called principal local truncation error

2.5 CONSISTENCY AND STABILITY

2.5.1 CONSISTENCY

Definition ***(2.2)*** according to Lambert (1973), the linear multistep methods ***(2.8)*** or ***(2.9)*** are consistent if and only if:

i. Order $p \geq 1$

ii. $\sum_{j=0}^{k} \alpha_j = 0$

iii. $\sum_{j=0}^{k} j\alpha_j = \sum_{j=0}^{k} \beta_j$

iv. $\rho(1) = 0$ and

v. $\rho^1(1) = \gamma(1)$

Where ρ and γ are first and second characteristics polynomials respectively.

2.5.2 STABILITY

Stability can be expressed in general terms to mean that a small perturbation in the initial values could cause only a bounded change in the solution, as h tends to zero.

Definitions 2.3 (zero – stable)

According to Lambert (1973). The linear multistep method (2.8 or 2.9) is said to be zero – stable if no root of the first characteristics polynomial $\rho(r)$ has modulus greater than one, and if every root with modulus one is simple.

For a one-step method the polynomial $\rho(r)$ has degree one, and if the method is consistent the only root r_1 is +1.

Thus a consistent one-step method is necessarily zero-stable.

Definition 2.4 (Absolute – stability)

A linear multistep method (2.8 or 2.9) is absolute stable in a region C of the complex plane defined by D = $(h\lambda / / \Pi (h\lambda) / < 0)$ if for all values $h\lambda \in C$, we have that all roots r_1 of the polynomial

$$\Pi(h\lambda) = \sum_{j=0}^{k} (\alpha_j - h\lambda \beta_j) r^j \text{ are such that:}$$

$|r_i| < 1, i = 1, 2$ ...., k, see Lambert (1973).

Definition 2.5 (A-stable).

A numerical method is said to be a A-stable if its region of absolute stability includes the whole of the left hand half of the complex plane, $R_e(h\lambda) < 0$, of the complex plane. See Dahlquist (1963), and fig. ***(2.1a)***.

The LMM which is A-stable is useful in solving ***(1.7)*** with large lipschitz constant (stiff system). However, A-stability is a severe requirement to ask of a numerical method, as the following theorems of Dahlquist (1963) show:

Theorem I

An explicit LMM cannot be A-stable.

Theorem II

The order of an A-stable implicit LMM cannot exceed two.

Theorem III

The second-order A-stable implicit LMM with smallest error constant is the Trapezoidal rule.

Definition 2.6

A numerical method is said to be stiffly stable if:

(i) Its region of absolute stability contain R_1 and R_2 and

(ii) It is accurate for all h $\in R_2$ when applied to the scalar test equation

$y^1 = \lambda y$, λ a complex constant with Re $\lambda < 0$,

$$Where\, R_1 = \{h\lambda / \mathrm{Re}\, h\lambda < -a\},$$
$$R_2 = \{h\lambda / -a \leq \mathrm{Re}\, h\lambda \leq b, -c \leq /mb\,\lambda \leq c\}, a, b, c$$

are positive constants see Dahlquist (1969)

Definition 2.7

A numerical scheme is said to be A(α) stable, $\alpha \in [(0, \Pi/2)]$, if its region of absolute stability contains infinite wedge S_α define by;

$$S_\alpha = [\hbar / Arg(-\hbar) / < \alpha, \hbar \neq 0)]$$

The largest α, is called the angle of absolute stability. The region of S_α is shown in Fig. ***(2.1b)*** below.

In order to apply A(α) stability conditions we verify whether the eigenvalues of the systems lie within a certain wedge S_α. It is said to be A(0) stable if it is A(α) stable for some $\alpha \in [0, \pi/2]$.

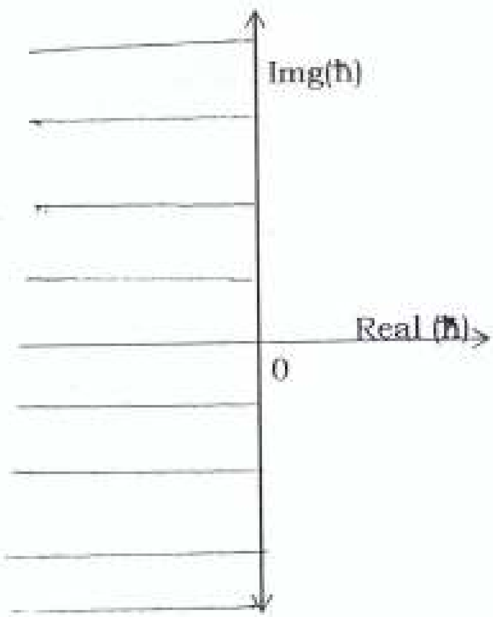

Fig. 2.1a: A stability region (shaded portion)

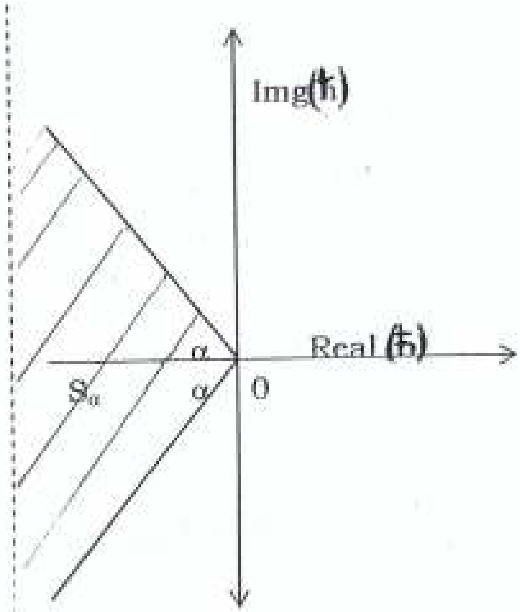

Fig. 2.1b: Region of A(α) stability,
(shaded portion)
Widlund (1967)

Theorem IV: (On zero-stable without proof).

No zero-stable linear multistep method of step number K can have order exceeding K+1 when K is odd, or exceeding K+2 when K is even.

2.6 EVALUATION OF METHODS

Evaluation of method can be divided into the followings:

(a) Reliability

(b) Accuracy

(c) Efficiency

(d) Convenience

(a) **RELIABILITY:** Simply saying, if a method is consistent and zero-stable, it is reliable.

(b) **ACCURACY:** When error term is small or when Numerical solution compared favorably with exact solution.

(c) **EFFICIENCY:** Efficiency of a method can be viewed from cost of solving a problem. Thus a method may be accurate but not efficient due to various involvement in the computation. For example, the Runge-Kutta method that can handle equations when discontinuity occurs involves a considerable number of function evaluations. It involves more work time to converge to the solution and though accurate but it is not efficient.

(d) CONVENIENCE: A method is said to be convenient, if it is not difficult to implement.

CHAPTER THREE

THE PROPOSED SCHEMES

3.1 DERIVATION OF THE NEW METHODS

In this chapter we discuss the development of our continuous collocation method for the solution of first order ordinary differential equations.

We find a real polynomial function of a single variable x as a basis function in the form:

$$y(x) = \sum_{j=0}^{m} x_j$$

yielding $y(x)=\sum_{j=0}^{m} a_j x^j$ ....(3.0)

as our approximate solution to equation ***(1.7)***

where all a_j's and m are real coefficients.

The first derivative is given as:

$y^1(x)=\sum_{j=0}^{m} j\, a_j x^{j-1}$ ....(3.1)

Two different methods are developed by collocation and interpolation will be considered in this thesis, and they shall be referred to as **Method I** and **Method II**.

3.1.1 Method I

In this case, considering equations (3.0) and (3.1), m = 2, j=0, 1, 2. Where X_n is the only interpolation point, and the two collocation points are at X_{n+u} and X_{n+v}.

And evaluating, letting $U = \frac{1}{3}$, $V = \frac{2}{3}$.

From ***(3.0)*** we will have:

$$y(x) = ao + a_1 x + a_2 x^2 \quad (3.3)$$

and from **(3.1)** we will have:

$$y^1(x) = a_1 + 2a_2 x \quad (3.4)$$

and from **(1.7)** and **(3.4)** we have

$$a_1 + 2a_2 x = f(x, y) \quad (3.5)$$

and we have the following non-linear system of equations as a result of collocating and interpolating as required.

$$a_1 + 2a_2 x_{n+u} = f_{n+u} \quad (3.6)$$

$$a_1 + 2a_2 x_{n+v} = f_{n+v} \quad (3.7)$$

$$a_0 + a_1 x_n + a_2 x_n^2 = y_n \quad (3.8)$$

Solving for a_j's we have.

$$a_2 = \frac{1}{2h(v-u)}\left[f_{n+v} - f_{n+u}\right] = K \quad (3.9)$$

$$a_1 = f_{n+v} - 2k\ x_{n+u} \quad (3.10)$$

$$a_n = y_n - x_n f_{n+u} + 2k\, x_n x_{n+u} - k\, x_n^2 \qquad (3.11)$$

Put ***(3.9)***, ***(3.10)***, ***(3.11)*** in ***(3.3)*** and simplify to determine our continuous method

$$y(x) = y_n + (x - x_n) f_{n+u} - 2k(x - x_n)(x_{n+u}) + k(x - x_n)(x + x_n)$$
$$= y_n + (x - x_n) f_{n+u} - 2k(x - x_n)(x_n + uh) + k(x - x_n)(x + x_n) \qquad (3.12)$$

From ***(3.9)*** and ***(3.12)***, we have

$$y(x) = y_n + \frac{(x - x_n)}{2h(v - u)}\left[2hv - (x - x_n)F_{n+v} + ((x - x_n) - 2uh) f_{n+v}\right] \qquad (3.13)$$

Evaluating (3.13) at x = X_{n+1}, X_{n+u} and X_{n+v} we obtained the following three finite difference schemes.

$$y(X_{n+1}) - y_n = \frac{h}{2(v - u)}\left[(2v - 1) f_{n+u} + (1 - 2u) f_{n+v}\right] \qquad (3.14a)$$

$$y(x_{n+u}) - y_n = \frac{Uh}{2(v - u)}\left[(2v - u) f_{n+u} + u f_{n+v}\right] \qquad (3.14b)$$

$$y(x_{n+v}) - y_n = \frac{Vh}{2(v - u)}\left[V f_{n+u} + (v - 2u) f_{n+v}\right] \qquad (3.14c)$$

Evaluating, letting

$u = 1/3$

$v = 2/3$

Our three finite difference schemes becomes:

$$y_{n+1} - y_n = \frac{h}{2}[f_{n+u} + f_{n+v}] \qquad (3.15a)$$

$$y_{n+u} - y_n = \frac{h}{6}[3f_{n+u} - f_{n+v}] \qquad (3.15b)$$

$$y_{n+v} - y_n = \frac{h}{3}[2f_{n+u}] \qquad (3.15c)$$

3.1.2 METHOD II

In this case, we consider equations (3.0) and (3.1) with m=3, j=0, 1, 2, 3, where x_n is the only interpolation point and the three collocation points are at x_{n+u}, x_{n+v} and x_{n+w}.

From (3.0) we will have:

$$y(x) = a_0 + a_1X + a_2x^2 + a_3X^3 \qquad (3.16)$$

and from (3.1) we will have:

$$y^1(x) = a_1 + 2a_2x + 3a_3x^2 \qquad (3.17)$$

and we have the following non-linear system of equations as a result of collocating and interpolating as required.

$$a_1 + 2a_2x_{n+u} + 3a_3x^2_{n+u} = f_{n+u} \qquad (3.18a)$$

$$a_1 + 2a_2\ x_{n+v} + 3a_3x^2_{n+v} = f_{n+v} \qquad (3.18b)$$

$$a_1 + 2a_2\ x_{n+w} + 3a_3x^2_{n+w} = f_{n+w} \qquad (3.18c)$$

$$a_0 + a_1\ x_n + a_2x^2_n + a_3x_n^3 = y_n \qquad (3.18d)$$

Solving for aj's we have,

$$a_3=\frac{1}{3h^2(v-u)(w-u)(w-v)}\left[(w-v)f_{n+u}+(u-w)f_{n+v}+(v-u)f_{n+w}\right]=K \quad \ldots(3.19)$$

$$a_2=\frac{1}{2h(v-u)}\left[f_{n+v}-f_{n+u}\right]-\frac{3}{2}k(x_{n+v}+x_{n+u}) \quad \ldots\ldots(3.20)$$

$$a_1=f_{n+u}-2x_{n+u}\left[\frac{1}{2h(v-u)}\left[f_{n+v}-f_{n+u}\right]-\frac{3}{2}k(x_{n+v}+x_{n+u})\right]-3x^2_{n+u}K \quad \ldots(3.21)$$

$$a_n=y_n-x_n\left[f_{n+u}-2x_{n+u}\left[\begin{array}{l}\frac{1}{2h(v-u)}\left[f_{n+v}-f_{n+u}\right]\\ -\frac{3}{2}K(x_{n+v}+x_{n+u})\end{array}\right]-3x^2_{n+u}K\right]$$
$$-x_n^2\left[\frac{1}{2h(v-u)}\left[f_{n+v}-f_{n+u}\right]-\frac{3}{2}K(x_{n+v}+x_{n+u})\right]-x_n^3[K] \quad \ldots\ldots(3.22)$$

Put (3.19), (3.20), (3.21), (3.22) in (3.16) and simplify to determine our continuous method

$$y(x)=y_n+(x-x_n)f_{n+v}-\frac{1}{h(v-u)}x_{n+u}(x-x_n)(f_{n+v}-f_{n+u})+3x_{n+v}k(x-x_n)(x_{n+v}+x_{n+u})$$
$$-3(x-x_n)k\,x^2_{n+u}+\frac{1}{2h(v-u)}(x-x_n)(x+x_n)(f_{n+v}-f_{n+u})-\frac{3}{2}K(x-x_n)(x+x_n)$$
$$(x_{n+v}+x_{n+u})+(x-x_n)(x^2+xx_n+x_n^2)k \quad \ldots\ldots(3.23)$$

as our continuous method.

Simplifying using Pascal triangle and some algebraic manipulations we obtain our simplified continuous method to be:

$$y(x) = y_n + (x - x_n) f_{n+u} + \frac{(x - x_n)}{2h(v - u)} \ (x - x_n - 2hu) \ (f_{n+v} - f_{n+u})$$

$$+ \frac{(x - x_n)k}{2} \left[3hvx_n + 3uhx_n + 6h^2vu - 4xx_n - 3uhx + 2x^2 + 2x_n^2 - 3xhv \right] \text{.......}(3.24)$$

From *(3.19)* and *(3.24)* we have:

$$y(x) = \frac{(x - x_n)}{6h^2 (v - u)(w - u)} \left[+ 6h^2 vw - 3hw(x - x_n) - 3hv(x - x_n) + 2(x - x_n)^2 \right] f_{n+u}$$

$$+ \frac{(x - x_n)}{6h^2 (v - u)(w - v)} \left[- 6h^2 uw + 3hw(x - x_n) + 3hu(x - x_n) - 2(x - x_n)^2 \right] f_{n+v}$$

$$+ \frac{(x - x_n)}{6h^2 (w - u)(w - v)} \left[6h^2 uv - 3hv(x - x_n) - 3hu(x - x_n) + 2(x - x_n)^2 \right] f_{n+w} \text{..........}(3.25)$$

Evaluating *(3.25)* at $x = x_{n+1}$, X_{n+u}, X_{n+v} and X_{n+w}, we obtained the following four finite difference schemes.

$$y(x_{n+1}) - y_n = \frac{h}{6(v - u)(w - u)} [6vw - 3w - 3v + 2] f_{n+u} + \frac{h}{6(v - u)(w - v)} [- 6uw + 3w + 3u - 2] f_{n+v}$$

$$+ \frac{h}{6(w - u)(w - v)} [6uv - 3v - 3u + 2] f_{n+w} \text{ ....}(3.26a)$$

$$y(x_{n+u}) - y_n = \frac{uh}{6(v - u)(w - u)} \left[6vw - 3uw - 3vu + 2u^2 \right] f_{n+u} + \frac{uh}{6(v - u)(w - v)} \left[- 3uw + u^2 \right] f_{n+v}$$

$$+ \frac{uh}{6(w - u)(w - v)} \left[3uv - u^2 \right] f_{n+w} \text{ ....}(3.26b)$$

$$y(x_{n+v}) - y_n = \frac{vh}{6(v - u)(w - u)} \left[3vw - v^2 \right] f_{n+u} + \frac{vh}{6(v - u)(w - v)} \left[- 6uw + 3wv + 3uv - 2v^2 \right] f_{n+v}$$

$$+ \frac{vh}{6(w - u)(w - v)} \left[3uv - v^2 \right] f_{n+w} \text{ ....}(3.26c)$$

$$y(x_{n+w}) - y_n = \frac{wh}{6(v-u)(w-u)}\left[3vw - w^2\right] f_{n+u} + \frac{wh}{6(v-u)(w-v)}\left[-3uw + w^2\right] f_{n+v}$$

$$+ \frac{wh}{6(w-u)(w-v)}\left[6uv - 3vw - 3uw + 2w^2\right] f_{n+w} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3.26d)$$

Our four finite differences schemes becomes

$$y_{n+1} - y_n = \frac{h}{3}\left[2f_{n+u} - f_{n+v} + 2f_{n+w}\right] \ldots\ldots\ldots\ldots\ldots\ldots (3.27a)$$

$$y_{n+u} - y_n = \frac{h}{48}\left[23f_{n+u} - 16f_{n+v} + 5f_{n+w}\right] \ldots\ldots\ldots\ldots\ldots\ldots (3.27b)$$

$$y_{n+v} - y_n = \frac{h}{12}\left[7f_{n+u} - 2f_{n+v} + f_{n+w}\right] \ldots\ldots\ldots\ldots\ldots\ldots (3.27c)$$

$$y_{n+w} - y_n = \frac{h}{16}\left[9f_{n+u} + 3f_{n+w}\right] \ldots\ldots\ldots\ldots\ldots\ldots (3.27d)$$

Also evaluating at Gaussian points, when

$$u = \frac{(5-\sqrt{15})}{10} \qquad v = 1/2 \qquad w = \frac{(5+\sqrt{15})}{10}$$

In (3.26a), (3.26b), (3.26c) and 3.26d).

Our four finite difference Schemes becomes

$$y_{n+1} - y_n = \frac{h}{18}\left[5f_{n+u} + 8f_{n+v} + 5f_{n+w}\right] \ldots\ldots\ldots\ldots\ldots\ldots (3.28a)$$

$$y_{n+u} - y_n = \frac{h}{180}\left[25f_{n+u} + (40 - 12\sqrt{15}) f_{n+v} + (25 - 6\sqrt{15}) f_{n+w}\right] \ldots\ldots\ldots (3.28b)$$

$$y_{n+v} - y_n = \frac{h}{72}\left[(10 + 3\sqrt{15}) f_{n+u} + 16f_{n+v} + (10 - 3\sqrt{15}) f_{n+w}\right] \ldots\ldots\ldots (3.28c)$$

$$y_{n+w} - y_n = \frac{h}{180}\left[(25 + 6\sqrt{15}) f_{n+u} + 16f_{n+v} + (10 + 12\sqrt{15}) f_{n+v} + 25f_{n+w}\right] \ldots\ldots\ldots (3.28d)$$

CHAPTER FOUR

ANALYSIS OF THE BASIC PROPERTIES OF THE NEW METHODS

Considering the nature of the process of development of the schemes, it is natural to expect that error will be associated with the method for solving initial value problems of ordinary differential equations (o.d.es)

Hence, there are needs to analyze the properties of the methods. These properties are: errors, the consistency, convergence, zero-stability and region of absolute stability.

These will enable us to know whether the new methods are capable of solving first order ordinary differential equations of our interest.

The results of these properties analysis determine the degree of accuracy and the general acceptability of the schemes.

Therefore the properties of the two methods are analyzed as follows:

4.1 METHOD I

4.1.1 ORDER AND ERROR TERM

Now, we find the order and the error constant of our three discrete schemes as follows:

For ***(3.15a)***, we have

$$y(x_{n+1}) = y(x_n + h) = y(x_n) + hy^{1}(x_n) + \frac{h^2}{2!}y^{11}(x_n) + \frac{h^3}{3!}y^{111}(x_n) + \frac{h^4}{4!}y^{iv}(x_n) + \ldots\ldots\ldots$$

$$f_{n+u} = y^{1}(x_{n+u}) = y^{1}(x_n + uh) = y^{1}(x_n) + uhy^{11}(x_n) + \frac{(uh)^2}{2!}y^{111}(x_n) + \frac{(uh)^3}{3!}y^{iv}(x_n) + \frac{(uh)^4}{4!}y^{v}(x_n) + \ldots\ldots$$

hence,

$$f_{n+1/3} = y^{1}(x_n) + \left(\frac{1}{3}\right)h\,y^{11}(x_n) + \frac{\left(\frac{1}{3}h\right)^2}{2!}y^{111}(x_n) + \frac{\left(\frac{h}{3}\right)^3}{3!}y^{iv}(x_n) + \ldots$$

$$f_{n+v} = y^{1}(x_{n+v}) = y^{1}(x_n + vh) = y^{1}(x_n) + vhy^{11}(x_n) + \frac{(vh)^2}{2!}y^{111}(x_n) + \frac{(vh)^3}{3!}y^{iv}(x_n) + \ldots\ldots +$$

hence

$$f_{n+2/3} = y^{1}(x_n) + \left(\frac{2}{3}\right)hy^{11}(x_n) + \frac{\left(\frac{2h}{3}\right)^2}{2!}y^{111}(x_n) + \frac{\left(\frac{2h}{3}\right)^3}{3!}y^{iv}(x_n) + \frac{\left(\frac{2h}{3}\right)^4}{4!}y^{v}(x_n) + \ldots\ldots +$$

$$but\ y(x_{n+1}) - y_n = \frac{h}{2}\left[f_{n+1/3} + f_{n+2/3}\right] = 0$$

we collect coefficient terms and solve as follows:

$$c_0 = 1 - 1 = 0$$

$$c_1 = h - \left(\frac{h}{2} + \frac{h}{2}\right) = h - h = 0$$

$$c_2 = \frac{h^2}{2!} - \left[\frac{h}{2}x\frac{h}{3} + \frac{h}{2}x\frac{h}{3}h\right] = \frac{h^2}{2!} - \left[\frac{h^2 + 2h^2}{6}\right] = \frac{h^2}{2} - \frac{3h^2}{6} = 0$$

$$c_3 = \frac{h^3}{3!} - \left[\frac{h}{2}x\frac{h^2}{9x2!} + \frac{h}{2}x\frac{4h^2}{9x2}\right]$$

$$= \frac{h^3}{6} - \frac{5h^3}{36} \neq 0.$$

Hence, $C_p = C_2$

and $C_{p+1} = C_{2+1} = C_3 = \frac{h^3}{6} - \frac{5h^3}{36} = \frac{6h^3 - 5h^3}{36} = \frac{h^3}{36} = \frac{1}{36}(h^3)$

hence its error constant $C_{p+1} = C_{2+1} = C_3 = \frac{1}{36}$

Similarly for ***(3.15b and 3.15c)***, all are summarized as follows:

$$\left.\begin{aligned} y_{n+1} - y_n &= \frac{h}{2}\left[f_{n+u} + f_{n+v}\right] \text{order } 2, \quad C_3 = \frac{1}{36} \\ y_{n+u} - y_n &= \frac{h}{6}\left[3f_{n+u} - f_{n+v}\right] \text{order } 2, \quad C_3 = \frac{15}{972} \\ y_{n+v} - y_n &= \frac{h}{3}\left[2f_{n+u}\right] \quad \text{order } 2, \quad C_3 = \frac{1}{81} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(4.0)$$

4.1.2 REGION OF ABSOLUTE STABILITY (RAS) OF THE DISCRETE SCHEMES

We apply the boundary locus method of Lambert (1973), given as:

$$h(\theta) = \frac{\rho(r)}{\gamma(r)} = \frac{\rho(\exp(i\theta))}{\gamma(\exp(i\theta))}$$

Where $r = e^{i\theta} = \cos\theta + i\sin\theta$ and ρ and γ are the first and second characteristic polynomial respectively.

Hence for ***(3.15a)*** we obtain

$y_{n+1} - y_{n+0}$ *as*:

$$r^1 - r^0 = r - 1 = \rho(r)$$

and

$$\frac{h}{2}\left[f_{n+u} + f_{n+v}\right] as:$$

$$\frac{1}{2}\left[r^u + r^v\right]$$

hence,

$$\hbar(\theta) = \frac{\rho(r)}{\gamma(r)} = \frac{2(r-1)}{r^{\frac{1}{3}} + r^{\frac{2}{3}}} = \frac{2(\cos\theta + i\sin\theta - 1)}{(\cos\theta + i\sin\theta)^u + (\cos\theta + i\sin\theta)^v}$$

$$= \frac{2(\cos\theta - 1) + i\,2\sin\theta}{\cos u\theta + \cos v\theta + i(\sin u\theta + \sin v\theta)}$$

Multiply by conjugate of the denominator and simplify, we have, the form:

$\hbar(\theta) = x(\theta) + i\,y(\theta)$, *where* $x(\theta)$ is the real part and $y(\theta)$ is the imaginary part.

Taking the real part, and simplify, we have

$$\frac{2(\cos\theta\cos u\theta+\sin\theta\sin u\theta)+2(\cos\theta\cos v\theta+\sin\theta\sin v\theta)-2(\cos u\theta+\cos v\theta)}{2(\cos u\theta\cos v\theta+\sin u\theta\sin v\theta)+2}$$

Setting $u=\frac{1}{3}, v=\frac{2}{3}$

and applying $\cos A\cos B+\sin A\sin B=\cos(A-B)$ *we have*

$$\frac{2\cos\left(\frac{2\theta}{3}\right)+2\cos\frac{\theta}{3}-2\cos\left(\frac{\theta}{3}\right)-2\cos\left(\frac{2\theta}{3}\right)}{2\cos\left(\frac{\theta}{3}\right)+2}$$

$$=\frac{0}{2\left(\cos\left(\frac{\theta}{3}\right)+1\right)}$$

Evaluate $x(\theta), 0\leq\theta\leq 180^0$, *we hve*

$x(\theta)=(0,0)$

Similarly we obtain for other two schemes (3.15b *and* 3.15c) *as shown below.*

For (3.15b), *we have*

$$x(\theta)=\frac{18-24\cos\left(\frac{\theta}{3}\right)+6\cos\left(\frac{2\theta}{3}\right)}{10-6\cos\left(\frac{\theta}{3}\right)}$$

Evaluate $x(\theta), \theta\leq\theta\leq 180^0$ *at interval of* 30^0 *is shown in the table below:*

θ	0^0	30^0	60^0	90^0	120^0	150^0	180^0
X(θ)	0	0.0007577	0.0100115	0.044422	0.1215626	0.2492488	0.4285712

$x(\theta)=(0,0.4285712)$

For (3.15c), *we have*

$$x(\theta)=\frac{6\left(1-\cos\left(\frac{\theta}{3}\right)\right)}{4}$$

and

θ	0°	30°	60°	90°	120°	150°	180°
X(θ)	0	0.02289	0.0905	0.2010	0.3510	0.5358	0.7500

Giving

x(θ) = (0,0.75)

4.1.3 CONSISTENCY, ZERO STABILITY AND CONVERGENCE

We apply section (2.5), to give the table below:

Table (4.1a): <u>Constitency, Zero-stable and Convergence of Method I</u>

Consistency					Zero-stable	Convergent
The scheme	$p \geq 1$	$\sum_{j=0}^{k} \alpha_j = 0$	$\rho^1(1) = \gamma(1)$	$\rho(1)=0$	One-step, since it is consistent	(i)consistent (ii)zero stable
3.15a	✓	✓	✓	✓	✓	✓
3.15b	✓	✓	✓	✓	✓	✓
3.15c	✓	✓	✓	✓	✓	✓

See section (4.2.2) in this chapter (chapter 4) for detail method of solving.

4.2 METHOD II

4.2.1 ORDER AND ERROR TERM

Now, we find the order and the error term as follows:

For **(3.27a)**, we have

$$y(x_{n+1}) = y(x_n+h) = y(x_n) + hy'(x_n) + \frac{h^2}{2!}y''(x_n) + \frac{h^3}{3!}y'''(x_n) + \frac{h^4}{4!}y^{iv}(x_n) + \frac{h^5}{5!}y^{v}(x_n) + \frac{h^6}{6!}y^{vi}(x_n)..$$

$$f_{n+u} = y'(x_{n+u}) = y'(x_n + uh) = y'(x_n) + uhy''(x_n) + \frac{(uh)^2}{2!}y'''(x_n) + \frac{(uh)^3}{3!}y^{iv}(x_n) + \frac{(uh)^4}{4!}y^{v}(x_n) +$$

$$\frac{(uh)^5}{5!}y^{vi}(x_n)\ldots\ldots$$

$$f_{n+1/4} = y'(x_{n+1/4}) = y'(x_n + \frac{h}{4}) = y'(x_n) + \frac{h}{4}y''(x_n) + \frac{h^2}{4^2 2!}y'''(x_n) + \frac{h^3}{4^3 3!}y^{iv}(x_n) + \frac{h^4}{4^4 4!}y^{v}(x_n) +$$

$$\frac{h^5}{4^5 5!}y^{vi}(x_n)\ldots\ldots$$

$$f_{n+v} = y'(x_{n+v}) = y'(x_n + vh) = y'(x_n) + vhy''(x_n) + \frac{(vh)^2}{2!}y'''(x_n) + \frac{(vh)^3}{3!}y^{iv}(x_n) + \frac{(vh)^4}{4!}y^{v}(x_n) +$$

$$\frac{(vh)^5}{5!}y^{vi}(x_n)\ldots\ldots$$

$$f_{n+1/2} = y'(x_{n+1/2}) = y'(x_n + \frac{1}{2}h) = y'(x_n) + \frac{h}{2}y''(x_n) + \frac{h^2}{2^2 2!}y'''(x_n) + \frac{h^3}{2^3 3!}y^{iv}(x_n) + \frac{h^4}{2^4 4!}y^{v}(x_n) +$$

$$\frac{h^5}{2^5 5!}y^{vi}(x_n)\ldots\ldots$$

$$f_{n+w} = y'(x_{n+w}) = y'(x_n + wh) = y'(x_n) + why''(x_n) + \frac{(wh)^2}{2!}y'''(x_n) + \frac{(wh)^3}{3!}y^{iv}(x_n) + \frac{(wh)^4}{4!}y^{v}(x_n) +$$

$$\frac{(wh)^5}{5!}y^{vi}(x_n)\ldots\ldots$$

$$f_{n+3/4} = y'(x_{n+3/4}) = y'(x_n + \frac{3}{4}h) = y'(x_n) + \frac{3}{4}hy''(x_n) + \frac{3^2h^2}{4^2 2!}y'''(x_n) + \frac{3^3h^3}{4^3 3!}y^{iv}(x_n) + \frac{3^4h^4}{4^4 4!}y^{v}(x_n) +$$

$$\frac{3^5h^5}{4^5 5!}y^{vi}(x_n)\ldots\ldots$$

$$[y(x_{n+1}) - y_n] - \frac{h}{3}[2f_{n+u} - f_{n+v} + 2f_{n+w}] = 0$$

We collect coefficient terms and solve as follows:

$C_0 = 1 - 1 = 0$

$$C_1 = h - \frac{h}{3}[2 \times 1 - 1 \times 1 + 2 \times 1] = h - \frac{h}{3} \times 3 = 0$$

$$C_2 = \frac{h^2}{2} - \frac{h}{3}\left[2 \times \frac{h}{4} - 1 \times \frac{h}{2} + 2\ \frac{3h}{4}\right] = \frac{h^2}{2} - \frac{h}{3}\left[\frac{2h}{4} - \frac{h}{2} + \frac{3h}{2}\right] = \frac{h^2}{2} - \frac{h}{3}\left[2h - \frac{h}{2}\right] = \frac{h^2}{2} - \frac{3h}{2} \times \frac{h}{3} = 0$$

$$C_3 = \frac{h^3}{3!} - \frac{h}{3}\left[2\times\frac{h^2}{16\times 2!} - \frac{h^2}{4\times 2!} + 2\times\frac{3^2 h^2}{4^2 2!}\right] = \frac{h^3}{3!} - \frac{h}{3}\left[\frac{h^2}{16} - \frac{h^2}{8} + \frac{9h^2}{16}\right] = \frac{h^3}{6} - \frac{h}{3}\times\frac{8h^2}{16} = \frac{h^3}{6} - \frac{h^3}{6} = 0$$

$$C_4 = \frac{h^4}{4!} - \frac{h}{3}\left[2\times\frac{h^3}{16\times4\times3!} - \frac{h^3}{8\times3!} + 2\times\frac{9\times3!}{16\times4\times3!}\right] = \frac{h^4}{4!} - \frac{h}{3}\left[\frac{h^3}{8\times4\times3!} - \frac{h^3}{8\times3!} + \frac{9\times3h^3}{8\times4\times3!}\right]$$

$$= \frac{h^4}{4!} - \frac{h}{3}\left[\frac{h^3}{8\times4!} - \frac{4h^3}{8\times4!} + \frac{9\times3h^3}{8\times4!}\right]$$

$$= \frac{h^4}{4!} - \frac{h}{3}\left[\frac{28h^3 - 4h^3}{8\times4!}\right] = \frac{h^4}{4!} - \frac{h}{3}\left[\frac{24h^3}{8\times4!}\right] = \frac{h^4}{4!} - \frac{h^4}{4!} = 0$$

$$C_5 = \frac{h^5}{5!} - \frac{h}{3}\left[2\times\frac{h^4}{16\times16\times4!} - \frac{h^4}{16\times4!} + 2\times\frac{9\times9h^4}{16\times16\times4!}\right] = \frac{h^5}{5!} - \frac{h}{3}\left[\frac{h^4}{8\times16\times4!} - \frac{h^4}{16\times4!!} + \frac{9\times9h^4}{8\times16\times4!}\right]$$

$$= \frac{h^5}{5!} - \frac{h}{3}\left[\frac{h^4 - 8h^4 + 9\times9h^4}{8\times16\times4!}\right] = \frac{h^5}{5!} - \frac{h}{3}\left[\frac{82h^4 - 8h^4}{8\times16\times4!}\right] = \frac{h^5}{5!} - \frac{h}{3}\left[\frac{74h^4}{8\times16\times4!}\right]$$

$$= \frac{h^5}{5!} - \frac{h}{3}\left[\frac{37h^4}{4\times16\times4!}\right] \neq 0 \quad \textit{hence } C_p = 4$$

$$= \frac{h^5}{120} - \frac{37h^5}{4608} = \frac{192h^5 - 185h^5}{5\times4608}$$

$$= \frac{7h^5}{23040} = \frac{h^5}{3291.4286} = \frac{1}{3291.4286}(h^5)$$

Hence its error constant

$$C_{p+1} = C_5 = \frac{1}{3291.4286} \ or \ \frac{7}{23040}$$

Similarly, for ***(3.27b – 3.27d)*** and all are summarized as follows:

$$\left.\begin{array}{llll} y_{n+1} - y_n = \frac{h}{3}\left[2f_{n+u} - f_{n+v} + 2f_{n+w}\right] & \text{order 4} & C_5 = \frac{7}{23040} \\ y_{n+u} - y_n = \frac{h}{48}\left[23f_{n+u} - 16f_{n+v} + 5f_{n+w}\right] & \text{order 3} & C_4 = \frac{-3}{2048} \\ y_{n+v} - y_n = \frac{h}{12}\left[7f_{n+u} - 2f_{n+v} + f_{n+w}\right] & \text{order 3} & C_4 = \frac{-1}{768} \\ y_{n+w} - y_n = \frac{h}{16}\left[9f_{n+u} + 3f_{n+w}\right] & \text{order 3} & C_4 = \frac{-3}{2048} \end{array}\right\} \quad \ldots (4.1)$$

4.2.2 INTERVAL OF ABSOLUTE STABILITY OF THE DISCRETE SCHEMES

We apply the boundary locus method of Lambert (1973), given as:

$$\bar{h}(\theta) = \frac{\rho(r)}{\gamma(r)} = \frac{\rho(\exp(i\theta))}{\gamma(\exp(i\theta))}$$

Where $r = e^{i\theta} = \text{Cos}\,\theta + i\text{Sin}\,\theta$ and ρ and γ are the first and second characteristic polynomial respectively.

Hence for ***(3.27a)*** we obtain

$$y_{n+1} - y_n = r - 1 = \rho(r)$$

$$\gamma(r) = \frac{1}{3}\left(2r^u - r^v + 2r^w\right)$$

$$\bar{h}(\theta) = \frac{\rho(r)}{\gamma(r)} = \frac{\rho(e^{i\theta})}{\gamma(e^{i\theta})} = \frac{3(\cos\theta + i\sin\theta - 1)}{2(Cos u\theta + i\sin\theta) - (Cos v\theta + i\sin\theta) + 2(Cos w\theta + i\sin w\theta)}$$

Multiply by conjugate of the denominator and simplify, we have, the form:

$\bar{h}(\theta) = x(\theta) + i\,y(\theta)$, where x (θ) is the real part, and y (θ) is the imaginary part.

Taking the real part, and simplify we have,

$$X(\theta) = \frac{\begin{array}{l} 6(\cos\theta\cos u\theta + \sin\theta\sin u\theta) + 6(\cos\theta\cos w\theta + \sin\theta\sin w\theta) - 3(\cos\theta\cos v\theta + \sin\theta\sin v\theta) \\ -6\cos u\theta + 3\cos v\theta - 6\cos w\theta) \end{array}}{\begin{array}{l} 4(\cos^2 u\theta + \sin^2 u\theta) + 4(\cos^2 w\theta + \sin^2 w\theta) + (\cos^2 v\theta + \sin^2 v\theta) - \\ 4(\cos u\theta\cos v\theta + \sin u\theta\sin v\theta + 8)(\cos 4\theta\cos w\theta + \sin u\theta\sin w\theta) - \\ 4(\cos v\theta\cos w\theta + \sin v\theta\sin w\theta) \end{array}}$$

applying, $\cos^2\theta + \sin^2\theta = 1$

CosA CosB + Sin A SinB = Cos (A-B)

U = ¼ , V = ½ , w = ¾ and simplify we have

$$X(\theta) = \frac{6(Cos\frac{1}{2}\theta - Cos\frac{3}{4}\theta)}{9 + 8(Cos\frac{1}{2}\theta - Cos\frac{1}{4}\theta)}$$

Evaluate x (θ), $0 \leq \theta \leq 180^\circ$ at intervals of 30°, is as shown in the table below:

θ	0°	30°	60°	90°	120°	150°	180°
X(θ)	0	0.0286	0.1163	0.2679	0.4941	0.8149	1.2690

The result from the table shows that the region of absolute stability of our discrete scheme (3.27a) is x (θ) = (1.2690,0).

Similarly we obtain for the other three schemes ***(3.27b) – (3.27d)***, as shown below:

(i) X (θ) = $$\frac{1104-1872Cos\left(\frac{\theta}{4}\right)+1008Cos\left(\frac{\theta}{2}\right)-240Cos\left(\frac{3\theta}{4}\right)}{810-931Cos\left(\frac{\theta}{4}\right)+230Cos\left(\frac{\theta}{2}\right)}$$

for ***(3.27b)***

Evaluate x(θ), 0 ≤ θ ≤ 180° at intervals of 30°, is as shown in the table below:

θ	0°	30°	60°	90°	120°	150°	180°
X(θ)	0	-0.000088	-0.008558	-0.041180	-0.110750	-0.217977	-0.329622

The result from the table shows that the region of absolute stability of our discrete scheme (3.27b) is x(θ) = (-0.329622, 0)

(ii) X (θ) = $$\frac{12\cos(\frac{\theta}{4})+24Cos(\frac{\theta}{2})-12Cos(\frac{3\theta}{4})-24}{54-32Cos\frac{\theta}{4}+14Cos\frac{\theta}{2}},$$

For (3.27c), we obtain as follows

θ	0°	30°	60°	90°	120°	150°	180°
X(θ)	0	-0.000235	-0.003135	-0.015588	-0.048306	-0.114027	-0.224063

X(θ) = (-0.224063,0)

(iii) X(θ)= $$\frac{48+144Cos\left(\frac{\theta}{2}\right)-144Cos\left(\frac{\theta}{4}\right)-48Cos\left(\frac{3\theta}{4}\right)}{90+54Cos\left(\frac{\theta}{2}\right)}$$

For (3.2d) we obtain as follows:

θ	0^0	30^0	60^0	90^0	120^0	150^0	180^0
$X(\theta)$	0	-0.000135	-0.002387	-0.012395	-0.040205	-0.102070	-0.220914

$X(\theta)$ = (-0.2240914,0)

4.2.3 CONSISTENCY, ZERO STABILITY AND CONVERGENCE

CONSISTENCY

Applying section ***(2.5.1)*** of chapter two to discrete scheme ***(3.27a)***.

We obtain as follows:

the scheme has order 3, and $3 > 1$, which satisfies $P \geq 1$

$$\sum_{j=0}^{k} \alpha_j = 0$$

$\alpha_1 = y_{n+v} \quad - \alpha_0 y_n$

$$\sum_{j=0}^{k} \alpha_j = \alpha_o + \alpha_1 = -1 + 1 = 0$$

$y_{n+v} \; - y_{n+o}$

The first characteristics polynomial of the finite difference scheme is :

$\rho(r) = r^v - r^o = r^v - 1$

Setting $r = 1$, we have,

$\rho\,(1) = 1^v - 1 = 0$

$\rho'\,(r\,) = vr^{v-1}$

$V = ½$

$\rho'\,(r) \;= ½\, r^{-1/2}$

$\rho'\,(1) \;= ½\, .1^{-1/2} \;= ½$

Similarly, the second characteristics polynomial of the finite difference scheme is :

$\Upsilon\,(r\,) = \; 1/12\,[7r^u - 2r^v + r^w]$

Setting r = 1, we have,

$\Upsilon(1) = 1/12\,[7(1^{u}) - 2(1^{v}) + 1^{w}]$

$= 1/12\,(7 - 2 + 1)$

$= 6/12 = ½$

$\rho'(1) = ½$ and $\Upsilon(1) = ½$

Hence

$\rho'(1) = \Upsilon(1)$.

The consistency properties are satisfied.

Similarly for (***3.27b - 3.27d***) and these are summarized in the table (4.1b) as shown below:

Table (4.1b): <u>Constitency, Zero-stable, and Convergence of Method II</u>

Consistency					Zero stable	convergent
The scheme	$P \geq 1$	$\sum_{j=0}^{k} \alpha_j = 0$	$P'(1) = \Upsilon(1)$	$\rho(1) = 0$	One step, since it is consistent	(i) consistent (ii) zero-stable
(3.27a)	√	√	√	√	√	√
(3.27b)	√	√	√	√	√	√
(3.27c)	√	√	√	√	√	√
(3.27d)	√	√	√	√	√	√

Zero Stability

We apply section ***(2.5.2)*** of chapter two and it is shown in table (4.1b) above.

(C) CONVERGENCE

According to Dalhquist (1962) a Linear multistep methods is convergent if :

(i) it is consistent

(ii) it is zero-stable

(see section (2.3) chapter two)

Since the four discrete schemes are consistent and zero-stable, they are convergent (see table (4.1b) above.

CHAPTER FIVE

COMPUTER IMPLEMENTATION AND NUMERICAL RESULTS

In order to demonstrate the applicability and suitability of the new method, there is the need to translate the new numerical formula ***(3.26a)*** into computer codes. This will involve the writing of the formula ***(3.26a)*** in the computer algorithm called pseudo-code, and implement on computer adopting computer programming language.

There are some various types of computer languages available. These include FORTRAN, BASIC, PASCAL, CLIPPER, DBASE.

In this thesis, we considered the FORTRAN programming language as the mode of implementation of the new method.

To achieve this, we adopted the following steps.

(i) Re-write the formula in an algorithmic form.

(ii) Translate the algorithm into a computer flow chart

(iii) Translate the flow chart into computer code.

(iv) Implement the code with sample problems on a digital computer

(v) Discuss the results

5.1 COMPUTATIONAL ALGORITHM

A set of steps taken to obtain the solution of a given problem is the algorithm of that problem.

In this section, we develop the numerical algorithm for implementing the method (3.26a) described in chapter three, and step by step size control measures, the error estimate were given. The algorithm is given in Appendix 1.

5.2 PROGRAM FLOW CHART

A computer flow chart is a diagrammatical representation of the algorithm or the plan of solution of a problem. It indicates the process of solution, the relevant operations and computations, the point of decision and other information at a point of solution.

Flow charts are of particular interest because of its documentary features. They are constructed by using special geometrical symbols, such as squares, rectangles, diamonds shapes or circles. Each symbol represent some activities which could be input/output of data, taking a decision, terminating the solution process and so on. The symbols are joined by directed lines segments to indicate direction of low. The flow chart of above algorithms is given in Appendix 2.

5.3 PROGRAMMING IMPLEMENTATION

The implementation is done in a variable step size fixed order method.

The flow chart and algorithm (see Appendix 1 and 2) were implemented for the program for the computational purposes in FORTRAN 77 language, in a mini-computer. The output were in double precision code.

5.4 NUMERICAL COMPUTATIONS AND RESULTS

To justify the desirability and the applicability of the derived methods, we have solved some examples and obtained the results as in table below.

TABLE 5.4.1: Examples and Results

EXAMPLE 1

$y^1 = 5y,\ y(0) = 1,\ 0 \le x \le 0.5$

$y(x) = e^{5x}$

h = 0.1

X	*EXACT*	*YC*	*EC*
0.1000	0.1648721271D+01	0.1648437500D+01	0.2837707001D-03
0.2000	0.2718281828D+01	0.271305385D+01	0.5228443042D-02
0.3000	0.4481689070D+01	0.4465233697D+01	0.1645537351D-01
0.4000	0.7389056099D+01	0.7349030459D+01	0.4002563956D-01
0.5000	0.1218249396D+02	0.1209527930D+02	0.8721466299D-01

h = 0.05

X	*EXACT*	*YC*	*EC*
0.1000	0.1648721271D+01	0.1648490482D+01	0.2307887937D-03
0.2000	0.2718281828D+01	0.2717176399D+01	0.1105429147D-02
0.3000	0.4481689070D+01	0.4478671649D+01	0.3017421329D-02
0.4000	0.7389056099D+01	0.7382111719D+01	0.6944380045D-02
0.5000	0.1218249396D+02	0.1216779833D+02	0.1469562653D-01

h = 0.025

X	*EXACT*	*YC*	*EC*
0.1000	0.1648721271D+01	0.1648675357D+01	0.4591387295D-04
0.2000	0.2718281828D+01	0.2718106031D+01	0.1757975177D-03
0.3000	0.4481689070D+01	0.4481234201D+01	0.4548692896D-03
0.4000	0.7389056099D+01	0.7388034071D+01	0.1022028129D-02
0.5000	0.1218249396D+02	0.1218036036D+02	0.2133599588D-02

h = 0.0125

X	EXACT	YC	EC
0.1000	0.1648721271D+01	0.1648714278D+01	0.6992532731D-05
0.2000	0.2718281828D+01	0.2718257148D+01	0.2468094070D-04
0.3000	0.4481689070D+01	0.4481626694D+01	0.6237620424D-04
0.4000	0.7389056099D+01	0.7388917507D+01	0.1385920239D-03
0.5000	0.1218249396D+02	0.1218220652D+02	0.2874428476D-03

EXAMPLE II

$y^1 = 3x^2y$ $y(0) = 1$, $0 \le x \le 0.5$

h = 0.1 *y(x) = exp(x³)*

X	EXACT	YC	EC
0.1000	0.1001000500D+01	0.1057838428D+01	0.5683792757D-01
0.2000	0.1008032086D+01	0.1045397948D+01	0.3736586200D-01
0.3000	0.1027367803D+01	0.1104497128D+01	0.7712932481D-01
0.4000	0.1066092399D+01	0.1126484565D+01	0.6039216672D-01
0.5000	0.1133148453D+01	0.1170051496D+01	0.3690304318D-01

h = 0.05

X	EXACT	YC	EC
0.1000	0.1001000500D+01	0.1022211365D+01	0.2121086504D-01
0.2000	0.1008032086D+01	0.1051101145D+01	0.4306905936D-01
0.3000	0.1027367803D+01	0.1064210270D+01	0.3684246703D-01
0.4000	0.1066092399D+01	0.1093837822D+01	0.2774542284D-01
0.5000	0.1133148453D+01	0.1148222811D+01	0.1507435808D-01

h = 0.025

X	EXACT	YC	EC
0.1000	0.1001000500D+01	0.1024168649D+01	0.2316814837D-01
0.2000	0.1008032086D+01	0.1029261692D+01	0.2122960606D-01
0.3000	0.1027367803D+01	0.1045316121D+01	0.1794831775D-01
0.4000	0.1066092399D+01	0.1079301280D+01	0.1320888094D-01
0.5000	0.1133148453D+01	0.1139768424D+01	0.6619971188D-02

h = 0.0125

X	*EXACT*	*YC*	*EC*
0.1000	0.1001000500D+01	0.1012545202D+01	0.1154470186D-01
0.2000	0.1008032086D+01	0.1018565478D+01	0.1053339264D-01
0.3000	0.1027367803D+01	0.1036218136D+01	0.8850333703D-02
0.4000	0.1066092399D+01	0.1072523675D+01	0.6431276285D-02
0.5000	0.1133148453D+01	0.1136218360D+01	0.3069906886D-02

h = 0.00625

X	*EXACT*	*YC*	*EC*
0.1000	0.1001000500D+01	0.1006762333D+01	0.5761832426D-02
0.2000	0.1008032086D+01	0.1013277776D+01	0.5202356214D-02
0.3000	0.1027367803D+01	0.1031761284D+01	0.4393481636D-02
0.4000	0.1066092399D+01	0.1069263777D+01	0.3171377816D-02
0.5000	0.1133148453D+01	0.1134621905D+01	0.1473451678D-02

EXAMPLE III

$y^{1} = x + y, \ y(0) = 1, 0 \leq x \leq 0.5$

$Y(x) = 2e^{x} - x - 1$

h = 0.1

X	*EXACT*	*YC*	*EC*
0.10000	0.1110341836D+01	0.1138675000D+01	0.2833316385D-01
0.20000	0.1242805516D+01	0.1263212599D+01	0.2040708229D-01
0.30000	0.1399717615D+01	0.1430946637D+01	0.3122902230D-01
0.40000	0.1583649395D+01	0.1604864734D+01	0.2121533854D-01
0.50000	0.1797442541D+01	0.1808641550D+01	0.1119900860D-01

h = 0.05

X	*EXACT*	*YC*	*EC*
0.10000	0.1110341836D+01	0.1124299611D+01	0.1395777475D-01
0.20000	0.1242805516D+01	0.1262178677D+01	0.1937316023D-01
0.30000	0.1399717615D+01	0.1414089231D+01	0.1437161572D-01
0.40000	0.1583649395D+01	0.1593019162D+01	0.1969766696D-02
0.50000	0.1797442541D+01	0.1801810108D+01	0.4367566176D-02

h = 0.025

X	*EXACT*	*YC*	*EC*
0.10000	0.1110341836D+01	0.1122216730D+01	0.1187489403D-01
0.20000	0.1242805516D+01	0.1252180243D+01	0.9374727000D-02
0.30000	0.1399717615D+01	0.1406592141D+01	0.6874526004D-02
0.40000	0.1583649395D+01	0.1588023681D+01	0.4374285750D-02
0.50000	0.1797442541D+01	0.1799316542D+01	0.1874000204D-02

h = 0.0125

X	*EXACT*	*YC*	*EC*
0.10000	0.1110341836D+01	0.1116201196D+01	0.5859359412D-02
0.20000	0.1242805516D+01	0.1247414854D+01	0.4609338090D-02
0.30000	0.1399717615D+01	0.1403076928D+01	0.3359312457D-02
0.40000	0.1583649395D+01	0.1585758677D+01	0.2109281840D-02
0.50000	0.1797442541D+01	0.1798301787D+01	0.8592454742D-03

h = 0.00625

X	*EXACT*	*YC*	*EC*
0.10000	0.1110341836D+01	0.1113251990D+01	0.2910154152D-02
0.20000	0.1242805516D+01	0.1245090668D+01	0.2285151459D-02
0.30000	0.1399717615D+01	0.1401377763D+01	0.1660148223D-02
0.40000	0.1583649395D+01	0.1584684540D+01	0.1035144359D-02
0.50000	0.1797442541D+01	0.179785268ID+01	0.4101397703D-03

EXAMPLE IV

$y^1 = -y/(2+2x)$, $y(0) = 1$, $0 \le x \le 0.5$

h = 0.1 $y(x) = (1+x)^{1/2}$

X	EXACT	YC	EC
0.1000	0.9534625892D+00	0.9614488664D+00	0.7986277176D-02
0.2000	0.9128709292D+00	0.9185629991D+00	0.5692069939D-02
0.3000	0.8770580193D+00	0.8842043900D+00	0.7146370728D-02
0.4000	0.8451542547D+00	0.8494636103D+00	0.4309355583D-02
0.5000	0.8164965809D+00	0.8185214268D+00	0.2024845868D-02

h = 0.05

X	EXACT	YC	EC
0.1000	0.9534625892D+00	0.9578786115D+00	0.4416022232D-02
0.2000	0.9128709292D+00	0.9177396196D+00	0.4868690398D-02
0.3000	0.8770580193D+00	0.8802514152D+00	0.3193395928D-02
0.4000	0.8451542547D+00	0.8470114147D+00	0.1857159967D-02
0.5000	0.8164965809D+00	0.8172734349D+00	0.7768539739D-03

h = 0.025

X	EXACT	YC	EC
0.1000	0.9534625892D+00	0.9568138288D+00	0.3351239514D-02
0.2000	0.9128709292D+00	0.9151892842D+00	0.2318355004D-02
0.3000	0.8770580193D+00	0.8785636890D+00	0.1505669655D-02
0.4000	0.8451542547D+00	0.8460104071D+00	0.8561523331D-03
0.5000	0.8164965809D+00	0.8168267226D+00	0.3301416495D-03

h = 0.0125

X	EXACT	YC	EC
0.1000	0.9534625892D+00	0.9551018173D+00	0.1639228008D-02
0.2000	0.9128709292D+00	0.9140018261D+00	0.1130896904D-02
0.3000	0.8770580193D+00	0.8777885078D+00	0.7304885097D-03
0.4000	0.8451542547D+00	0.8455644272D+00	0.4101724798D-03
0.5000	0.8164965809D+00	0.8166471398D+00	0.1505588484D-03

h = 0.00625

X	*EXACT*	*YC*	*EC*
0.1000	0.9534625892D+00	0.9542732408D+00	0.8106515665D-03
0.2000	0.9128709292D+00	0.9134293901D+00	0.5584609448D-03
0.3000	0.8770580193D+00	0.8774177245D+00	0.3597051835D-03
0.4000	0.8451542547D+00	0.8453548886D+00	0.2006338424D-03
0.5000	0.8164965809D+00	0.8165682392D+00	0.7165822396D-04

A look at the examples show that, with decreasing h, ($h \to 0$) the numerical values are moving closer and closer to the corresponding exact solutions, i.e. the method converges.

It is also noted that in all examples the appropriate solutions are approaching the exact solutions. This shows the convergence of the new method.

5.5 COMPARING TWO OF THE NUMERICAL EXAMPLES WITH PREVIOUS METHODS.

We compare two of the numerical results with Butcher's 4th order method, as shown below.

Errors of numerical solutions fro example II and example III with h = 0.1 are as below:

Example II, $y^1 = 3x^2y$, $y(0) = 1$, $0 \le x \le 0.5$

$y(x) = \exp(x^3)$

Table 5.4.2: Comparing Results on Example II.

Mesh values (x)	Butcher's 4^{th} order method	Our new 4^{th} order method
0.1	-5.9997×10^{-4}	0.0568
0.1	-1.8107×10^{-3}	0.0347
0.3	-3.6776×10^{-2}	0.07713
0.4	-6.3088×10^{-2}	0.0604
0.5	-9.9115×10^{-2}	0.0369

Example III

$y^1 = x + y$, $y(0) = 1$, $0 \le x \le 0.5$

$y(x) = 2e^x - x - 1$,

Table 5.4.3: Comparing Results on Example III.

Mesh values (x)	Butcher's 4^{th} order method	Our new 4^{th} order method
0.1	-0.04016870	0.02833316
0.1	0.019151047	0.02040708
0.3	0.01845324	0.03122902
0.4	0.026395160	0.02121539
0.5	0.040799140	0.01119901

From table ***(5.4.2)*** and ***(5.4.3),*** as the number of mesh points increases our new 4^{th} order methoc based on hybrid collocation method produces a better results than Butcher's 4^{th} order method. And our new method converges faster than Butchers 4^{th} Order Method.

Our new method, also based on one-step collocation methods, has it approximate solutions approaching the exact ones from one side i.e. the errors are all positive or negative. Therefore one-step collocation in nature has this added advantage to studying and solving many practical problems.

On the computer implementation, our new one-step Hybrid collocation method ***(3.26a)*** is simple, and it is easier to program than butcher's or Yakubu's 4^{th} and 6^{th} order, Ruge-Kutta method respectively.

NOTE:

The method (3.26a) when implemented on a mini-computer has the same accuracy at

U = ¼, v = ½ , w = ¾ See (3.27a)

And at Gaussian points, when,

$u = \frac{(5-\sqrt{15})}{10}$ $v = 1/2$ $w = \frac{(5+\sqrt{15})}{10}$ see (3.28a)

CHAPTER SIX

GENERAL CONCLUSION

6.1 SUMMARY

In this thesis, we have developed two hybrid method based on continuous collocation method, for solving first order differential equations.

They were analyzed, one of it, (method **(3.26a**), was computerized and implemented with some sample problems on a micro computer.

The results show that the method is capable of solving first order ordinary differential equations of both non stiff and moderately stiff initial value problems.

6.2 LIMITATIONS

Since the method were based on Taylor series expansions, they are subject to point to point error and possible error propagation.

Also, several iteration are involve with difficulties in handling them moderately in some stiff initial value problems. Hence the method will be restricted to non stiff and moderately stiff initial value problems.

6.3 RECOMMENDATIONS

Based on the limitation, the study of the one-step collocation Hybrid methods developed in relationship to the existing one-step methods will lead to greater interest in continuous solution of equations, hence higher order and better efficient collocation points should be exploited for more useful general purpose code for solution of ordinary differential equations.

6.4 CONTRIBUTION TO KNOWLEDGE

The methods are suitable for solving first order differential equations for both non stiff and moderately stiff initial value problems.

There are indications of improved accuracy over 4^{th} order Butcher's method with decreasing h.

The new methods are simple and easy to implement.

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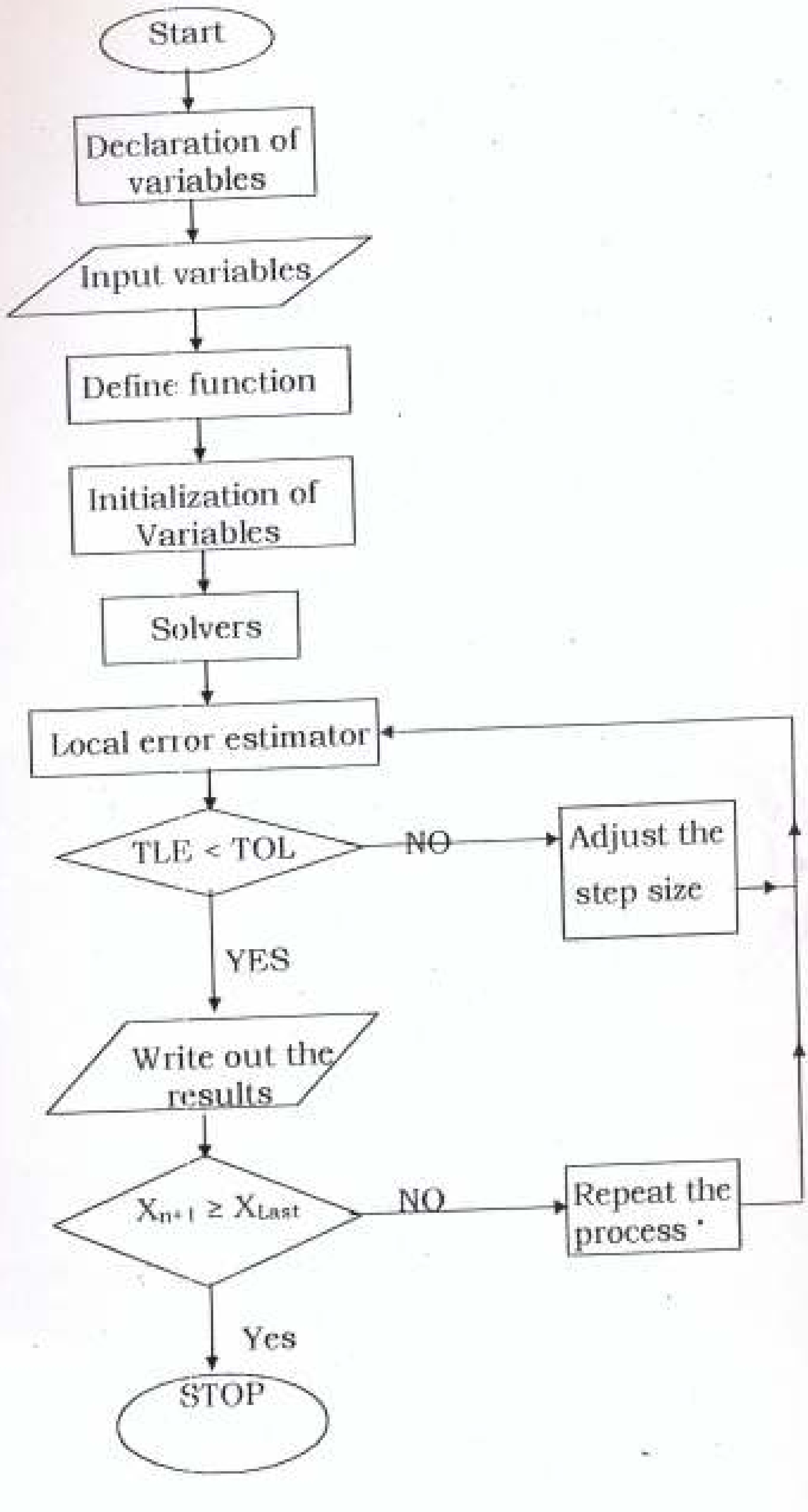
APPENDIX 1

```
NAME OF FILE: EDAM1.FOR
SOLUTION OF FIRST ORDER INITIAL VALUE PROBLEMS Y'=F(X,Y)
BY A FAMILY OF ONE-STEP SYMMETRIC HYBRID METHODS
IMPLICIT DOUBLE PRECISION(A-H,O-Z)
DIMENSION YN1C(80,80),YEX(80,80),ERC(80,80),TT(80,80)
F(X,Y)=5.D0*Y
Y(X)=DEXP(5.D0*X)
OPEN(6,FILE='EDAM1.OUT')
N=80
NSTEP=80
A=0.D0
H=0.00625D0
B=0.00625D0
DX=H/FLOAT(N)
D=.5D0
XN=A
YN=1.D0
U=0.25D0
V=0.5D0
W=0.75D0
XN1=XN+H
XNU=XN+0.25D0*U
XNV=XN+0.5D0*V
XNW=XN+0.75D0*W
WRITE(6,5)
FORMAT(8X,'X',12X,'EXACT',20X,'YC ,20X,'EC',22X,'YD',22X,'ED'/)
CALCULATE PREDICTOR
DO 1 I=1,N
CALCULATE FP
FF=F(XN,YN)
DFX=0.D0
DFY=5.D0
FP=DFX+FF*DFY
DFXX=0.D0
DFXY=0.D0
DFYY=0.D0
FPP=DFXX+2.D0*FF*DFXY+FF*FF*DFYY+DFX*DFY+FF*DFY*DFY
YN1=YN+H*FF+((H*H)/2.D0)*FP+((H**3)/6.D0)*FPP
F1=F(XN1,YN1)
YNU=YN+U*H*FF+U*U*H*H*FP/2.D0+((U*H)**3)*FPP/6.D0
FU=F(XNU,YNU)
YNV=YN+V*H*FF+V*V*H*H*FP/2.D0+((V*H)**3)*FPP/6.D0
FV=F(XNV,YNV)
YNW=YN+W*H*FF+W*W*H*H*FP/2.D0+((W*H)**3)*FPP/6.D0
FW=F(XNW,YNW)
CALCULATE COEFFICIENTS OF CONTINUOUS METHOD
DO 2 J=1,NSTEP
TT(I,J)=XN+DX*FLOAT(J)
X=TT(I,J)
P=X-XN
A1=P/(6.D0*(V-U)*(W-U)*H*H)
A2=6.D0*H*H*V*W-P*(3.D0*H*W+3.D0*H*V-2.D0*P)
B1=A1*A2
A3=P/(6.D0*H*H*(V-U)*(W-V))
A4=-6.D0*H*H*U*W+P*(3.D0*H*W+3.D0*H*U-2.D0*P)
B2=A3*A4
A5=P/(6.D0*H*H*(W-U)*(W-V))
A6=6.D0*H*H*U*V-P*(3.D0*H*V+3.D0*H*U-2.D0*P)
B3=A5*A6
YN1C(I,J)=YN+B1*FU+B2*FV+B3*FW
YC=YN1C(I,J)
CALCULATE EXACT SOLUTION AND ERROR OF THE METHOD
IF(X.GE.B) THEN
YEX(I,J)=Y(X)
```

```
YE=YEX(I,J)
ERC(I,J)=DABS(YN1C(I,J)-YEX(I,J))
ER=ERC(I,J)
WRITE(6,10)X,YE,YC,ER
FORMAT(1X,F8.5,3X,3D20.10)
CHANGE  VARIABLES
XN=XN1
YN=YN1
XNU=XNV
YNU=YNV
XNV=XNW
YNV=YNW
XN1=XN1+H
ELSE
GO TO 2
ENDIF
IF(B.GE.D) GO TO 1
B=B+H
GO TO 1
CONTINUE
CONTINUE
STOP
END
```

APPENDIX 2


Start
Declaration of variables
Input variables
Define function
Initialization of Variables
Solvers
Local error estimator
TLE < TOL
NO
Adjust the step size
YES
Write out the results
X n+1 ≥ X Last
NO
Repeat the process
Yes
STOP